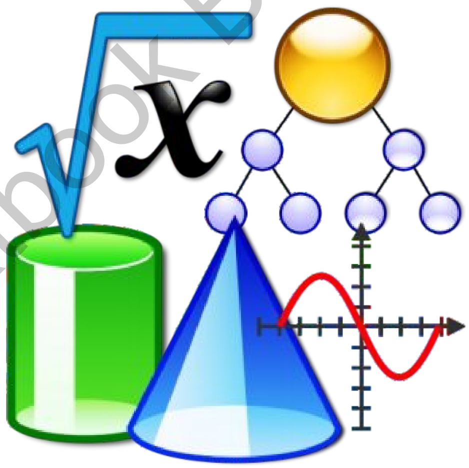
TEST EDITION

THE TEXTBOOK OF

MATHEMATICS

For Class - X

SHAHEED BENAZIR BHUTTO FREE TEXTBOOKS SERIES



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PREFACE

The Sindh Textbook Board, is assigned with preparation and publication of the textbooks to equip our new generation with knowledge, skills and ability to face the challenges of new millennium in the fields of Science, Technology and Humanities. The textbooks are also aimed at inculcating the ingredients of universal brotherhood and to reflect the valiant deeds of our forebears and portray the illuminating patterns of our rich cultural heritage and tradition.

The textbook, in your hand, of Mathematics for class X has been developed according to provincial curriculum 2017 in the continuity of Mathematics for class IX. The curriculum contains 30 units, first 16 units were included in Mathematics for class IX (previous year). Hence, remaining units (from 17 to 30) are included in this textbook of Mathematics for class X .

The Sindh Textbook Board has taken great pains and incurred expenditure in publishing this book inspite of its limitations. A textbook is indeed not the last word and there is always room for improvement. While the authors have tried their level best to make the most suitable presentation, both in terms of concept and treatment, there may still have some deficiencies and omissions. Learned teachers and worthy students are, therefore, requested to be kind enough to point out the shortcomings of the text or diagrams and to communicate their suggestions and objections for the improvement of the next edition of this book.

In the end, I am thankful to our learned authors, editors and specialist of Board for their relentless service rendered for the cause of education.

Chairman
Sindh Textbook Board

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SETS AND FUNCTIONS

Unit 17

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Recall the sets denoted by $\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{E}, \mathbb{O}, \mathbb{P}, \mathbb{Q}$ and $\mathbb{R}$
- Type of sets and representation of sets
- Operations on sets
  - Union,
  - Intersection,
  - Difference,
  - Complement.
- Symmetric difference of two sets
- Give formal proof of the following fundamentalproperties of union and intersection of two or three sets.
  - Commutative property of union,
  - Commutative property of intersection,
  - Associative property of union,
  - Associative property of intersection,
  - Distributive property of union over intersection,
  - Distributive property of intersection over union,
  - De Morgan's laws.
- Verify the fundamental property of given sets.
- Use Venn diagrams to represent
  - Union and intersection of sets,
  - Complement of a set.
  - Symmetric difference of two sets.
- Use Venn diagram to verify
  - Commutative property of union overintersection of sets,
  - De Morgan's laws,
  - Associative laws,
- Distributive laws.
- Recognize order pairs.
- To form Cartesian products.
- Define a binary relation and identify its domain and range.
- Define a function and identify its domain, co-domain and range.
- Demonstrate the following:
  - Into and one-one function (injective function),
  - Onto function (subjective function),
  - One-one and onto function (bijective function).
- Examine whether a given relation is a function ornot.
- Differentiate between one-one correspondence andone-one function.

17.1 Operations on Sets:

17.1(i) Recall the sets denoted by $\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{E}, \mathbb{O}, \mathbb{P}, \mathbb{Q}$ and $\mathbb{R}$

As a matter of fact, set is one of the fundamental concepts of Mathematics which is useful in formulating and analyzing many mathematical notions. The concept of set, in detail, has already been discussed in previous classes. Let us recall some important sets.

Set of Natural Numbers: $\mathbb{N} = \{1,2,3,...\}$

Set of Whole Numbers: $\mathbb{W} = \{0,1,2,3,...\}$

Set of Integers: $\mathbb{Z} = \{0,\pm1,\pm2,\pm3,...\}$

Set of Even Numbers: $\mathbb{E} = \{0,\pm2,\pm4,\pm6,...\}$

Set of Odd Numbers: $\mathbb{O} = \{\pm1,\pm3,\pm5,...\}$

Set of Prime Numbers: $\mathbb{P} = \{2,3,5,7,...\}$

Set of Rational Numbers: $\mathbb{Q} = \left\{x \mid x = \frac{p}{q} \text{ where } p,q \in Z \wedge q \neq 0\right\}$

Set of Irrational Numbers: $\mathbb{Q}' = \left\{x \mid x \neq \frac{p}{q} \text{ where } p,q \in Z \wedge q \neq 0\right\}$

Set of Real Numbers: $\mathbb{R} = \left\{x \mid x = \frac{p}{q} \vee x \neq \frac{p}{q} \text{ where } p,q \in Z \wedge q \neq 0\right\}$

i.e. $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$

Note:
- The above sets can also be represented by N,W,Z,E,O,P,Q and R.
- $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the set of positive and negative real numbers respectively.
- The set of all rational, irrational and real numbers cannot be written in tabular form

17.1 (ii) Types of sets and representation of sets

Firstly, we discuss the representation of sets. We are already familiar with the three methods or forms of representing a set which we studied in previous classes. These are

1. Descriptive form.
2. Tabular form or Roster form.
3. Set Builder form.

Recall that, in descriptive form, a set is described by common characteristics of its elements in any common language, for instance

A = Set of natural numbers between 5 and 10.

In tabular form, a set is described by listing its elements within braces.

The above set is written in tabular form as $A = \{6,7,8,9\}$ while in set builder form a set is described by common characteristics of its elements using symbols.

The given set A can be written in set builder form as $A = \{x \mid x \in \mathbb{N} \wedge 5 < x < 10\}$

Example 1: Write the set $A=\{6,8,10,12\}$ in descriptive and set builder form.

Solution:

Descriptive form: A = Set of even numbers between 5 and 13

Set builder form: $A=\{x \mid x \in \mathbb{E} \wedge 5<x<13\}$

Example 2: Write the set $B=\{y \mid y \in \mathbb{P} \wedge y<10\}$ in tabular and descriptive form.

Solution:

Tabular form: $B=\{2,3,5,7\}$

Descriptive form: B = Set of first four prime numbers

Now, we discuss some types of sets.

Empty set or Null set:

A set having no element is called an empty or null set. It is denoted by $\varnothing$ or $\{\ \}$.

For example: (i) A = Set of even numbers between 5 and 6.

(ii) $B=\{x \mid x \in \mathbb{N} \wedge x<1\}$

Finite Set:

A set having limited number of elements is called a finite set.

For example: (i) $A=\{10,12,14,...,50\}$

(ii) B = Set of all countries of the world

Remember that the empty set is considered as a finite set.

Infinite Set:

A set having unlimited number of elements is called an infinite set.

For example: (i) $P=\{10,20,30,...\}$

(ii) $Q=\{x \mid x \in \mathbb{Q} \wedge 1 \leq x \leq 2\}$

Subset:

If every element of set A is also an element of set B then A is called a subset of B. Symbolically we write as $A \subseteq B$.

For example: If $A=\{2,3,5,7\}$, $B=\{1,2,3,4,5,6,7\}$ and $C=\{6,7,8,9\}$

then $A \subseteq B$ but $A \not\subseteq C$ (A is not subset of C)

Note: (i) Every set is subset of itself.
(ii) Empty set is a subset of every set.
(iii) Every non-empty set has at least two subsets.
(iv) Number of all subsets of a set of n elements is 2^n.

Superset:

If set A is subset of set B then B is called superset of A.

Symbolically we write as $B \supseteq A$

For example: If $X=\{a,e,i,o,u\}$ and $Y=\{a,b,c,...,z\}$ then $Y \supseteq X$

Equal Sets:

Two sets A and B of same order are called equal sets if all the elements of both the sets are same. Symbolically we write as $A = B$.

Thus $A = B$ iff $A \subseteq B$ and $B \subseteq A$ also $A = B$ iff $A \supseteq B$ and $B \supseteq A$

Example: Let $A = \{1,2,3,6\}, B = \{x \mid x \in N \wedge x \leq 6\}$ and C = Set of divisions of 6

Here $A = C$ but $A \neq B$

Note: Order of set A means the number of elements of A. It is denoted by O(A) or n(A) or |A|.

Equivalent Sets:

Two sets A and B are said to be equivalent sets if their orders are equal. Symbolically, we write as $A \sim B$, i.e., $A \sim B$ iff $O(A) = O(B)$

Thus, if $A \sim B$ then one-one correspondence between their elements can be established.

For example:

Let $A = \{x \mid x \in \mathbb{R} \wedge x^2 = 64\}$ and B = Set of prime numbers less than 5.

Here $A \sim B$ because $O(A) = O(B) = 2$

Proper Subset:

Let set A is a subset of set B. The set A is called proper subset of B if $A \neq B$. Symbolically we write as $A \subset B$.

For example: If $A = \{2,4,6\}$ and $B = \{2,4,6,8\}$

then $A \subset B$ because A is subset of B and $A \neq B$.

Improper Subset:

Let set A is a subset of set B. The set A is called improper subset of B iff $A = B$

For example: If $A = \{a,b,c\}$ and B = Set of first three letters of English alphabet

then A is improper subset of B. i.e. $A = B$

Power Set:

The set of all the subsets of set A is called power set of A. It is denoted as P(A).

For example:

If $A = \{x,y,z\}$ then

$$P(A) = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{y,z\}, \{x,z\}, \{x,y,z\}\}$$

Note: Power set of an empty set is non-empty.

Singleton or Unit Set:

A set having single element is called singleton or unit set.

For example: $A = \{x \mid x \in W \wedge x < 1\}$ is singleton.

Universal Set:

The superset of all the sets under consideration is called universal set. It is denoted by U or X.

For example: If $A = \{1,2,3,4\}$ and $B = \{2,4,6\}$ then $U = \{1,2,3,4,5,6\}$

Further types of sets will be discussed in section 17.1(iv).

Exercise 17.1

1. **Write the following sets in tabular form.**

(i) A = Set of all integers between -3 and 3

(ii) B = Set of composite numbers less than 11.

(iii) $C=\{x \mid x \in \mathbb{P} \wedge 5<x \leq 13\}$

(iv) $D=\{y \mid y \in \mathbb{O} \wedge 7<y<17\}$

(v) $E=\{z \mid z \in \mathbb{R} \wedge z^2=121\}$

(vi) $F=\{p \mid p \in \mathbb{Q} \wedge p^2=-1\}$

2. **Write the following sets in set builder form.**

(i) A = Set of all rational numbers between 5 and 6 (ii) $B=\{1,2,3,4,6,12\}$

(iii) $C=\{0, \pm 1, \pm 2, \ldots, \pm 40\}$ (iv) $D=\{-4,-2,0,2,4\}$

(v) $E=\{1,4,9,16,25\}$ (vi) $F=\{-1,-3,-5,-7, \ldots\}$

3. **Write any five examples of empty set.**

4. **Classify the following as finite and infinite sets.**

(i) Set of Asian countries.

(ii) Set of all the medical universities of the world.

(iii) Set of all real numbers between 6 and 9.

(iv) Set of all the even prime numbers.

(v) Set of all odd numbers less than 5.

5. **Write an equivalent set, an improper subset and three proper subsets of each of the following sets.**

(i) $P=\{a, e, i, o, u\}$

(ii) $Q=\{x \mid x \in \mathbb{Z} \wedge -2 \leq x \leq 2\}$

6. **Write any two examples of singleton in set builder form.**

7. **Write power set of the following sets.**

(i) $A=\{5,10,15\}$

(ii) $B=\{x \mid x \in Z \wedge -1<x<4\}$

8. **Find a set which has only**

(i) two proper subsets.

(ii) one proper subset.

(iii) no proper subset.

17.1 (iii) Operations on sets

- Union
- Intersection
- Difference
- Complement

Union of two sets

The union of two sets X and Y, denoted as $X \cup Y$, is the set of all those elements which belong to X or to Y or to both X and Y.

i.e., $X \cup Y = \{x \mid x \in X \vee x \in Y\}$

For example: If $X = \{1,3,5\}$ and $Y = \{1,2,3,4\}$ then $X \cup Y = \{1,2,3,4,5\}$

Intersection of two sets

The intersection of two sets X and Y, denoted as $X \cap Y$, is the set of all those elements which belong to both X and Y.

i.e., $X \cap Y = \{x \mid x \in X \wedge x \in Y\}$

For example: If $X = \{2,4,6,8\}$ and $Y = \{1,2,3,6\}$ then $X \cap Y = \{2,6\}$

Difference of two sets

For any two sets X and Y, the difference X–Y is the set of all the elements which belong to X but do not belong to Y. It is also denoted as X\Y.

i.e., $X - Y = \{x \mid x \in X \wedge x \notin Y\}$

Similarly, the difference $Y - X$ is the set of all the elements which belong to Y but do not belong to X. It is also denoted as Y\X. i.e., $Y - X = \{y \mid y \in Y \wedge y \notin X\}$

For example:

If $X = \{4,6,8,9,10\}$ and $Y = \{2,4,6,8\}$

then $X - Y = \{9,10\}$

and $Y - X = \{2\}$

Complement of a set

If set A is a subset of universal set U then the complement of A is the set of all the elements of U which are not in A. It is denoted by A' or A^c.

Thus $A' = U - A$

i.e., $A' = \{x \mid x \in U \wedge x \notin A\}$

For example: If $U = \{1,2,3,...,20\}$ and $A = \{1,3,5,...,19\}$ then $A' = \{2,4,6,...,20\}$

17.1 (iv) Symmetric difference of two sets

The symmetric difference of two sets A and B, denoted $A \Delta B$, is the set of all the elements of A or B which are not common in both the sets.

i.e., $A \Delta B = \{x \mid x \in A \cup B \wedge x \notin A \cap B\}$

or $A \Delta B = (A \cup B) - (A \cap B)$

For example: If $A=\{1,2,3,4,5\}$ and $B=\{1,3,5,7\}$ then $A\Delta B=\{2,4,7\}$

In continuation of section 17.1(ii), further types of sets are given below:

Disjoint Sets:

Two sets A and B are called disjoint sets if they have no element common

i.e., Two sets A and B are disjoint sets if $A\cap B=\varnothing$

For example: The sets $A=\{1,3,5\}$ and $B=\{2,4,6\}$ are disjoint sets.

Overlapping Sets:

Two sets A and B are called overlapping sets if there is at least one element common in both. Moreover neither of them is subset of other.

i.e., Two sets A and B are overlapping sets if $A\cap B\neq\varnothing$ and $A\not\subseteq B$ or $B\not\subseteq A$

For example: The sets $A=\{1,2,3,4\}$ and $B=\{2,4,6,8\}$ are overlapping sets.

Exhaustive Sets:

If two sets A and B are subsets of universal set U then A and B are called exhaustive sets if $A\cup B=U$.

For example: If $A=\{1,2,3,4\}$, $B=\{1,3,5\}$ and $U=\{1,2,3,4,5\}$ then A and B are exhaustive sets because $A\cup B=U$

Cells:

If A and B are two non-empty subsets of universal set U then A and B are called cells if they are disjoint as well as exhaustive sets.

i.e., A and B are called cells if A and B are non-empty subsets of U and

$A\cap B=\varnothing$

Also $A\cup B=U$

For example: If $A=\{1,3,5,7,9\}$, $B=\{2,4,6,8,10\}$ and $U=\{1,2,3,...,10\}$ then A and B are cells.

Some important laws related to the operations on sets are as under.

Identity Laws:

For any set A

(i) $A\cup\varnothing=A$ (ii) $A\cup U=U$ (iii) $A\cap U=A$ (iv) $A\cap\varnothing=\varnothing$

Idempotent Laws:

For any set A

(i) $A\cup A=A$ (ii) $A\cap A=A$

Laws of the Complement:

For any set A

(i) $A\cup A'=U$ (ii) $A\cap A'=\varnothing$

(iii) $(A')'=A$ (iv) $U'=\varnothing$ and $\varnothing'=U$

De Morgan's Laws:

For any two sets A and B.

(i) $(A\cup B)'=A'\cap B'$ (ii) $(A\cap B)'=A'\cup B'$

Exercise 17.2

1. **Which of the following sets are disjoint, overlapping, exhaustive and cells.**

(i) $\{1,2,3,5,7\}$ and $\{4,6,8,9,10\}$

(ii) $\{1,2,3,6\}$ and $\{1,2,4,8\}$

(iii) E and O when $U = Z$ (U denotes universal set)

(iv) N and W when $U = \{x \mid x \in Z \wedge x \geq 0\}$

(v) Q and Q' when $U = \mathbb{R}$

2. **If $A = \{1,2,3,4,5,6\}$ and $B = \{2,4,6,8,10\}$ then find**

(i) $A \cup B$ (ii) $B \cap A$ (iii) $A - B$
(iv) $B - A$ (v) $A \Delta B$

3. **If $U = \{1,2,3,...,10\}$, $A = \{1,2,3,4,5\}$ and $B = \{1,3,5,7,9\}$ then find:**

(i) A' (ii) B' (iii) $A' \cup B'$ (iv) $A' \cap B'$
(v) $(A \cup B)'$ (vi) $(A \cap B)'$ (vii) $A' \Delta B'$ (viii) $(A \Delta B)'$
(ix) $A - B'$ (x) $A' - B$

4. **If $U = \{x \mid x \in Z \wedge -4 < x < 6\}$, $P = \{p \mid p \in E \wedge -4 < p < 6\}$**
$Q = \{q \mid q \in P \wedge q < 6\}$ then show that:

(i) $P - Q = P \cap Q'$ (ii) $Q - P = Q \cap P'$
(iii) $(P \cup Q)' = P' \cap Q'$ (iv) $(P \cap Q)' = P' \cup Q'$

5. **If $A = \{2n \mid n \in N\}$, $B = \{3n \mid n \in N\}$ and $C = \{4n \mid n \in N\}$ then find:**

(i) $A \cap B$ (ii) $A \cup C$ (iii) $B \cap C$

17.1.2(i) Properties of Union and Intersection.

Give formal proof of the following fundamental properties of union and intersection of two or three sets.

➢ **Commutative Property of Union**

We know that for any two sets A and B, $\boxed{A \cup B = B \cup A}$

This property is called commutative property of union.

Proof:

L.H.S $= A \cup B$

$= \{x \mid x \in A \text{ or } x \in B\}$ (By definition of union)

$= \{x \mid x \in B \text{ or } x \in A\}$ $\because$ Order of elements in a set is not preserved

$= B \cup A$ (By definition of union)

$=$ R.H.S

$\because$ L.H.S = R.H.S

$\therefore$ $A \cup B = B \cup A$

Hence proved.

➤ Commutative Property of Intersection

We know that for any two sets A and B, $\boxed{A \cap B = B \cap A}$

This property is called commutative property of intersection.

Proof:

L.H.S $= A \cap B$

$= \{x \mid x \in A \text{ and } x \in B\}$ (By definition of intersection)

$= \{x \mid x \in B \text{ and } x \in A\}$ $\because$ Order of elements in a set is not preserved

$= B \cap A$ (By definition of intersection)

$=$ R.H.S

$\because$ L.H.S $=$ R.H.S

$\therefore$ $A \cap B = B \cap A$

Hence proved

➤ Associative property of union

We are already familiar with associative property of union which is as follows.

For any three sets A, B and C, $\boxed{A \cup (B \cup C) = (A \cup B) \cup C}$

Proof:

L.H.S $= A \cup (B \cup C)$

$= \{x \mid x \in A \text{ or } x \in B \cup C\}$ (By definition of union)

$= \{x \mid x \in A \text{ or } x \in B \text{ or } x \in C\}$

$= \{x \mid x \in A \cup B \text{ or } x \in C\}$ (By definition of union)

$= (A \cup B) \cup C$ (By definition of union)

$=$ R.H.S

$\because$ L.H.S $=$ R.H.S

$\therefore$ $A \cup (B \cup C) = (A \cup B) \cup C$

Hence proved.

➤ Associative property of intersection

We already know the associative property of intersection which states that, for any three sets A,B and C, $\boxed{A \cap (B \cap C) = (A \cap B) \cap C}$

Proof:

L.H.S $= A \cap (B \cap C)$

$= \{x \mid x \in A \text{ and } x \in B \cap C\}$ (By definition of intersection)

$= \{x \mid x \in A \text{ and } x \in B \text{ and } x \in C\}$

$= \{x \mid x \in A \cap B \text{ and } x \in C\}$ (By definition of intersection)

$= (A \cap B) \cap C$ (By definition of intersection)

$=$ R.H.S

$\because$ L.H.S $=$ R.H.S

$\therefore$ $A \cap (B \cap C) = (A \cap B) \cap C$

Hence proved.

Distributive property of union over intersection

We have already studied distributive property of union over intersection in previous classes which is as follows,

For any three sets A,B and C, $\boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

Proof:

$$
\begin{aligned}
\text{L.H.S} &= A \cup (B \cap C) \\
&= \{x \mid x \in A \text{ or } x \in B \cap C\} && (\text{By definition of union}) \\
&= \{x \mid x \in A \text{ or } (x \in B \text{ and } x \in C)\} && (\text{By definition of intersection}) \\
&= \{x \mid (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\} \\
&= \{x \mid x \in A \cup B \text{ and } x \in A \cup C\} && (\text{By definition of union}) \\
&= (A \cup B) \cap (A \cup C) && (\text{By definition of intersection}) \\
&= \text{R.H.S}
\end{aligned}
$$

$\because$ L.H.S = R.H.S

$\therefore$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Hence proved.

Distributive property of intersection over union

We are already familiar with distributive property of intersection over union which is as under.

For any three sets A,B and C, $\boxed{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

Proof:

$$
\begin{aligned}
\text{L.H.S} &= A \cap (B \cup C) \\
&= \{x \mid x \in A \text{ and } x \in B \cup C\} && (\text{By definition of intersection}) \\
&= \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\} && (\text{By definition of union}) \\
&= \{x \mid (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\} \\
&= \{x \mid x \in A \cap B \text{ or } x \in A \cap C\} && (\text{By definition of intersection}) \\
&= (A \cap B) \cup (A \cap C) && (\text{By definition of union}) \\
&= \text{R.H.S}
\end{aligned}
$$

$\because$ L.H.S = R.H.S

$\therefore$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Hence proved.

De Morgan's Laws

We have already studied in section 17.1(iv) that there are two De Morgan's Laws which are as under.

For any two sets A and B

(i) $(A\cup B)' = A'\cap B'$ (ii) $(A\cap B)' = A'\cup B'$

Proof: (i) $(A\cup B)' = A'\cap B'$

$L.H.S = (A\cup B)'$

$= \{x \mid x\in U \text{ and } x\notin A\cup B\}$ (By definition of complement)

$= \{x \mid x\in U \text{ and } (x\notin A \text{ and } x\notin B)\}$

$= \{x \mid (x\in U \text{ and } x\notin A) \text{ and } (x\in U \text{ and } x\notin B)\}$

$= \{x \mid x\in A' \text{ and } x\in B'\}$ (By definition of complement)

$= A'\cap B'$ (By definition of intersection)

$= R.H.S$

$\because$ $L.H.S = R.H.S$

$\therefore$ $(A\cup B)' = A'\cap B'$

Hence proved.

Proof: (ii) $(A\cap B)' = A'\cup B'$

$L.H.S = (A\cap B)'$

$= \{x \mid x\in U \text{ and } x\notin A\cap B\}$ (By definition of complement)

$= \{x \mid x\in U \text{ and } (x\notin A \text{ or } x\notin B)\}$

$= \{x \mid (x\in U \text{ and } x\notin A) \text{ or } (x\in U \text{ and } x\notin B)\}$

$= \{x \mid x\in A' \text{ or } x\in B'\}$ (By definition of complement)

$= A'\cup B'$ (By definition of union)

$= R.H.S$

$\because$ $L.H.S = R.H.S$

$\therefore$ $(A\cap B)' = A'\cup B'$

Hence proved.

17.1.2(ii) Verify the fundamental properties of given sets.

Let us verify the fundamental properties with the help of the following examples.

Example 1:

If $A = \{1,2,3,4,6,12\}$ and $B = \{4,6,8,9,10,12\}$ then verify commutative property of union and intersection.

Verification:

(a) Commutative property of union

i.e. $A \cup B = B \cup A$

$$\begin{aligned} L.H.S &= A \cup B \\ &= \{1,2,3,4,6,12\} \cup \{4,6,8,9,10,12\} \\ &= \{1,2,3,4,6,8,9,10,12\} \end{aligned}$$

$$\begin{aligned} R.H.S &= B \cup A \\ &= \{4,6,8,9,10,12\} \cup \{1,2,3,4,6,12\} \\ &= \{1,2,3,4,6,8,9,10,12\} \end{aligned}$$

$\because$ L.H.S = R.H.S

$\therefore$ $A \cup B = B \cup A$

Hence verified.

(b) Commutative property of intersection

i.e. $A \cap B = B \cap A$

$$\begin{aligned} L.H.S &= A \cap B \\ &= \{1,2,3,4,6,12\} \cap \{4,6,8,9,10,12\} \\ &= \{4,6,12\} \end{aligned}$$

$$\begin{aligned} R.H.S &= B \cap A \\ &= \{4,6,8,9,10,12\} \cap \{1,2,3,4,6,12\} \\ &= \{4,6,12\} \end{aligned}$$

$\because$ L.H.S = R.H.S

$\therefore$ $A \cap B = B \cap A$

Hence verified.

Example 2:

If $A = \{1,2,5,10\}$, $B = \{2,3,5,7\}$ and $C = \{1,3,5,7,9\}$ then verify associative property of union and intersection.

Verification:

(a) Associative property of union

i.e. $A \cup (B \cup C) = (A \cup B) \cup C$

$$\begin{aligned} L.H.S &= A \cup (B \cup C) \\ &= \{1,2,5,10\} \cup [\{2,3,5,7\} \cup \{1,3,5,7,9\}] \\ &= \{1,2,5,10\} \cup \{1,2,3,5,7,9\} \\ &= \{1,2,3,5,7,9,10\} \end{aligned}$$

$$\begin{aligned} R.H.S &= (A \cup B) \cup C \\ &= [\{1,2,5,10\} \cup \{2,3,5,7\}] \cup \{1,3,5,7,9\} \\ &= \{1,2,3,5,7,10\} \cup \{1,3,5,7,9\} \\ &= \{1,2,3,5,7,9,10\} \end{aligned}$$

$\because$ L.H.S = R.H.S

$\therefore$ $A \cup (B \cup C) = (A \cup B) \cup C$

Hence verified.

(b) Associative property of intersection

i.e. $A \cap (B \cap C) = (A \cap B) \cap C$

$$\begin{aligned} L.H.S &= A \cap (B \cap C) \\ &= \{1,2,5,10\} \cap [\{2,3,5,7\} \cap \{1,3,5,7,9\}] \\ &= \{1,2,5,10\} \cap \{3,5,7\} \\ &= \{5\} \end{aligned}$$

$$\begin{aligned} R.H.S &= (A \cap B) \cap C \\ &= [\{1,2,5,10\} \cap \{2,3,5,7\}] \cap \{1,3,5,7,9\} \\ &= \{2,5\} \cap \{1,3,5,7,9\} \\ &= \{5\} \end{aligned}$$

$\because$ L.H.S = R.H.S

$\therefore$ $A \cap (B \cap C) = (A \cap B) \cap C$

Hence verified.

Example 3: If $A=\{1,2,3,...,10\}$, $B=\{2,4,6,8,10\}$ and $C=\{3,6,9\}$ then verify

(a) Distributive property of union over intersection.
(b) Distributive property of intersection over union.

(a) Verification:

(a) Distributive property of union over intersection

i.e., $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

$$\begin{aligned}
L.H.S &= A\cup(B\cap C)\\
&=\{1,2,3,...,10\}\cup[\{2,4,6,8,10\}\cap\{3,6,9\}]\\
&=\{1,2,3,...,10\}\cup\{6\}\\
&=\{1,2,3,...,10\}
\end{aligned}$$

$$\begin{aligned}
R.H.S &= (A\cup B)\cap(A\cup C)\\
&=[\{1,2,3,...,10\}\cup\{2,4,6,8,10\}]\cap[\{1,2,3,...,10\}\cup\{3,6,9\}]\\
&=\{1,2,3,...,10\}\cap\{1,2,3,...,10\}\\
&=\{1,2,3,...,10\}
\end{aligned}$$

$\because$ L.H.S = R.H.S

$\therefore$ $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

Hence verified.

(b) Distributive property of Intersection over union

i.e., $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

$$\begin{aligned}
L.H.S &= A\cap(B\cup C)\\
&=\{1,2,3,...,10\}\cap[\{2,4,6,8,10\}\cup\{3,6,9\}]\\
&=\{1,2,3,...,10\}\cap\{2,3,4,6,8,9,10\}\\
&=\{2,3,4,6,8,9,10\}
\end{aligned}$$

$$\begin{aligned}
R.H.S &= (A\cap B)\cup(A\cap C)\\
&=[\{1,2,3,...10\}\cap\{2,4,6,8,10\}]\cup[\{1,2,3,...,10\}\cap\{3,6,9\}]\\
&=\{2,4,6,8,10\}\cup\{3,6,9\}\\
&=\{2,3,4,6,8,9,10\}
\end{aligned}$$

$\because$ L.H.S = R.H.S

$\therefore$ $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

Hence verified.

Example 4: If $U=\{1,2,3,...,20\}$, $A=\{1,3,5,...,19\}$ and $B=\{2,4,6,...,20\}$ then verify De Morgan's laws.

i.e., (a) $(A\cup B)'=A'\cap B'$ (b) $(A\cap B)'=A'\cup B'$

Verification:

(a) $(A\cup B)'=A'\cap B'$

$$
\begin{aligned}
\text{L.H.S} &= (A\cup B)' \\
&= U-(A\cup B) \\
&= \{1,2,3,...,20\}-[\{1,3,5,...,19\}\cup\{2,4,6...,20\}] \\
&= \{1,2,3,...,20\}-\{1,2,3,...,20\} \\
&= \varnothing
\end{aligned}
$$

$$
\begin{aligned}
\text{R.H.S} &= A'\cap B' \\
&= (U-A)\cap(U-B) \\
&= [\{1,2,3,...,20\}-\{1,3,5,...,19\}]\cap[\{1,2,3,...,20\}-\{2,4,6,...,20\}] \\
&= \{2,4,6,...,20\}\cap\{1,3,5,...,19\} \\
&= \varnothing
\end{aligned}
$$

$\because$ L.H.S = R.H.S

$\therefore$ $(A\cup B)'=A'\cap B'$

Hence verified.

(b) $(A\cap B)'=A'\cup B'$

$$
\begin{aligned}
\text{L.H.S} &= (A\cap B)' \\
&= U-(A\cap B) \\
&= \{1,2,3,...,20\}-[\{1,3,5,...,19\}\cap\{2,4,6...,20\}] \\
&= \{1,2,3,...,20\}-\{\ \} \\
&= \{1,2,3,...,20\}
\end{aligned}
$$

$$
\begin{aligned}
\text{R.H.S} &= A'\cup B' \\
&= (U-A)\cup(U-B) \\
&= [\{1,2,3,...,20\}-\{1,3,5,...,19\}]\cup[\{1,2,3,...,20\}-\{2,4,6,...,20\}] \\
&= \{2,4,6,...,20\}\cup\{1,3,5,...,19\} \\
&= \{1,2,3,...,20\}
\end{aligned}
$$

$\because$ L.H.S = R.H.S

$\therefore$ $(A\cap B)'=A'\cup B'$

Hence verified.

Exercise 17.3

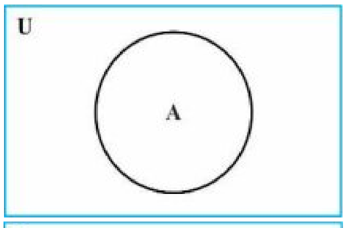
1. **Verify the commutative property of union and intersection for the following sets.**
   (i) $A=\{a,b,c,d,e\}$ and $B=\{a,e,i,o,u\}$
   (ii) $P=\{x \mid x\in Z \wedge -3<x<3\}$ and $Q=\{y \mid y\in E^{+} \wedge y\leq 4\}$
2. **Verify the associative property of union and intersection for the following sets.**
   (i) $A=\{1,2,4,5,10,20\}$, $B=\{5,10,15,20\}$ and $C=\{1,2,5,10\}$
   (ii) $A=\mathbb{N}$, $B=\mathbb{P}$ and $C=\mathbb{Z}$
3. **Verify**
   (a) Distributive property of union over intersection.
   (b) Distributive property of intersection over union for the following sets
   (i) $A=\{1,2,3,...,10\}$, $B=\{2,3,5,7\}$ and $C=\{1,3,5,7,9\}$
   (ii) $A=\mathbb{N}$, $B=\mathbb{P}$ and $C=\mathbb{W}$
4. **Verify De Morgan's laws if**
   $U=\{1,2,3...,12\}$, $A=\{1,2,3,4,6,12\}$ and $B=\{2,4,6,8\}$.
5. **If, A and B are subset of U then prove the following using properties.**
   (i) $A\cup(A\cap B)=A\cap(A\cup B)$ (ii) $A\cup B=A\cup(A'\cap B)$
   (iii) $B=(A\cap B)\cup(A'\cap B)$ (iv) $B=A\cup(A'\cap B)$, if $A\subseteq B$

17.1.3 Venn Diagrams:

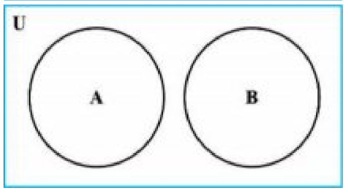
We have already studied in previous classes that the sets, their relations and operations can also be represented geometrically. This geometrical representation is called Venn Diagram named after the English mathematician John Venn who introduced it in 1881 AD. In Venn Diagram, a rectangle is usually used to represent the universal set whereas circles or ovals represent sets.

Recall some Venn diagrams

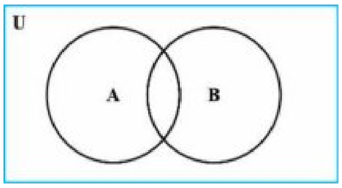
(i) Venn diagram showing a set A inside universal set U.



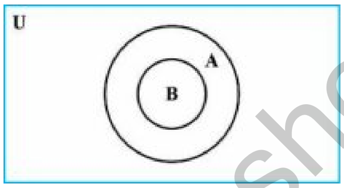
(ii) Venn diagram showing two disjoint sets A and B.



(iii) Venn diagram showing two overlapping sets A and B.



(iv) Venn diagram showing a subset B of A. i.e. $B \subseteq A$



17.1.3(i) Use Venn diagrams to represent

- ➢ **Union and intersection of sets**
- ➢ **Complement of a set**
- ➢ **Symmetric difference of two sets**

➢ **Union and intersection of sets**

In Venn diagram, the union of two sets A and B is represented by the entire region of both sets A and B. In Fig (i) the shaded or coloured region represents $A \cup B$.

Whereas the intersection of two sets A and B is represented by the common region of both A and B. In Fig (ii), the shaded or coloured region shows $A \cap B$.

$A \cup B$

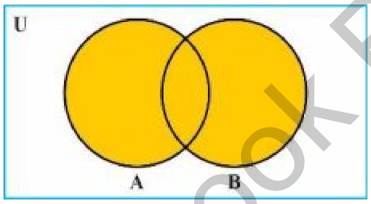


Fig (i)

$A \cap B$

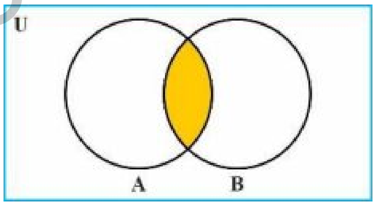


Fig (ii)

Example 1: If $U = \{1, 2, 3, ..., 10\}$, $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8\}$ then use Venn diagram to represent $A \cup B$ and $A \cap B$.

Solution:

In Fig (a), coloured or shaded region represents $A \cup B$.

From the Venn diagram, we have

$A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$

In Fig (b), the coloured or shaded region represents $A \cap B$

From the Venn diagram, we have

$A \cap B = \{2, 4\}$.

$A \cup B$

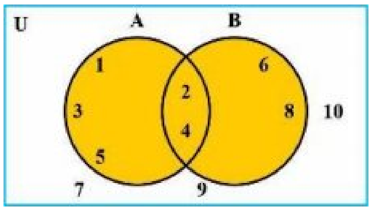


Fig (a)

$A \cap B$

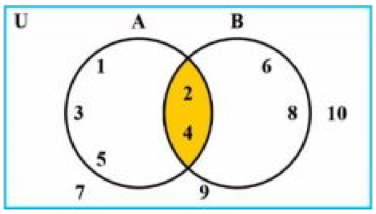
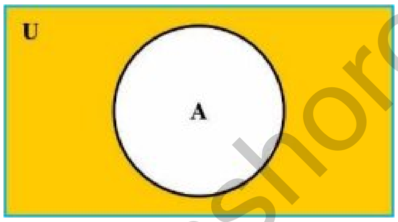


Fig (b)

➢ Complement of a set

In Venn diagram, the complement of set A is represented by the region of universal set excluding the region of A. In Fig (i) shaded or coloured region represents A'.

A'



For Example:

If $U = \{1, 2, 3, ..., 8\}$ and $A = \{1, 2, 3, 4\}$ then use Venn diagram to represent A'

Solution:

In the adjacent Venn diagram Fig (ii) the shaded or coloured region represents A'.

So we have $A' = \{5, 6, 7, 8\}$

A'

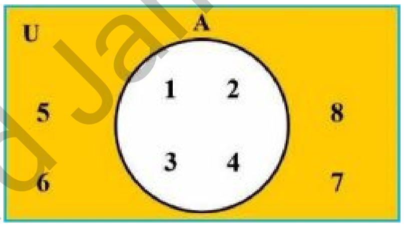


Fig (ii)

➢ Symmetric difference of two sets

In Venn diagram, the symmetric difference of two sets A and B is represented by the entire region of both A and B except the common region.

In Fig (i), the coloured or shaded region represents $A \Delta B$.

$A \Delta B$

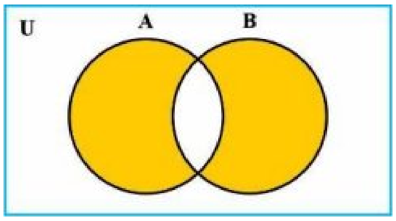


Fig (i)

Example:

If $U = \{a, b, c, d, e, f\}, A = \{a, c, e\}$ and $B = \{a, b, c\}$ then use Venn diagram to represent $A \Delta B$.

Solution:

In adjacent Venn diagram fig (ii), the shaded or coloured region represents $A \Delta B$.

i.e., $A \Delta B = \{b, e\}$

$A \Delta B$

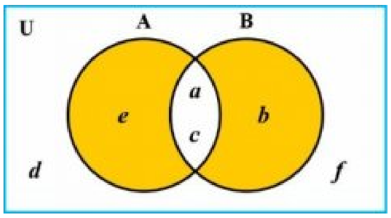


Fig (ii)

17.1.3(ii) Use Venn diagram to verify

- Commutative property of union and intersection
- Associative laws
- Distributive laws
- De Morgan's laws

Commutative property of union and intersection

Let us verify the commutative property of union and intersection using Venn diagram with the help of the following example.

Example: If $A=\{1,2,3,4,5,6\}$ and $B=\{4,6,8\}$ then verify commutative property of union and intersection using Venn diagram.

Verification: (i) Commutative property of union, i.e., $A \cup B = B \cup A$

$A \cup B$

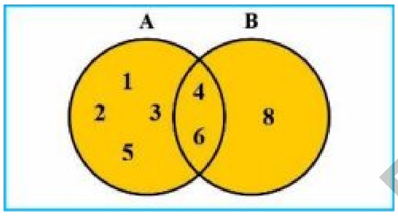


Fig (i)

From Venn diagram of Fig. i

$A \cup B=\{1,2,3,4,5,6,8\}$

$B \cup A$

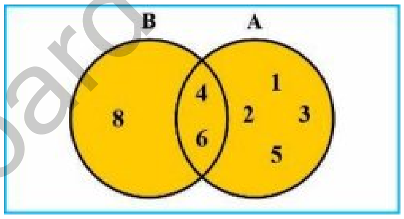


Fig (ii)

From Venn diagram of Fig. ii

$B \cup A=\{1,2,3,4,5,6,8\}$

$\because$ Shaded or coloured regions and their elements are equal as shown in the Fig (i) and Fig(ii).

$\therefore$ $A \cup B = B \cup A$

Hence verified.

Verification: (ii) Commutative property of intersection i.e. $A \cap B = B \cap A$.

$A \cap B$

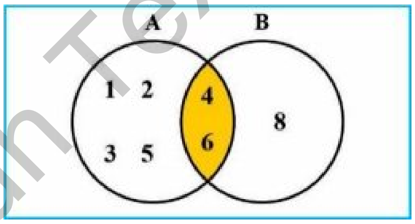


Fig (i)

From Venn diagram of Fig. i

$A \cap B=\{4,6\}$

$B \cap A$

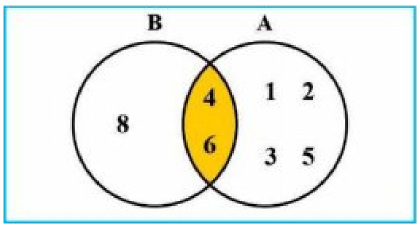


Fig (ii)

From Venn diagram of Fig. ii

$B \cap A=\{4,6\}$

$\because$ Shaded or coloured regions and their elements are equal as shown in Fig (i) and Fig (ii)

$\therefore$ $A \cap B = B \cap A$

Hence verified.

➢ Associative Laws

In order to verify associative laws using Venn diagram, we take the following example.

Example:

If $A=\{1,2,3,4,5,6\}$, $B=\{6,8,10\}$ and $C=\{5,6,7,8\}$ then verify associative laws of union and intersection.

Verification: Associative law of union i.e. $A\cup(B\cup C)=(A\cup B)\cup C$

$\underline{\text{L.H.S}=A\cup(B\cup C)}$

$B\cup C$

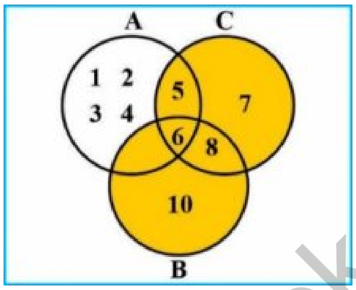


Fig (i)

$A\cup(B\cup C)$

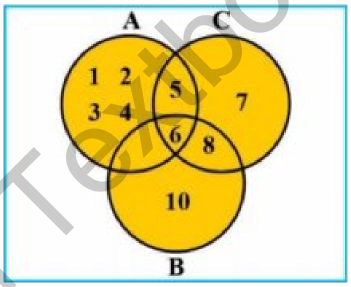


Fig (ii)

From Venn diagram of Fig: (ii)

$A\cup(B\cup C)=\{1,2,3,4,5,6,7,8,10\}$

$\underline{\text{R.H.S}=(A\cup B)\cup C}$

$A\cup B$

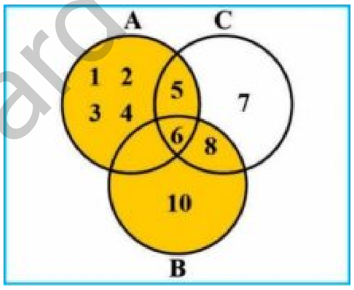


Fig (iii)

$(A\cup B)\cup C$

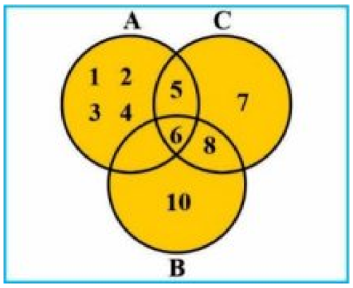


Fig (iv)

From Venn diagram of Fig. (iv)

$(A\cup B)\cup C=\{1,2,3,4,5,6,7,8,10\}$

$\because$ Shaded or coloured regions and their elements as shown in Fig: (ii) and Fig: (iv) are equal

$\therefore$ $A\cup(B\cup C)=(A\cup B)\cup C$

Hence verified.

Verification: Associative law of intersection i.e. $A \cap (B \cap C) = (A \cap B) \cap C$

$\underline{L.H.S = A \cap (B \cap C)}$ $\underline{R.H.S = (A \cap B) \cap C}$

$B \cap C$

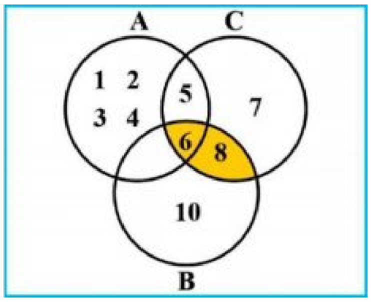


Fig (i)

$A \cap B$

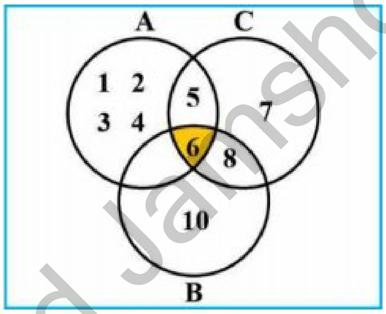


Fig (iii)

$A \cap (B \cap C)$

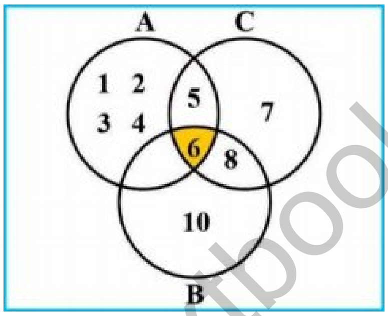


Fig (ii)

$(A \cap B) \cap C$

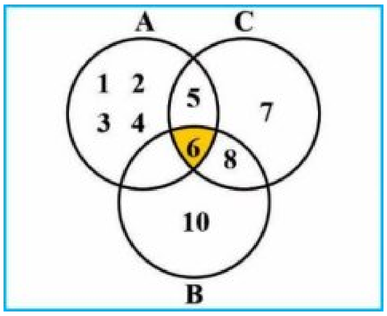


Fig (iv)

From Venn diagram of Fig. (ii)

$A \cap (B \cap C) = \{6\}$

From Venn diagram of Fig. (iv)

$(A \cap B) \cap C = \{6\}$

$\because$ Coloured or shaded regions and their elements of Fig:(ii) and Fig:(iv) are equal

$\therefore$ $A \cap (B \cap C) = (A \cap B) \cap C$

Hence verified.

- **Distributive Laws**

Now we verify distributive laws using Venn diagram with the help of the following example.

Example:

Verify distributive laws using Venn diagram

if $A = \{1,2,3,4,5,6\}$, $B = \{3,4,5,6,7\}$ and $C = \{2,4,6,8\}$.

Verification:

(i) Distributive law of union over intersection

i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$\underline{L.H.S = A \cup (B \cap C)}$

$B \cap C$

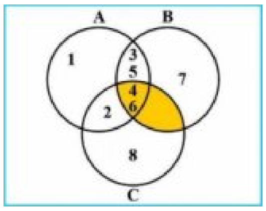


Fig (i)

$\underline{R.H.S = (A \cup B) \cap (A \cup C)}$

$A \cup B$

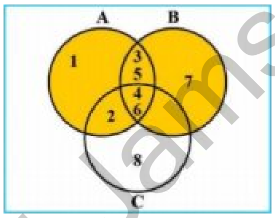


Fig (iii)

$A \cup C$

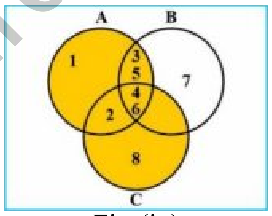


Fig (iv)

$A \cup (B \cap C)$

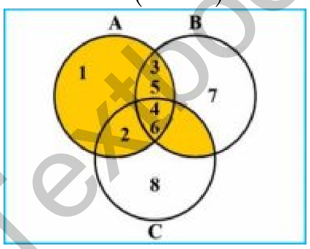


Fig (ii)

From Venn diagram of Fig.(ii)

$A \cup (B \cap C) = \{1, 2, 3, 4, 5, 6\}$

$(A \cup B) \cap (A \cup C)$

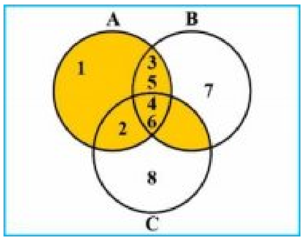


Fig (iv)

From Venn diagram of Fig.(v)

$(A \cup B) \cap (A \cup C) = \{1, 2, 3, 4, 5, 6\}$

$\because$ Shaded or coloured region and their elements are equal as shown in Fig (ii) and Fig (v)

$\therefore$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Hence verified.

Verification:

(ii) Distributive law of intersection over union

i.e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

L.H.S $= A \cap (B \cup C)$

$B \cup C$

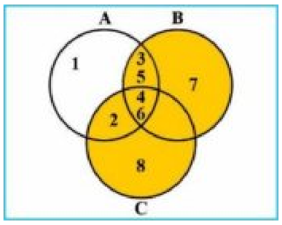


Fig (i)

R.H.S $= (A \cap B) \cup (A \cap C)$

$A \cap B$

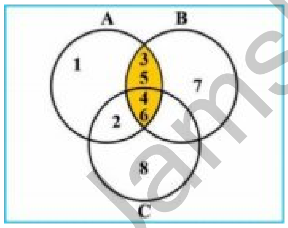


Fig (iii)

$A \cap C$

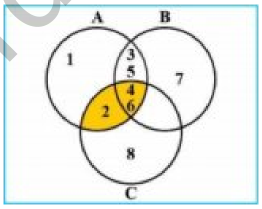


Fig (iv)

$A \cap (B \cup C)$

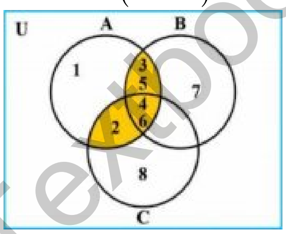


Fig (ii)

$(A \cap B) \cup (A \cap C)$

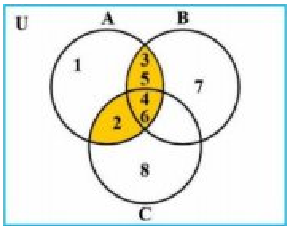


Fig (v)

From Venn diagram of Fig. (ii)

$A \cap (B \cup C) = \{2, 3, 4, 5, 6\}$

From Venn diagram of Fig. (v)

$(A \cap B) \cup (A \cap C) = \{2, 3, 4, 5, 6\}$

$\because$ Shaded or coloured regions and their elements as shown in Fig. (ii) and Fig. (v) are equal

$\therefore$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

Hence verified.

➢ De Morgan's Laws

Let us verify De Morgan's laws using Venn diagram with the help of the following example.

Example: Verify De Morgan's laws using Venn diagram.

If $U=\{1,2,3,...,10\}$, $A=\{1,2,3,4,5,6\}$ and $B=\{2,4,6,8,10\}$

Verification: (i) $(A\cup B)'=A'\cap B'$

<u>$L.H.S=(A\cup B)'$</u>

$A\cup B$

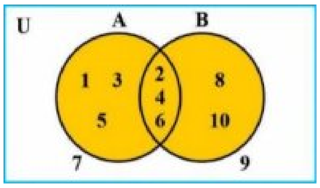


Fig (i)

$(A\cup B)'$

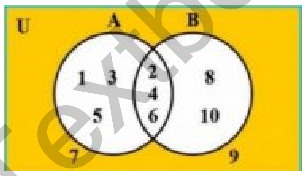


Fig (ii)

From Venn diagram of Fig. (ii)

$(A\cup B)'=\{7,9\}$

<u>$R.H.S=A'\cap B'$</u>

A'

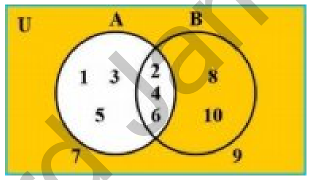


Fig (iii)

B'

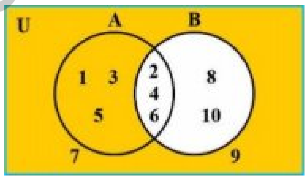


Fig (iv)

$A'\cap B'$

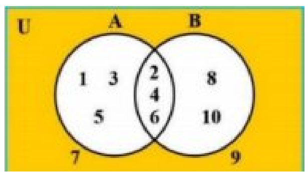


Fig (v)

From Venn diagram of Fig. (v)

$A'\cap B'=\{7,9\}$

$\because$ Shaded or coloured regions and their elements as shown in Fig. (ii) and Fig. (v) are equal

$\therefore$ $(A\cup B)'=A'\cap B'$

Hence verified.

Verification: (ii) $(A \cap B)' = A' \cup B'$

<u>L.H.S $= (A \cap B)'$</u>

$A \cap B$

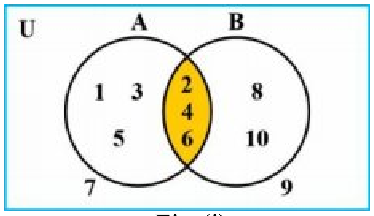


Fig (i)

$(A \cap B)'$

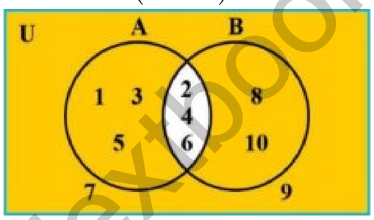


Fig (ii)

From Venn diagram of Fig. (ii)

$(A \cap B)' = \{1,3,5,7,8,9,10\}$

<u>R.H.S $= A' \cup B'$</u>

A'

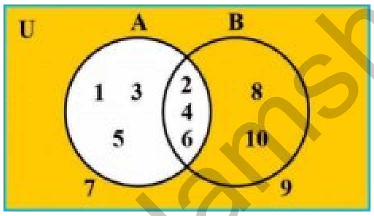


Fig (iii)

B'

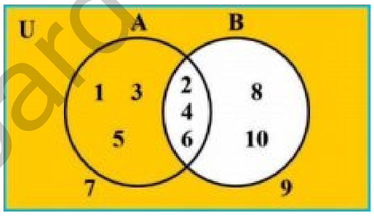


Fig (iv)

$A' \cup B'$

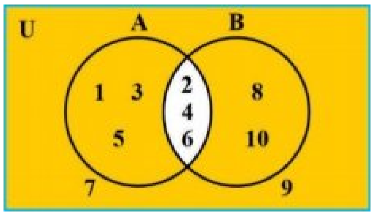


Fig (v)

From Venn diagram of Fig. (v)

$A' \cup B' = \{1,3,5,7,8,9,10\}$

$\because$ Shaded or coloured regions and their elements as shown in Fig: (ii) and Fig: (v) are equal

$\therefore$ $(A \cap B)' = A' \cup B'$

Hence verified.

Exercise 17.4

1. **If A and B are any two subsets of U then draw Venn diagrams in general of $A \cup B$, $A \cap B$, $A \Delta B$ and $A - B$, if**

(i) A and B are disjoint sets.
(ii) A and B are overlapping sets.
(iii) A is subset of B.

2. **Verify commutative property of union and intersection using Venn diagram if**

(i) $A = \{a, b, c, d, e\}$ and $B = \{a, e, i, o, u\}$

(ii) $P = \{1, 2, 3, ..., 10\}$ and $Q = \{2, 4, 6, 8, 10\}$

3. **Verify De Morgan's laws using Venn diagram if**
$A = \{1, 3, 5, 7, 9\}$, $B = \{5, 6, 7, 8\}$ and $U = \{1, 2, 3, ..., 10\}$

4. **Verify associative laws and distributive laws using Venn diagram**
if $A = \{1, 2, 3, ..., 8\}$, $B = \{4, 6, 7, 8, 9, 10\}$ and $C = \{5, 7, 9, 11\}$

17.1.4 Ordered pairs and Cartesian products

Cartesian product is named after French mathematician Rene Descartes who introduced Analytic Geometry. In Analytic Geometry ordered pairs are used to locate points in plane. Let us discuss ordered pair and Cartesian product in detail.

17.1.4 (i) Recognize ordered pair

A pair of numbers in which order is maintained is called an ordered pair. For any two real numbers *a* and *b,* if we regard ***a*** as first and ***b*** as second then the ordered pair of *a* and *b* is denoted as (a, b). In the ordered pair (a, b), *a* is called first component or element and b is called second component or element.

Two ordered pairs are equal if their corresponding components are equal

i.e., $(a, b) = (c, d)$ iff $a = c$ and $b = d$

Example: Find *x* and *y* if $(x + 5, 8)$ and $(9, y - 6)$ are equal

Solution:

We have $(9, y - 6) = (x + 5, 8)$

$\Rightarrow$ $9 = x + 5$ and $y - 6 = 8$
$\Rightarrow$ $9 - 5 = x$ $\Rightarrow$ $y = 8 + 6$
or $x = 4$ or $y = 14$

So the value of *x* and *y* are 4 and 14 respectively.

17.1.4 (ii) To form Cartesian products

If A and B are any two non-empty sets then the Cartesian product of A with B, denoted as $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

i.e. $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

Similarly, Cartesian product of B with A is defined as

$B \times A = \{(b, a) \mid b \in B \text{ and } a \in A\}$

For example: If $P=\{1,2,4\}$ and $Q=\{5,10\}$

then $P\times Q=\{(1,5),(1,10),(2,5),(2,10),(4,5),(4,10)\}$

and $Q\times P=\{(5,1),(5,2),(5,4),(10,1),(10,2),(10,4)\}$

Note:

(i) $A\times B$ is read as "A cross B".

(ii) If $O(A)=m$ and $O(B)=n$ then $O(A\times B)=mn$

(iii) $A\times B\neq B\times A$ in general

17.2 Binary Relations

Define a binary relation and identify its domain and range.

- **Binary Relation**

If A and B are any two non-empty sets then any subset R of the Cartesian product $A\times B$ is called a binary relation from A to B.

Example: If $A=\{a,b,c\}$ and B= $\{2,4\}$ then find

(a) Two relations from A to B.
(b) Three relations from B to A.
(c) Four binary relations in B.

Solution (a) Two relations from A to B

Here, $A\times B=\{(a,2),(a,4),(b,2),(b,4),(c,2),(c,4)\}$

Now $R_1=\{(a,2),(b,4)\}$ and $R_2=\{(b,2),(c,2),(c,4)\}$

are any two relations from A to B.

(b) Three relations from B to A

Here, $B\times A=\{(2,a),(2,b),(2,c),(4,a),(4,b),(4,c)\}$

Now $R_1=\{(2,a)\}$, $R_2=\{(2,a),(2,b)\}$ and $R_3=\{(4,a),(4,b),(4,c)\}$

are any three relations from B to A.

(c) Four binary relations in B

Here, $B\times B=\{(2,2),(2,4),(4,2),(4,4)\}$

Now $R_1=\varnothing$, $R_2=\{(2,2)\}$, $R_3=\{(2,4),(4,2)\}$ and $R_4=\{(2,2),(2,4)\}$

are any four binary relations in B.

- **Domain and Range of a Binary Relation**

Let R be a binary relation from set A to set B. Then domain of R, denoted as Dom R, is the set of first components of all the ordered pairs of R.

The range of R, denoted as Range R, is the set of second components of all the ordered pairs of R. For example If $A=\{1,2,3,4\}$, $B=\{4,5,6,7\}$ and $R=\{(1,4),(2,4),(3,5),(3,6)\}$ is a binary relation from A to B.

then, Dom $R=\{1,2,3\}$ and Range $R=\{4,5,6\}$

Example:

If $x, y \in$ N and a binary relation R in N is given as $R = \{(x, y) \mid x + y = 5\}$ then write R in tabular form. Also find its domain and range.

Solution:

In tabular form, $R = \{(1,4),(2,3),(3,2),(4,1)\}$

Here Dom $R = \{1,2,3,4\}$

and Range $R = \{1,2,3,4\}$

17.3 Functions:

Function is one of the basic concepts of calculus, the branch of Mathematics, which has revolutionized the field of science. Function, in fact, is a rule or relation which relates two sets or quantities, for instance, area "A" of circle is related with radius "r" by rule $A = \pi r^2$. At this level, we will only discuss function on the basis of sets.

17.3 (i) Define a function and identify its domain, co-domain and range

A function is a binary relation in which for each element of domain there is one and only one element of range. For example: $\{(1,3),(2,4),(3,5),(4,6)\}$ is a function.

Whereas $\{(1,5),(2,6),(2,7),(3,9)\}$ is not a function because the element "2" of domain is associated with more than one element of range.

- **Function from set A to set B**

Let A and B be any two non-empty sets and R be a binary relation from A to B. Then R is called a function from A to B if

(i) Domain of R = A

(ii) Every element of A is associated with unique element of B under R i.e. if $(a,b) \in$ R and $(a,c) \in$ R then $b = c$. Functions are usually denoted by letters of English and Greek alphabets like f, g, h etc and α, β, γ etc.

If f is a function from A to B then we write it as $f : A \to B$, and for every $(a,b) \in f$, b is called the image of a under f and we write it as $b = f(a)$.

Example: Let $A = \{1,2,3,4\}$ and $B = \{2,3,4,5\}$. Identify which one of the following is a function from A to B.

(i) $R_1 = \{(1,2),(2,3),(3,4)\}$

(ii) $R_2 = \{(1,2),(2,3),(2,4),(3,4),(4,5)\}$

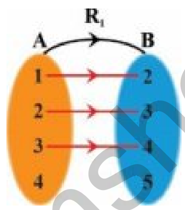
(iii) $R_3 = \{(1,2),(2,3),(3,4),(4,5)\}$

Solution:

Mapping Diagram

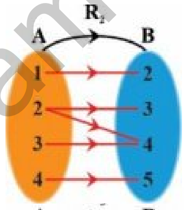
(i) $R_1 = \{(1,2),(2,3),(3,4)\}$

R_1 is not a function because $\text{Dom } R_1 \neq A$ as shown in the adjacent mapping diagram.



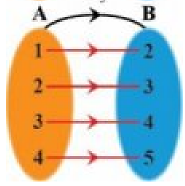
(ii) $R_2 = \{(1,2),(2,3),(2,4),(3,4),(4,5)\}$

R_2 is not a function because an element "2" of domain is not associated with unique element of B as shown in the adjacent mapping diagram.



(iii) $R_3 = \{(1,2),(2,3),(3,4),(4,5)\}$

R_3 is a function because $\text{Dom } R_3 = A$ and each element of domain is associated with unique element of B as shown in the adjacent mapping diagram.



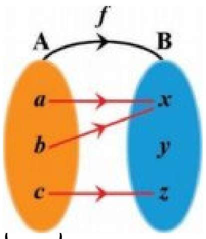
- **Domain, Co-domain and Range**

  If f is a function from A to B i.e. $f: A \rightarrow B$ then
  A is called its domain (First set)
  and B is called its co-domain (Second set)

Whereas,
Range is the set of all the images of f.

Example:

If f is a function from A to B as shown in the given mapping diagram then write down its domain, co-domain and range. Also write down the values of f(*a*) and f(*c*).



Solution:

From mapping diagram of f

$\text{Dom f} = A = \{a, b, c\}$, Co-domain of $f = B = \{x, y, z\}$ and Range $f = \{x, z\}$

Also $f(a) = x$ and $f(c) = z$

Note: (i) Range of f is subset of its co-domain.
(ii) Every function is a relation but converse is not true.

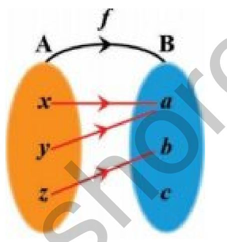
17.3 (ii) Demonstrate the following

- **Into and one-one function (Injective function)**
- **Onto function (Surjective function)**
- **One-one and onto function (Bijective function)**

First of all we define into function and one-one function.

- **Into function:**

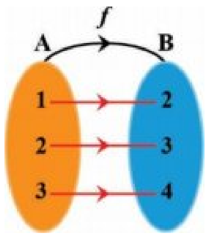
For any two sets A and B, a function $f: A \rightarrow B$ is called into function if Range of f is proper subset of B (i.e. Range $f \subset B$.)



For example: For $A = \{x, y, z\}$ and $B = \{a, b, c\}$, a function $f: A \rightarrow B$ defined as $f = \{(x, a), (y, a), (z, b)\}$ is an into function because Range $f \subset B$ as shown in the figure where Range of $f = \{a, b\}$.

- **One-one function:**

For any two sets A and B, a function $f: A \rightarrow B$ is called one-one function if distinct elements of A have distinct images in B.



For example: For $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, a function $f: A \rightarrow B$ defined as $f = \{(1, 2), (2, 3), (3, 4)\}$ is one-one function because distinct elements of A have distinct images in B as shown in the mapping diagram.

- **Into and one-one function or injective function:**

A function which is into as well as one-one, is called an injective function.

For example: For $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$ a function $f: A \rightarrow B$ defined as $f = \{(1, 2), (2, 3), (3, 4)\}$ is an injective function because it is into as well as one-one as shown in the mapping diagram Fig (i).

A function $g: A \rightarrow B$ defined as $g = \{(1, 3), (2, 5), (3, 4)\}$ is also an injective function as shown in mapping diagram Fig (ii).

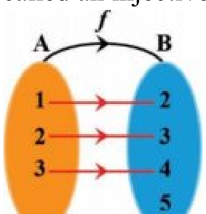


Fig (i)

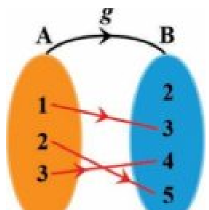


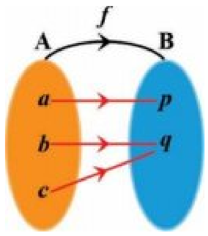
Fig (ii)

- **Onto function or surjective function:**

For any two sets A and B, a function $f: A \rightarrow B$ is called onto function or surjective function if Range $f = B$

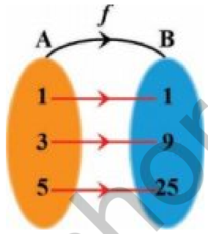
For example:

For $A = \{a, b, c\}$ and $B = \{p, q\}$ a function $f: A \rightarrow B$ defined as $f = \{(a, p), (b, q), (c, q)\}$ is an onto function because Range $f = B$ as shown in the adjacent figure.



- **One-one and onto function or bijective function:**

A function which is one-one as well as onto, is called one-one and onto function or bijective function.



For example: For $A = \{1, 3, 5\}$ and $B = \{1, 9, 25\}$ a function $f : A \rightarrow B$ defined as
$f = \{(1,1), (3,9), (5,25)\}$ is a bijective function because it is one-one as well as onto as shown in the adjacent figure.

17.3 (iii) Examine whether a given relation is a function or not

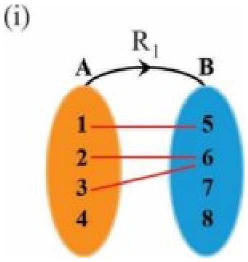
In order to examine whether a given relation is a function or not we have to concentrate on domain of the relation.

If there is any element of domain without image or any element of domain with more than one image in a relation then this relation is not a function.

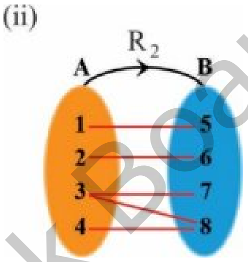
Example:

Examine the given relations from A to B whose mapping diagrams are given decide which one is function and which one is not.

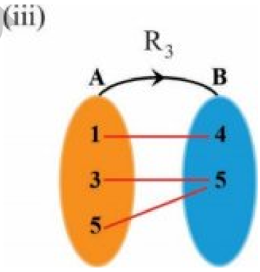
(i)



(ii)



(iii)



Solution:

R_1 is not a function because an element "4" of domain is without image.

R_2 is also not a function because an element "3" of domain has more than one images.

R_3 is a function because each element of domain has unique image.

17.3 (iv) Differentiate between one-one correspondence and one-one function.

Let A and B be two non-empty sets. In one-one correspondence between A and B, each element of either set is associated with exactly one element of the other set.

i.e. one-one correspondence always represents a bijective function.

Whereas one-one function is not always bijective.

For example, In figure (i) there is a one-one correspondence.

Whereas in Figure (ii), there is no one-one correspondence but it represents one-one function.

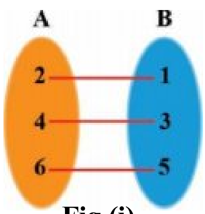


Fig (i)

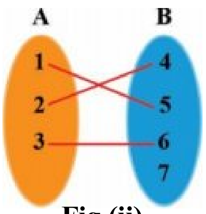


Fig (ii)

Exercise 17.5

1. **Find the values of x and y if:**

(i) $(x-5,10)=(11,y-7)$

(ii) $(5x+8,5y-4)=(3x+10,2y+2)$

(iii) $(2x-3y,5x+y)=(3,16)$

2. **If set P has 10 elements and Q has 15 elements. Find the number of elements of $P\times Q$, $Q\times P$ and $P\times P$**

3. **If $A=\{1,2\}, B=\{2,3\}$ and $C=\{1,3,5\}$ then find:**

(i) $A\times B$ (ii) $B\times C$ (iii) $A\times(B\cup C)$

(iv) $B\times(A\cup C)$ (v) $(A\cap B)\times(B\cap C)$

4. **If $A=\{5,6\}$ and $B=\{1,2,3\}$ then find:**

(i) Three relations in $A\times B$ (ii) Four relations in $B\times A$

(iii) Five relations in B (iv) All relations in A

5. **For two sets A and B, if $O(A)=3$ and $O(B)=4$, then find number of all binary relations in $A\times B$.**

6. **Write the following binary relations of $A\times B$ in tabular form where $A=\{0,1,2,3\}$ and $B=\{2,4,6,8\}$ such that $a\in A$ and $b\in B$**

(i) $R_1=\{(a,b)\mid b<5\}$ (ii) $R_2=\{(a,b)\mid a+b=9\}$

(iii) $R_3=\{(a,b)\mid a-b=1\}$

7. **If relation $R=\{(x,y)\mid y=2x+5\}$ is in Z then:**

(i) find range if domain is $\{-2,-1,0,1,2\}$

(ii) find domain if range is $\{11,13,15,17\}$

8. **If x,y represent elements of W, find domain and range of following relations**

(i) $\{(x,y)\mid 3x+y=11\}$ (ii) $\{(x,y)\mid x-y=6\}$

9. **If $f:A\rightarrow B$ is a function given by $f=\{(1,5),(2,6),(3,7),(4,8)\}$** where **$A=\{1,2,3,4\}$ and $B=N$** then write its domain, co-domain and range. Also write the values of **$f(2)$ and $f(4)$.**

10. **If $A=\{0,1,2,3,4\}$ and $B=\{2,4,6,8,10\}$** then find whether the following relation from A to B are functions or not. If they are functions find their types:

(i) $R_1=\{(0,2),(1,4),(2,6),(3,8)\}$

(ii) $R_2 = \{(0,10),(1,8),(2,6),(2,4),(3,4),(4,2)\}$

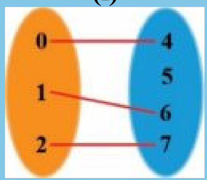
(iii) $R_3 = \{(0,2),(1,4),(2,6),(3,8),(4,8)\}$

(iv) $R_4 = \{(0,2),(1,2),(2,2),(3,2),(4,8)\}$

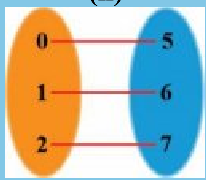
(v) $R_5 = \{(0,2),(1,4),(2,6),(3,8),(4,10)\}$

11. Which of the following is one-one function or one-one correspondence or neither of them

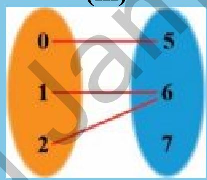
(i)



(ii)



(iii)



12. If $P = \{a, b, c\}$, $Q = \{x, y, z\}$ and $R = \{p, q, r, s\}$ then find:

(i) a function f from P into Q.
(ii) a function g from R onto P.
(iii) a function h from P to R which is injective.
(iv) a function k from Q to P which is bijective.

Review Exercise 17

1. Choose the correct option:

(i) Which one is subset of N
(a) Z (b) Q (c) E (d) P

(ii) Tabular form of $A = \{x \mid x \in Z \wedge x^2 = 25\}$ is ________.
(a) $\{1,5\}$ (b) $\{-5,5\}$
(c) $\{1,5,25\}$ (d) $\{-5\}$

(iii) For two sets A and B, if $O(A) = O(B)$ then
(a) $A \subseteq B$ (b) $B = A$
(c) $A \quad B$ (d) $A \subset B$

(iv) $\{x \mid x \in N \wedge x < 1\}$ is ______
(a) Singleton (b) Infinite set
(c) Empty set (d) Super set

(v) If $A = \{1,2,3,4\}$ and $B = \{2,4,6\}$ then $A \Delta B =$ ________.
(a) $\{1,2\}$ (b) $\{6\}$
(c) $\{1,3\}$ (d) $\{1,3,6\}$

(vi) Which operation is not commutative

(a) Symmetric difference (b) Union

(c) Difference (d) Intersection

(vii) $\{x \mid x \in P \vee x \in Q\} =$ ________.

(a) $P \cap Q$ (b) $P \cup Q$

(c) $P - Q$ (d) $P \Delta Q$

(viii) $\{x \mid x \in U \wedge x \notin A \cap B\} =$ ________.

(a) $A \cap B$ (b) $(A \cup B)'$

(c) $(A \cap B)'$ (d) $A' \cap B'$

(ix) For any two non-empty sets A and B, if $A \cup B = U$ and $A \cap B = \varnothing$ then A and B are ________.

(a) Cells (b) Overlapping sets

(c) Equal sets (d) None

(x) $\left((A')'\right)' =$ ________.

(a) $(A')'$ (b) A'

(c) A (d) B

(xi) If $A \supseteq B$ then $A \cup B =$ ________.

(a) A (b) B

(c) $\varnothing$ (d) U

(xii) In Venn diagram, ________ is used to represent universal set.

(a) rectangle (b) circle

(c) oval (d) all of these

(xiii) If $(x, 6) = (2, y - 6)$ then $x + y =$ ________.

(a) 8 (b) 10 (c) 12 (d) 14

(xiv) If $O(A \times B) = 100$ and $O(B) = 5$ then $O(A) =$ ________.

(a) 10 (b) 15 (c) 20 (d) 25

(xv) A function is called surjective if range ________ co-domain

(a) $\supseteq$ (b) $\subset$

(c) $=$ (d) all of these

(xvi) A function is called into if range ________ co-domain

(a) $\supseteq$ (b) $\subset$

(c) $=$ (d) none of these

(xvii) Which one is a function

(a) $\{(2,5),(2,7),(3,8)\}$ (b) $\{(6,7),(7,6),(6,8)\}$

(c) $\{(0,5),(6,0),(5,6)\}$ (d) none of these

(xviii) Which one is always bijective

(a) Onto function (b) One-one function

(c) Into function (d) One-one correspondence

(xix) If $f:\mathbb{N}\to\mathbb{R}$ then which one is not possible

(a) $f(5)=6$ (b) $f(7)=8$

(c) $f(-2)=6$ (d) $f(9)=0$

(xx) Which one is commutative

(a) Difference (b) Union

(c) Cartesian product (d) None

2. If $A=\{1,2,5\}$, $B=\{2,4,6\}$, $U=\{1,2,3,4,5,6\}$, then find the following

(a) $A\cup B$ (b) $A\cap B$ (c) $A\Delta B$

(d) $A-B$ (e) A' (f) B'

3. Verify De Morgan's laws for the sets of question 2.

4. Verify commutative properties of union and intersection for the sets.

$A=\{a,b,d\}$, $B=\{a,d,e\}$

5. For sets $P=\{1,3,5\}$ and $Q=\{1,3,7\}$ represent the following set by using Venn diagram.

(a) $P\cup Q$ (b) $P\cap Q$ (c) $P\Delta Q$ (d) $P-Q$

6. If $A=\{a,c\}$ and $B=\{b,d\}$ then find

(a) Two relations in A (b) Three relations in B

(c) Three relations in $A\times B$ (d) Two relations in $A\cup B$

7. If $A=\{1,2,5\}$ and $A=\{6,8\}$ then find function from A to B, which is

(a) Onto function (b) Into function

Summary

- Tabular, descriptive and set builder forms are the method of representation of a set.
- Empty set has no element.
- If A is subset of B then B is superset of A.
- If $A \subseteq B$ and $B \subseteq A$ then $A = B$.
- Equivalent sets have one-one correspondence.
- If $A \subseteq B$ and $A \neq B$ then $A \subset B$.
- Union, intersection, difference, symmetric difference, complementation and Cartesian product are operation on sets.
- If $A \cap B = \varnothing$ then A and B are disjoint sets.
- If $A \cup B = U$ then A and B are exhaustive sets.
- Cells are always disjoint and exhaustive.
- Difference and Cartesian product are not commutative.
- Union, intersection and symmetric difference are commutative and associative.
- Union and intersection are distributive over each other.
- De Morgan's Laws: (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
- Geometrical representation of sets, their relations and operations is called Venn diagram.
- Venn diagrams were introduced by John Venn in 1881 AD.
- Sets and universal set in Venn diagram are represented by circles or ovals and rectangle respectively.
- If $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$
- $O(A \times B) = O(A) \cdot O(B)$
- $A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$
- Every subset of Cartesian product is called binary relation.
- Function is a binary relation in which each element of domain has unique image.
- If $f : X \to Y$ is a function then X and Y are domain and co-domain respectively whereas range is the set of all images.
- A function is called (i) Into if range $\subset$ co-domain (ii) Onto if range = co-domain
  (iii) One-one if distinct elements of domain have distinct images in range.
- A function is called
  (i) Injective if it is one-one and into. (ii) Surjective if it is onto.
  (iii) Bijective if it is one-one and onto
- One-one correspondence is always bijective.
- One-one function is not always bijective.

VARIATIONS

Unit 18

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Define ratios, proportions and variations (direct and inverse).
- Find 3^{rd}, 4^{th} proportionals in proportion and mean proportional in a continued proportion.
- Apply theorems of:
  - invertendo,
  - alternando,
  - componendo,
  - dividendo and componendo
  - dividendo to find proportions.
- Define joint variations.
- Solve problems related to joint variations.
- Use ***k***- Method to prove conditional equalities involving proportions.
- Solve real life problems based on variations.

18.1: Ratio, Proportions and variations

18.1.1 Define ratio, proportion and variations (direct and inverse)

(a) Ratio:

Ratio is a comparison of two quantities having same units. It is the relation between two quantities of the same kind. In other words, ratio means what part of one quantity is of the other. If a and b are two quantities of the same kind and b is not zero, then the ratio of a and b is written as a $a:b$ or $\frac{a}{b}$.

For example: If in a class there are 13 boys and 8 girls, then the ratio of the number of boys to the number of girls can be expressed as $13:8$ or in fraction $\frac{13}{8}$.

Note that:

(i) The order of the elements in a ratio is important.

(ii) A ratio has no unit.

(iii) In a : b the first term a is called antecedent and the second term b called consequent.

Example 1: Find the ratio

(i) 400m to 900m (ii) 700gm to 2kg

(iii) 30sec to 2min (iv) Rs 200 to 300gm

Solution:

(i): Ratio of 400m to 900m

$400:\ 900 = \frac{400}{900} = \frac{4}{9} = 4:9$

$4:9$ is the simplest (lowest) form of the ratio $400:900$

(ii) Ratio of 700gm to 2 kg

Since, 1kg = 1000gm

2kg = 2000gm

therefore,

$700:\ 2000 = \frac{700}{2000} = \frac{7}{20} = 7:20$

(iii) Ratio of 30 sec to 2 min

Since, 1 min = 60sec

2 min = 120sec

therefore,

$30:\ 120 = \frac{30}{120} = \frac{1}{4} = 1:4$

(iv) Ratio of Rs 200 to 300gm

Since, the quantities are not of same kind, so ratio between Rs 200 to 300gm cannot be found.

Example 2: Find the ratio $5a+2b:4a+3b$ if $a:b=4:5$

Solution: Given that $a:b=4:5$ or $\frac{a}{b}=\frac{4}{5}$

Now $5a+2b:4a+3b=\frac{5a+2b}{4a+3b}$

$$=\frac{\frac{5a+2b}{b}}{\frac{4a+3b}{b}}=\frac{5(\frac{a}{b})+2(\frac{b}{b})}{4(\frac{a}{b})+3(\frac{b}{b})}$$ (Dividing numerator and denominator by "b")

$$=\frac{5(\frac{4}{5})+2}{4(\frac{4}{5})+3}=\frac{4+2}{\frac{16}{5}+3}=\frac{6}{\frac{31}{5}}=\frac{30}{31} \quad \because \left(\frac{a}{b}\right)=\left(\frac{4}{5}\right)$$

Hence, $5a+2b:4a+3b=30:31$

Example 3: Find the value of m, when the ratios $3m+5:4m+3$ and $2:3$ are equal.

Solution: According to the given condition

$\Rightarrow \frac{3m+5}{4m+3}=\frac{2}{3}$

$\Rightarrow 3(3m+5)=2(4m+3)$

$\Rightarrow 9m+15=8m+6$

$\Rightarrow 9m-8m=6-15$

$\Rightarrow m=-9$

Thus the required value of m is -9.

Example 4: What number must be added to each term of the ratio $4:15$ to make it equal to $\frac{2}{3}$.

Solution: Let the required number be a.

According to the given condition

$\Rightarrow \frac{4+a}{15+a}=\frac{2}{3}$

$\Rightarrow 3(4+a)=2(15+a)$

$\Rightarrow 12+3a=30+2a$

$\Rightarrow 3a-2a=30-12$

$\Rightarrow a=18$

Thus the required number is 18.

(b) Proportion:

The equality of two ratios is called proportion or a proportion is a statement, which is expressed as an equivalence of two ratios. If $\frac{a}{b}=\frac{c}{d}$ then a, b, c and d are in proportion and we can write it as $a:b::c:d$ where quantities a and d are called extremes where b and c are called means. The proportion of a, b, c and d is written as

$a:b::c:d$

or $a:b=c:d$

or $\frac{a}{b}=\frac{c}{d}$

$ad=bc$

Note that: The product of extremes = product of means

Example 5: Find the value of x , if $40:60=50:x$

Solution:

Given that $40:60=50:x$

Product of extremes = product of means

$$40x=60\times 50$$

$$x=\frac{60\times 50}{40}=75$$

i.e., $x = 75$

Exercise 18.1

1. Find the ratio of the following.
   (i) 70kg and 28kg (ii) 60cm and 1m (iii) 40sec, 3min
   (iv) 200 *ml* and 2*l* (v) 135° and 360° (vi) 3.5kg, 5kg 200*gm*
2. In a factory, there are 120 workers in which 45 are women and remaining are men. Find the ratio of
   (i) men to women (ii) women to men
   (iii) women to total worker (iv) men to total workers
3. If $5(4x-2y)=3x-4y$, find $x:y$
4. Find the value of 'a' if the ratios $3a+4:2a+5$ and $4:3$ are equal
5. What number must be added to each term of the ratio 5:27 to make it equal to 1:3?
6. If $a:b=5:8$, find the value of $3a+4b:5a+7b$
7. Find the value of x in the following.

   (i) $2x+5:5::3x-2:7$ (ii) $\frac{4x-3}{5}:\frac{3}{4}::\frac{4x}{3}:\frac{7}{2}$

   (iii) $\frac{x-3}{2}:\frac{5}{x-1}::\frac{x-1}{3}:\frac{4}{x+4}$ (iv) $(a^2-ab+b^2):x::\frac{a^3+b^3}{a-b}:(a+b)^2$

   (v) $11-x:8-x::25-x:16-x$

(e) Variation:

Variation is defined as the change in one quantity due to change in other quantity. There are two types of variations.

- Direct variation
- Inverse variation

Direct Variation:

If two quantities A and B are so related that when A increases (or decreases) in given ratio, B also increases (or decreases) in the same ratio then this variation is called direct variation. If a quantity y varies directly with regard to a quantity x , we say that y is directly proportional to x and is written as $y\propto x$

or $y=k\,x$ where $k\neq 0$

The sign $\propto$ is called the sign of proportionality or variation and k is called constant of proportionality or variation.

For example:

(i) Length of side of square and its area are in direct proportion.

(ii) The speed of cycle and covered distance are in direct variation.

Example 1: If y varies directly as x, find

(a) the equation connecting x and y.

(b) the relation between x and y when $x=3$ and $y=7$

(c) the value of y when $x=24$

(d) the value of x when $y=21$

Solution: (a) $\because$ y is directly proportional to x

$\therefore$ $y \propto x$

i.e. $y = kx$ ... (i)

where k is constant of variation

(b) By putting $x=3$ and $y=7$ in equation (i),

we get 7=3k $\Rightarrow k=\frac{7}{3}$

By putting the value of k in equation (i), we get $y=\frac{7}{3}x$ ...(ii)

(c) By putting $x=24$ in equation (ii), we get $y=\frac{7}{3}(24)$ =56

(d) By putting $y=21$, in equation (ii),

we get, $21=\frac{7}{3}x \Rightarrow x=\frac{3\times 21}{7}=9$

Example 2: If y varies directly to the square root of x, $y=10$ when $x=16$, then find y when $x=36$

Solution: Given that y varies directly as square root of x

i.e. $y \propto \sqrt{x}$

or $y=k\sqrt{x}$ ... (i)

where k is constant of variation

By putting $x=16$ and $y=10$ in equation (i)

we get $10=k\sqrt{16} \Rightarrow k=\frac{10}{4}=\frac{5}{2}$

By putting $k=\frac{5}{2}$ in eq (i)

$$y=\frac{5}{2}\sqrt{x} \quad \text{...(ii)}$$

By putting $x=36$ in eq (ii) we get

$$y=\frac{5}{2}\sqrt{36}= \quad y=\frac{5}{2}(6) = 15$$

Example 3: Given that V varies directly as a cube of r and $V=\frac{792}{7}$ when $r=3$

Find the value of V when $r=7$

Solution: Since V varies directly as cube of r.

$\therefore \quad V \propto r^3$

$V = kr^3 \quad$ (i) $\qquad$ (where k is constant of variation)

By putting $r=3$ and $V=\frac{792}{7}$ in eq (i)

we get $\quad \frac{792}{7}=k(3)^3 \Rightarrow k=\frac{792}{7\times 27}=\frac{88}{21}$

Hence $\quad V=\frac{88}{21}r^3 \quad$ (ii)

By putting $r=7$ in eq: (ii)

we get $\quad V=\frac{88}{21}(7)^3=\frac{4312}{3}$

Inverse Variation:

If two variables (quantities) are related to each other in such a way that increase in one variable causes decrease in another variable and vice versa then the variables are in inverse variation to each other. If a quantity y varies inversely with regard to quantity x, we say that y is inversely proportional to x and we write it as

$y \propto \frac{1}{x}$ or $y=\frac{k}{x}$ $\quad$ where $k \neq 0$ (constant of variation)

$\Rightarrow \quad yx=k$

Example 1: If y varies inversely as x and $y=7$ when $x=2$. Find y when $x=126$

Solution: Here y varies inversely as x

i.e. $\quad y \propto \frac{1}{x}$

$\Rightarrow \quad y=k(\frac{1}{x})=\frac{k}{x}$

$\Rightarrow \quad k=xy \quad$ ... (i)

By puting $y=7$ and $x=2$ in eq (i)

we get

$k=(2)(7)=14$

equation (i) becomes

$xy=14 \quad$ ... (ii)

By putting $x=126$ in eq (ii)

we get

$(126)y=14$

$y=\frac{14}{126}$

$y=\frac{1}{9}$

Example 2: If y varies inversely as the cube root of x and $y=27$ when $x=8$. Find x when $y=36$

Solution: Here y varies inversely as $\sqrt[3]{x}$

i.e. $\quad y \propto \frac{1}{\sqrt[3]{x}}$

or $\quad y=\frac{k}{\sqrt[3]{x}} \Rightarrow y\left(\sqrt[3]{x}\right)=k \quad$ ... (i)

By putting $x=8$ and $y=27$ in eq (i)

we get $k=(27)(\sqrt[3]{8})=54$

Now by putting $k=54$ in eq (i)

we get $y\sqrt[3]{x}=54$ ... (ii)

Now, by putting $y=36$ in eq (ii)

we get $(36)\left(\sqrt[3]{x}\right)=54$

$\sqrt[3]{x}=\frac{54}{36}=\frac{3}{2}$

$\Rightarrow \quad x=\frac{27}{8}$

Exercise 18.2

1. If y varies directly as x, and y =10 when $x=3$, find
   (i) y in terms of x (ii) y when x =6 (iii) x when y =15
2. If $V \propto T$ and $V=15$ when T= 24, find
   (i) The equation connecting V and T. (ii) V when $T=30$
   (iii) T when $V=10$
3. If $u \propto \sqrt[3]{v}$ and $u=4$ when $v=64$, find the value of u when $v=216$ and the value of v when $u=5$
4. If F varies directly as m^3 when $m=3$ and $F=81$, find F when $m=5$.
5. If y varies inversely as x and y=10 when $x=3$, find y when $x=10$.
6. The volume V of a gas varies inversely as square root of the pressure P of the gas, if V=12 when P=9, find P when V = 4
7. If $F \propto \frac{1}{r^2}$ and F=8 when $r=2$ then find:
   (i) F when $r=5$ (ii) r when F=24
8. The cube root of x varies as the square of y if $x=8$ when $y=3$ find x when $y=\frac{3}{2}$
9. The force F between two bodies is inversely proportional to the square of the distance between their centers. If F=2 and d=3 then find d when F=72
10. If y varies inversely as $(x-5)$ when $y=6$ and $x=8$, find y when $x=10$

18.1(ii) Find 3rd, 4th, proportionals of proportion and mean proportional in a continued proportion

We are already familiar with proportion that if *a*, *b*, *c* and *d* are in proportion then
$$a:b::c:d$$
Here *a*, *b*, *c*, and *d* are called first, second, third and fourth proportional respectively

(a) Third proportional

Example 1:
If 4,7 and 14 are the first, second and fourth proportional in a proportion respectively then find its third proportional.

Solution: let x be the third proportional
then,
$$4:7::x:14$$
$$\Rightarrow \quad 7x=56$$
$$\Rightarrow \quad x=8$$

Example 2:
If $(a-b), (a^2+ab+b^2)$ and (a^3-b^3) are the first, second and fourth proportional respectively then find the third proportional.

Solution: let x be the third proportional
then,
$$(a-b):(a^2+ab+b^2)::x:a^3-b^3$$
$$\Rightarrow \quad (a^2+ab+b^2)x=(a-b)(a^3-b^3)$$
$$(a^2+ab+b^2)\,x=(a-b)^2(a^2+ab+b^2)$$
$$\Rightarrow \quad x=\frac{(a-b)^2(a^2+ab+b^2)}{(a^2+ab+b^2)}$$
$$\Rightarrow \quad x=(a-b)^2$$

(b) fourth proportional

Example 1: Find the fourth proportional if first three proportional are $a^3+b^3, a-b$ and a^2-ab+b^2

Solution: let x be the fourth proportional

then $a^3+b^3 : a-b :: (a^2-ab+b^2) : x$

i.e. $x\,(a^3+b^3) = (a-b)(a^2-ab+b^2)$

$$\Rightarrow \quad x = \frac{(a-b)(a^2-ab+b^2)}{(a+b)(a^2-ab+b^2)}$$

$$\Rightarrow \quad x = \frac{a-b}{a+b}$$

Continued Proportion and mean proportional

The quantities *a*, *b*, *c*, are said to be in continued proportion if $a:b=b:c$

or $a:b::b:c \quad \Rightarrow b^2 = ac$

Here *b* is called mean proportional

Example 1: Find a mean proportional of x^2-y^2 *and* $\frac{x-y}{x+y}$

Solution: Let z be the mean proportional

then

$$x^2-y^2 : z :: z : \frac{x-y}{x+y}$$

i.e., $z^2 = (x^2-y^2)\dfrac{(x-y)}{x+y} = (x-y)^2$

$\Rightarrow \quad z = (x-y)$

Example 2: Find the value of a if $7, a-3$ and 28 are in continued proportion

Solution: Since $7, a-3$ and 28 are in continued proportion.

$7 : a-3 :: a-3 : 28$

i.e. $(a-3)^2 = 7\times 28$

$\Rightarrow$ $(a-3)^2 = 196$

$\Rightarrow$ $a-3 = 14$

$\Rightarrow$ $a = 14+3$

$\Rightarrow$ $a = 17$

Exercise 18.3

1. Find the third proportional if first, second and fourth proportional are

(i) 6, 18 and 54 (ii) $a^2-b^2, a+b$ and $a-b$

(iii) $(x+y)^2, x^3+y^3$ and $x+y$ (iv) $\dfrac{a^3+b^3}{a^2-b^2}, \dfrac{a^2-ab+b^2}{a-b}$ and $a+b$

2. Find the fourth proportional to

(i) 8, 4, 2 (ii) $a^3+b^3,\ a^2-b^2,\ a^2-ab+b^2$

(iii) $a^2-8a+12,\ a-2,\ 2a^3-12a^2$

(iv) $(a^2-b^2)(a^2-ab+b^2), a^3+b^3, a^3-b^3$

3. Find the mean proportional to

(i) 8, 18 (ii) $5ab^2, 28a^3b^2$

(iii) $a^4-b^4, \dfrac{a^2-b^2}{a^2+b^2}$ (iv) $a^3-b^3, \dfrac{a-b}{a^2-ab+b^2}$

4. Find the value of x in the following continued proportions.

(i) 45, x, 5 (ii) 16, x, 9

(iii) 12, 3x -6, 27 (iv) 7, x -3, 112

18.2 Theorems on proportions

18.2.1: Apply theorems of invertendo, alternando, componendo, dividendo to solve the problem of proportions.

(i). Theorem of invertendo:

If $a:b=c:d$ then $b:a=d:c$

The above statement is called theorem of invertendo.

Proof: Since $a:b=c:d$

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{b}{a}=\frac{d}{c} \qquad \text{(Taking reciprocal of both sides)}$$

i.e. $b:a=d:c$ **Hence proved.**

For example: (i) If 2:3 = 4:6 then by theorem of invertendo

$3:2=6:4$

(ii) If $4p:5q=2r:5s$ then by theorem of invertendo

$5q:4p=5s:2r$

(ii). Theorem of alternando:

If $a:b=c:d$ then $a:c=b:d$

This statement is called theorem of alternando.

Proof: Since $a:b=c:d$

or $\dfrac{a}{b}=\dfrac{c}{d}$

$\Rightarrow$ $ad=bc$

Dividing both sides by cd

we get

we get $\dfrac{a}{c}=\dfrac{b}{d}$

i.e. $a:c=b:d$ **Hence proved.**

Example:

(i) If $2:5=6:15$ then by theorem of alternando

$2:6=5:15$

(ii) If $4p-1:2-3q=5+2r:2s+1$ then by theorem of alternando

$4p-1:5+2r=2-3q:2s+1$

(iii) Theorem of componendo:

If $a:b=c:d$ then according to theorem of componendo

(i) $a+b:b=c+d:d$ (ii) $a:a+b=c:c+d$

Since $a:b=c:d$

or $\frac{a}{b}=\frac{c}{d}$

Adding 1 to both sides

$\frac{a}{b}+1=\frac{c}{d}+1$

$\Rightarrow \frac{a+b}{b}=\frac{c+d}{d}$

or $a+b:b=c+d:d$

Hence proved

(ii) $\frac{a}{b}=\frac{c}{d}$

$\Rightarrow \frac{b}{a}=\frac{d}{c}$ (By invertendo theorem)

By adding 1 to both sides

$\frac{b}{a}+1=\frac{d}{c}+1$

$\frac{b+a}{a}=\frac{d+c}{c}$

$\Rightarrow \frac{a}{b+a}=\frac{c}{c+d}$

or $a:a+b=c:c+d$

Hence proved

If $p+2:q=r:s-3$ then by theorem of componendo (i)

$p+2+q:q=r+s-3:s-3$

Similarly by theorem of componendo (ii)

$p+2:p+2+q=r:r+s-3$

(iv). Theorem of Dividendo:

If $a:b=c:d$ then according to theorem of dividendo

(i) $a-b:b=c-d:d$ (ii) $a:a-b=c:c-d$

Proof: (i) Since $a:b=c:d$

or $\frac{a}{b}=\frac{c}{d}$

Subtracting 1 from both sides.

$$\frac{a}{b}-1=\frac{c}{d}-1$$

$$\frac{a-b}{b}=\frac{c-d}{d}$$

or $a-b:b=c-d:d$

Hence proved.

(ii) Since $a:b=c:d$

or $\frac{a}{b}=\frac{c}{d}$

$\Rightarrow \frac{b}{a}=\frac{d}{c}$ (By invertendo theorem)

Subtracting 1 from both sides.

$$\frac{b}{a}-1=\frac{d}{c}-1$$

$$\frac{b-a}{a}=\frac{d-c}{c}$$

$\Rightarrow \frac{a}{b-a}=\frac{c}{d-c}$ (By invertendo theorem)

or $a:b-a=c:d-c$

Hence proved.

Example:

If $m+5:n-3=4p+7:3q+2$ then show that $m-n+8:n-3=4p-3q+5:3q+2$

Solution: Since

$$m+5:n-3=4p+7:3q+2$$

or $\frac{m+5}{n-3}=\frac{4p+7}{3q+2}$

By dividendo theorem

$$\frac{(m+5)-(n-3)}{n-3}=\frac{(4p+7)-(3q+2)}{3q+2}$$

$$\frac{m-n+8}{n-3}=\frac{4p-3q+5}{3q+2}$$

i.e. $m-n+8:n-3=4p-3q+5:3q+2$

Hence shown.

(v). Theorem of componendo and dividendo:

If $a:b=c:d$ then according to theorem of componendo and dividendo

(i) $a+b:a-b=c+d:c-d$

(ii) $a-b:a+b=c-d:c+d$

Example 1: If $m:n=p:q$ then show that $3m-2n:3m+2n=3p-2q:3p+2q$

Solution: Since $m:n=p:q$

or $\frac{m}{n}=\frac{p}{q}$

Multiplying both sides by $\frac{3}{2}$

$$\frac{3m}{2n}=\frac{3p}{2q}$$

By using componendo and dividendo theorem (ii)

$$\frac{3m-2n}{3m+2n}=\frac{3p-2q}{3p+2q}$$

or $3m-2n:3m+2n=3p-2q:3p+2q$

Hence shown.

Example 2: If $3p+4q:3p-4q=3r+4s:3r-4s$ then show that $p:q=r:s$

Solution: Since $3p+4q:3p-4q=3r+4s:3r-4s$

By using componendo and dividendo theorem (i)

$$\therefore \quad \frac{(3p+4q)+(3p-4q)}{(3p+4q)-(3p-4q)}=\frac{(3r+4s)+3r-4s}{(3r+4s)-(3r-4s)}$$

$$\frac{3p+4q+3p-4q}{3p+4q-3p+4q}=\frac{3r+4s+3r-4s}{3r+4s-3r+4s}$$

$$\frac{6p}{8q}=\frac{6r}{8s} \qquad \text{(by cancelling } \frac{6}{8} \text{ from both side)}$$

$$\Rightarrow \quad \frac{p}{q}=\frac{r}{s}$$

or $p:q=r:s$

Hence shown.

Example 3: Using theorem of componendo and dividend theorem find the value of x.

if $\dfrac{\sqrt{x+6}-\sqrt{x-6}}{\sqrt{x+6}+\sqrt{x-6}}=\dfrac{2}{5}$.

Solution: We have $\dfrac{\sqrt{x+6}-\sqrt{x-6}}{\sqrt{x+6}+\sqrt{x-6}}=\dfrac{2}{5}$

By componendo and dividendo theorem

$$\frac{(\sqrt{x+6}-\sqrt{x-6})+(\sqrt{x+6}+\sqrt{x-6})}{(\sqrt{x+6}-\sqrt{x-6})-(\sqrt{x+6}+\sqrt{x-6})}=\frac{2+5}{2-5}$$

$$\frac{\sqrt{x+6}-\sqrt{x-6}+\sqrt{x+6}+\sqrt{x-6}}{\sqrt{x+6}-\sqrt{x-6}-\sqrt{x+6}-\sqrt{x-6}}=\frac{7}{-3}$$

$$\frac{2\sqrt{x+6}}{-2\sqrt{x-6}}=\frac{7}{-3} \quad \Rightarrow \quad \frac{\sqrt{x+6}}{\sqrt{x-6}}=\frac{7}{3}$$

squaring both sides

$$\frac{x+6}{x-6}=\frac{49}{9}$$

$$9x+54=49x-294$$

$$9x-49x=-294-54$$

$$-40x=-348$$

$$x=\frac{348}{40}=\frac{87}{10}$$

Example 4: Solve the equation: $\dfrac{(x+3)^2+(x-1)^2}{(x+3)^2-(x-1)^2}=\dfrac{5}{4}$ by using componendo and dividendo theorem.

Solution: We have $\dfrac{(x+3)^2+(x-1)^2}{(x+3)^2-(x-1)^2}=\dfrac{5}{4}$

By componendo and dividendo theorem

$$\frac{(x+3)^2+(x-1)^2+(x+3)^2-(x-1)^2}{(x+3)^2+(x-1)^2-(x+3)^2+(x-1)^2}=\frac{5+4}{5-4}$$

$$\Rightarrow \quad \frac{2(x+3)^2}{2(x-1)^2}=\frac{9}{1}$$

$$\Rightarrow \quad \left(\frac{x+3}{x-1}\right)^2=9$$

$$\Rightarrow \quad \frac{x+3}{x-1}=\pm 3$$

$\dfrac{x+3}{x-1}=3$ or $\dfrac{x+3}{x-1}=-3$

$x+3=3x-3$ $\quad$ $x+3=-3x+3$

$-2x=-6$ $\quad$ $4x=0$

$x=3$ $\quad$ $x=0$

$\therefore$ The solution set is $\{0,3\}$

Example 5: Prove that $a:b=c:d$ if $\dfrac{ac^2-bd^2}{ac^2+bd^2}:\dfrac{c^3-d^3}{c^3+d^3}$

Solution:

Since $\dfrac{ac^2-bd^2}{ac^2+bd^2}=\dfrac{c^3-d^3}{c^3+d^3}$

By componendo and dividendo theorem

$$\frac{ac^2-bd^2+ac^2+bd^2}{ac^2-bd^2-ac^2-bd^2}=\frac{c^3-d^3+c^3+d^3}{c^3-d^3-c^3-d^3}$$

$$\frac{2ac^2}{-2bd^2}=\frac{2c^3}{-2d^3}$$

$$\Rightarrow \quad \frac{a}{b}=\frac{c}{d}$$

or $\quad a:b=c:d$

Hence Proved.

Exercise 18.4

1. If $x : y = z : w$ then prove that

(i) $\dfrac{4x+3y}{4x-3y} = \dfrac{4z+3w}{4z-3w}$ (ii) $\dfrac{5x-3y}{5x+3y} = \dfrac{5z-3w}{5z+3w}$

(iii) $\dfrac{x^3+y^3}{x^3-y^3} = \dfrac{z^3+w^3}{z^3-w^3}$ (iv) $\dfrac{3x+2y}{3x-2y} = \dfrac{3z+2w}{3z-2w}$

(v) $\dfrac{2x+3y+2z+3w}{2x+3y-2z-3w} = \dfrac{2x-3y+2z-3w}{2x-3y-2z+3w}$

2. Prove that $a:b=c:d$ if

(i) $a^2-b^2 : a^2+b^2 = ac-bd : ac+bd$

(ii) $\dfrac{a+b+c+d}{a+b-c-d} = \dfrac{a-b+c-d}{a-b-c+d}$

3. Solve the following equation by using componendo – dividendo theorem

(i) $\dfrac{(x+3)^2-(x-5)^2}{(x+3)^2+(x-5)^2} = \dfrac{4}{5}$ (ii) $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = \dfrac{1}{2}$

(iii) $\dfrac{\sqrt{x+5}-\sqrt{x-5}}{\sqrt{x-5}+\sqrt{x-5}} = \dfrac{1}{10}$ (iv) $\dfrac{(x+3)^2+(x-1)^3}{(x+3)^2-(x-1)^3} = \dfrac{11}{7}$

18.3 Joint variation

18.3(i) Define joint variation

When a variable depends on the product or quotient of two or more variables is called the joint variation. In other words, joint variation occurs when a variable varies directly or inversely with multiple variables.

For example

(i) If x varies directly with both y and z we have $x \propto yz \Rightarrow x = kyz$

(ii) If x varies directly with y and inversely with z we have $x \propto \dfrac{y}{z} \Rightarrow x = \dfrac{ky}{z}$

18.3 (ii) Solve problem related to joint variation.

Example 1:

If y varies directly as $x\,z^2$ and $y=4$ when $x=6$ and $z=3$. Find the value of y when $x=-81$ and $z=5$

Solution . Since y varies directly as $x\,z^2$

$\therefore \quad y \propto xz^2$

i.e. $\quad y = k\,xz^2 \quad$ (i)

By putting, $y=4$, $x=6$ and $z=3$ in eq (i)

$4 = k\ (6)\ (3)^2 \ \Rightarrow\ k = \dfrac{4}{54} = \dfrac{2}{27}$

By putting, $k = \frac{2}{27}$ in eq (i)

we get $y = \frac{2}{27} x z^2$ ...(ii)

Now by putting $x = -81$ and $z = 5$ in eq (ii)

$$y = \frac{2}{27}(-81)(5)^2 = 2(-3)(25) = -150$$

Example 2. If s varies directly as v^3 and inversely as $\sqrt{t}$ u and s = 16 when v = 4, t = 9 and u = 24; find the value of s when u = 25, t = 36 and v = 5.

Solution. Since s varies directly as v^3 and inversely as $\sqrt{t}$ u.

$$\therefore \quad s \propto \frac{v^3}{u\sqrt{t}} \quad \Rightarrow \quad s = k\frac{v^3}{u\sqrt{t}} \quad \text{... (i)}$$

By putting s = 16, v = 4, t = 9 and u = 24 in eq (i)

$$16 = \frac{k(4)^3}{\sqrt{9}(24)}$$

$$16 = \frac{k(64)}{72} \quad \Rightarrow \quad k = 16 \times \frac{72}{64} = 18$$

By putting k = 18 in eq (i)

$$s = \frac{18\,v^3}{u\sqrt{t}} \quad \text{... (ii)}$$

By putting u = 25, t = 36 and v = 5 in eq (ii) we get

$$s = \frac{18\,(5)^3}{\sqrt{36}\,(25)} = \frac{18 \times 125}{6 \times 25} = 15$$

Exercise 18.5

1. If y varies directly as x^2 and z and y=6 when $x = 4, z = 9$. Write y as a function of x and z and determine the value of y when $x = -8$ and $z = 12$
2. If y varies directly as x and u^2 and inversely as v and t. y=40 when $x = 8$, u = 5, v = 3 and t = 2 find y in terms of x, u, v and t. Also find the value of y when $x = -2$, u = 4, v = 3, t = −1.
3. If w varies directly as u^2 and inversely as cubic root of v and w = 216 when u = 6 and v=27. Find the value of w when u = 10 and v = 125.
4. If the time period T of simple pendulum is directly proportional to the square root of its length L and inversely proportional to the square root of acceleration due to gravity 'g' if T = 2sec L=100cm and g = 9.8m/s. Find time period of simple pendulum when L = 200m and g = 7.6 at a particular height.
5. The volume V of a particular gas is directly proportional to the temperature T and inversely proportional to the square root of pressure. If V=100, T = 30 and P = 64. Find the value of V when T=60 and P=81.

18.4 k – Method

When two or more ratios are equal many useful proportion may be proved by introducing a single symbol 'k' to denote each of the equal ratios. This method of using symbol k to each ratio is called k-method, the k method is a very useful method in solving many problems.

18.4 (i) Use k-method to prove conditional equalities involving proportions

If $a:b::c:d$ is a proportion.

Then by k – method

i.e $\frac{a}{b} = \frac{c}{d} = k$

$\Rightarrow \frac{a}{b} = k$ and $\frac{c}{d} = k$

$a = b\,k$ ... (i) and $c = d\,k$ ... (ii)

By using these equations, we can solve problem relating to proportion.

Example1. If $a:b::c:d$ then show that $\frac{8a-5b}{8a+5b} = \frac{8c-5d}{8c+5d}$

Solution: Since $a : b = c : d$

Let $\frac{a}{b} = \frac{c}{d} = k$

Then $a = bk$ and $c = dk$.

Now L H.S $= \frac{8a-5b}{8a+5b} = \frac{8bk-5b}{8bk+5b} = \frac{b(8k-5)}{b(8k+5)} = \frac{8k-5}{8k+5}$

R.H.S $= \frac{8c-5d}{8c+5d} = \frac{8dk-5d}{8dk+5d} = \frac{d(8k-5)}{d(8k+5)} = \frac{8k-5}{8k+5}$

$\because$ L.H.S = R.H.S

$\therefore \frac{8a-5b}{8a+5b} = \frac{8c-5d}{8c+5d}$

Hence shown.

Example 2: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ then prove then $\frac{(a+c+e)^3}{(b+d+f)^3} = \frac{ace}{bdf}$.

Proof: we have $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

Then $\frac{a}{b} = k\ ,\ \frac{c}{d} = k\ ,\ \frac{e}{f} = k$

$\Rightarrow a = bk \quad c = dk$ and $e = fk$

L.H.S $= \frac{(a+c+e)^3}{(b+d+f)^3} = \frac{(bk+dk+fk)^3}{(b+d+f)^3} = \frac{[k(b+d+f)]^3}{(b+d+f)^3} = k^3$

$$\text{R.H.S} = \frac{ace}{bdf} = \frac{(bk)(dk)(fk)}{bdf} = \frac{k^3 bdf}{bdf} = k^3$$

L.H.S = R.H.S

$$\frac{(a+c+e)^3}{(b+d+f)^3} = \frac{ace}{bdf}$$

Hence proved.

Example 3:

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ then show that $(a^2+c^2+e^2)(b^2+d^2+f^2) = (ab+cd+ef)^2$

Proof: Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

$\Rightarrow \quad a = bk, \quad c = dk$ and $e = fk$

$$\begin{aligned} \text{L.H.S} &= (a^2+c^2+e^2)\,(b^2+d^2+f^2) \\ &= (b^2k^2+d^2k^2+f^2k^2)\,(b^2+d^2+f^2) \\ &= k^2(b^2+d^2+f^2)\,(b^2+d^2+f^2) \\ &= k^2(b^2+d^2+f^2)^2 \end{aligned}$$

$$\begin{aligned} \text{R.H.S} &= (ab+cd+ef)^2 \\ &= \left[(bk)b+(dk)d+(fk)f\right]^2 \\ &= \left[k\,b^2+k\,d^2+k\,f^2\right]^2 \\ &= \left[k(b^2+d^2+f^2)\right]^2 \\ &= k^2(b^2+d^2+f^2)^2 \end{aligned}$$

L.H.S = R.H.S

$\therefore \quad (a^2+c^2+e^2)(b^2+d^2+f^2) = (ab+cd+ef)^2$

Hence shown.

Exercise 18.6

1. If $p:q = r:s$ then show that

(i) $\frac{8p-3q}{8p+3q} = \frac{8r-3s}{8r+3s}$

(ii) $\sqrt[3]{\frac{p^3+r^3}{q^3+s^3}} = \frac{p}{q}$

(iii) $(p^2+q^2) : \frac{p^3}{p+q} = (r^2+s^2) : \frac{r^3}{r+s}$

(iv) $p^5+r^5 : q^5+s^5 = p^3r^2 : q^3s^2$

(v) $\frac{p-q}{p} : \frac{q}{p+q} = \frac{r-s}{r} : \frac{s}{r+s}$

2. If $a:b = c:d = e:f$ then show that

(i) $\frac{a^4b^2+a^2e^2-e^4f}{b^6+b^2f^2-f^5} = \frac{a^4}{b^4}$

(ii) $\frac{a^2b+c^2d+e^2f}{ab^2+cd^2+ef^2} = \frac{a+c+e}{b+d+f}$

(iii) $\frac{ac}{bd}+\frac{ce}{df}+\frac{ea}{fb} = \frac{a^2}{b^2}+\frac{c^2}{d^2}+\frac{e^2}{f^2}$

18.4 (ii) Solve real life problems based on variations:

Example 1.

The potential energy of a body varies jointly as the mass "m" of the body and the height "h" If potential energy is 1960 joules when m=2kg and h=100 then determine potential energy of the body having mass 5kg and height 300m.

Solution: Since potential energy "p" varies jointly as the mass m and the height h

i.e. $p \propto mh$

p = kmh … (i) when k is constant of proportionality

By putting m = 2, h = 100 and p = 1960 in equation (i)

We get $1960 = k(2)(100) \Rightarrow k = \dfrac{1960}{200} = 9.8$

By putting the value of k in equal (i),

$p = 9.8mh$ … (ii)

Now by putting m=5 and h = 300 in eq: (ii)

We get $p = 9.8(5)(300) = 14700$ joules.

Example 2.

According to Hook's law the force F applied to stretch a spring varies directly as the amount of elongation S when F=64 *lb* and S=3.2 inches.

Find. (i) S when F=100 *lb*

(ii) F when S=0.4 inches.

Solution: Since $F \propto S$

F=kS… (i)

By Putting F=64 & S=3.2

we get

64=3.2 k

$\Rightarrow \quad k = \dfrac{64}{3.2} = 20$

By putting k = 20 in eq (i)

we get F=20 S … (ii)

(i) By putting F=100 in eq (ii)

we get 100=20 S

$\Rightarrow$ S=5 inch

(ii) By putting S=0.4 in eq (ii)

We get F=20 (0.4) = 8 *lb*

Example 3:

Labour cost C varies jointly as the number of workers "*n*" and the average number of days *d*, if the cost of 200 workers for 10 days is Rs.450,000. Find the labour cost of 300 workers for 16 days.

Solution: Since cost varies directly as the number of workers and the days.

i.e., $C \propto nd$

$C = knd$ (i)

By putting $C = 450,000, n = 200$ and $d = 10$ in eq (i)

$$450,000 = k(200)(10)$$

$$\Rightarrow \quad k = \frac{450,000}{2000} = 225$$

By putting $k = 225$ in eq (i)

we get $C = 225 \cdot nd$ (ii)

By putting $n = 300$ and $d = 16$ in eq (ii)

we get $C = 225 \times 300 \times 16$

$C = 1080000$

Hence the cost for 300 workers for 16 days is Rs.1080000.

Example 4:

The expenses of a hostel are partly constant and partly vary according to the number of students. When the number of students are 140 the expenditure is Rs 19300 and for 200 students the expenditure is Rs 26500. Find the expenses when number of students are 300.

Solution: Let the constant expenses be denoted as C, Q denotes partly expense, x is the number of students and P denotes the total expense.

Since the expenses varies directly to number of students.

i.e., $Q \propto x$

$Q = nx$ where n is constant

$P = C + Q$

$P = C + nx$ where n is constant

According to given condition

Hence $19300 = C + 140n$ ...(i)

and $26500 = C + 200n$ ...(ii)

also $P = C + 300n$ ...(iii)

From (i) and (ii), we get $n = 120$ and $C = 2500$

Now, equation (iii) becomes

$$P = 2500 + 300(120) = 38500$$

The total expense for 300 students is 38500.

Exercise 18.7

1. The current in a wire is directly proportional to the potential difference and inversely proportional to the resistance R. If I=6 amperes, when V=220 volts and R=5 ohms. Find I when V=180 volts and R=8 ohms
2. The intensity of light measured in foot candles varies inversly with the square of the distance from the light source. Suppose the intensity of a high bulb is 0.08 foot candles at a distance of 3 meters .find the intensity level at 8 meters.
3. The strength S of a rectangular beam can varies directly as the breadth b and the square of the depth d. if a bean 9 cm wide a 12 cm deep with support 1200 pound. What weight a bean of 12 cm wide and 9cm deep will support.
4. Labor cots C varies jointly as the number of worker n and the average number of days d, if the cost of 100 workers for 15 days is 9000, then find the labour cost of 300 workers for 20 days.
5. The sales tax on the purchase of a new car varies directly as the price of a car. If a new car is purchased in Rs 2000000, then the sale tax is Rs 40000. How much sale tax is charged if the new car is priced at Rs 2800000.

Review Exercise 18

1. Multiple Choice Question

Select the correct option in the following

(i) In a proportion $p:q::r:s$, p is called

(a) third proportional (b) mean (c) fourth proportional (d) first proportional

(ii) In a ratio $l:m$, l is called

(a) consequent (b) antecedent (c) relation (d) non of these

(iii) In a ratio $u:v$, v is called

(a) antecedent (b) consequent (c) relation (d) non of these

(iv) If a,b,c are in continued proportion then b is called ____________ proportion between a & c

(a) 1^{st} (b) mean (c) 3^{rd} (d) None of these

(v) The mean proportional between a^2 and b^2 is ______________

(a) $\sqrt{ab}$ (b) ab (c) $\frac{a}{b}$ (d) $-ab$

(vi) If $x+5:x+7=5:7$ then x is equal to____________

(a) 2 (b) -1 (c) 0 (d) 1

(vii) If 1,9, x and 45 are in proportion, then $x=$

(a) 27 (b) $\frac{1}{5}$ (c) 5 (d) 405

(viii) If $p:q=r:s$ then $p:r=q:s$ this property is called

(a) compodendo (b) invertendo (c) dividendo (d) alternando

(ix) If $x:y=z:w$ then according to componendo

(a) $\frac{x}{x+y}=\frac{z}{z+w}$ (b) $\frac{x}{x-y}=\frac{z}{z-w}$

(c) $\frac{x-y}{x+y}=\frac{z-w}{z+w}$ (d) $\frac{x-y}{y}=\frac{z-w}{w}$

(x) If $a:b=c:d$ then according to alternando property

(a) $\frac{a}{b}=\frac{c}{d}$ (b) $\frac{a+b}{b}=\frac{c+d}{c}$ (c) $\frac{b}{a}=\frac{d}{c}$ (d) $\frac{a}{c}=\frac{b}{d}$

(xi) The fourth proportional to 3,5,12 is

(a) 20 (b) 15 (c) 60 (d) 36

(xii) If $2x$, $3y$ and $6z$ are in continued proportion then

(a) $y^2=12xy$ (b) $9y^2=xz$ (c) $9y^2=12xz$ (d) $3y^2=4xz$

(xiii) If $\frac{x}{y}=\frac{w}{z}$ then according to dividendo property is

(a) $\frac{x-y}{y}=\frac{w-z}{z}$ (b) $\frac{x+y}{y}=\frac{w+z}{z}$

(c) $\frac{x}{x+y}=\frac{w}{w+z}$ (d) None of these

(xiv) Force and acceleration are in

(a) direct proportion (b) joint proportion

(c) inverse proportion (d) None of these

(xv) If $a:4::15:5$ then a =____________

(a) 20 (b) 15 (c) 12 (d) 10

2. Find the ratios of the

(i) 100m and 500cm (ii) 50kg and 300g

3. Find the value of x in the following

(i) $5:x-3=x+11:3$ (ii) $9:x-10=x+13:12$

4. If y varies directly as x and $y = 25$ when $x = 75$ then find y when $x = 144$.

5. If y varies inversely as x and $y = 100$ when $x = \frac{1}{2}$ then find y when $x = 4$.

6. If $x:y=z:w$ then prove that $\frac{7x+5y}{7x-5y}=\frac{7z+5w}{7z+5w}$.

7. Solve $\frac{(x-3)(x-5)}{(x-7)(x-2)}=\frac{(x-6)(x-2)}{(x-1)(x-8)}$ by componendo – dividendo theorem.

8. If x varies directly as y and inversely as z. If $x = 30$ when $y = 15$ and $z = 2$. Find x if $y = 30$ when $z = 12$.

9. The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is 40 ampere, the resistance in 10ohms. Find the current if resistance is 12 ohm.

Summary

- The comparison between two quantities of the same kind is called ratio
- If two ratio $a:b$ and $c:d$ are equal then we can write.

  $a:b::c:d$
- Equality of two ratios is called proportion.
- If two quantities are related in such a way that increase (decrease) in one quantity causes increase (decrease) respectively in the other quantity is called direct variation.
- If two quantities are related in such a way that when one quantity increase, then the other decreases and vice versa is called the quantities are in direct variation.
- Theorems on proportion

  (i) Theorems invertando If $a:b=c:d$ then $b:a=d:c$

  (ii) Theorems of alternendo If $a:b=c:d$ then $a:c=b:d$

  (iii) Theorems of componendo

  If $a:b=c:d$ then (a) $a+b:b=c+d:d$ (b) $a:a+b=c:c+d$

  (iv) Theorem of dividend

  If $a:b=c:d$ then $a-b:b=c-d:d$ and $a:a-b=c:c-d$

  (v) Theorem componendo and dividendo

  If $a:b=c:d$ then

  $a+b:a-b=c-d:c-d$
- If one variable varies directly or inversely with two or more than two variables then it is called joint variation

MATRICES AND DETERMINANTS

Unit 19

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Define
  - A matrix with real entries and relate its rectangular layout (formation) with representation in real life aswell.
  - Know about the rows and columns of a matrix,
  - The order/size of a matrix,
  - Equality of two matrices.
- Define and identify row matrix, column matrix, rectangular matrix, square matrix, zero/null matrix, identity/unit matrix, scalar matrix, diagonal matrix, transpose of a matrix, symmetric (upto three by three,3 x 3) and skew-symmetric matrices.
- Know whether the given matrices are conformable for addition and subtraction.
- Scalar multiplication of a matrix by a real number.
- Add and subtract matrices
- Verify commutative and associative laws w.r.t.addition.
- Define additive identity of a matrix.
- Find additive inverse of a matrix.
- Know whether the given matrices are conformable for multiplication.
- Multiply two (or three) matrices.
- Verify associative law under multiplication.
- Verify distributive laws.
- Verify, with the help of an example that commutative law w.r.t. multiplication does not hold, in general. (i.e., $AB \neq BA$),
- Define multiplicative identity of a matrix.
- Verify the result $(AB)^t = B^t A^t$.
- A matrix with real entries and relate its rectangular layout (formation) with
- Define the determinant of a square matrix.
- Finding the value of determinant of a matrix.
- Define singular and non-singular matrices.
- Define minors and co-factors
- Define adjoint of a matrix.
- Find the multiplicative inverse of a non-singular matrix A and verify that $AA^{-1} = I = A^{-1}A$, where, I isthe identity matrix.
- Use the adjoint method to calculate inverse of a non- singular matrix.
- Verify the result $(AB)^{-1} = B^{-1}A^{-1}$.
- Solve the system of two linear equations, related to real life problems, in two unknowns using
  - Matrix inversion method.
  - Cramer's rule.

19.1 Introduction to Matrices

The matrices belong to a field of mathematics, that is, linear algebra. It is mainly used in business, engineering, physics and computer science. **Arthur Cayley** (1821-1895) was the first mathematician who developed the theory of matrices.

19.1 (i) Define a matrix with real entries and relate its rectangular layout (formation) with representation in real life as well.

A matrix is a rectangular array of elements. The elements can be symbolic expressions or numbers. Matrix is usually denoted by capital letter of English alphabet and each entry (or "element") is shown by a lower-case letter. Elements of a matrix are enclosed within [ ] or ( ). For example:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ or } A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad (1)$$

Here a_{11}, a_{22} and a_{33} are the elements of main diagonal and are called diagonal elements.

Let us take an example of matrix in real life, following table shows the number of residents, televisions and computers in three houses as

	Residents	**Televisions**	**Computers**
House A	4	2	1
House B	6	2	3
House C	2	1	0

The above data can be written in matrix form as

$$\begin{bmatrix} 4 & 2 & 1 \\ 6 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

Note: The plural of word "Matrix" is "Matrices".

19.1 (ii) Know about the rows and columns of a matrix

The horizontal line of entries in a matrix is called row of a matrix and the vertical line of entries in a matrix is called column of a matrix.

For Example, matrix $A=\begin{bmatrix} 2 & 5 & 1 & 4 \\ 6 & 3 & -2 & 0 \end{bmatrix}$ has two horizontal lines of entries so it consists of two rows. It has four vertical lines of entries so it consists of four columns.

19.1.(iii) The order/size of a matrix

The order or size of a matrix is defined by the number of rows and columns it contains. The order of matrix is represented by $m\times n$ (read as m by n) where m is the number of rows, and n is the number of columns in the given matrix. For example,

If $A=\begin{bmatrix} 3 & 4 & 9 \\ 12 & 11 & 35 \end{bmatrix}$ then order of A is 2×3

19.1.(iv) Equality of two matrices

Two matrices A and B are called equal matrices if they have the same order and their corresponding elements are equal. Symbolically, we write as A = B. For example,

$A=\begin{pmatrix}2 & 6 & 1\\ 0 & 5 & 3\end{pmatrix}$ and $B=\begin{pmatrix}2 & 6 & 1\\ 0 & 5 & 3\end{pmatrix}$ are equal matrices

$P=\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 4\end{pmatrix}$ and $Q=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$ are not equal matrices because their orders are not same.

19.2 Types of Matrices

19.2.(i) Define and identify row matrix, column matrix, rectangular matrix, square matrix, zero/null matrix, identity/unit matrix, scalar matrix, diagonal matrix, transpose of a matrix, symmetric (up to three by three, (3×3) and skew-symmetric matrices.

Row matrix

A matrix having one row is called row matrix.

For example, $\begin{bmatrix}1 & 4 & 7\end{bmatrix}$ and $\begin{bmatrix}4 & 3 & 8 & 6\end{bmatrix}$ are row matrices.

Column matrix

A matrix having one column is called column matrix.

For example, $\begin{bmatrix}7\\ 5\\ 9\end{bmatrix}$ and $\begin{bmatrix}1\\ 2\\ 0\\ 5\end{bmatrix}$ are column matrices.

Rectangular matrix

A matrix in which number of rows is not equal to number of columns is called rectangular matrix. i.e. $m\neq n$. For example,

$A=\begin{bmatrix}3\\ -1\end{bmatrix}$; $B=\begin{bmatrix}5 & 3 & 9\\ 1 & -7 & 10\end{bmatrix}$; $C=\begin{bmatrix}2 & 1\\ 7 & 2\\ -1 & 1\end{bmatrix}$ are rectangular matrices.

Square matrix

A matrix in which number of rows is equal to number of columns, is called square matrix. i.e. $m=n$. For example

$A=\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}9 & 5 & 2\\ 1 & 8 & 5\\ 3 & 1 & 6\end{bmatrix}$ and $C=\begin{bmatrix}x & y & z & 1\\ a & b & c & 1\\ p & q & r & 1\\ m & n & o & 1\end{bmatrix}$.

Zero/null matrix

A zero/null matrix is a matrix in which each element is equal to zero. It is generally denoted by capital letter "O". Following are few examples of zero matrix.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Diagonal matrix

A square matrix is called diagonal matrix if all the elements of matrix are zero except at least one element of main diagonal.

For example, $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ are diagonal matrices.

Scalar matrix

A diagonal matrix in which diagonal elements are same. For example,

$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ are scalar matrices.

Identity/unit matrix

A diagonal matrix in which each element of main diagonal is 1. It is generally denoted by capital letter "I". For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Transpose of a matrix

A matrix that is obtained by interchanging the rows into columns or the columns into rows, is called transpose of the matrix. It is denoted by A^t.

For example, transpose of $\begin{bmatrix} 1 & 5 \\ 4 & 6 \\ 0 & 7 \end{bmatrix}$ is $\begin{bmatrix} 1 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}$.

Symmetric matrix

A square matrix is said to be symmetric matrix if its transpose is equal to itself. i.e. $A^t = A$.

For example, if $A = \begin{bmatrix} 1 & 2 & 17 \\ 2 & 5 & -11 \\ 17 & -11 & 9 \end{bmatrix}$ then $A^t = \begin{bmatrix} 1 & 2 & 17 \\ 2 & 5 & -11 \\ 17 & -11 & 9 \end{bmatrix}$. As $A^t = A$, so A is a symmetric matrix.

Skew-symmetric matrix

A square matrix is said to be skew-symmetric matrix if its transpose is equal to negative of itself i.e., $A^t = -A$.

For example, if $A = \begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$ then $A^T = \begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix} = -A$.

As $A^t = -A$, so A is skew-symmertic matrix.

19.3 Addition and Subtraction of Matrices

19.3.(i) Know whether the given matrices are conformable for addition and subtraction

If two matrices A and B have the same order, then they are conformable for addition and subtraction. For example, $\begin{bmatrix} 1 & 4 \\ 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 6 \\ 5 & 9 \end{bmatrix}$ are conformable for addition and subtraction.

19.3.(ii) Scalar multiplication of a matrix by a real number

The scalar multiplication of a matrix refers to the product of a real number and the matrix. In scalar multiplication, each entry in the matrix is multiplied by the given scalar. Here, if c is a scalar and A is a matrix then scalar multiplication of a matrix is denoted by cA. For example,

If $A = \begin{pmatrix} 0 & -1 & 5 \\ -3 & 2 & 1 \\ 2 & 0 & -4 \end{pmatrix}$ and $c = -5$ then $cA = -5\begin{pmatrix} 0 & -1 & 5 \\ -3 & 2 & 1 \\ 2 & 0 & -4 \end{pmatrix}$

i.e. $cA = \begin{pmatrix} 0 & 5 & -25 \\ 15 & -10 & -5 \\ -10 & 0 & 20 \end{pmatrix}$

19.3.(iii) Add and Subtract Matrices

The addition and subtraction of two matrices A and B are obtained by adding or subtracting of the corresponding elements of both matrices.

For example, If $A = \begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 3 & 7 \end{pmatrix}$ then $A+B = \begin{pmatrix} 2+1 & 5+4 \\ -1+3 & 3+7 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 2 & 10 \end{pmatrix}$.

For subtraction,

if $P = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 6 & 1 \\ -2 & 9 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 10 & 4 & 2 \\ 2 & 7 & 1 \\ 1 & 9 & 4 \end{pmatrix}$ then $P - Q = \begin{pmatrix} 1-10 & 0-4 & 2-2 \\ 4-2 & 6-7 & 1-1 \\ -2-1 & 9-9 & 0-4 \end{pmatrix} = \begin{pmatrix} -9 & -4 & 0 \\ 2 & -1 & 0 \\ -3 & 0 & -4 \end{pmatrix}$.

19.3.(iv) Verify commutative and associative laws w.r.t addition

Commutative law w.r.t addition

If A and B are two matrices of the same order, then the commutative law of addition is defined as A + B = B + A.

Example:

Verify commutative law w.r.t addition for $A=\begin{pmatrix}1 & 0 & 2\\ 4 & 6 & 1\\ -2 & 9 & 0\end{pmatrix}$ and $B=\begin{pmatrix}10 & 4 & 2\\ 2 & 7 & 1\\ 1 & 9 & 4\end{pmatrix}$.

Here,

$$A+B=\begin{pmatrix}1+10 & 0+4 & 2+2\\ 4+2 & 6+7 & 1+1\\ -2+1 & 9+9 & 0+4\end{pmatrix}=\begin{pmatrix}11 & 4 & 4\\ 6 & 13 & 2\\ -1 & 18 & 4\end{pmatrix}$$

Now,

$$B+A=\begin{pmatrix}10+1 & 4+0 & 2+2\\ 2+4 & 7+6 & 1+1\\ 1-2 & 9+9 & 4+0\end{pmatrix}=\begin{pmatrix}11 & 4 & 4\\ 6 & 13 & 2\\ -1 & 18 & 4\end{pmatrix}.$$

Since $A+B=B+A$. Therefore, commutative law w.r.t addition is verified.

Associative law w.r.t addition

If A, B and C are three matrices of the same order, then the associative law of addition is defined as (A + B) +C = A+ (B + C).

Example:

Verify associative law w.r.t addition for $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}, B=\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}$ and $C=\begin{pmatrix}9 & 10\\ 11 & 12\end{pmatrix}$.

Here,

$$(A+B)+C=\left[\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}+\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}\right]+\begin{pmatrix}9 & 10\\ 11 & 12\end{pmatrix}=\begin{pmatrix}6 & 8\\ 10 & 12\end{pmatrix}+\begin{pmatrix}9 & 10\\ 11 & 12\end{pmatrix}$$

$$(A+B)+C=\begin{pmatrix}15 & 18\\ 21 & 24\end{pmatrix}$$

Now

$$A+(B+C)=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}+\left[\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}+\begin{pmatrix}9 & 10\\ 11 & 12\end{pmatrix}\right]=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}+\begin{pmatrix}14 & 16\\ 18 & 20\end{pmatrix}$$

$$A+(B+C)=\begin{pmatrix}15 & 18\\ 21 & 24\end{pmatrix}$$

Since $(A+B)+C=A+(B+C)$

Therefore, Associative law w.r.t addition is verified.

19.3.(v) Define additive identity of a matrix

If A and O are two matrices of the same order and A + O = A = O + A.Here O is null matrix and called additive identity of the matrix A.

19.3.(vi) Find additive inverse of a matrix

If A and B are two matrices of the same order such that A + B = O = B + A then A and B are called additive inverses of each other. The additive inverse of matrix A is denoted by −A For example,

$$\text{If } A = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 6 & 1 \\ -2 & 9 & 0 \end{pmatrix}$$

$$\text{then } -A = -\begin{pmatrix} 1 & 0 & 2 \\ 4 & 6 & 1 \\ -2 & 9 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -2 \\ -4 & -6 & -1 \\ 2 & -9 & 0 \end{pmatrix}$$

A and − A are additive inverse of each other.

19.4 Multiplication of Matrices (up to 2 by 2)

19.4.(i) Know whether the given matrices are conformable for multiplication

Two matrices A and B are said to be conformable for multiplication, if the number of columns of the first matrix is equal the number of rows of the second matrix. For example, $\begin{bmatrix} 1 & -4 \\ 8 & 2 \end{bmatrix}$ and $\begin{bmatrix} 4 & -7 \\ 2 & 7 \end{bmatrix}$ are conformable for multiplication.

19.4.(ii) Multiply two (or three) matrices

If two matrices are conformable for multiplication then the element a_{ij} of the product AB is obtained by multiplying the ith row of A by the jth column of B.

Example:1

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$ then compute AB.

Solution:

Here, AB is possible because the number of columns of A is 2 and the number of rows of B.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(0)+2(6) & 1(-1)+2(7) \\ 3(0)+4(6) & 3(-1)+4(7) \end{bmatrix}$$

$$AB = \begin{bmatrix} 0+12 & -1+14 \\ 0+24 & -3+28 \end{bmatrix} = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}$$

$$AB = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}$$

Example 2:

Compute the product: $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 4 & 5 \\ 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 6 & 8 \\ 5 & 1 \end{pmatrix}$.

Solution:

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 4 & 5 \\ 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 6 & 8 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4+12 & 5+0 \\ 8+16 & 10+0 \end{pmatrix} \cdot \begin{pmatrix} 6 & 8 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 5 \\ 24 & 10 \end{pmatrix} \cdot \begin{pmatrix} 6 & 8 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 96+25 & 128+5 \\ 144+50 & 192+10 \end{pmatrix} = \begin{pmatrix} 121 & 133 \\ 194 & 202 \end{pmatrix}$$

19.4.(iii) Verify associative law under multiplication

Let A, B and C are three matrices then associative law under multiplication is defined as (AB)C=A(BC).

Example 1: Verify associative law under multiplication when

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} -2 & 3 \\ 4 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} -1 & 5 \\ 1 & 2 \end{pmatrix}.$$

Verification:

Taking L.H.S $(AB)C$

$$\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 4 & 2 \end{bmatrix}\right)\begin{bmatrix} -1 & 5 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 13 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 36 \\ -5 & 4 \end{bmatrix}$$

Now taking R.H.S: $A(BC)$

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} -2 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 1 & 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 5 & -4 \\ -2 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 36 \\ -5 & 4 \end{bmatrix}$$

Hence, associative law under multiplication is verified.

19.4.(iv) Verify distributive laws

Let A, B and C are three matrices then the distributive laws are

(i) A(B+C) = (AB)+(AC) (Left distributive law)

(ii) (B+C) A = (BA)+(CA) (Right distributive law)

Example: If $A=\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}, B=\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}$ and $C=\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}$ then verify the following:

(i) A(B+C) = (AB)+(AC) (ii) (B+C) A = (BA)+(CA)

Verification:

(i) A(B+C) = (AB)+(AC)

Taking L.H.S=A(B+C)

$$\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\left(\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}+\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}\right)$$

$$=\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}-2 & -1\\ 1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}0 & 3\\ -1 & -2\end{bmatrix}$$

Now, R.H.S = AB + AC

$$\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}+\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}+\begin{bmatrix}-2 & 2\\ 0 & -1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 3\\ -1 & -2\end{bmatrix}$$

Hence, A(B+C) = (AB)+(AC) is verified. While, applying similar procedure, (B+C) A =(BA)+(CA) can be verified.

19.4.(v) Verify with the help of example that commutative law w.r.t. multiplication does not hold in general (i.e., AB ≠ BA)

Example: Let $A=\begin{bmatrix}5 & 2\\ -2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 2\\ 3 & -2\end{bmatrix}$ then verify that AB ≠ BA.

Verification:

$$AB=\begin{bmatrix}5 & 2\\ -2 & -1\end{bmatrix}\cdot\begin{bmatrix}-1 & 2\\ 3 & -2\end{bmatrix}$$

$$AB=\begin{bmatrix}(-5+6) & (10-4)\\ (2-3) & (-4+2)\end{bmatrix}=\begin{bmatrix}1 & 6\\ -1 & -2\end{bmatrix}$$

Now,

$$BA=\begin{bmatrix}-1 & 2\\ 3 & -2\end{bmatrix}\cdot\begin{bmatrix}5 & 2\\ -2 & -1\end{bmatrix}$$

$$BA=\begin{bmatrix}(-5-4) & (-2-2)\\ (15+4) & (6+2)\end{bmatrix}=\begin{bmatrix}-9 & -4\\ 19 & 8\end{bmatrix}$$

$\because$ AB ≠ BA $\therefore$ the commutative law w.r.t multiplication does not hold, in general.

19.4.(vi) Define multiplicative identity of a matrix.

Let A be a square matrix of order n and I_n is unit matrix such that $AI_n = A$ or $I_nA = A$ here I_n is called the multiplicative identity of A.

Example:

Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, verify that $I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is the multiplicative identity.

Solution, given that

$$AI_2=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=A$$

$$I_2A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=A.$$

19.4(vii) Verify the result $(AB)^t = B^tA^t$

Example: If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}$ verify that $(AB)^t=B^tA^t$

Verification:

$$AB=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}=\begin{bmatrix}5 & -2\\ 11 & -4\end{bmatrix}$$

and the transpose of AB is:

$$(AB)^t=\begin{bmatrix}5 & 11\\ -2 & -4\end{bmatrix}.$$

Now,

$$B^tA^t=\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}=\begin{bmatrix}5 & 11\\ -2 & -4\end{bmatrix}$$

$$(AB)^t=B^tA^t.$$

Hence verified.

Example 2: Verify the result: $(AB)'=B'A'$, If $A=\begin{pmatrix}6 & 5 & 2\\ 4 & 6 & 3\\ 0 & 1 & 7\end{pmatrix}$ and $B=\begin{pmatrix}6 & 8 & 0\\ 5 & 1 & 2\\ 3 & 6 & 4\end{pmatrix}$.

Verification:

$$AB=\begin{pmatrix}6 & 5 & 2\\ 4 & 6 & 3\\ 0 & 1 & 7\end{pmatrix}\cdot\begin{pmatrix}6 & 8 & 0\\ 5 & 1 & 2\\ 3 & 6 & 4\end{pmatrix}$$

$$=\begin{pmatrix}36+25+6 & 48+5+12 & 0+10+8\\ 24+30+9 & 32+6+18 & 0+12+12\\ 0+5+21 & 0+1+42 & 0+2+28\end{pmatrix}=\begin{pmatrix}67 & 65 & 18\\ 63 & 56 & 24\\ 26 & 43 & 30\end{pmatrix}$$

$$(AB)^t = \begin{pmatrix} 67 & 65 & 18 \\ 63 & 56 & 24 \\ 26 & 43 & 30 \end{pmatrix}^t = \begin{pmatrix} 67 & 63 & 26 \\ 65 & 56 & 43 \\ 18 & 24 & 30 \end{pmatrix}$$

$$\text{Now, } B^t A^t = \begin{pmatrix} 6 & 8 & 0 \\ 5 & 1 & 2 \\ 3 & 6 & 4 \end{pmatrix}^t \cdot \begin{pmatrix} 6 & 5 & 2 \\ 4 & 6 & 3 \\ 0 & 1 & 7 \end{pmatrix}^t$$

$$= \begin{pmatrix} 6 & 5 & 3 \\ 8 & 1 & 6 \\ 0 & 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 6 & 4 & 0 \\ 5 & 6 & 1 \\ 2 & 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 36+25+6 & 24+30+9 & 0+5+21 \\ 48+5+12 & 32+6+18 & 0+1+42 \\ 0+10+8 & 0+12+12 & 0+2+28 \end{pmatrix}$$

$$= \begin{pmatrix} 67 & 63 & 26 \\ 65 & 56 & 43 \\ 18 & 24 & 30 \end{pmatrix}$$

$(AB)^t = B^t A^t.$

Hence verified.

EXERCISE: 19.1

1. Specify the type of each of the following matrices.

(i). $\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$ (ii). $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ (iii). $\begin{bmatrix} -2 & 0 \end{bmatrix}$ (iv). $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ (v). $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.

2. What is the order of each of the following matrix?

(i). $\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 3 \\ -1 & -3 & 2 \end{bmatrix}$ (ii). $\begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 3 \end{bmatrix}$ (iii). $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ (iv). $\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ (v). $\begin{bmatrix} 0 \end{bmatrix}$

3. Check whether the following matrices or symmetric matrix or skew symmetric matrix

(i). $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$ (ii). $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ (iii). $\begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ (iv). $\begin{bmatrix} 0 & -6 & 4 \\ -6 & 0 & 7 \\ 4 & 7 & 0 \end{bmatrix}$

4. If $A=\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}, B=\begin{bmatrix}2 & -1\\1 & 0\end{bmatrix}, C=\begin{bmatrix}2 & 3\\-1 & 2\end{bmatrix}, D=\begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\1 & 2 & 3\end{bmatrix}$ and $E=\begin{bmatrix}1 & 2 & -1\\-1 & 0 & 2\\2 & 1 & 0\end{bmatrix}$ then

compute the following (if possible)

(i) $A+C$ (ii) $A+E$ (iii) $B-D$ (iv) $2B+3A$

(v) $B-C$ (vi) A^2 (vii) B^2 (vii) $D+E$

5. For the matrices $A=\begin{bmatrix}-1 & 0 & 1\\2 & 1 & 0\\3 & 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 2 & 3\\-2 & 4 & 1\\3 & 2 & 1\end{bmatrix}$. Find

(i) $A+B$ (ii) $A-B$ (iii) $3A+2B$

(iv) AB (v) BA (vi) A^2

6. Verify that: (i) $A+B=B+A$ and (ii) $C-D\neq D-C$ for the following matrices

$A=\begin{pmatrix}2 & -1\\-7 & 4\end{pmatrix}, B=\begin{pmatrix}-3 & 0\\7 & -4\end{pmatrix}, C=\begin{pmatrix}3 & 1 & -4\\4 & 3 & 1\\1 & 4 & -3\end{pmatrix}$ and $D=\begin{pmatrix}2 & 7 & -5\\-2 & 1 & 0\\6 & 3 & 4\end{pmatrix}$.

7. Show that $AB\neq BA$ when $A=\begin{pmatrix}8 & 2 & -1\\2 & 1 & 2\\2 & 3 & 0\end{pmatrix}$ and $B=\begin{pmatrix}1 & 4 & -4\\2 & 5 & 7\\0 & 4 & -3\end{pmatrix}$.

8. If $A=\begin{pmatrix}-3 & 0\\7 & -4\end{pmatrix}, B=\begin{pmatrix}2 & -1\\-7 & 4\end{pmatrix}$ and $C=\begin{pmatrix}1 & 0\\-2 & -4\end{pmatrix}$, find $2A-3B+4C$.

9. If $\begin{bmatrix}a & b\\c & d\end{bmatrix}+\begin{bmatrix}3 & -1\\1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, then find a,b,c and d.

10. If $\begin{bmatrix}x & -1 & y\\2 & 0 & 3\\z & 3 & 2\end{bmatrix}=\begin{bmatrix}3 & a & x\\b & 0 & c\\2 & 3 & d\end{bmatrix}$, then evaluate a,b,c,d,x,y and z.

11. Evaluate possible products of the following matrices,

$A=\begin{bmatrix}1 & -2\\3 & 1\end{bmatrix}, B=\begin{bmatrix}2 & 1 & 0\\1 & 0 & -1\end{bmatrix}$ and $C=\begin{bmatrix}0 & -1 & 2\\-1 & 0 & -4\\3 & 1 & 2\end{bmatrix}$.

12. Determine: $\begin{pmatrix}1 & 0 & 3\\2 & 1 & 2\\1 & 3 & 1\end{pmatrix}\begin{pmatrix}2 & 2 & 0\\1 & 3 & 2\\3 & 2 & 0\end{pmatrix}$.

13. If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}, B=\begin{bmatrix}2 & -1 & 0\\ 3 & 1 & 2\end{bmatrix}$ and $C=\begin{bmatrix}1 & 1 & 2\\ -2 & 2 & 3\end{bmatrix}$

then verify that (i). $A(B+C)=AB+AC$ (ii).$(B+C)A=BA+CA$.

14. If $A=\begin{pmatrix}6 & 5 & 2\\ 4 & 6 & 3\\ 0 & 1 & 7\end{pmatrix}, B=\begin{pmatrix}6 & 8 & 0\\ 5 & 1 & 2\\ 3 & 6 & 4\end{pmatrix}$ and $C=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$ then verify the following:

(a) $A+B=B+A$ (b) $(A+B)+C=A+(B+C)$

(c) $(A+B)C=AC+BC$ (d) $A(B+C)=AB+AC$

15. If $A=\begin{bmatrix}2 & 3 & 1\\ 1 & -1 & 1\\ 0 & 2 & 2\end{bmatrix}, B=\begin{bmatrix}1 & 2 & 1\\ -1 & 2 & -1\\ 2 & 0 & 2\end{bmatrix}$ and $C=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 0 & 2\end{bmatrix}$ then

show that $A(BC)=(AB)C$.

19.5 Determinant of a Matrix

19.5.(i) Define the determinant of a square matrix

We can associate with every square matrix A over R, a number |A|, known as the determinant of the matrix A.

If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then determinant of the matrix A is denoted by $|A|=\begin{vmatrix}a & b\\ c & d\end{vmatrix}$.

19.5.(ii) Finding the value of determinant of a matrix

For a 2×2 matrix $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $|A|=\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$ is called value of its determinant.

For example, $\begin{vmatrix}3 & 7\\ 1 & -4\end{vmatrix}=3\times(-4)-7\times 1=-19$.

Now, The Leibniz formula or laplacian expression for the determinant of 3×3 matrix $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ is the following

$$|A|=a_{11}\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}.$$

Note. The above evaluation of the determinant is said to be the expansion by row one (R_1).

Example: Find $|A|$ if $A=\begin{bmatrix}4 & -3 & 5\\ 1 & 0 & 3\\ -1 & 5 & 2\end{bmatrix}$

Solution: To find $|A|$, let us expand the determinant by R_1.

$$|A| = 4\begin{vmatrix}0 & 3\\ 5 & 2\end{vmatrix} - (-3)\begin{vmatrix}1 & 3\\ -1 & 2\end{vmatrix} + 5\begin{vmatrix}1 & 0\\ -1 & 5\end{vmatrix}$$

$$|A| = 4(0-15)+3(2+3)+5(5+0) = -20.$$

Hence, the determinant a matrix has a unique value.

19.5.(iii) Define singular and non-singular matrices

A square matrix A is said to be singular if its determinant is equal to zero. i.e., $|A| = 0$.

A square matrix A is said to be non-singular if its determinant is not equal to zero. i.e, $|A| \neq 0$.

Note: A singular matrix is also called non invertible matrix, and a non-singular matrix is also called invertible matrix.

Example: Determine whether $A=\begin{bmatrix}2 & 4 & 6\\ 2 & 0 & 2\\ 6 & 8 & 14\end{bmatrix}$ is a singular matrix or not.

Solution:

$$|A| = \begin{vmatrix}2 & 4 & 6\\ 2 & 0 & 2\\ 6 & 8 & 14\end{vmatrix} = 2(0-16)-4(28-12)+6(16-0) = -32-64+96 = 0.$$

$\because$ $|A| = 0$

$\therefore$ A is singular martix.

19.5.(iv) Define minors and co-factors

Let us consider a square matrix A of order 3×3 then the minor of an element a_{ij}, denoted by M_{ij} is the determinant of the matrix obtained by deleting the i[th] row and the j^{th} column of A. For example, if

$A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ then the matrix obtained by deleting the first row and second column of A

is $\begin{bmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{bmatrix}$ and its determinant is the minor as defined below:

$M_{12} = \begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}$ while, cofactor of an element a_{ij}, is denoted A_{ij} and can be computed by

$A_{ij} = (-1)^{i+j} M_{ij}$ Here, M_{ij} is the minor of an element a_{ij} of the square matrix A.

Example: Let $A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$ then compute M_{12} and A_{12}.

Solution: Here $M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = 6\,(-7) - 4(1) = -46$

and $A_{ij} = (-1)^{i+j}\, M_{ij}$,

so $A_{12} = (-1)^{1+2}\, M_{12}$

$A_{12} = (-1)^3(-46)$

$A_{12} = (-1)(-46) = 46.$

19.6 Multiplicative Inverse of a Matrix

19.6.(i) Define adjoint of a matrix

Adjoint of a matrix with order 2 × 2

The adjoint of a matrix A is the transpose of the matrix of cofactors of A.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the adjoint of matrix A is denoted by $\text{Adj}(A)$ or $\text{adj}(A)$.

and $\text{Adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\text{Adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t.$$

Where $A_{11}, A_{12}, \ldots A_{33}$ are the cofactors of A.

Example: Find the adjoint of the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Solution: First we, find all the cofactors of matrix A

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -2 & 0 \\ 2 & -1 \end{vmatrix} = 2$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = 2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} = 6$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix} = -2$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -5$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -2$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} = -8$$

Now, $\text{Adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$.

$$\text{Adj}(A) = \begin{bmatrix} 2 & 2 & 6 \\ -1 & -2 & -5 \\ -2 & -2 & -8 \end{bmatrix}^t = \begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix}.$$

19.6.(ii) Multiplicative inverse of a non-singular matrix A and verify that $AA^{-1} = I = A^{-1}A$, where, I is the identity matrix.

If A and B are $n \times n$ non singular matrices such that $AB = BA = I_n$, then A and B are multiplicative inverses of each other. The multiplicative inverse of A is denoted by A^{-1}

Example: Check whether A and B are multiplicative inverses of each other.

If $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 15-14 & -21+21 \\ 10-10 & -14+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly,

$$BA = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 15-14 & 35-35 \\ -6+6 & -14+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

as $AB = BA = I_2$

Hence A and B are multiplicative inverses of each other.

19.6.(iii) Use the adjoint method to calculate inverse of a non- singular matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a non-singular matrix then A^{-1} can be found as, $A^{-1} = \dfrac{Adj(A)}{|A|}$ where $|A| \neq 0$.

The method of finding inverse in this way is called adjoint method

Example 1: Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ by adjoint method.

Solution: Here $|A| = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 2$. Since $|A| \neq 0$ therefore, inverse of A exists.

Now $Adj(A) = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$ and

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}.$$

Example 2: Find the inverse of the matrix $A = \begin{bmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{bmatrix}$ by adjoint method.

Solution:

Here

$$|A| = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix}$$

$$= 9(2-0) - 2(-10-24) + 1(0+4)$$

$$= 18 + 68 + 4$$

$$= 90.$$

Since $|A| \neq 0$, Here A^{-1} inverse exists. Now, we find all cofactors of A

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 6 \\ 0 & -2 \end{vmatrix} = 2 - 0 = 2$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 6 \\ 4 & -2 \end{vmatrix} = -(-10-24) = 34$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}5 & -1\\ 4 & 0\end{vmatrix}=0+4=4$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}2 & 1\\ 0 & -2\end{vmatrix}=-(-4-0)=4$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}9 & 1\\ 4 & -2\end{vmatrix}=(-18-4)=-22$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}9 & 2\\ 4 & 0\end{vmatrix}=-(0-8)=8$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}2 & 1\\ -1 & 6\end{vmatrix}=12+1=13$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}9 & 1\\ 5 & 6\end{vmatrix}=-(54-5)=-49$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}9 & 2\\ 5 & -1\end{vmatrix}=-9-10=-19$$

$$\text{Adj}(A)=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}^t. \text{ Thus, Adj}(A)=\begin{bmatrix}2 & 4 & 13\\ 34 & -22 & -49\\ 4 & 8 & -19\end{bmatrix}$$

Now $A^{-1}=\dfrac{\text{Adj}(A)}{|A|}$

$$A^{-1}=\frac{1}{90}\begin{bmatrix}2 & 4 & 13\\ 34 & -22 & -49\\ 4 & 8 & -19\end{bmatrix}=\begin{bmatrix}\frac{1}{45} & \frac{2}{45} & \frac{13}{90}\\ \frac{17}{45} & \frac{-11}{45} & \frac{-49}{90}\\ \frac{2}{45} & \frac{4}{45} & \frac{-19}{90}\end{bmatrix}$$

19.6.(iv) Verify the result $(AB)^{-1}=B^{-1}A^{-1}$.

Let $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$. Verify that $(AB)^{-1}=B^{-1}A^{-1}$

Solution:

Given $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$.

Inverse of AB:

$$AB=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$$

$$AB=\begin{bmatrix}18+49 & 24+63\\ 12+35 & 16+45\end{bmatrix}=\begin{bmatrix}67 & 87\\ 47 & 61\end{bmatrix}.$$

Now, $|AB| = 4087 - 4089 = -2$

Since, $|AB| \neq 0$

Therefore, $(AB)^{-1}$ exists.

Now

$$(AB)^{-1}=\frac{\text{adj }(AB)}{|AB|}$$

$$=\frac{1}{-2}\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix} \quad ...(i)$$

Now, Inverse of A

We have $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$

$|A| = 15 - 14 = 1$

Since, $|A| \neq 0$

Hence, A^{-1} exists.

$$A^{-1}=\frac{\text{adj } A}{|A|}=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

Inverse of B:

We have $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$

Now

$|B| = 54 - 56 = -2$

Since, $|B| \neq 0$

Therefore, B^{-1} exists.

$$B^{-1}=\frac{\text{adj } B}{|B|}=\frac{1}{-2}\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}$$

Now

$$B^{-1}A^{-1}=\frac{1}{-2}\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

$$=\frac{1}{-2}\begin{bmatrix}45+16 & -63-24\\ -35-12 & 49+18\end{bmatrix}$$

$$B^{-1}A^{-1}=\frac{1}{-2}\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix} \quad ...(ii)$$

From equation (i) and (ii),

$(AB)^{-1} = B^{-1}A^{-1}$

Hence verified.

19.7 Solution of Simultaneous Linear Equations

Consider a system of two linear equations in two variables:

$$\left.\begin{aligned}a_1x+b_1y=c_1\\ a_2x+b_2y=c_2\end{aligned}\right\} \quad (i)$$

The system can be written in matrix form as

$$AX = B \quad (ii)$$

Where, $A=\begin{bmatrix}a_1 & b_1\\ a_2 & b_2\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

The matrix A is called the matrix of the coefficients of system i, matrix X is the column matrix of unknown and B is the column matrix of constants. An ordered pair (x, y) is called the solution of given system satisfied by these two values. The set of all solutions of the given system is called the solution set of simultaneous linear equation.

19.7.(i) Solve the system of two linear equations related to real life problems in two unknown using

Matrix inversion method:

Consider the matrix equation as discussed in previous section 19.7.

$$AX = B, \qquad (i)$$

Where, A is a non-singular matrix. Multiplying both sides of (ii) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow \quad (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow \quad X = A^{-1}B \quad (ii)$$

To find X, the matrix of unknown by eq (ii) is called matrix inversion method.

Example: Solve the following system of linear equations by using matrix inversion method.

$$5x + 2y = 3$$
$$3x + 2y = 5$$

Solution:

Writing the given system in matrix form:

$$\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$AX = B$$

$$\text{where } A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

First, we find A^{-1}

$$A^{-1} = \frac{Adj(A)}{|A|} = \frac{\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}}{10-6} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

Now, by matrix inversion method

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

The solution is $x = -1$ and $y = 4$. The solution set is $\{(-1, 4)\}$.

Example 2: A civil engineer needs 6000 m^3 of sand and 9000 m^3 of coarse gravel for a building project. There are two pits from which these materials can be obtained. The number of these pits is shown in table:

	Sand	Coarse Gravel
Pit - I	3	8
Pit - II	4	11

By using inverse matrix method to compute volume of materials (sand and coarse gravel) that must be hauled from each pit in order to meet the engineer's need.

Solution:

Let x represents volume and and y represents coarse gravel. The inverse matrix method is

$$X = A^{-1}B\text{............(i)}$$

Writing the given data in matrix form:

$$AX = B$$

$$\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6000 \\ 9000 \end{bmatrix}$$

First, we need to calculate A^{-1}

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = \frac{\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}}{3(11) - 8(4)} = \frac{1}{1}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

so, $$A^{-1} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

Now, from equation (i),

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 6000 \\ 9000 \end{bmatrix} = \begin{bmatrix} -6000 \\ 3000 \end{bmatrix}$$

By ignoring '–' sign, we have $x = 6000$ and $y = 3000$.

Hence 6000cm^3 can be haulded of sand and 3000 m^3 can be haulded of coarse graved.

19.2.(ii) Cramer's rule

Cramer's rule is one of the methods used to solve a system of linear equations. Let us consider a system of linear equations in two variables x and y written as

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \qquad \text{(i)}$$

System (i) is written in the matrix form:

$$AX = B,$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Where $A=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

System (i) has exactly one solution if $|A| \neq 0$.

Which is obtained by $x=\dfrac{|A_1|}{|A|}=\dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$ and $y=\dfrac{|A_2|}{|A|}=\dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$.

Where $A_1=\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}$ and $A_2=\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}$

Example 1: Solve the system using Cramer's Rule:

$$3x+2y=10$$
$$-6x+4y=4$$

Given system can be written in matrix format as:

Here $|A|=\begin{bmatrix} 3 & 2 \\ -6 & 4 \end{bmatrix}$

$$12+12=24$$

$\therefore \quad |A| \neq 0$

$x=\dfrac{|A_1|}{|A|}=\dfrac{\begin{vmatrix} 10 & 2 \\ 4 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ -6 & 4 \end{vmatrix}}$ is by cramers rule (This method is called crammers rule)

$$=\frac{40-8}{24}$$

$$=\frac{32}{24}$$

$$x=\frac{4}{3}$$

$$y=\frac{|A_2|}{|A|}=\frac{\begin{vmatrix} 3 & 10 \\ -6 & 4 \end{vmatrix}}{24}$$

$$=\frac{12+60}{24}$$

$$=\frac{72}{24}$$

$y=3$ Hence, The solution set is $\left\{\left(\frac{4}{3},3\right)\right\}$

EXERCISE 19.2

1. Evaluate each of the following determinants:

i) $\begin{vmatrix} -5 & -3 \\ 3 & -4 \end{vmatrix}$ ii) $\begin{vmatrix} -4 & 3 \\ 1 & -3 \end{vmatrix}$ iii) $\begin{vmatrix} -1 & -5 \\ 2 & 3 \end{vmatrix}$

iv) $\begin{vmatrix} 2 & -5 \\ 2 & -1 \end{vmatrix}$ v) $\begin{vmatrix} 3 & -3 & -4 \\ 4 & 1 & -5 \\ 0 & -1 & -4 \end{vmatrix}$ vi) $\begin{vmatrix} -3 & 4 & -5 \\ 2 & -3 & -5 \\ 1 & 3 & 5 \end{vmatrix}$

vii) $\begin{vmatrix} 0 & 5 & -4 \\ -3 & 4 & -5 \\ 1 & 0 & -5 \end{vmatrix}$ viii) $\begin{vmatrix} 4 & 3 & -5 \\ -5 & -1 & -5 \\ 2 & 4 & 4 \end{vmatrix}$ ix) $\begin{vmatrix} 1 & 0 & 4 \\ -2 & 3 & -5 \\ 3 & 5 & 3 \end{vmatrix}$

x) $\begin{vmatrix} 0 & a & b \\ 0 & c & d \\ 0 & x & y \end{vmatrix}$ xi) $\begin{vmatrix} -2 & 0 & 0 \\ -6 & x & 1 \\ -4 & 0 & -1 \end{vmatrix}$

2. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$ then compute M_{12}, M_{22}, M_{21}, A_{12}, A_{22}, and A_{21}.

3. For what value of x, the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular?

4. Classify the square matrices as singular or non-singular given by

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

5. Find the adjoint of the following:

$$A = \begin{pmatrix} 1 & 6 \\ 4 & 7 \end{pmatrix}, B = \begin{pmatrix} -3 & 2 & -5 \\ -1 & 0 & -2 \\ 3 & -4 & 1 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

6. Verify $A(\text{adj } A) = |A| I$, where $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$.

7. Find the inverse of the following matrices by adjoint method if exist

i) $A = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$ ii) $B = \begin{bmatrix} 3 & 6 \\ 5 & 10 \end{bmatrix}$

iii) $C=\begin{bmatrix}2&1&1\\3&2&1\\2&1&2\end{bmatrix}$ iv) $D=\begin{bmatrix}1&0&1\\-4&1&-1\\6&-2&1\end{bmatrix}$ v) $E=\begin{bmatrix}1&-1&3\\2&1&2\\-2&-2&1\end{bmatrix}$.

8. Find the solution by matrix inversion method and cramer's rule:

1) $2x+3y=14$, $-4+y=28$ 2) $2x-4y=-12$, $2y+3x=0$

REVIEW EXERCISE 19

1. Tick the correct option

i. If m denotes the number of rows and n denotes the number of columns such that $m=n$, then matrix is called __________ matrix..

(a) Rectangular (b) Equal (c) Square (d) Null

ii. If $A=\begin{bmatrix}i&0\\0&-i\end{bmatrix}$, then A^2 is

(a) I_2 (b) I_3 (c) $-I_2$ (d) O

iii. If A is any square matrix such that $A^t=-A$, then A is said to be:

(a) Diagonal matrix (b) Scalar matrix
(c) Symmetric matrix (d) Skew Symmetric matrix

iv. If A , B and C are matrices of same order then $(ABC)^t=$

(a) $A^t.B^t.C^t$ (b) $C^tB^tA^t$
(c) $C^tA^tB^t$ (d) $(B^tA^t)C^t$

v. If $A=\begin{bmatrix}i&0\\0&-i\end{bmatrix}$, $B=\begin{bmatrix}0&i^2\\-i^2&0\end{bmatrix}$, $C=\begin{bmatrix}0&i\\i&0\end{bmatrix}$ then

(a) $A^2=-I$ (b) $B^2=-I$
(c) $C^2=-I$ (d) All of them

vi. For two matrices A and B if $A=\begin{bmatrix}\sin\theta&-\cos\theta\\\cos\theta&\sin\theta\end{bmatrix}$ and $B=\begin{bmatrix}\sin\theta&\cos\theta\\-\cos\theta&\sin\theta\end{bmatrix}$ then AB =

(a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ (b) $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$
(c) $\begin{bmatrix}1&1\\1&1\end{bmatrix}$ (d) $\begin{bmatrix}0&0\\0&0\end{bmatrix}$

vii. If $2\begin{bmatrix}x\\3\\4\end{bmatrix}+3\begin{bmatrix}-1\\y\\0\end{bmatrix}=4\begin{bmatrix}0\\4\\z\end{bmatrix}$ then the values of x, y and z are_____

(a) $2,\frac{3}{2},\frac{10}{3}$ (b) $\frac{3}{2},2,\frac{10}{3}$
(c) $\frac{3}{2},\frac{10}{3},2$ (d) 1,2,3

viii. For matrix A, $\left(A^{-1}\right)^{-1} =$ ——

(a) A^{-2} (b) A (c) A^{-1} (d) A^2

ix. Find x if $\begin{vmatrix} 5 & 1 \\ 2 & x \end{vmatrix} = x+4$

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) 0 (d) None of them

x. If the matrix $\begin{bmatrix} \lambda & -3 & 4 \\ -3 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix}$ is invertible then $\lambda \neq$ ——

(a) −15 (b) −17 (c) −16 (d) None of these

2. Define the rows and column of matrix.

3. Find the order of the matrices:

i) $\begin{bmatrix} 1 & 5 & 8 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 8 \end{bmatrix}$ iii) $\begin{bmatrix} 7 & 2 \\ 9 & 3 \\ 8 & 1 \end{bmatrix}$

4. Define the following matrices:

i) Square matrix ii) Rectangle matrix
iii) Diagonal matrix iv) Scalar matrix
v) Symmetric matrix vi) Skew Symmetric matrix

5. If $A = \begin{bmatrix} 1 & 8 & 9 \\ 2 & 1 & 0 \\ -2 & 1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 & 8 \\ 1 & 3 & 4 \\ 7 & 2 & 4 \end{bmatrix}$ then find

i) $A+B$ ii) $A-B$ iii) AB iv) BA
v) 5A vi) 7B vii) $8A-9B$

6. Evaluate $\begin{vmatrix} 1 & 8 & 9 \\ 2 & 0 & -1 \\ -7 & 8 & -10 \end{vmatrix}$

7. Define singular and non singular matrices with examples.

8. If $A = \begin{bmatrix} 1 & 4 & 2 \\ 7 & 0 & 9 \\ 0 & 2 & -3 \end{bmatrix}$ then find A^{-1}

Q8: Solve $2x+5y=27$
by $7x+y=12$

i) matrix inversion method ii) Cramer's rule

Summary

- A matrix is a rectangular array of elements.
- The horizontal line of entries in a matrix is called the row of a matrix and vertical line of entries in a matrix is called column of a matrix.
- Two matrices are said to be equal if they have same order and same corresponding elements.
- A matrix having one row is called row matrix, having one column is called column of matrix.
- A matrix in which number of rows is equal to number of columns is known as square matrix otherwise it is rectangular matrix.
- A square matrix is diagonal matrix if all the elements are zero except at least one diagonal element.
- A diagonal matrix in which diagonal elements are same is called scalar matrix.
- A diagonal matrix in which each element of main diagonal is 1 is called identity matrix.
- A square matrix is said to be symmetric matrix if its transpose is equal to itself.
- A square matrix is said to be skew-symmetric matrix if its transpose is equal to negative of itself.
- The addition and subtraction of two matrices are obtained by adding or subtracting of the corresponding elements of both matrices.
- If A and B are two matrices of the same order, then the commutative law of addition is defined as $A+B=B+A$.
- If A, B and C are three matrices of the same order, then the associative law of addition is defined as $(A+B)+C=A+(B+C)$.
- Two matrices A and B are said to be conformable for multiplication, if the number of columns of the first matrix is equal the number of rows of the second matrix.
- For every square matrix A over R, a number $|A|$, known as the determinant of the matrix A.
- If A and B are $n\times n$ non singular matrices such that $AB=BA=I$, then A and B are multiplicative inverse of each other.
- System of linear equations can be solved by matrix inversion method and Cramer's rule.

THEORY OF QUADRATIC EQUATIONS

Unit 20

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Define discriminant ($b^2 - 4ac$) of the quadratic expression $ax^2 + bx + c$.
- Find the discriminant of a given quadratic equation.
- Discuss the nature of the roots of a quadratic equation through discriminant.
- Determine the nature of the roots of a given quadratic equation and verify the result by solvingthe equation.
- Determine the values of an unknown involved in agiven quadratic equation when the nature of its roots is given.
- Find cube roots of unity.
- Recognize complex cube roots of unity as w and w^2.
- Verify the properties of cube roots of unity.
- Use properties of cube roots of unity to solve alliedproblems.
- Find the relation between the roots and the coefficients of a quadratic equation.
- Find the sum and the product of the roots of a given quadratic equation without solving it.
- Find the value(s) of the unknown involved in a given quadratic equation when
  - Sum of roots is equal to a multiple of the product of the roots,
  - Sum of the squares of the roots is equal to a given number,
  - Roots differ by a given number,
  - Roots satisfying a given relation (e.g., the relation $2\alpha + 5\beta = 7$, where a and β are the roots of the given equations),
  - Both sum and product of the roots are equal to a given number.
- Define symmetric functions of the roots of a quadratic equation.
- Represent a symmetric function graphically
- Evaluate a symmetric function of the roots of a quadratic equation in terms of its coefficients.
- Establish the formula, x^2 – (sum of the roots) x + (product of the roots) = 0, to find the quadratic equation of the given roots.
- form the quadratic equation whose roots, for examples are of the type:
  - $2\alpha+1, 2\beta+1$
  - α^2, β^2
  - $\frac{1}{\alpha}, \frac{1}{\beta}$
  - $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$
  - $\alpha+\beta, \frac{1}{\alpha}+\frac{1}{\beta}$

  where, a , β are the roots of a quadratic equation
- Find the values of α , β , where the roots of an equation are $\frac{1}{\alpha}, \frac{1}{\beta}$
- Solve the cubic equation if one root of the equation is given,
- Solve a biquadratic (quartic) equation if two of the real roots of the equation are given.
- Solve a system of two equations in two variables, when
  - One equation is linear and the other is quadratic,
  - Both the equations are quadratic.
- Solve the real life problems leading to quadratic equations.

20.1 Nature of the Square Roots of a Quadratic Equation:

20.1.(i) Define discriminant $\left(b^2-4ac\right)$ of the quadratic expression $ax^2+bx+c, a\neq 0$.

A polynomial of second degree is called Quadratic Expression. General form of quadratic expression is $ax^2+bx+c, (a\neq 0)$ where a and b are coefficients of x^2, and x respectively. While c is a constant term.

The discriminant of a quadratic expression

For quadratic expression $ax^2+bx+c, (a\neq 0)$ the expression b^2-4ac is called its discriminant and is denoted by Δ or D i.e. $\Delta = b^2-4ac$

20.1.(ii) Find the discriminant of a given quadratic equation

We already know that solution of the quadratic equation $ax^2+bx+c, a\neq 0$ is $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. Here , b^2-4ac enables us to know the nature of the roots either they are real or complex. The expression b^2-4ac is the discriminate of quadratic equation.

$$ax^2+bx+c, a\neq 0$$

Example: Find the discriminant of $2x^2+5x-4=0$

Solution: $2x^2+5x-4=0$

Here, $a=2, b=5$ and $c=-4$

$$\Delta = b^2-4ac = (5)^2-4(2)(-4)$$
$$= 25+32 = 57$$

20.1.(iii) Discuss the nature of the root of a quadratic equation through discriminant.

We can find the nature of the roots of the quadratic equation $ax^2+bx+c, a\neq 0$, using discriminant i.e. $\Delta = b^2-4ac$.

The roots of the quadratic equation $ax^2+bx+c=0$ are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$. The nature of the roots depends on the value of b^2-4ac, where a, b and c being real numbers.

i. If $\Delta = b^2-4ac>0$, then the roots are real and unequal.

ii. If $\Delta = b^2-4ac<0$, then the roots are non-real (complex or imaginary).

iii. If $\Delta = b^2-4ac=0$, then the roots are rational and equal, each being equal to $-\dfrac{b}{2a}$.

iv. If a,b,c are rational and $\Delta = b^2-4ac$ is perfect square, then roots are rational and unequal otherwise irrational.

20.1.(iv) Determine the nature of the roots of a given quadratic equation and verify the result by solving the equation.

Example 1:

Use the discriminant to find the nature of the roots of the following equations and verify by solving the equation:

(i) $x^2+5x-14=0$ (ii) $5x^2+3x+1=0$

(iii) $x^2-9x+5=0$ (iv) $9x^2=6x-1$

Solution:

(i) $x^2+5x-14=0$

Here $a=1, b=5$ and $c=-14$

$\Delta=b^2-4ac$

$\therefore$ $\Delta=(5)^2-4(1)(-14)$

$\Rightarrow$ $\Delta=25+56$

$\Rightarrow$ $\Delta=81=(9)^2>0,$

perfect square and positive.

Hence roots are real, rational and unequal

Verification:

$x^2+5x-14=0$

$x^2+7x-2x-14=0$

$x(x+7)-2(x+7)=0$

$x+7=0$ or $(x-2)=0$

$\Rightarrow$ $x=-7$ or $x=2$

Here, roots are real, rational and unequal

Hence verified.

Solution:

(ii) $5x^2+3x+1=0,$

Here $a=5, b=3$ and $c=1$

$\Delta=b^2-4ac$

$\therefore$ $\Delta=(3)^2-4(5)(1)$

$\Rightarrow$ $\Delta=9-20$

$\Rightarrow$ $\Delta=-11<0,$ i.e negative.

Hence roots are complex.

Verification:

By using Quadratic formula

$$x=\frac{-3\pm\sqrt{9-20}}{2}$$

$$x=\frac{-3\pm\sqrt{11}\,i}{2} \qquad (i^2=-1)$$

Here, roots are complex.

Hence verified

Solution:

(iii) $x^2-9x+5=0,$

Here $a=1, b=-9$ and $c=5$

$\Delta=b^2-4ac$

$\therefore$ $\Delta=(-9)^2-4(1)(5)$

$\Rightarrow$ $\Delta=81-20$

$\Rightarrow$ $\Delta=61>0,$ not perfect square.

Hence the roots are real, irrational and unequal.

Verification:

$x^2-9x+5=0,$

By using Quadratic formula

$$x=\frac{9\pm\sqrt{81-20}}{2}$$

$$x=\frac{-3\pm\sqrt{61}}{2}$$

Here the roots are real, irrational and unequal.

Hence verified.

Solution:

(iv) $9x^2 = 6x - 1$

$\Rightarrow 9x^2 - 6x + 1 = 0$

Here $a = 9, b = -6$ and $c = 1$

$\because \quad \Delta = b^2 - 4ac$

$\therefore \quad \Delta = (6)^2 - 4(9)(1)$

$\Rightarrow \quad \Delta = 36 - 36$

$\Rightarrow \quad \Delta = 0,$ perfect square and equal to 0.

Hence roots are real and equal.

Verification:

$9x^2 - 6x + 1 = 0$

$(3x)^2 - 2(3x)(1) + (1)^2 = 0$

$(3x - 1)^2 = 0$

$3x - 1 = 0 \quad$ or $\quad 3x - 1 = 0$

$\Rightarrow \quad x = \frac{1}{3} \quad$ or $\quad x = \frac{1}{3}$

Hence roots are real and equal.

Hence verified.

Example 2:

Show that the roots of the equation $abc^2x^2 + c(3a^2 + b^2)x + 3a^2 + b^2 - ab = 0$, are rational.

Solution: $abc^2x^2 + c(3a^2 + b^2)x + 3a^2 + b^2 - ab = 0$

Here $\text{A} = abc^2, \text{B} = c(3a^2 + b^2)$ and $\text{C} = 3a^2 + b^2 - ab$

$\Delta = \text{B}^2 - 4\text{AC}$

$\Delta = \left[c(3a^2 + b^2)\right]^2 - 4(abc^2)(3a^2 + b^2 - ab)$

$= c^2(3a^2 + b^2)^2 - 4abc^2(3a^2 + b^2 - ab)$

$= c^2\left[(3a^2 + b^2)^2 - 4ab(3a^2 + b^2) + 4a^2b^2\right]$

$= c^2\left[(3a^2 + b^2)^2 - 2(3a^2 + b^2)(2ab) + (2ab)^2\right] \quad \because \ 4ab = 2(2ab)$

$= c^2\left[(3a^2 + b^2 - 2ab)^2\right]$

$= \left[c(3a^2 + b^2 - 2ab)\right]^2$ which is perfect square. Hence roots are rational.

20.1.(v) Determine the values of an unknown involved in a given quadratic equation when the nature of its root is given.

The following examples will help us to find the unknown involved in the quadratic equation when nature of roots is given.

Example 1:

For what value of p will $3x^2 + 5x + p = 0$ have

(i) Equal roots (ii) Rational roots (iii) Complex roots.

Solutions (i):

$3x^2 + 5x + p = 0$

Here $a = 3, b = 5$ and $c = p$?

$\Delta = b^2 - 4ac$

$\therefore \quad \Delta = (5)^2 - 4(3)(p) = 25 - 12p,$

For equal roots

$$\Delta = 0$$

i.e. $25 - 12p = 0$

$\Rightarrow$ $-12p = -25,$

$\Rightarrow$ $p = \frac{25}{12},$

Thus, at $p = \frac{25}{12}$, given equation has equal roots.

(ii) For rational roots,

Δ should be perfect square

i.e. $25 - 12p$ must be perfect square.

when $p = 0, 2$ and -2 etc.

Thus, at $p = 0, 2, -2$ and some others values also make the discriminant perfect square.

(iii) For complex roots,

i.e. $25 - 12p < 0$

$\Rightarrow$ $12p > 25$

$\Rightarrow$ $p > \frac{25}{12}$

Thus, when $p > \frac{25}{12}$, given equation has complex roots.

Example 2:

If the equation $x^2 - 15 - k(2x - 8) = 0$ has equal roots, find the values of k.

Solution:

$x^2 - 15 - k(2x - 8) = 0$, we get rewrite it in the standard form of the quadratic equation

i.e. $x^2 - 2kx + 8k - 15 = 0$

Here $a = 1, b = -2k$ and $c = 8k - 15$

$$\Delta = b^2 - 4ac$$

$\therefore$ $\Delta = (-2k)^2 - 4(1)(8k - 15)$

$\Rightarrow$ $\Delta = 4k^2 - 4(8k - 15)$

For equal roots

$$\Delta = 0$$

$\therefore$ $4k^2 - 4(8k - 15) = 0$

$\Rightarrow$ $k^2 - 8k + 15 = 0$ (dividing both sides by 4)

$\Rightarrow$ $k^2 - 5k - 3k + 15 = 0$

$\Rightarrow$ $k(k - 5) - 3(k - 5) = 0$

$\Rightarrow$ $(k - 5)(k - 3) = 0$

i.e. $k - 5 = 0$ or $k - 3 = 0$

$\Rightarrow$ $k = 5$ or $k = 3$

Thus, $k = 3$ and 5.

Example 3:

For what value of p, the equation $2x^2 - px + 18 = 0$, has real and unequal roots.

Solution:

$2x^2 - px + 18 = 0,$

Here $a = 2, b = -p$ and $c = 18$

$\Delta = b^2 - 4ac$

$\therefore \quad \Delta = (-p)^2 - 4(2)(18) = p^2 - 144$

For real and unequal root

$\Delta > 0$

$\therefore \quad p^2 - 144 > 0$

$\Rightarrow \quad p^2 > 144$

$\Rightarrow \quad p > 12$ or $p < -12$

$\Rightarrow$ Thus, for $p > 12$ or $p < -12$ given equation has real root.

EXERCISE 20.1

1. Use discriminant find the nature of the roots of the following quadratic equations:
   (i) $x^2 + 5x - 6 = 0$ (ii) $x^2 = 6x$ (iii) $2x^2 + 8 = 6x$
   (iv) $1 + 6x + 9x^2 = 0$ (v) $24x^2 + 12x + 36 = 0$ (vi) $x = x^2 + 1$
   (vii) $3x^2 + 6 = 5x$ (viii) $3x^2 + 9 = 0$
2. Find the value(s) of k that ensure that following quadratic equations have
   (a) Same solution (b) Different real solutions
   (Hint: for same solution $\Delta = 0$ and for different solutions $\Delta > 0$).
   (i) $x^2 - 3x + k = 0$ (ii) $x^2 + k = 4$
   (iii) $x^2 + kx + 2 = 0$ (iv) $(k-1)x^2 - 4x + 2 = 0$
   (v) $x^2 + kx + 4 = 0$ (vi) $9x^2 + kx = -16$
   (vii) $(k-2)x^2 = 4x + (k+2)$ (viii) $x^2 + 1 = kx$
3. Determine the value of m in each of the following quadratic equations that will make the roots equal.
   (i) $(m+1)x^2 + 2(m+3)x + (2m+3) = 0,$ provided $m \neq -1$
   (ii) $9x^2 + mx + 16 = 0$
4. Show that the roots of the following quadratic equations are real.
   (i) $x^2 - 2x\left(k + \frac{1}{k}\right)x + 3 = 0, \ \forall k \in \mathbb{R} - \{0\}$
   (ii) $2nx^2 + 2(l + m + n)x + (l + m) = 0, \ \forall l, m, n \in \mathbb{R}$ and $m \neq 0$
5. Show that the roots of the following quadratic equations are rational.
   (i) $(l - m)x^2 + (m + n - l)x - n = 0, \ \forall l, m, n \in \mathbb{R}$ and $l \neq m$
   (ii) $(a + c - b)x^2 + 2cx + (b + c - a) = 0, \ \forall a, b, c \in \mathbb{R}$

6. For what values of p and q the roots of quadratic equation $x^2+(2p-4)x-(3q+5)=0$ vanish?

7. Show that the roots of the quadratic equation given by $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ are real and they can not be equal unless $a=b=c$.

20.2 Cube Roots of Unity and their properties

20.2.(i) Find the cube roots of unity

Let x be cube root of unity

i.e., $x^3=1$

$\Rightarrow \quad (x)^3-(1)^3=0$

$\Rightarrow \quad (x-1)(x^2+x+1)=0, \qquad \left[\, a^3-b^3=(a-b)(a^2+ab+b^2)\right]$

i.e. $x-1=0$

$\Rightarrow \quad x=1$

or $x^2+x+1=0,$

Here $a=b=c=1,$

Using quadratic formula, we have,

$$\therefore \quad x=\frac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2(1)}=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm\sqrt{-1\times 3}}{2}=\frac{-1\pm i\sqrt{3}}{2}, \quad (i^2=-1)$$

$$\Rightarrow \quad x=\frac{-1+i\sqrt{3}}{2} \text{ and } \frac{-1-i\sqrt{3}}{2}$$

Thus cube root of unity are

$$1, \frac{-1+i\sqrt{3}}{2} \text{ and } \frac{-1-i\sqrt{3}}{2}.$$

20.2.(ii) Recognize complex cube roots of unity as ω and ω^2.

Two complex cube roots are:

$\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$, for these two complex roots we use a Greek alphabet "ω" and read as omega and let us assume $\omega=\frac{-1+i\sqrt{3}}{2}$ then $\omega^2=\frac{-1-i\sqrt{3}}{2}$.

Hence cube roots of unity now are $1, \omega$ and ω^2.

20.2.(iii) Verify the properties of cube roots of unity.

Properties of the cube roots of unity are:

(i) Each of the complex cubes of unity is square of other.

Verification:

If $\omega=\frac{-1+i\sqrt{3}}{2}$ is one complex cube root of unity,

$\therefore \quad \omega^2 = \left(\frac{-1+i\sqrt{3}}{2}\right)^2 = \frac{1-2i\sqrt{3}+3i^2}{4} = \frac{1-2i\sqrt{3}+3(-1)}{4}$

$\Rightarrow \quad \omega^2 = \frac{-2-2i\sqrt{3}}{4} = \frac{2(-1-i\sqrt{3})}{4} \qquad (\because i^2=-1)$

$\Rightarrow \quad \omega^2 = \frac{-1-i\sqrt{3}}{2}$

Now $\quad (\omega^2)^2 = \left(\frac{1-i\sqrt{3}}{2}\right)^2 = \frac{-1+2i\sqrt{3}+3i^2}{4} = \frac{1+2i\sqrt{3}+3(-1)}{4} \qquad (\because i^2=-1)$

$\Rightarrow \quad (\omega^2)^2 = \frac{-2+2i\sqrt{3}}{4} = \frac{2(-1+i\sqrt{3})}{4} = \frac{-1+i\sqrt{3}}{2}$

$\Rightarrow \quad (\omega^2)^2 = \omega.$

Hence each complex cube root of unity is square of the other
Verified.

(ii) Sum of three cube roots of unity is zero, i.e. $1+\omega+\omega^2=0$

Verification: L.H.S

$1+\omega+\omega^2$

$= 1+\left(\frac{-1+i\sqrt{3}}{2}\right)+\left(\frac{-1-i\sqrt{3}}{2}\right)$, where $\omega = \frac{-1+i\sqrt{3}}{2}$ and $\omega^2 = \frac{-1-i\sqrt{3}}{2}$

$= \frac{2-1+i\sqrt{3}-1-i\sqrt{3}}{2}$

$= \frac{2-2}{2} = \frac{0}{2} = 0 = \text{R.H.S.}$

Verified.

(iii) Product of three cube roots of unity is 1, i.e, $\omega\cdot\omega^2\cdot 1 = 1$

Verification: L.H.S $\omega\cdot\omega^2\cdot 1$

$= 1\left(\frac{-1+i\sqrt{3}}{2}\right)\left(\frac{-1-i\sqrt{3}}{2}\right)$

$= \frac{1-i^2 3}{4}$

$= \frac{1-(-1)(3)}{4}, \qquad (\; i^2=-1)$

$= \frac{1+3}{4} = \frac{4}{4} = 1$

$=$ R.H.S $\qquad$ i.e. $1\cdot\omega\cdot\omega^2 = 1$

Verified.

(iv) Each complex cube root of unity is reciprocal of the other.

Proof: One complex cube root is ω,

$\Rightarrow \quad \omega^3 = 1,$

$\Rightarrow \quad \omega \cdot \omega^2 = 1,$

$\Rightarrow \quad \omega^2 = \dfrac{1}{\omega}$ or $\omega = \dfrac{1}{\omega^2}.$

(v) Every integral power of ω^3 is unity.

$\because \quad \omega^3 = 1,$

$\therefore \quad \left(\omega^3\right)^m = 1, \qquad \forall m \in \mathbb{Z}.$

$\Rightarrow \quad \omega^{3m} = 1.$

Proved.

20.2.(iv) Use properties of cube roots of unity to solve allied problems.

Following are the allied problem related to the cube root of unity.

Example 2: Find the cube roots of -27.

Solution:

Let x be the cube root of -27,

$\therefore \quad x = (-27)^{\frac{1}{3}}$

Cubing on both the sides, we have,

$x^3 = -27$

$\Rightarrow \quad x^3 + 27 = 0$

$\Rightarrow \quad (x)^3 + (3)^3 = 0$

$\Rightarrow \quad (x+3)(x^2 - 3x + 9) = 0, \qquad \left[a^3 + b^3 = (a+b)(a^2 - ab + b^3)\right]$

i.e, $\quad x + 3 = 0 \Rightarrow x = -3,$

Now $\quad x^2 - 3x + 9 = 0$

Here, $a = 1, b = -3$ and $c = 9.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$\Rightarrow \quad x = \frac{-(-3) \pm \sqrt{9 - 36}}{2}$$

$$\Rightarrow \quad x = \frac{-(-3) \pm \sqrt{-27}}{2}$$

$$\Rightarrow \quad x = \frac{-(-3) \pm \sqrt{-1 \times 27}}{2}$$

$$\Rightarrow \quad x = \frac{-(-3) \pm \sqrt{i^2 \times 27}}{2} \qquad (\because i^2 = -1)$$

$$\Rightarrow \quad x=\frac{-(-3)\pm i3\sqrt{3}}{2}$$

$$\Rightarrow \quad x=\frac{-3\left(-1\quad i\sqrt{3}\right)}{2}$$

$$\Rightarrow \quad x=\frac{-3\left(-1+i\sqrt{3}\right)}{2} \text{ and } \frac{-3\left(-1-i\sqrt{3}\right)}{2}$$

$$\Rightarrow \quad x=-3\omega,-3\omega^2$$

Hence, three cube roots of -27 are $-3,-3\omega$ and $-3\omega^2$.

Example 3: Show that:

(a) $2+\omega=\dfrac{3}{2+\omega^2}$

(b) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$... to $2n$ factors $=1$, where $n\in\mathbb{N}$

(a) Solution: L.H.S $=2+\omega$

$$=\frac{(2+\omega)(2+\omega^2)}{(2+\omega^2)} \qquad \text{Multiply and divide by } (2+\omega^2)$$

$$=\frac{4+2\omega+2\omega^2+\omega^3}{2+\omega^2}$$

$$=\frac{4+2(\omega+\omega^2)+1}{2+\omega^2} \qquad (\because \omega^3=1)$$

$$=\frac{5+(-1)}{2+\omega^2} \qquad (\because \omega+\omega^2=-1)$$

$$=\frac{5-2}{2+\omega^2}=\frac{3}{2+\omega^2}$$

Hence shown.

(b) Solution: L.H.S $=(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$... to $2n$ factors $\quad \forall n\in\mathbb{N}$

$$=\left[(-\omega^2)(-\omega)\right]\left[(1+\omega^3\cdot\omega)(1+\omega^6\cdot\omega^2)\right]\text{... to } n \text{ factors,} \qquad \left(\begin{array}{l}\because 1+\omega=-\omega^2\\ 1+\omega^2=-\omega\\ \omega^3=\omega^6=1\end{array}\right)$$

$$=\left[(-1)^2\cdot(\omega^3)\right]\left[(1+\omega)(1+\omega^2)\right]\text{... to } n \text{ factors,}$$

$$=\left[(-1)^2\cdot 1\right]\left[(-\omega^2)(-\omega)\right]\text{... to } n \text{ factors,}$$

$$=\left[(-1)^2\right]\left[(-1^2)\cdot\omega^3\right]\text{... to } n \text{ factors,}$$

$$=(-1)^2\cdot(-1)^2\text{ ... to } n \text{ factors,}$$

$$=(1)(1)\text{... to } n \text{ factors,}$$

$=(1)^n=1$. **Hence Shown.**

Example 4: Prove that

$$(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)=x^3+y^3+z^3-3xyz, \text{where}$$

ω and ω^2 are complex cube roots of unity.

Proof: L.H.S $=(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)$

$$=(x+y+z)\left[(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)\right]$$

$$=(x+y+z)\left[x^2+xy\omega^2+xz\omega+xy\omega+y^2\omega^3+yz\omega^2+xz\omega^2+yz\omega^4+z^2\omega^3\right]$$

$$=(x+y+z)\left[x^2+y^2(1)+z^2(1)+xy(\omega^2+\omega)+yz(\omega^2+\omega^4)+zx(\omega+\omega^2)\right]$$

$$=(x+y+z)\left[x^2+y^2+z^2+xy(-1)+yz(-1)+zx(-1)\right],$$

$$=(x+y+z)\left[x^2+y^2+z^2-xy-yz-zx\right] \qquad \left(\begin{array}{l}\because \omega^3=1, \omega^2+\omega=-1\\ \text{and } \omega^2+\omega^4=\omega+\omega^2=-1\end{array}\right)$$

$$=x^3+y^3+z^3-3xyz.$$

= R.H.S

Hence Proved.

EXERCISE 20.2

1. Find all the cube roots of:

   (i) 64 (ii) -125 (iii) 216

2. Evaluate the following:

   (i) $(1+\omega^2)^4$ (ii) $(1-\omega+\omega^2)(1+\omega-\omega^2)$

   (iii) $(2+5\omega+2\omega^2)^6$ (iv) $(-1+\sqrt{-3})^4+(-1-\sqrt{-3})^4$

3. Show that:

   (i) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)=(\omega+\omega^2)^4$

   (ii) $(a+b)(a\omega+b\omega^2)(a\omega^2+b\omega)=a^3+b^3$

   (iii) $(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)=a^2+b^2+c^2-ab-ba-ca$

   (iv) $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^9+\left(\dfrac{-1-i\sqrt{3}}{2}\right)^9-2=0$

   (v) $(1+\omega-\omega^2)^3-(1-\omega+\omega^2)^3=0$

20.3 Roots and Coefficient of a Quadratic Equation

20.3.(i) Find the relation between the roots and the coefficient of a quadratic equation

Let the two roots of $ax^2+bx+c=0, a\neq 0$ be denoted by α and β, then

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad \alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad \alpha+\beta=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=\frac{-2b}{2a}=\frac{-b}{a} \qquad (1)$$

$$\text{and} \quad \alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$\Rightarrow \quad \alpha\beta=\frac{(-b)^2-\left(b^2-4ac\right)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a} \qquad (2)$$

Therefore (1) represents the sum of the roots $=-\dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

and (2) represents the product of the roots $=\dfrac{\text{Constant term}}{\text{Coefficient of } x^2}$

Hence, we have the important result:

If α,β be the roots of the equation $ax^2+bx+c=0, a\neq 0,$ then

$\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$.

20.3.(ii) Find the sum and product of the roots of a given quadratic equation without solving it.

Example: Without solving, find the sum and product of roots in each of the following equations.

(i) $4x^2+6x+1=0$ (ii) $3\left(5x^2+1\right)=17x$

Solutions (i):

$4x^2+6x+1=0$

Here $a=4, b=6$ and $c=1$

$\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

$\therefore \quad \alpha+\beta=-\dfrac{6}{4}$ and $\alpha\beta=\dfrac{1}{6}$,

$\Rightarrow \quad \alpha+\beta=-\dfrac{3}{2}$ and $\alpha\beta=\dfrac{1}{6}$

Solution (ii): $3\left(5x^2+1\right)=17\,x$

$\Rightarrow \quad 15x^2-17x+3=0$

Here $a=15, b=-17$ and $c=3$

$\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

$\therefore \quad \alpha+\beta=-\dfrac{(-17)}{15}$ and $\alpha\beta=\dfrac{3}{15}=\dfrac{1}{5}$

$\Rightarrow \quad \alpha+\beta=\dfrac{17}{15}$ and $\alpha\beta=\dfrac{1}{5}$

20.3.(iii) Find the value(s) of the unknown involved in a given quadratic equation when:

(a) Sum of the roots is equal to a multiple of the product of the roots.

(b) Sum of the squares of the roots is equal to a given number.

(c) Roots differ by a given number.

(d) **Roots satisfying a given relation. (e.g, the relation $2\alpha+5\beta=7$, where α and β are the roots of the equation)**

(e) **Both sum and product of the roots are equal to a given number.**

The all above mentioned conditions can be explained through examples.

(a) **Sum of the roots is equal to a multiple of the product of the roots.**

Example 1: Find the values of k, if the sum of the roots of $6x^2-3kx+5=0$ is equal to the product of its roots.

Solution: Let α,β be the roots of the equation $6x^2-3kx+5=0$

Here, $a=6$, $b=-3$ and $c=5$

$$\alpha+\beta=-\frac{b}{a}=-\frac{(-3k)}{6}=\frac{3k}{6} \text{ and } \alpha\beta=\frac{c}{a}=\frac{5}{6}$$

It is given that

Sum of the roots = product of the roots

i.e, $\alpha+\beta=\alpha\beta$

$\therefore$ $\frac{3k}{6}=\frac{5}{6}$ $\Rightarrow$ $k=\frac{5}{3}$.

Example 2: Find the value of p, if the sum of the roots is equal to two times the product of the roots of the equation $2x^2+(8-4p)x+3p=0$.

Solution: Let α,β be the roots of the equation $2x^2+(8-4p)x+3p=0$

Here $a=2$, $b=8-4p$ and $c=3p$

$$\alpha+\beta=-\frac{b}{a}=-\frac{(8-4p)}{2}=2p-4 \text{ and } \alpha\beta=\frac{c}{a}=\frac{3p}{2}$$

As per condition in the problem, we have

$\alpha+\beta=2(\alpha\beta)$

$\therefore$ $2p-4=2\left(\frac{3p}{2}\right)=3p$

$\Rightarrow$ $p=-4$.

(b) **Sum of the squares of the roots is equal to a given number.**

Example 3:

Find k, if the sum of the squares of the roots of the equation $2x^2+3kx+k^2=0$ is 5.

Solution: Let α,β be the roots of the equation

$2x^2+3kx+k^2=0$

Here, $a=2$, $b=3k$ and $c=k^2$

$$\alpha+\beta=-\frac{b}{a}=-\frac{3k}{2} \text{ and } \alpha\beta=\frac{c}{a}=\frac{k^2}{2}$$

According to given condition

$\alpha^2+\beta^2=5$

$\Rightarrow \quad (\alpha+\beta)^2 - 2\alpha\beta = 5 \qquad \because [a^2+b^2=(a+b)^2-2ab]$

$\Rightarrow \quad \left(\frac{-3k}{2}\right)^2 - 2\left(\frac{k^2}{2}\right) = 5$

$\Rightarrow \quad \frac{9k^2}{4} - k^2 = 5$

$\Rightarrow \quad \frac{9k^2-4k^2}{4} = 5$

$\Rightarrow \quad 5k^2 = 5\times 4$

$\Rightarrow \quad k^2 = 4$

$\Rightarrow \quad k = \pm 2$

(c) Roots differ by a given number.

Example 4: Find p, if the roots of the equation $x^2 - px + 8 = 0$ differ by 2.

Let α,β be roots of the equation $x^2 - px + 8 = 0$

Here, $a=1, b=-p$ and $c=8$

$$\alpha+\beta = -\frac{b}{a} = -\frac{(-p)}{1} = p \quad \text{and} \quad \alpha\beta = \frac{c}{a} = \frac{8}{1} = 8$$

According to given condition,

$\alpha - \beta = 2$

Squarring on both sides, we have

$(\alpha-\beta)^2 = 4$

$\Rightarrow \quad (\alpha+\beta)^2 - 4\alpha\beta = 4 \qquad \left[\because (a-b)^2 = (a+b)^2 - 4ab\right]$

$\Rightarrow \quad p^2 - 4(8) = 4$

$\Rightarrow \quad p^2 - 32 = 4$

$\Rightarrow \quad p^2 = 36$

$\Rightarrow \quad p = \pm 6$

(d) Roots satisfying a given relation (e.g, the relation $2\alpha+5\beta=7$, where α and β are the roots of the equation)

Example 5: Find k, if the roots α and β of the equation $x^2-5x+k=0$ satisfy the condition $2\alpha+5\beta=7$

Solution: In the given equation $x^2-5x+k=0$,

Here $a=1, b=-5$ and $c=k$ and let α,β be the roots of given equation.

$\because \quad \alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = -\frac{c}{a}$

$\therefore \quad \alpha+\beta = -\frac{(-5)}{1} = 5$

$\Rightarrow \quad \alpha = 5-\beta \qquad$ (i)

and $\quad \alpha\beta = \frac{k}{1} = k$

$\Rightarrow \quad \alpha\beta = k \qquad$ (ii)

Given that

$\because \quad 2\alpha + 5\beta = 7$

$\therefore \quad 2(5-\beta) + 5\beta = 7$

$\Rightarrow \quad 10 - 2\beta + 5\beta = 7$

$\Rightarrow \quad 3\beta = -3$

$\Rightarrow \quad \beta = -1$

$\therefore$ From equation (i), we have,

$\alpha = 5 - (-1) = 5 + 1 = 6,$

To find the value of k, substitute the values of α and β in equation (ii), we get

$k = 6(-1) = -6$

$\Rightarrow \quad k = -6$

(e) Both sum and product of the roots are equal to a given number:

Example 6: Find k, if sum and product of the roots of the equation $6x^2 - 3kx + 5 = 0$ is equal to $\frac{5}{6}$.

Solution: Let α, β be the roots of the equation $6x^2 - 3kx + 5 = 0$, then

$$\alpha + \beta = -\frac{b}{a} = -\frac{(-3k)}{6} = \frac{k}{2} \qquad \text{(i)}$$

and $$\alpha\beta = \frac{c}{a} = \frac{5}{6} \qquad \text{(ii)}$$

According to the given condition that sum and product of the roots equal to $\frac{5}{6}$.

i.e., $\alpha + \beta = \alpha\beta = \frac{5}{6}$

i.e., $\frac{k}{2} = \frac{5}{6}$

$\Rightarrow \quad k = \frac{5}{3},$

Hence, at $k = \frac{5}{3}$, given equation has sum of the roots equal to its product of the roots equal to $\frac{5}{6}$.

EXERCISE 20.3

1. Without solving, find the sum and product of the roots in each of the following quadratic equations

(i) $x^2 - x - 1 = 0$ (ii) $2x^2 - 3x - 4 = 0$

(iii) $x^2 - \frac{3}{4}a^2 = ax$ (iv) $7x^2 - 5kx + 7k = 0$

2. Find the value of m, if,

(i) Sum of the roots of the equation $x^2 + (3m - 7)x + 5m = 0$ is $\frac{3}{2}$ times the product of its roots.

(ii) Sum of the roots of the equation $2x^2 - 3x + 4m = 0$ is 6 times the product of its roots.

3. Find the value of p, if

(i) Sum of the squares of the roots of the equation of $x^2 - px + 6 = 0$ is 13.

(ii) Sum of the squares of the roots of the equation of $x^2 - 2px + (2p - 3) = 0$ is 6.

4. Find the value of m, if

(i) the roots of the equation $x^2 - 5x + 2m = 0$ differ by 1.

(ii) the roots of the equation $x^2 - 8x + m + 2 = 0$ differ by 2.

5. Find the value of k, if

(i) The roots of the equation on $5x^2 - 7x + k - 2 = 0$ satisfy the relation $2\alpha + 5\beta = 1$

(ii) The roots of the equation on $3x^2 - 2x + 7k + 2 = 0$ satisfy the relation $7\alpha - 3\beta = 18$.

6. Find the value of p, if sum and product of the roots of the following quadratic equation are equal to a given number 4.

(i) $(2p+3)x^2 + (7p-5)x + (3p-10) = 0$ (ii) $4x^2 - (5p+3)x + 17 - 9p = 0$

20.4 Symmetric Function of Roots of a Quadratic Equation.

20.4.(i) Define symmetric function of the roots of a Quadratic Equation.

Let α and β be the roots of the quadratic equation, then a function in α and β is said to be symmetric function if the function remains the same when α and β are interchanged. i.e. $f(\alpha,\beta) = f(\beta,\alpha)$.

$\alpha + \beta$ and $\alpha\beta$ are known as the primary symmetric functions of α and β.

It is noted that $\alpha - \beta \neq \beta - \alpha$, so that $\alpha - \beta$ is not symmetric function of roots.

The value of a symmetric function of roots in terms of the coefficients of the quadratic equation can be obtained using $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ by expressing in terms of $\alpha + \beta$ and $\alpha\beta$.

Examples of symmetric functions are as under:

- $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
- $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$
- $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2=\left[(\alpha+\beta)^2-2\alpha\beta\right]-2(\alpha\beta)^2$ etc.

Example: Find the value of $\alpha^2+\beta^2+\alpha\beta$, when $\alpha=2$ and $\beta=3$ and show that it is symmetric function of α and β, where α and β are the roots of quadratic equation.

Solution: Let $f(\alpha,\beta)=\alpha^2+\beta^2+\alpha\beta$,

Given that $\alpha=2$ and $\beta=3$

$\therefore \quad f(2,3)=2^2+3^2+(2)(3)=4+9+6=19,$

Now,

$f(\beta,\alpha)=\beta^2+\alpha^2+\beta\alpha=\alpha^2+\beta^2+\alpha\beta=f(\alpha,\beta)$

$\because \quad f(\alpha,\beta)=f(\beta,\alpha)$

$\therefore \quad f$ is symmetric function.

Given expression is symmetric function of roots of α and β.

Hence Shown.

20.4.(ii) Represent a symmetric function graphically.

In article 20.4.1 we have already defined symmetric functions of roots of quadratic equations, e.g, $\alpha+\beta,\alpha\beta,\alpha^2+\beta^2,\alpha^2+\beta^2+2\alpha\beta,\alpha^3+\beta^3$ etc.

When a symmetric function (expression) is equated to constant $c\in\mathbb{R}$, we obtain a symmetric equation, so that symmetric equation of the roots can be written as $f(\alpha,\beta)=c$. Every equation can be expressed graphically as the set of all points denoted by $G(f)$ defined as

$$G(f)=\{(\alpha,\beta)\mid f(\alpha,\beta)=c\wedge\alpha,\beta\in\mathbb{R}\}$$

Example: Represent symmetric equation $\alpha^2+\beta^2=9$ graphically and sketch the graph.

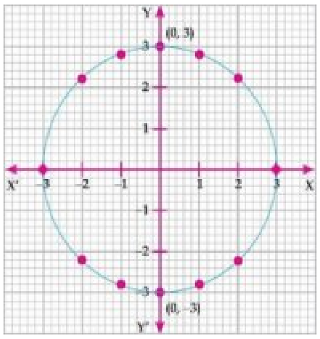
Solution: Given that

$\therefore \quad \alpha^2+\beta^2=9$

$\Rightarrow \quad \beta=\pm\sqrt{9-\alpha^2}$

To plot the graph, we take some points. By assigning different values of α and get corresponding values of β.

α	0	1	2	3	−1	−2	−3
β	±3	±2.828	±2.2360	0	±2.828	±2.2360	0



Thus the symmetric function $\alpha^2+\beta^2$ represents a circle when equated to constant $c=9$.

20.4.(iii) Evaluate a Symmetric functions of roots of a quadratic equation in terms of its coefficients.

Let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0, a \neq 0$ (i)

Then $\alpha + \beta = -\dfrac{b}{a}$ (ii)

and $\alpha\beta = \dfrac{c}{a}$ (iii)

The above equations (ii) and (iii) represent the primary symmetric functions of the quadratic equation (i).

Example 1: If α, β are the root of $x^2 - px + q = 0,$ find the value of :

(i) $\alpha^2 + \beta^2$ (ii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

Solution: Given that α, β are the root of $x^2 - px + q = 0$

Here $a = 1, b = -p$ and $c = q$

$\therefore \quad \alpha + \beta = -\dfrac{b}{a} = -\dfrac{(-p)}{1} = p$

and $\quad \alpha\beta = \dfrac{c}{a} = \dfrac{q}{1} = q$

Now

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (p)^2 - 2q \qquad (\because \alpha + \beta = p \text{ and } \alpha\beta = q)$$

$$= p^2 - 2q$$

(ii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{p^2 - 2q}{q} \qquad (\because \alpha + \beta = p \text{ and } \alpha\beta = q)$$

EXERCISE 20.4

1. If α,β are the roots of a quadratic equation. Express the following symmetric functions in terms of $\alpha+\beta$ and $\alpha\beta$.

   (i) $(\alpha-\beta)^2$ (ii) $(\alpha+\beta)^3$ (iii) $\alpha^2\beta^{-1}+\beta^2\alpha^{-1}$

   (iv) $\alpha^3\beta+\alpha\beta^3$ (v) $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$ (vi) $\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2$

2. If α,β are the roots of the equation $2x^2-3x+7=0$, find the value of following symmetric functions.

   (i) $\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2$ (ii) $\frac{\alpha}{\beta^2}+\frac{\beta}{\alpha^2}$ (iii) $\frac{1}{a\alpha+1}+\frac{1}{a\beta+1}$

3. If α,β are the roots of the equation $px^2+qx+q=0$, $p\neq 0$, find the value of $\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}$

4. Represent symmetric equation $\alpha+\beta=8$ graphically when α and β are the root of a quadratic equation and also plot the graph.

20.5 Formation of Quadratic Equations

20.5.(i) Establish the formula $x^2-(\text{sum of the roots})\,x+(\text{product of the roots})=0$, to find the quadratic equation of the given roots.

Let α,β be the roots of the quadratic equation $ax^2+bx+c=0, a\neq 0$, (i)

then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

Now, rewrite the equation (i) as under

$$x^2+\frac{b}{a}x+\frac{c}{a}=0, \quad \text{provided } a\neq 0$$

$$\Rightarrow \quad x^2-\left(-\frac{b}{a}\right)x+\frac{c}{a}=0$$

$$\Rightarrow \quad x^2-(\text{sum of the roots})\,x+(\text{product of the roots})=0,$$

where $\frac{-b}{a}$ and $\frac{c}{a}$ represent the sum and product of the roots respectively.

i.e., Now it can be written precisely as under

$x^2-Sx+P=0$, where S and P respectively denote the sum and product of the roots of the given quadratic equation.

Example 1: Form the quadratic equation whose roots are:

(i) $-\frac{4}{5},\frac{3}{7}$ (ii) $7\pm 2\sqrt{5}$

Solution (i): Let $\alpha = -\frac{4}{5}$ and $\beta = \frac{3}{7}$

$\therefore \quad S = \alpha + \beta = -\frac{4}{5} + \frac{3}{7} = \frac{-28+15}{35} = \frac{-13}{35}$

and $\quad P = \alpha\beta = \left(-\frac{4}{5}\right)\left(\frac{3}{7}\right) = \frac{-12}{35}$

When roots are known then equation is

$x^2 - Sx + P = 0$

i.e, $\quad x^2 - \left(\frac{-13}{35}\right)x + \left(\frac{-12}{35}\right) = 0$

$\Rightarrow \quad 35x^2 + 13x - 12 = 0.$

is the required equation

Solution (ii): Let $\alpha = 7 + 2\sqrt{5}$ and $\beta = 7 - 2\sqrt{5}$

$\therefore \quad S = \alpha + \beta = 7 + 2\sqrt{5} + 7 - 2\sqrt{5} = 14$

and $\quad P = \alpha\beta = \left(7 + 2\sqrt{5}\right)\left(7 - 2\sqrt{5}\right) = 49 - 20 = 29$

When roots are known, then equation is

$x^2 - Sx + P = 0$

$\Rightarrow \quad x^2 - 14x + 29 = 0,$ is the required equation.

20.5.(ii) Form the quadratic equation whose roots are of the types:

(a) $2\alpha+1, 2\beta+1$ (b) α^2, β^2 (c) $\frac{1}{\alpha}, \frac{1}{\beta}$

(d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (e) $\alpha+\beta, \frac{1}{\alpha}+\frac{1}{\beta},$

where α, β are the root of a quadratic equation.

Example 1: If α, β are the roots of the equation $x^2 - 5x + 6 = 0$, form the equation whose roots are

(i) $2\alpha+1, 2\beta+1$ (ii) α^2, β^2 (iii) $\frac{1}{\alpha}, \frac{1}{\beta}$

(iv) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (v) $\alpha+\beta, \frac{1}{\alpha}+\frac{1}{\beta}$

Solution (i): Since α, β are the roots of the equation $x^2 - 5x + 6 = 0$

Here $a = 1, b = -5$ and c=6

$\therefore \quad \alpha + \beta = -\frac{b}{a} = -\frac{(-5)}{1} = 5$ and $\alpha\beta = \frac{c}{a} = \frac{6}{1} = 6,$

The roots of required equation are $2\alpha+1$ and $2\beta+1$.

Now find the sum and product of these roots.

$S = (2\alpha + 1) + (2\beta + 1)$
$S = 2(\alpha + \beta) + 2$
$S = 2(5) + 2 \qquad (\alpha + \beta = 5)$
$S = 10 + 2 = 12$
$P = (2\alpha + 1)(2\beta + 1)$
$P = 4\alpha\beta + 2(\alpha + \beta) + 1$
$P = 4(6) + 2(5) + 1 \qquad (\alpha + \beta = 5 \text{ and } \alpha\beta = 6)$
$P = 24 + 10 + 1 = 35$

The required equation is $x^2 + 12x + 35 = 0$

Solution (ii): The roots of required equation are α^2, β^2.

$S = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (5)^2 - 2(6) = 25 - 12 = 13, (\because \alpha + \beta = 5 \text{ and } \alpha\beta = 6)$

and $P = \alpha^2\beta^2 = (\alpha\beta)^2 = (6)^2 = 36$

$x^2 - Sx + P = 0$

$\therefore \quad x^2 - 13x + 36 = 0$, is the requied equation.

Solution (iii): The roots of required equation are $\frac{1}{\alpha}, \frac{1}{\beta}$.

$S = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{5}{6}, \qquad (\alpha + \beta = 5 \text{ and } \alpha\beta = 6)$

and $P = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{1}{\alpha\beta} = \frac{1}{6}$

$x^2 - Sx + P = 0$

$\therefore \quad x^2 - \frac{5}{6}x + \frac{1}{6} = 0$

$\Rightarrow \quad 6x^2 - 5x + 1 = 0$, is the requied equation.

Solution (iv): The roots of required equation are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$.

$S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} \qquad (\because \alpha + \beta = 5 \text{ and } \alpha\beta = 6)$

$= \frac{25 - 2(6)}{6} = \frac{25 - 12}{6} = \frac{13}{6},$

and $P = \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = \frac{\alpha\beta}{\alpha\beta} = \frac{6}{6} = 1$

$x^2 - Sx + P = 0$

$\therefore \quad x^2 - \frac{13}{6}x + 1 = 0$

$\Rightarrow \quad 6x^2 - 13x + 6 = 0$, is the required equation.

Solution (v): The roots of required equation are $\alpha+\beta$ and $\frac{1}{\alpha}+\frac{1}{\beta}$

$$S=(\alpha+\beta)+\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$$
$$=(\alpha+\beta)+\left(\frac{\alpha+\beta}{\alpha\beta}\right)$$
$$=(\alpha+\beta)\left(1+\frac{1}{\alpha\beta}\right)$$
$$=5\left(1+\frac{1}{6}\right)=5\left(\frac{7}{6}\right)=\frac{35}{6} \qquad (\because \alpha+\beta=5 \text{ and } \alpha\beta=6)$$
$$P=(\alpha+\beta)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$$
$$=(\alpha+\beta)\left(\frac{\alpha+\beta}{\alpha\beta}\right)=\frac{(\alpha+\beta)^2}{\alpha\beta}=\frac{25}{6} \qquad (\because \alpha+\beta=5 \text{ and } \alpha\beta=6)$$
$$x^2-Sx+P=0$$

$\therefore$ $x^2-\frac{35}{6}x+\frac{25}{6}=0$

$\Rightarrow$ $6x^2-35x+25=0$, is the required equation.

20.5.(iii) Find the values of α,β, where the roots of an equation are $\frac{1}{\alpha},\frac{1}{\beta}$.

Example: If $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of the $x^2-6x+8=0$, find the value(s) of α and β.

Solution: Since $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of the equation $x^2-6x+8=0$.

Here $a=1$, $b=-6$ and $c=8$

$\therefore$ $\frac{1}{\alpha}+\frac{1}{\beta}=-\frac{b}{a}=-\left(\frac{-6}{1}\right)=6,$

$\Rightarrow$ $\frac{\alpha+\beta}{\alpha\beta}=6$

$\Rightarrow$ $\alpha+\beta=6\alpha\beta$ ... (i)

and $\frac{1}{\alpha}\times\frac{1}{\beta}=\frac{c}{a}=\frac{8}{1}=8$

$\Rightarrow$ $\alpha\beta=\frac{1}{8}$ (ii)

From equation (i), $\beta=6\left(\frac{1}{8}\right)-\alpha=\frac{3-4\alpha}{4}$ (iii)

By substituting the value of β in equation (ii), we have,

$$\alpha\left(\frac{3-4\alpha}{4}\right)=\frac{1}{8}$$

$\Rightarrow \quad \frac{8\alpha}{4}(3-4\alpha)-1=0$

$\Rightarrow \quad 2\alpha(3-4\alpha)-1=0$

$\Rightarrow \quad 6\alpha-8\alpha^2-1=0$

$\Rightarrow \quad 8\alpha^2-6\alpha+1=0$

$\Rightarrow \quad 8\alpha^2-4\alpha-2\alpha+1=0$

$\Rightarrow \quad 4\alpha(2\alpha-1)-1(2\alpha-1)=0$

$\Rightarrow \quad (2\alpha-1)(4\alpha-1)=0$

i.e, $\quad 2\alpha-1=0 \quad$ or $\quad 4\alpha-1=0$

$\Rightarrow \quad \alpha=\frac{1}{2} \quad$ or $\quad \alpha=\frac{1}{4},$

then from equation (iii), we have,

$$\beta=\frac{3-4\left(\frac{1}{2}\right)}{4}=\frac{1}{4} \quad \text{or} \quad \beta=\frac{3-4\left(\frac{1}{4}\right)}{4}=\frac{2}{4}=\frac{1}{2},$$

Hence, $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{4}$ or $\alpha=\frac{1}{4}$ and $\beta=\frac{1}{2}$.

EXERCISE 20.5

1. Form the equations whose roots are:
   (i) $2,-3$ (ii) ω,ω^2 (iii) $2+i,2-i$ (iv) $2\sqrt{2},-2\sqrt{2}$
2. If α,β are the roots of the equation $6x^2-3x+1=0$. Form the equations whose roots are:
   (i) $2\alpha+1, 2\beta+1$ (ii) α^2,β^2 (iii) $\frac{1}{\alpha},\frac{1}{\beta}$
   (iv) $\frac{\alpha}{\beta},\frac{\beta}{\alpha}$ (v) $\alpha+\beta,\frac{1}{\alpha}+\frac{1}{\beta}$
3. Find the equation whose roots are the reciprocals of the roots of $px^2-qx+r=0, p\neq 0$.
4. Find the equation whose roots are double the roots of $x^2-px+q=0$.
5. Find the equation whose roots exceed by 2 the roots of $px^2+qx+r=0$.
6. Find the condition that one root of $ax^2+bx+c=0, a\neq 0,$ may be:
   (i) 3 times the other (ii) square of the other
   (iii) additive inverse of the other (iv) multiplicative inverse of the other.

20.6 Higher Dgree Equations reducible to Quadratic Form.

20.6.(a) Solve the cubic equation if one root of the equation is given

Let $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$, be the cubic equation and its one root be α.

Suppose that

$$f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 = 0$$

and $f(x) = 0$ has a root say α, then,

$f(x) = (x-\alpha)\cdot f_1(x)$, where $f_1(x)$ is the quadratic expression.

Example : Solve $x^3 - 6x^2 + 11x - 6 = 0$, if one root of this equation is 1.

Solution: $x^3 - 6x^2 + 11x - 6 = 0$,

It is given that one root is 1, i.e, $x = 1$, is the multiplier factor.

By synthetic division method

1	1	−6	11	−6
	↓	1	−5	6
	1	−5	6	0 = remainder

Thus, we have quadratic equation as under

$$x^2 - 5x + 6 = 0$$
$$\Rightarrow \quad x^2 - 2x - 3x + 6 = 0$$
$$\Rightarrow \quad x(x-2) - 3(x-2) = 0$$
$$\Rightarrow \quad (x-2)(x-3) = 0$$

i.e, $x - 2 = 0$ or $x - 3 = 0$

$\Rightarrow$ $x = 2$ or $x = 3$

$\therefore$ Solution set $= \{1, 2, 3\}$.

20.6.(b) Solve a biquadratic (quartic) equation if two of the real roots of the equation are given

Let $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, a_0 \neq 0$, be the biquadratic equation and its two real roots be α and β.

If $f(x) = 0$, has two roots say α and β, then

$$f(x) = (x-\alpha)(x-\beta)\cdot f_1(x),$$

where $f_1(x)$ is the quadratic expression.

Example 1: Find the remaining two roots of the biquadratic equation $x^4 - 9x^3 + 19x^2 + 9x - 20 = 0$, if its two roots are 5 and 1.

Solution: $x^4 - 9x^3 + 19x^2 + 9x - 20 = 0$, is the given quartic equation whose given two roots are 1 and 5 i.e., multipliers 5 and 1.

By synthetic division method, we have,

5	1	−9	19	9	−20
	↓	5	−20	−5	20
1	1	−4	−1	4	0 = remainder
	↓	1	−3	−4	
	1	−3	−4	0 = remainder	

Thus, we have quadratic equation

$$x^2 - 3x - 4 = 0$$
$$\Rightarrow \quad x^2 - 4x + x - 4 = 0$$
$$\Rightarrow \quad x(x-4) + 1(x-4) = 0$$
$$\Rightarrow \quad (x-4)(x+1) = 0$$

i.e., $x - 4 = 0$ or $x + 1 = 0$

$\Rightarrow$ $x = 4$ or $x = -1$

Therefore, remaining two roots are 4 and -1.

EXERCISE 20.6

1. Find the remaining two roots of the following cubic equations, when one root is given:
   (i) $2x^3 - x^2 - 2x + 1 = 0$, and $x = 1$
   (ii) $x^3 - 4x^2 + x + 6 = 0$, and $x = 3$
   (iii) $x^3 - 28x + 48 = 0$, and $x = 2$
2. Find the remaining two roots of the following biquadratic equations, when its two roots are given:
   (i) $12x^4 - 8x^3 - 7x^2 + 2x + 1 = 0$; and $x = 1, -\frac{1}{2}$
   (ii) $x^4 + 4x^2 - 5 = 0$; and $x = 1, -1$
   (iii) $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$; and $x = 3, -4$
3. Find the value of m and remaining two roots of the equation $2x^3 - 3mx^2 + 9 = 0$, if its one root is 3.
4. If one root of $x^3 - 3ax^2 - x + 6 = 0$ is 1, then find the value of a. Also find its remaining roots.
5. Find the value of a and b, if two roots of the equation $x^4 - ax^2 + bx + 252 = 0$ are 6 and -2, also find its remaining two roots.

20.7 Simultaneous Equations

Two or more equations taken together is called a system of simultaneous equations. To determine the values of two unknown variables we need a pair of equations.

The set of all ordered pairs (x, y) which satisfies the system of equation is called the solution set of the system.

20.7.(i) (a) Solve a system of two equations in two variables, when one equation is linear and other is quadratic.

The complete method (procedure) to solve the system of equation when one is quadratic other is linear is illustrated by solving the examples given below.

Example 1: Solve the system of equations $2x + y = 10$ and $4x^2 + y^2 = 68$.

Solution:

$2x + y = 10$ (i)

and $4x^2 + y^2 = 68$ (ii)

From equation (i), we have,

$y = 10 - 2x$ (iii)

Substitute this value of y in equaion (ii), we have,

$$4x^2 + (10 - 2x)^2 = 68$$

$$\Rightarrow \quad 4x^2 + 100 - 40x + 4x^2 - 68 = 0$$

$$\Rightarrow \quad 8x^2 - 40x + 32 = 0$$

$\Rightarrow \quad x^2 - 5x + 4 = 0$

$x^2 - 4x - x + 4 = 0$

$(x-4)(x-1) = 0$

i.e. $\quad x - 4 = 0 \quad$ or $\quad x - 1 = 0$

$\Rightarrow \quad x = 4 \quad$ or $\quad x = 1$

Putting these values of x in equation (iii)

when $x = 4$, then $y = 10 - 2(4) = 2$

when $x = 1$, then $y = 10 - 2(1) = 8$

Thus, solution set is $\{(4,2),(1,8)\}$.

Example 2: Solve the system of equations $3x - 2y = 7$ and $xy = 20$.

Solution:

$3x - 2y = 7 \quad$ (i)

and $\quad xy = 20 \quad$ (ii)

From equation (i), we have, $\quad x = \dfrac{7+2y}{3} \quad$ (iii)

Substitute this value of x in equaion (ii), we have

$$\left(\frac{7+2y}{3}\right)y = 20$$

$\Rightarrow \quad 7y + 2y^2 = 20 \times 3 = 60$

$\Rightarrow \quad 2y^2 + 7y - 60 = 0$

$\Rightarrow \quad 2y^2 - 8y + 15y - 60 = 0$

$\Rightarrow \quad 2y(y-4) + 15(y-4) = 0$

$\Rightarrow \quad (y-4)(2y+15) = 0$

i.e. $\quad y - 4 = 0 \quad$ or $\quad 2y + 15 = 0$

$\Rightarrow \quad y = 4 \quad$ or $\quad y = -\dfrac{15}{2}$

Putting these values of x in equation (iii)

when $y = 4$, then $x = \dfrac{7+2(4)}{3} = \dfrac{15}{3} = 5$

when $y = -\dfrac{15}{2}$, then $x = \dfrac{7 + 2\left(\dfrac{-15}{2}\right)}{3} = -\dfrac{8}{3}$

Thus solution set is $\left\{(5,4),\left(-\dfrac{8}{3}, -\dfrac{15}{2}\right)\right\}$.

20.7.(i) (b) When both the equations are quadratic

We explain the method of solution of system when both equations are quadratic by the following example.

Example 3: Solve the system of equations $x^2 + y^2 = 4$ and $2x^2 - y^2 = 8$.

Solution:

$x^2 + y^2 = 4$ (i)

and $2x^2 - y^2 = 8$ (ii)

To eliminate y^2 adding equations (i) and (ii), we have

$\therefore$ $3x^2 = 12$

$\Rightarrow$ $x^2 = 4$

$\Rightarrow$ $x = \pm 2,$

$\therefore$ By using $x = \pm 2,$ in equation (i), we get

$(2)^2 + y^2 = 4$	$(-2)^2 + y^2 = 4$
$\Rightarrow$ $4 + y^2 = 4$	$\Rightarrow$ $4 + y^2 = 4$
$\Rightarrow$ $y^2 = 4 - 4 = 0$	$\Rightarrow$ $y^2 = 4 - 4 = 0$
$\Rightarrow$ $y = 0$	$\Rightarrow$ $y = 0$

Thus, solution set is $\{(-2,0),(2,0)\}$.

Example 4: Solve the system of equations $x^2 + y^2 = 13$ and $xy = 6$.

Solution:

$x^2 + y^2 = 13$ ... (i)

and $xy = 6$ ... (ii)

In this type of equations we eliminate constant by multiplying equation (i) by 6 and equation (ii) by 13, we get

$\therefore$ $6x^2 + 6y^2 = 78$ ... (iii)

and $13xy = 78$ ... (iv)

By substracting equation (iv) from equation (iii)

$6x^2 - 13xy + 6y^2 = 0$

$\Rightarrow$ $6x^2 - 9xy - 4xy + 6y^2 = 0$

$\Rightarrow$ $3x(2x - 3y) - 2y(2x - 3y) = 0$

i.e., $2x - 3y = 0$ or $3x - 2y = 0$

$\Rightarrow$ $x = \dfrac{3y}{2}$ or $x = \dfrac{2y}{3}$

Thus, we have following two systems

$$\left.\begin{aligned} xy &= 6 \\ x &= \frac{3y}{2} \end{aligned}\right\} \text{System (A)}$$

and $$\left.\begin{aligned} xy &= 6 \\ x &= \frac{2y}{3} \end{aligned}\right\} \text{System (B)}$$

In system (A),

$$\left(\frac{3y}{2}\right)y = 6, \qquad \left(\because\ x = \frac{3y}{2}\right)$$

$$\Rightarrow \quad 3y^2 = 2\times 6$$

$$\Rightarrow \quad y^2 = \frac{2\times 6}{3} = 4$$

$$y = \pm 2$$

when $y = \pm 2$ then $x = \frac{3}{2}(\pm 2) = \pm 3$

In system (B),

$$\left(\frac{2y}{3}\right)\cdot y = 6 \qquad \left(\because\ x = \frac{2y}{3}\right)$$

$$\Rightarrow \quad 2y^2 = 3\times 6$$

$$\Rightarrow \quad y^2 = \frac{3\times 6}{2} = 9$$

$$\Rightarrow \quad y = \pm 3,$$

when $y = \pm 3$, then $x = \frac{2}{3}(\pm 3) = \pm 2$

Thus, the solution set is

$\{(3,2),(-3,-2),(2,3),(-2,-3)\}$.

20.7.(ii) Solve the real life problems leading to quadratic equations

Example 1:

Find two consecutive positive integers whose product is 72.

Solution:

Let x and $x+1$ be the two consecutive integers.

According to given condition.

$x(x+1) = 72$

$\Rightarrow \quad x^2 + x - 72 = 0$

$\Rightarrow \quad (x+9)(x-8) = 0$ (By factoriziation)

i.e., $\quad x+9 = 0 \quad$ or $\quad x-8 = 0$

$\Rightarrow \quad x = -9 \quad$ or $\quad x = 8$

$\because$ Neglecting $x = -9$ being negative integer

$\therefore$ Required consecutive integers are 8 and 9.

Example 2:

The area of a rectangular plot is $320m^2$. The width of the plot is less $4m$ than the length of the plot. Find the length and width of the plot.

Solution:

Let x be the length of the plot then, the width is $x-4$.

Given that

$\because$ Area of the plot $= 320m^2$

$\therefore \quad x(x-4) = 320 \qquad [\because\ \text{Area} = \text{Length} \times \text{Width}]$

$\Rightarrow \quad x^2 - 4x - 320 = 0$

$\Rightarrow \quad x^2 - 20x + 16x - 320 = 0$

$\Rightarrow \quad x(x-20) + 16(x-20) = 0$

$\Rightarrow \quad (x-20)(x+16) = 0$

i.e. $\quad x - 20 = 0 \quad$ or $\quad x + 16 = 0$

$\Rightarrow \quad x = 20 \quad$ or $\quad x = -16$

Neglecting $x = -16$ because length is always positive.

Thus, length of the plot is $20m$ and width of the plot $= 20 - 4 = 16m$.

EXERCISE 20.7

Solve the following systems of equations:

1. $2x - y = 3$ and $x^2 + y^2 = 2$
2. $2x + y = 4$ and $x^2 - 2x + y^2 = 0$
3. $\frac{4}{x} + \frac{3}{y} = 2$ and $4x + 3y = 25$
4. $(x-1)^2 + (y+3)^2 = 25$ and $x^2 + (y+1)^2 = 10$
5. $x^2 + y^2 = 25$ and $(4x - 3y)(x - y - 5) = 0$
6. $x^2 + y^2 = 16$ and $2x^2 - 3xy + y^2 = 0$
7. $x^2 + y^2 = 5$ and $xy = 2$
8. $x^2 + xy = 5$ and $x^2 - 2xy = 2$
9. $x + \frac{4}{y} = 1$ and $y + \frac{4}{x} = 25$
10. Divide 12 into two parts such that the sum of their squares is greater than twice their product by 4.
11. The length of a prayer hall is 5 meters more than its width. If the area of the hall is $36m^2$, find the length and width of hall.
12. The sum of the squares of two positive numbers is 100. One number is 2 more than the other find the numbers.
13. The length of the base of the right triangle exceeds the length of the perpendicular by 3. When hypotenuse of the triangle is 34. Find base and perpendicular.
14. The perimeter of an isosceles triangle is 36cm. The altitude to unequal side of the triangle is 36cm. Find the length of the three sides of the triangle.
15. The difference of two numbers is 5, and difference of their squares is 275. Find the numbers.

REVIEW EXERCISE 20

1. Multiple Choice Question MCQs.

Encircle the correct answer.

i. If p, q are the roots of $2x^2+5x+3=0$, then $p+q=$ ________

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $\frac{5}{2}$ (d) $-\frac{5}{2}$

ii. If $\frac{1}{\alpha}, \frac{1}{\beta}$ are the roots of the $ax^2+bx+c=0$, $a\neq 0$, then $\alpha+\beta=$ ______

(a) $-\frac{b}{a}$ (b) $\frac{b}{c}$ (c) $-\frac{b}{c}$ (d) $-\frac{c}{b}$.

iii. If the sum of the roots of $(p+1)x^2+(2p+3)x+(3p+4)=0$ is -1, then product of the root is

(a) 0 (b) 1 (c) 2 (d) 3

iv. The nature of the roots of $ax^2+bx+c=0$, $a\neq 0$ is determined by

(a) sum of the roots (b) product of the roots
(c) discriminant (c) none of these

v. If sum of the roots of a quadratic equation is $\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$, then equation is

(a) $ax^2+bx+c=0$ (b) $ax^2+bx-c=0$
(c) $ax^2-bx+c=0$ (d) $ax^2-bx-c=0$

vi. The required equation whose roots are the reciprocal of the root of $ax^2+bx+c=0$ is

(a) $ax^2+bx+c=0$ (b) $cx^2+bx+a=0$
(c) $cx^2+ax+b=0$ (d) $cx^2-bx-a=0$

vii. If α,β are the roots of $x^2-2x-15=0$, then the value of $\alpha^2+\beta^2=$ ______

(a) 34 (b) -34 (c) 26 (d) -26

viii. If $\Delta=b^2-4ac$, of a quadratic equation with real coefficient is perfect square, then roots are

(a) real and equal (b) real, rational and un equal
(c) real, irrational and unequal (d) imaginary

ix. If one root of quadratic equation is $2+\sqrt{3}$, then other root will be

(a) 2 (b) $-2+\sqrt{3}$ (c) $2-\sqrt{3}$ (d) $-2-\sqrt{3}$

x. The quadratic equation whose roots are complex cube roots.

(a) $x^2-x-1=0$ (b) $x^2-x+1=0$
(c) $x^2+x+1=0$ (d) $x^2+x-1=0$

2. Discuss the nature of the following quadratic equation:

i) $x^2 - 7x + 12 = 0$ ii) $x^2 - 14x + 49 = 0$

iii) $x^2 - x + 7 = 0$ iv) $x^2 - 5 = 0$

3. For what value of k, the equation $x^2 + kx + 4 = 0$

i) has equal roots ii) has complex roots

iii) has real roots iv) has rational roots

4. Find the cube roots of 729.

5. Find the sum and product of roots of the following quadratic equation.

i) $x^2 - 7x + 29 = 0$ ii) $x^2 - px + q = 0$

iii) $7x - 8 = 5x^2$ iv) $11x = 9x^2 - 28$

6. Define symmetric function of quadratic equations.

7. Form the quadratic equation whose roots are $1-\sqrt{3}$ and $1+\sqrt{3}$.

8. Find the equation whose roots are the reciprocals of the roots of $x^2 - 10x + 16 = 0$

9. Solve the following system of equation

i) $x^2 + y^2 = 10$, $x + y = 3$ ii) $x^2 + y^2 = 37$, $2x + 5y = 32$

SUMMARY

- Discriminant of the quadratic expression $ax^2 + bx + c, a \neq 0$ is $\Delta = b^2 - 4ac$.
  - i. If $\Delta = b^2 - 4ac > 0$, then the roots are real and unequal.
  - ii. If $\Delta = b^2 - 4ac < 0$, then the roots are non-real (complex or imaginary).
  - iii. If $\Delta = b^2 - 4ac = 0$, then the roots are rational and equal, each being equal to $-\frac{b}{2a}$.
  - iv. If a,b,c are rational and $\Delta = b^2 - 4ac$ is perfect square, then roots are rational and unequal otherwise irrational.
- The cube roots of unity are $1, \omega$ and ω^2.
- Properties of the cube roots of unity are:
  - i. Each of the complex cubes of unity is square of other.
  - ii. Sum of three cube roots of unity is zero.
  - iii. Product of three cube roots of unity is 1.
  - iv. Each complex cube root of unity is reciprocal of the other.
- If α and β are the roots of quadratic equation $ax^2 + bx + c, a \neq 0$, then

  $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$
- Symmetric functions of the roots of quadratic equation are those functions which remain unchanged when roots are interchanged, i.e. $f(\alpha,\beta) = f(\beta,\alpha)$.

PARTIAL FRACTION

Unit 21

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Define proper, improper and rational fractions.
- Resolve an algebraic fraction into its partial fractionswhen its denominator consist of
  - Non- repeated linear factors,
  - Repeated linear factors,
  - Non- repeated quadratic factors,
  - Repeated quadratic factors.

Introduction:

To split a rational fraction into the sum or difference of two or more fractions, such fractions are known as partial fractions. Partial fractions can only be found if the degree of the polynomial in numerator is strictly less than the degree of the polynomial in denominator. For instance,

(i) $\dfrac{2x+3}{(x-1)(x+4)} = \dfrac{1}{x-1} + \dfrac{1}{x+4}$

(ii) $\dfrac{-(4x^2+x+11)}{(x^2+1)(x-3)} = \dfrac{x+2}{x^2+1} - \dfrac{5}{x-3}$

21.1 Define Proper, Improper and Rational Fractions

Rational Fraction:

We know that, a number of the form $\dfrac{p}{q}$, where $p, q \in Z$ and $q \neq 0$ is called a rational number. Similarly, the quotient of two polynomials $\dfrac{P(x)}{Q(x)}$, $Q(x) \neq 0$ is called a rational algebraic expression. It is commonly known as rational fraction.

Example:

(i) $\dfrac{2x^3-5x^2-3x-10}{x^2-1}$ (ii) $\dfrac{5x+8}{3x^2-2x-1}$

Proper fractions:

A rational fraction $\dfrac{P(x)}{Q(x)}$ is said to be proper fraction if the degree of numerator $P(x)$ is less than the degree of denominator $Q(x)$.

Example:

(i) $\dfrac{9x^2-9x+6}{(x-1)(2x-1)(x+2)}$ (ii) $\dfrac{6x+27}{3x^3-9x}$

Improper fractions:

A rational fraction $\dfrac{P(x)}{Q(x)}$ is said to be improper fraction if the degree of numerator $P(x)$ is equal or greater than the degree of denominator $Q(x)$.

Example:

(i) $\dfrac{2x^3-5x^2-3x-10}{x^2-1}$ (ii) $\dfrac{6x^3-5x^2-7}{3x^2-2x-1}$

(iii) $\dfrac{x^2+1}{x^2-1}$

Additionally, any improper fraction can be resolved into the sum of polynomial and proper fraction.

e.g. $\frac{3x^2-2x+1}{x+2}=3x-8+\frac{17}{x+2}$.

Here, $\frac{3x^2-2x+1}{x+2}$ is improper fraction, $3x-8$ is polynomial and $\frac{17}{x+2}$ is proper fraction. To resolve the improper fraction into the sum of polynomial and proper fraction, one needs to divide the numerator with denominator.

21.2 Resolution of fraction into its partial fractions

To resolve the rational fraction into partial fractions, it is necessary that the rational fraction must be a proper fraction. If it is not, then it must be converted into proper fraction by division.

21.2.(i) Resolve an algebraic fraction into its partial fractions when its denominator consist of

- Non- repeated linear factors,
- Repeated linear factors,
- Non- repeated quadratic factors,
- Repeated quadratic factors.

Case-I: Denominator Consists of Non-repeated linear factors

Let $R(x)=\frac{P(x)}{Q(x)}$ is a rational fraction, where, its denominator $Q(x)$ is the product of non-repeated linear factors which can be written as

$Q(x)=(x-a_1)(x-a_2)(x-a_2).....(x-a_n)$.

Now $R(x)=\frac{P(x)}{Q(x)}$ is resolved as under:

$$\frac{P(x)}{Q(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\frac{A_3}{x-a_3}+...+\frac{A_n}{x-a_n}.$$

Here, the constants $A_1, A_2, A_3, ..., A_n$ are to be found. The method is explained by following examples.

Example 1:

Resolve $\frac{11-3x}{(x-1)(x+3)}$ into partial fractions.

Solution:

Let $$\frac{11-3x}{(x-1)(x+3)}=\frac{A_1}{(x-1)}+\frac{A_2}{(x+3)} \quad \text{(i)}$$

Here, A_1 and A_2 are unknown constants to be determined.

From equation (i),

$$\frac{11-3x}{(x-1)(x+3)}=\frac{A_1(x+3)+A_2(x-1)}{(x-1)(x+3)}$$

Multiplying both sides by $(x-1)(x+3)$

$\Rightarrow 11-3x = A_1(x+3)+A_2(x-1)$ (ii)

To determine constants A_1 and A_2, values of x are chosen. To get A_1, put $x=1$, on both sides in equation (ii),

we get $11-3(1)=A_1(1+3)+A_2(0)$

$\Rightarrow \quad 8=4A_1$

$\Rightarrow \quad A_1=2$

To get A_2, put $x=-3$, in equation (ii),

we get $11-3(-3)=A_1(0)+A_2(-3-1)$

$\Rightarrow \quad 20=-4A_2$

$\Rightarrow \quad A_2=-5$

Finally, by putting the values of constants A_1 and A_2 in equation (ii),

we get $\dfrac{11-3x}{(x-1)(x+3)}=\dfrac{2}{x-1}-\dfrac{5}{x+3}$.

Example 2:

Resolve $\dfrac{6x^3+5x^2-7}{2x^2-x-1}$ into partial fractions.

Solution:

The given rational fraction is an improper fraction; hence it is converted in the proper fraction by dividing numerator with denominator.

$$\frac{6x^3+5x^2-7}{2x^2-x-1}=\frac{6x^3+5x^2-7}{(x-1)(2x+1)}=3x+4+\frac{7x-3}{(x-1)(2x+1)} \qquad \text{(i)}$$

Consider the expression $\dfrac{7x-3}{(x-1)(2x+1)}$ for resolving into partial fraction.

Let $\dfrac{7x-3}{(x-1)(2x+1)}=\dfrac{A_1}{x-1}+\dfrac{A_2}{2x+1}$ (ii)

Here, A_1 and A_2 are unknown constants to be determined.

From equation (ii),

$$\frac{7x-3}{(x-1)(2x+1)}=\frac{A_1(2x+1)+A_2(x-1)}{(x-1)(2x+1)}$$

By multiplying $(x-1)(2x+1)$ to both sides

we get

$$\Rightarrow \quad 7x-3 = A_1(2x+1) + A_2(x-1) \qquad \text{(iii)}$$

To determine constants A_1 and A_2, values of x are chosen. To get A_1, put $x=1$, in equation (iii),

we get

$$7(1)-3 = A_1(2+1) + A_2(0)$$

$$\Rightarrow \quad 4 = 3A_1$$

$$\Rightarrow \quad A_1 = \frac{4}{3}$$

To get A_2, put $x = -\frac{1}{2}$, in equation (iii), we get

$$7\left(-\frac{1}{2}\right) - 3 = A_1(0) + A_2\left(-\frac{1}{2}-1\right)$$

$$\Rightarrow \quad -\frac{13}{2} = -\frac{3}{2}A_2$$

$$\Rightarrow \quad A_2 = \frac{13}{3}$$

By putting the values of constants A_1 and A_2 in equation (ii), we get

$$\frac{7x-3}{(x-1)(2x+1)} = \frac{4}{3(x-1)} + \frac{13}{3(2x+1)}.$$

Finally, equation (i) becomes

$$\frac{6x^3+5x^2-7}{2x^2-x-1} = 3x+4+\frac{4}{3(x-1)} + \frac{13}{3(2x+1)}.$$

EXERCISE: 21.1

Resolve the following into partial fractions

1. $\dfrac{12}{x^2-9}$
2. $\dfrac{4(x-4)}{x^2-2x-3}$
3. $\dfrac{x^2-3x+6}{x(x-2)(x-1)}$
4. $\dfrac{3(2x^2-8x-1)}{(x+4)(x-1)(2x-1)}$
5. $\dfrac{x^2+9x+8}{x^2+x-6}$
6. $\dfrac{x^2-x-14}{x^2-2x-3}$
7. $\dfrac{3x^3-2x^2-16x+20}{(x-2)(x+2)}$

Case-II: Denominator consists of repeated linear factors

Let $R(x)=\dfrac{P(x)}{Q(x)}$ is a rational fraction, where, its denominator $Q(x)$ is the product of repeated linear factors which can be written as $Q(x)=(x-a)^n$.

Now, $R(x)=\dfrac{P(x)}{Q(x)}$ is resolved as:

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a)^1}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\ldots+\frac{A_n}{(x-a)^n}.$$

Here, the constants $A_1, A_2, A_3, \ldots, A_n$ are to be found. The method is explained by following examples.

Example 1:

Resolve $\dfrac{2x+3}{(x-2)^2}$ into partial fractions.

Solution:

Let $\dfrac{2x+3}{(x-2)^2}=\dfrac{A_1}{(x-2)}+\dfrac{A_2}{(x-2)^2}$ (i)

Here, A_1 and A_2 are unknown constants to be determined.

From of equation (i)

$$\frac{2x+3}{(x-2)^2}=\frac{A_1(x-2)+A_2}{(x-2)^2}$$

By multiplying $(x-2)^2$ both sides

we get $2x+3=A_1(x-2)+A_2$

$\Rightarrow$ $2x+3=A_1x-2A_1+A_2$ (ii)

To determine constants A_1 and A_2, by equating the like-terms on both sides of equation (ii). Now, by equating the coefficients of x in equation (ii), we get,

$$2=A_1$$

Again, by equating the constants in equation (ii), we get

$3=-2A_1+A_2$

$3=-2(2)+A_2$ $(\because A_1=2)$

$7=A_2$

Finally, by putting the values of constants A_1 or A_2 in equation (i), we get

$$\frac{2x+3}{(x-2)^2}=\frac{2}{(x-2)}+\frac{7}{(x-2)^2}.$$

Example 2:

Resolve $\frac{5x^2-2x-19}{(x+3)(x-1)^2}$ into partial fractions.

Solution:

Let $$\frac{5x^2-2x-19}{(x+3)(x-1)^2}=\frac{A_1}{(x+3)}+\frac{A_2}{(x-1)}+\frac{A_3}{(x-1)^2} \quad \text{(i)}$$

Here, A_1, A_2 and A_3 are unknown constants to be determined.

By multiplying $(x+3)(x-1)^2$ to both sides of eq (i)

$\Rightarrow$ $5x^2-2x-19=A_1(x-1)^2+A_2(x+3)(x-1)+A_3(x+3)$ (ii)

$5x^2-2x-19=A_1x^2+2A_1x+A_1+A_2x^2+2A_2x-3A_2+A_3x+3A_3$

$\Rightarrow$ $5x^2-2x-19=(A_1+A_2)x^2+(A_3+2A_2-A_1)x+(A_1-3A_2+3A_3)$ (iii)

To determine constants A_1, A_2 and A_3, values of x are chosen. To get A_1, put $x=-3$, in equation (ii), we get.

$$5(-3)^2-2(-3)-19=A_1(-4)^2+A_2(0)(-4)+A_3(0)$$
$$\Rightarrow \quad 32=16A_1$$
$$\Rightarrow \quad A_1=2$$

By equating the like-terms on both sides of equation (iii). Now, by equating the coefficients of x^2 in equation (iii), we get,

$$5=A_1+A_2$$
$$\Rightarrow \quad 5=(2)+A_2 \qquad (\because A_1=2)$$
$$\Rightarrow \quad A_2=3$$

Again, by equating the coefficients of x in equation (iii)

$$-2=A_3+2(3)-2(2)$$
$$-2=A_3+2$$
$$A_3=-4$$

Finally, by putting the values of constants A_1, A_2 and A_3 in equation (i), we get

$$\frac{5x^2-2x-19}{(x+3)(x-1)^2}=\frac{2}{(x+3)}+\frac{3}{(x-1)}-\frac{4}{(x-1)^2}.$$

EXERCISE: 21.2

Resolve the following into partial fractions

1. $\frac{4x-3}{(x+1)^2}$
2. $\frac{x^2+7x+3}{x^2(x+3)}$
3. $\frac{5x^2-30x+44}{(x-2)^3}$
4. $\frac{18+21x-x^2}{(x-5)(x+2)^2}$
5. $\frac{x^2-x+3}{(x-1)^3}$

Case-III: Denominator consists of non-repeated quadratic factors

Let $R(x)=\dfrac{P(x)}{Q(x)}$ is a rational fraction, where, its denominator $Q(x)$ is the product of non-repeated irreducible quadratic factors.

Now $R(x)=\dfrac{P(x)}{Q(x)}$ is resolved as:

$$\frac{P(x)}{Q(x)}=\frac{A_1x+A_2}{a_1x^2+b_1x+c_1}+\frac{A_3x+A_4}{a_2x^2+b_2x+c_2}+\ldots+\frac{A_{2n-1}x+A_{2n}}{a_nx^2+b_nx+c_n}$$

Here, the constants $A_1, A_2, A_3, \ldots, A_{2n}$ are to be found. The method is explained by following examples.

Example 1:

Resolve $\dfrac{7x^2+5x+13}{(x+1)(x^2+2)}$ into partial fractions.

Solution:

Let $$\frac{7x^2+5x+13}{(x+1)(x^2+2)}=\frac{A_1}{x+1}+\frac{A_2x+A_3}{x^2+2} \qquad \text{(i)}$$

Here, A_1, A_2 and A_3 are unknown constants to be determined.

By multiplying $(x+1)(x^2+2)$ to both sides of eq: (i)

$$7x^2+5x+13=A_1(x^2+2)+(A_2x+A_3)(x+1) \qquad \text{(ii)}$$
$$7x^2+5x+13=A_1x^2+2A_1+A_2x^2+A_2x+A_3x+A_3$$
$$7x^2+5x+13=(A_1+A_2)x^2+(A_2+A_3)x+(2A_1+A_3) \qquad \text{(iii)}$$

To determine constants A_1, A_2 and A_3, values of x are chosen. To get A_1, put $x=-1$, in equation (ii), we get

$$7(-1)^2+5(-1)+13=A_1((-1)^2+2)+(A_2(-1)+A_3)(-1+1)$$
$$\Rightarrow \quad 15=3A_1$$
$$\Rightarrow \quad A_1=5$$

By equating the like-terms on both sides of equation (iii), we get,

$$7=(A_1+A_2) \qquad\qquad 5=(A_2+A_3)$$
$$\Rightarrow \quad 7=(5+A_2) \qquad\qquad \Rightarrow \quad 5=(2+A_3)$$
$$\Rightarrow \quad 2=A_2 \qquad\qquad \Rightarrow \quad 3=A_3$$

Finally, by putting the values of constants A_1, A_2 and A_3 in equation (i), we get

$$\frac{7x^2+5x+13}{(x+1)(x^2+2)}=\frac{5}{(x+1)}+\frac{2x+3}{(x^2+2)}.$$

Example 2:

Resolve $\dfrac{2x+7}{(x^2+3)(x^2+6)}$ into partial fractions.

Solution:

Let $\dfrac{2x+7}{(x^2+3)(x^2+6)} = \dfrac{A_1x+A_2}{x^2+3} + \dfrac{A_3x+A_4}{x^2+6}$ (i)

Here, A_1, A_2, A_3 and A_4 are unknown constants to be determined.

By multiplying $(x^2+3)(x^2+6)$ to both sides

$\Rightarrow 2x+7 = (x^2+6)(A_1x+A_2)+(x^2+3)(A_3x+A_4)$

$\Rightarrow 2x+7 = (A_1+A_3)x^3+(A_2+A_4)x^2+(6A_1+3A_3)x+(6A_2+3A_4)$ (ii)

By equating the like-terms on both sides of equation (ii), we get,

$A_1+A_3=0$ (iii)
$A_2+A_4=0$ (iv)
$6A_1+3A_3=2$ (v)
$6A_2+3A_4=7$ (vi)

solving equations (i) and (iv) simultaneously, we get $A_1=\dfrac{2}{3}$, and $A_3=-\dfrac{2}{3}$.

similarly, the values are $A_2=\dfrac{7}{3}$ and $A_4=-\dfrac{7}{3}$.

Finally, by putting the values of constants A_1, A_2, A_3 and A_4 in equation (i), we get

$$\frac{2x+7}{(x^2+3)(x^2+6)} = \frac{\frac{2}{3}x+\frac{7}{3}}{x^2+3} + \frac{-\frac{2}{3}x-\frac{7}{3}}{x^2+6} = \frac{2x+7}{3(x^2+3)} + \frac{-2x-7}{3(x^2+6)}.$$

EXERCISE: 21.3

Resolve the following into partial fractions

1. $\dfrac{x^2-x-13}{(x^2+7)(x-2)}$
2. $\dfrac{6x-5}{(x^2+10)(x+1)}$
3. $\dfrac{15+5x+5x^2-4x^3}{x^2(x^2+5)}$
4. $\dfrac{3x^2-x+1}{(x+1)(x^2-x+3)}$
5. $\dfrac{x^2-x+2}{(x+1)(x^2+3)}$

Case-IV: Denominator consists of repeated quadratic factors

Let $R(x)=\dfrac{P(x)}{Q(x)}$ is a rational fraction, where, its denominator $Q(x)$ has the repeated irreducible quadratic factors. For the sake of simplicity, it is taken that $Q(x)=(ax^2+bx+c)^n$.

Now $R(x)=\dfrac{P(x)}{Q(x)}$ is resolved as:

$$\frac{P(x)}{Q(x)}=\frac{A_1x+A_2}{ax^2+bx+c}+\frac{A_3x+A_4}{(ax^2+bx+c)^2}+\dots+\frac{A_{2n-1}x+A_{2n}}{(ax^2+bx+c)^n}.$$

Here, the constants $A_1, A_2, A_3, \dots, A_n$ are to be found. The method is explained by following examples.

Example 1:

Resolve $\dfrac{5x^2+2}{\left(x^2+x+1\right)^2}$ into partial fractions.

Solution:

Let $$\frac{5x^2+2}{\left(x^2+x+1\right)^2}=\frac{A_1x+A_2}{x^2+x+1}+\frac{A_3x+A_4}{\left(x^2+x+1\right)^2} \qquad \text{(i)}$$

Here, A_1, A_2, A_3 and A_4 are unknown constants to be determined.

By multiplying $\left(x^2+x+1\right)^2$ to both sides

$$5x^2+2=\left(x^2+x+1\right)\left(A_1x+A_2\right)+A_3x+A_4$$

$$5x^2+2=x^3A_1+x^2A_1+x^2A_2+xA_1+xA_2+xA_3+A_2+A_4$$

$$0.x^3+5x^2+0.x+2=x^3A_1+x^2\left(A_1+A_2\right)+x\left(A_1+A_2+A_3\right)+\left(A_2+A_4\right) \qquad \text{(ii)}$$

By equating the like-terms on both sides of equation (ii), we get,

$A_1=0$ (iii)

$A_1+A_2=5$ (iv)

$A_1+A_2+A_3=0$ (v)

$A_2+A_4=2$ (vi)

Put the value of $A_1=0$, in eq (iv) we get $A_2=5$.

Similarly, by putting the values of A_2 and A_1 in eq (v)

We get

$$0+5+A_3=0$$
$$\Rightarrow \quad A_3=-5$$

Again by putting the values of A_2 in eq (vi)

$$5+A_4=2$$
$$\Rightarrow \quad A_4=-3$$

Finally, by putting the values of constants A_1, A_2, A_3 and A_4 in equation (i), we get

$$\frac{5x^2+2}{\left(x^2+x+1\right)^2}=\frac{0.x+5}{x^2+x+1}+\frac{-5x-3}{\left(x^2+x+1\right)^2}.$$

Example: 2

Resolve $\dfrac{x^4+x^3+x^2+x+1}{x\left(x^2+1\right)^2}$ into partial fractions.

Solution

$$\text{Let}\quad \frac{x^4+x^3+x^2+x+1}{x\left(x^2+1\right)^2}=\frac{A_1}{x}+\frac{A_2x+A_3}{\left(x^2+1\right)}+\frac{A_4x+A_5}{\left(x^2+1\right)^2} \qquad (i)$$

Here, A_1, A_2, A_3, A_4 and A_5 are unknown constants to be determined.

By multiplying $x\left(x^2+1\right)^2$ to both sides of eq (i)

$$x^4+x^3+x^2+x+1=A_1\left(x^2+1\right)^2+x\left(A_2x+A_3\right)\left(x^2+1\right)+x\left(A_4x+A_5\right)$$
$$x^4+x^3+x^2+x+1=A_1x^4+2A_1x^2+A_1+A_2x^4+A_2x^2+A_3x^3+A_3x+A_4x^2+A_5x$$
$$x^4+x^3+x^2+x+1=\left(A_1+A_2\right)x^4+A_3x^3+\left(2A_1+A_2+A_4\right)x^2+\left(A_3+A_5\right)x+A_1 \qquad \text{(ii)}$$

To determine constants A_1, A_2, A_3, A_4 and A_5, we equate the like-terms of equation (ii), we have

$$1=A_1+A_2 \qquad \text{(iii)}$$
$$1=A_3 \qquad \text{(iv)}$$
$$1=2A_1+A_2+A_4 \qquad \text{(v)}$$
$$1=A_3+A_5 \qquad \text{(vi)}$$
$$1=A_1 \qquad \text{(vii)}$$

Put the value of A_1 in equation (iii), we get

$$1=A_1+A_2$$
$$\Rightarrow\quad 1=\left(1+A_2\right)$$
$$\Rightarrow\quad 0=A_2$$

Put the value of A_3 in equation (vi), we get

$$1=1+A_5$$
$$0=A_5$$

Put the value of A_1 and A_2 in equation (v), we get

$$1=2(1)+(0)+A_4$$
$$-1=A_4$$

Finally, putting all values of constants A_1, A_2, A_3, A_4 and A_5 in equation (i), we get

$$\frac{x^4+x^3+x^2+x+1}{x\left(x^2+1\right)^2}=\frac{1}{x}+\frac{1}{\left(x^2+1\right)}-\frac{x}{\left(x^2+1\right)^2}.$$

EXERCISE: 21.4

Resolve the following into partial fractions

1. $\frac{x^2}{(x^2+1)^2(1-x)}$
2. $\frac{x^2+x+2}{x^2(x^2+3)^2}$
3. $\frac{x^2}{(1+x)(1+x^2)^2}$
4. $\frac{7}{(x+1)(2+x^2)^2}$
5. $\frac{49}{(x-2)(3+x^2)^2}$

REVIEW EXERCISE 21

1. Tick the correct option

i. An improper fraction can be reduced into proper fraction by
   (a) addition (b) multiplication
   (c) subtraction (d) division

ii. Partial fractions of $\frac{x}{(x-a)(x-b)(x-c)}$ can have a form
   (a) $\frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{x+c}$ (b) $\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
   (c) $\frac{A}{x+a}+\frac{B}{x-b}+\frac{C}{x+c}$ (d) None of these

iii. Find the partial fractions of $\frac{x-3}{x^3+3x}$ are ____________.
   (a) $\frac{-1}{x}-\frac{x-1}{x^2+3}$ (b) $\frac{1}{x}+\frac{x+1}{x^2+3}$
   (c) $\frac{1}{x}-\frac{x+1}{x^2+3}$ (d) $\frac{-1}{x}+\frac{x+1}{x^2+3}$

iv. $\frac{x^3+1}{(x-1)(x+2)}$ is
   (a) Proper fraction (b) An improper fraction
   (c) An identity (d) A constant term

v. The fraction $\frac{2x+5}{x^2+5x+6}$ is known as:
   (a) Proper (b) Improper
   (c) Both proper and improper (d) None of these

2. Define proper, improper and rational fraction.

3. Resolve the following fractions into partial functions.

i) $\dfrac{5x+8}{(x-1)(x+2)}$ ii) $\dfrac{9x^2+5x+7}{x(x+2)(x-5)}$

iii) $\dfrac{x^2+2x+3}{(x^2+1)(x-2)}$ iv) $\dfrac{x^3+8x^2+9}{(x^2+x+1)(x+1)}$

v) $\dfrac{x+5}{(x^2+1)^2(x-3)}$ vi) $\dfrac{7x+3}{(x-1)^2(x+2)}$

BASIC STATISTICS

Unit 22

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Know about frequency distribution, and learn how to:
  - Construct a grouped frequency table.
  - Construct histograms with equal class intervals.
  - Construct histograms with unequal class intervals.
  - Construct a frequency polygon.
- Know about cumulative frequency distribution, and learn how to:
  - Construct a cumulative frequency table.
  - Draw a cumulative frequency polygon.
- Know about measures of central tendency, and learn how to:
  - Calculate (for ungrouped and grouped data):
    - Arithmetic mean by definition and using deviations from assumed mean.
    - Median, mode, geometric mean and harmonic mean.
  - Recognize properties of arithmetic mean.
  - Calculate weighted mean and moving averages.
  - Estimate median, quartiles and mode, graphically.
- Know about measures of dispersion, and learn how to:
  - Define, identify and measure range, calculate variance, mean deviation and standard deviation.

Introduction:

Statistics is an important branch of Mathematics which deals with the collection, organization, representation, interpretation and analysis of data to derive meaningful insights and patterns which help take wiser decisions and frame efficient policies. We already know basic concepts of classification, graphical representation and some measures of central tendency of simpler data. Here, we explore these in more detail and extend the concepts towards moving averages and measures of dispersion.

Data are basis of any statistical investigation, and are gathered on a variable of interest by asking questions, counting and measuring, referring reports, news, articles. The very basic form of data are the **raw data**, which are **unclassified/ ungrouped**, and it is difficult to directly get any meaningful conclusion from it. Such data are obtained using surveys, interviews or questionnaires, also referred as **primary sources** of data. After classifying and organizing raw/ungrouped data, we can get useful information through **grouped/classified data**. Newspapers, reports and articles are referred as **secondary sources** of data because these lead to classified data.

Data are divided into two types: **qualitative (categorical)** or **quantitative (numeric)**. Data on gender, blood group, eyebrow color, grades, roll numbers, etc. are qualitative data. Data on height, weight, salary, pH values of solutions, annual profits of a company, marks of students in a subject etc. are quantitative.

Observations in qualitative data are purely attributes/categories without any numerical significance, but these may or may not possess ranking/ordering (ascending or descending). Qualitative data which does not possess ordering are **nominal**, otherwise **ordinal**. Gender, blood group, eye brow color, religion lead to nominal data. For example, we cannot rank male/female observations as to which is higher/lower. Data on grades of students (A, B, C, Fail), quality of food (excellent, good, fair, bad, worst) can be ordered/ranked, and lead to ordinal data.

Observations in quantitative data are numbers with numerical significance and ordering. The numbers may be integers only or also in decimal form. If observations in a quantitative data are only integers, then data are **discrete**, otherwise **continuous**. Data on salary (in Rs.), number of female students per class in a school, number of heads while tossing 4 coins simultaneously lead to discrete data. Data on measuring heights, weights, pH values of solutions are continuous data.

It is extremely important to distinguish between types of data to appropriately apply some statistical tests or to measure any indicators for further analysis of data.

22.1. Frequency distribution:

To extract meaningful information from ungrouped data with higher number of observations, we divide data into smaller **classes/groups**. The classes are constructed to include all similar observations at one place. The **discrete classes** comprise of a single number/attribute, whereas **continuous classes** contain a range of numbers. The number of observations falling in a particular class is called its **frequency**.

A **frequency distribution** is a tabular description of a grouped data, consisting of classes and corresponding frequencies. Data arranged into a frequency distribution provide clearer understanding for the basic analysis than the ungrouped/raw data.

The **relative** and **percentage frequencies** of a class with frequency "f" are: $\left(\frac{f}{n}\right)$ and $\left(\frac{f}{n}\times 100\right)$, respectively, where "$n$" is total number of observations. The sum of all relative frequencies is 1, and of percentage frequencies is 100.

22.1(i) Construct grouped frequency table:

The procedure to construct a grouped frequency table or a frequency distribution depends on the way we form groups/classes. We discuss the following two ways of construction.

(a) Frequency table with discrete classes:

Discrete classes refer to the classes with only a single number/attribute to include similar observations. We use discrete classes only when a fewer observations repeat more number of times in the data. Frequency table with discrete classes is also referred as discrete frequency table. Sometimes, the relative and percentage frequencies are also added in the table in last if required.

Given a dataset, we follow the following steps to construct its frequency table with discrete classes comprising of three columns.

1. Identify distinct observations as **classes/groups** and write these in the first column in any order if data are nominal. For ordinal and quantitative data, write the classes in ascending order.
2. Use **tally marks** to distribute data into the classes in the second column.
3. Count tally marks to write **frequency** of each class in third column.

Example 1:

Construct grouped frequency table of the blood groups of 30 students of a class.

A+ O– B+ A+ A– AB+ B+ O+ O+ A– A+ AB– A+ O+ B+
B+ O+ B+ O– AB+ O– A+ AB+ A+ O+ B– O+ A– AB– O+

Solution:

Here, n = 30 and data are nominal. The eight distinct blood groups: A+, B+, AB+, O+, A–, B–, AB– and O– are the eight classes which are written in first column. Using tally marks in the second column to distribute data and then counting these to write frequencies in the third column, we get the required grouped frequency table for the distribution of blood groups of students.

Grouped Frequency Table

Classes/Groups "Blood groups"	Tally marks	Frequency "Number of students"
A+	𝍸 |	6
B+	𝍸	5
AB+	|||	3
O+	𝍸 ||	7
A–	|||	3
B–	|	1
AB–	||	2
O–	|||	3
Total	---	n = 30

Example 2:

The responses of 40 randomly selected customers in a shopping mall when asked "How satisfied are you with the services?" are given below. Construct a grouped frequency table of their responses. Also compute relative and percentage frequencies.

Very	Not very	Not very	Very	Somewhat	Not at all	Very	Not very
Very	Not sure	Very	Very	Very	Not very	Very	Somewhat
Not sure	Very	Not at all	Very	Very	Very	Not very	Somewhat
Not at all	Not sure	Not very	Not very	Not sure	Very	Very	Very
Very	Somewhat	Very	Somewhat	Not very	Not sure	Very	Somewhat

Solution:

Here, $n = 40$, and data are ordinal. The five distinct satisfaction levels in ascending order (lowest to highest) are classes. With tally marks and frequencies we get the grouped frequency table. The relative and percentage frequencies are also computed in fourth and fifth columns.

Classes/Groups "Satisfaction levels"	Tally marks	Frequency (f)	Relative frequency $\left(\frac{f}{n}\right)$	Percentage frequency (%) $\left(\frac{f}{n}\times 100\right)$
Not at all	III	3	$\frac{3}{40} = 0.075$	$\frac{3}{40}\times 100 = 7.5$
Not very	卌 III	8	0.2	20
Not sure	卌	5	0.125	12.5
Somewhat	卌 I	6	0.15	15
Very	卌 卌 卌 III	18	0.45	45
Total	---	$n = 40$	1	100

Note that:

a. From grouped frequency table of Example 2, we can easily conclude that highest number of customers were "Very satisfied", where as a fewer number of customers were "Not satisfied at all".

b. It is appropriate and understandable to say that 45% customers were very satisfied instead of 18 out of 40 were very satisfied.

Example 3:

The responses of 45 families in a city, when asked about number of mobiles they used, were noted as follows. Construct a discrete frequency table of the responses. Also obtain relative and percentage frequencies of the responses.

3 1 3 2 2 2 2 1 2 1 2 2 3 3 3 3 4 1 3 0 2 4 3

3 3 2 3 2 2 5 1 6 1 6 2 1 5 3 2 4 2 4 7 4 2

Solution:

Here, $n = 45$ and data are quantitative, and in particular discrete.

The distinct numbers in the data in ascending order are: 0, 1, 2, 3, 4, 5, 6, 7 which are fewer numbers repeating again and again. These are the 8 classes in the first column.

Writing and counting the tally marks in second column, we compute frequencies in the third column to get the required discrete frequency table of distribution of number of mobiles possessed by 45 families.

Discrete Frequency Table

Classes/Groups "Number of mobiles"	Tally marks	Frequency "Number of families"
0	|	1
1	~~||||~~ ||	7
2	~~||||~~ ~~||||~~ ~~||||~~	15
3	~~||||~~ ~~||||~~ ||	12
4	~~||||~~	5
5	||	2
6	||	2
7	|	1
Total	---	$n = 45$

Example 4:

The amount of cold drink was measured (in liters, L) in 20 randomly selected 1.5L bottles of a company as given below. Construct a discrete frequency table of the measured amount of cold drink. Also compute relative and percentage frequencies.

1.48	1.51	1.50	1.49	1.49	1.49	1.51	1.48	1.49	1.52
1.51	1.49	1.51	1.50	1.50	1.50	1.51	1.49	1.49	1.50

Solution:

The data are quantitative, and in particular continuous. The distinct numbers: 1.48, 1.49, 1.50, 1.51 and 1.52 (in ascending order) are five classes. With tally marks and frequencies, we get the discrete frequency table. The relative and percentage frequencies are also computed in fourth and fifth columns.

Class limits "Amount of cold drink (L)"	Tally marks	Frequency "Number of bottles"	Relative frequency $\left(\frac{f}{n}\right)$	Percentage frequency (%) $\left(\frac{f}{n}\times 100\right)$
1.48	||	2	$\frac{2}{20} = 0.1$	$\frac{2}{20}\times 100 = 10$
1.49	~~||||~~ ||	7	0.35	35
1.50	~~||||~~	5	0.25	25
1.51	~~||||~~	5	0.25	25
1.52	|	1	0.05	5
Total	---	$n = 20$	1	100

(b) Frequency table with continuous classes:

When data are quantitative (discrete or continuous) and observations are split over a range of number with lesser repetitions and more variations, then we use **continuous classes**. These classes are not single numbers but a range of numbers (intervals) each consisting of all numbers within the limits of the class. A **continuous frequency table** comprises of continuous classes together with respective frequencies.

We have already studied frequency distribution for discrete data in previous class. Here, we will generalize and use the basic concepts for discrete and continuous data with any number of decimal places. We first recall and define some important terms which are actively used in the procedure.

Range:

It is the difference between highest and lowest observations in the given data.

Range = [Highest observation] – [Lowest observation] (1)

Class limits:

These are numbers used to identify a class. For each class, there is a smallest number, called **lower class limit** (LCL), and a largest number called **upper class limit** (UCL). The observations starting from the LCL up to the UCL fall in a particular class.

Class interval/ width (h):

It is defined as the **size / length** of a class, and computed by finding difference between any two consecutive LCLs or UCLs. We usually use constant class width/interval, denoted by h for all classes and calculated as:

$$h = \frac{1}{10^m}\left\lceil \frac{R}{K} \times 10^m \right\rceil \quad (2)$$

where, K is number of classes, R is range, and m is number of maximum decimal places in the observations.

If data are discrete, then m = 0, if data are continuous with values up to one decimal places, then m = 1, and so on. $\lceil . \rceil$ denotes the **ceiling approximation** to get final integer value of h. The ceiling of an integer is the same integer, whereas of a decimal fraction is an integer immediately greater than it.

For example $\lceil 4 \rceil = 4, \lceil 1.6 \rceil = 2, \lceil 1.9 \rceil = 2, \lceil 7.5 \rceil = 8$.

Number of classes (K):

The **number of classes** is denoted by K and it ranges from 5 to 15 depending on number of observations and range. It should be carefully chosen/calculated, if not given. Much higher value of K results in poor grouping and loss of information. H. Sturges (1926) suggested a rule to calculate desirable number of classes K by using the numbers of observations n. The Sturges' rule is defined as:

$$K = \lceil 1 + 3.322 \log(n) \rceil \quad (3)$$

where, log (n) is logarithm of n with base 10, and $\lceil . \rceil$ is ceiling approximation. It should be noted that Sturges' rule is mostly appropriate for $15 \le n \le 200$. For example, if $n = 25$, then: $K = \lceil 1 + 3.322 \log(25) \rceil = \lceil 5.6439 \rceil = 6$ classes.

Spacing between the classes (d):

The constant difference between the LCL of a class and UCL of the next class is **spacing between the classes (*d*)**. If m is number of maximum decimal digits, then:

$$d = \frac{1}{10^m}, \; m = 0, 1, 2, \ldots \quad (4)$$

For discrete data, $m = 0$ and $d = 1$. If data are given up to one digit after the decimal, then $m = 1$ and $d = 0.1$. For data up to 2 decimal places, $m = 2$ and $d = 0.01$.

Procedure to construct continuous frequency table:

The steps to construct continuous frequency table for quantitative data are:

1. Note number of observations (n) and identify type of data: discrete or continuous. Also, note the maximum number of decimal places in data (m).
2. Calculate range (R) using equation (1), number of classes (K) using Sturges' rule in equation (3), if not given.
3. Calculate class interval (h) using equation (2) and spacing between the classes (d) using equation (4). In extreme conditions, we may have to take *"h + d"* as class width or add one more class.
4. Determine class limits to assure that all observations are included into the K classes. We can also use the lowest observation as starting LCL. The starting UCL is: $\text{UCL} = \text{LCL} + h - d$. Write LCLs and UCLs of other classes by continuously adding h in starting LCL and UCL.
5. Write class limits as classes in the first column of continuous frequency table.
6. Use **tally notation** / **list entries** to distribute data into classes in second column. Tally marks are preferred.
7. Count tally marks or entries to write frequencies of all classes in third column.

Example 1:

The electricity consumption (in kWh) in a shop was noted for 60 consecutive days as mentioned below. Construct frequency distribution using tally method.

106	107	76	82	109	107	115	93	187	95	139	119	115	128	115
123	125	111	92	86	70	126	68	130	129	194	82	90	158	118
123	146	80	136	137	110	141	152	104	111	140	184	204	178	75
113	162	131	99	185	181	84	486	100	98	148	90	110	107	78

Solution:

Here, $n = 60$, data are discrete with $m = 0$. $\text{R} = 204 - 68 = 136$. Using Sturges' rule for number of classes:

$$\text{K} = \lceil 1 + 3.322\log(60) \rceil = \lceil 6.9070 \rceil = 7.$$

With

$$h = \left\lceil \frac{136}{7} \right\rceil = \lceil 19.4285 \rceil = 20, \text{ and } d = 1.$$

the starting LCL=68, the starting UCL=68+20–1=87. Adding h=20 in these, we get all other limits. Writing class limits in first column, tally marks in second, and frequencies in third, we get the required frequency table.

Frequency Table

Class limits	Tally marks	Frequency
68 – 87	卌 卌	10
88 – 107	卌 卌 |||	13
108 – 127	卌 卌 卌	15
128 – 147	卌 卌	10
148 – 167	||||	4
168 – 187	卌 |	6
188 – 207	||	2
Total	---	n = 60

Example 2:

The diameters (in mm) of a reel of wire measured at 24 places correct to nearest 0.01mm are: 2.10, 2.29, 2.32, 2.21, 2.14, 2.22, 2.28, 2.18, 2.17, 2.20, 2.23, 2.13, 2.26, 2.10, 2.21, 2.17, 2.28, 2.15, 2.34, 2.27, 2.11, 2.23, 2.25, 2.16. Construct frequency distribution.

Solution:

$n = 24$, data are continuous up to two digits after decimal, so: $m = 2$.

$R = 2.34 - 2.10 = 0.24$, and $K = \lceil 1 + 3.322\log(24) \rceil = 6$.

$h = \frac{1}{100}\left\lceil \frac{R}{K} \times 100 \right\rceil = 0.04$, $d = \frac{1}{100} = 0.01$. Starting with LCL=2.10 and $h = 0.04$, six classes are: 2.10-2.13, 2.14-2.17, 2.18-2.21, 2.22-2.25, 2.26-2.29, 2.30-2.33. But, 2.34 cannot be included in any class. In this extreme case, we use class width of $h + d = 0.05$. The required six classes are: 2.10 - 2.14, 2.15 - 2.19, 2.20 - 2.24, 2.25 - 2.29, 2.30 - 2.34, 2.35 - 2.39, and the required frequency distribution of the diameters is:

Frequency Table

Class limits	2.10 – 2.14	2.15 – 2.19	2.20 – 2.24	2.25 – 2.29	2.30 – 2.34	2.35 – 2.39	Total
Tally mark	$\vert\vert\vert$	$\not{\vert\vert\vert\vert}$	$\not{\vert\vert\vert\vert}\ \vert$	$\not{\vert\vert\vert\vert}\ \vert$	$\vert\vert\vert$		---
Frequency	3	5	6	6	3	0	$n = 24$

Note that:

In Example 2, we could also add just one more class to include 2.34. The table becomes:

Class limits	2.10 – 2.13	2.14 – 2.17	2.18 – 2.21	2.22 – 2.25	2.26 – 2.29	2.30 -2.33	2.34 – 2.37
Tally mark	$\vert\vert\vert\vert$	$\not{\vert\vert\vert\vert}$	$\vert\vert\vert\vert$	$\vert\vert\vert\vert$	$\not{\vert\vert\vert\vert}$	$\vert$	$\vert$
Frequency	4	5	4	4	5	1	1

Some more important terms in grouped frequency tables with continuous classes are:

Class boundaries:

The numbers which separate a class form adjoining classes are called **class boundaries**. For a class, its **lower class boundary** (LCB) and **upper class boundary** (UCB) are obtained from the LCL and UCL of that class, respectively as:

For a class: $\text{LCB} = \text{LCL} - \frac{d}{2}$ and $\text{UCB} = \text{UCL} + \frac{d}{2}$

where, d is the spacing between the classes.

Class marks/midpoints:

Class marks or **mid-points**, denoted as x, in a grouped data with continuous classes are the arithmetic means of the class limits or class boundaries of the classes.

For a class: $x = \frac{\text{LCL+UCL}}{2}$ or $x = \frac{\text{LCB+UCB}}{2}$

Cumulative Frequency:

The word **cumulative** means something increasing/growing by successive additions. If we go on adding frequencies of classes, one after one, each time adding frequency of next class in the total, we get cumulative frequencies. For a particular class, the **cumulative frequency** is the sum of all frequencies up to that class.

Example 3:

Compute class boundaries, class marks and cumulative frequencies for the frequency table given below.

Class limits	7.1-7.3	7.4-7.6	7.7-7.9	8.0-8.2	8.3-8.5	8.6-8.8	8.9-9.1
Frequencies	3	5	9	14	11	6	2

Solution:

Here, $n = 50$ and $d = 0.1$. Class boundaries, class marks and cumulative frequencies are computed in the following table.

Class limits	Class boundaries	Class marks	Frequencies	Cumulative frequencies
7.1-7.3	7.05-7.35	7.2	3	3
7.4-7.6	7.35-7.65	7.5	5	3+5=8
7.7-7.9	7.65-7.95	7.8	9	17
8.0-8.2	7.95-8.25	8.1	14	31
8.3-8.5	8.25-8.55	8.4	11	42
8.6-8.8	8.55-8.85	8.7	6	48
8.9-9.1	8.85-9.15	9.0	2	n = 50
Total	----	----	**n = 50**	----

Note that:

1. We can observe that the UCB of a class and LCB of the next class are same.
2. Adding h successively in starting LCB, we get all remaining class boundaries.
3. We can get all remaining class marks by adding "h" in the starting class marks.
4. The cumulative frequency of the last class is the total number of observations.

EXERCISE 22.1

1. Identify type of data (discrete/ continuous/ nominal/ ordinal) and total number of observations in the following:
   a. Number of visitors in a shopping mall per day for 30 days.
   b. Time (in minutes per day) spent by a person in GYM for a month.
   c. Life (in years) of 45 car batteries.
   d. Electricity consumption (in kWh/day) for 2 months.
   e. Weight (in Kg) of 55 students in a class.
   f. Eyebrow colors of 20 individuals.
   g. Grades obtained by 70 students in a class.
   h. Result (Positive/ Negative) of COVID diagnosis test of 10 people.
   i. Marital status of 30 teachers of a school.
   j. District of domicile of students of your class.
2. Construct grouped frequency table of the educational levels (current class of study) of 20 students going to a country tour from a school. The educational levels are: VIII, X, X, VIII, VI, IX, X, IX, X, VIII, VII, VIII, VI, X, VI, V, IV, VII, IX, IX.
3. The mode of transport used by 20 students to attend a language course were noted as given below. Construct grouped frequency table with percentage frequencies.

Walk	Walk	Bike	Bus	Car	Walk	Bike	Walk	Walk	Bus
Car	Bus	Car	Walk	Walk	Bus	Car	Car	Bus	Bus

4. Classify responses of 40 families on how frequently they spend vacations out of city in a grouped frequency distribution. Also compute relative frequencies.

Never	Rarely	Often	Sometimes	Never	Rarely	Always	Often
Sometimes	Never	Sometimes	Always	Sometimes	Rarely	Never	Sometimes
Sometimes	Sometimes	Always	Always	Sometimes	Often	Often	Never
Often	Rarely	Sometimes	Always	Rarely	Always	Often	Often
Always	Never	Always	Sometimes	Never	Never	Often	Always

5. Organize the colors of Caps of prefects of various schools in frequency table.

Black	Blue	Black	Brown	Green	Blue	Black	Brown	Brown	Green
Blue	Green	Red	Black	Red	Blue	Red	Green	Black	Red

6. The marks secured by 25 students is a class test out of 5 were: 4.0, 0.5, 4.5, 1.0, 3.5, 3.5, 5.0, 2.5, 4.0, 2.5, 1.0, 2.0, 4.0, 3.5, 0.0, 3.0, 5.0, 1.0, 2.0, 2.0, 4.5, 2.5, 3.0, 3.5, 1.5. Obtain frequency table with discrete classes and relative frequencies.
7. The number of times the lectures of a teacher were downloaded per hour from a website was noted for a day before test, and results were: 1, 1, 2, 1, 3, 3, 4, 1, 4, 1, 2, 3, 3, 2, 4, 2, 3, 2, 3, 2, 1, 0, 0, 1. Classify data into a discrete frequency table.
8. Construct frequency distribution with continuous classes of the number of absentees in a class for 18 days: 4, 3, 0, 1, 2, 5, 6, 8, 10, 7, 11, 15, 13, 14, 3, 4, 12, 12.
9. Construct continuous frequency distribution of the weights recorded to nearest pound of 40 college students: 138, 148, 146, 146, 119, 164, 152, 173, 158, 154, 150, 144, 142, 140, 165, 132, 150, 147, 147, 153, 144, 156, 135, 126, 140, 125, 168, 157, 138, 135, 149, 136, 135, 142, 161, 145, 176, 163, 145, 128. Also compute class boundaries, class marks and cumulative frequencies.
10. The value of resistance in ohms of a batch of 48 resistors of similar value are: 21.0, 22.4, 22.8, 21.5, 22.6, 21.1, 21.6, 22.3, 22.9, 20.5, 21.8, 22.2, 21.0, 21.7, 22.5, 20.7, 23.2, 22.9, 21.7, 21.4, 22.1, 22.2, 22.3, 21.3, 22.1, 21.8, 22.0, 22.7, 21.7, 21.9, 21.1, 22.6, 21.4, 22.4, 22.3, 20.9, 22.8, 21.2, 22.7, 21.6, 22.2, 21.6, 21.3, 22.1, 21.5, 22.0, 23.4, 21.2. Form continuous frequency table. Also, compute cumulative and percentage frequencies.
11. The mass (in Kg) of 50 blocks of metal were measured collect to nearest 0.1kg and are listed below. Construct frequency distribution of the masses and compute relative and percentage frequencies.

8.0	8.3	7.7	8.1	7.4	8.6	7.1	8.4	7.4	8.2	8.4	8.8	7.9	8.1	8.2	7.5	8.3
8.8	8.0	7.7	8.3	8.2	7.9	8.5	7.9	8.0	8.4	7.2	8.7	8.0	9.1	8.5	7.6	8.2
7.8	7.8	8.7	8.5	8.4	8.5	8.1	7.8	8.2	7.7	7.5	8.5	8.1	7.3	9.0	8.6	

22.1(ii). Construct histograms with equal class intervals:

Histograms are used for graphical representation of quantitative data by using the frequency tables with discrete and continuous classes and related concepts. A **histogram** is drawn by using adjacent rectangles with bases of rectangles marked by distribution of numbers in the classes and areas/heights of rectangles being proportional to the class frequencies. In histograms with equal class intervals, both areas and heights of rectangles are proportional to frequency.

In histograms corresponding to discrete frequency tables, the bases of rectangles represent the distinct numbers in the data and the heights are usually taken as the frequencies of classes. The width of rectangles are taken with a fixed interval.

In histograms corresponding to continuous frequency tables with equal class intervals, the bases of rectangles represents class boundaries and the heights are usually taken as the frequencies of classes. The width of rectangles is equal to the class interval.

While drawing histograms, axes should be labelled clearly and scales should be defined. X-axis does not have to start from 0. Y-axis must start from 0. Frequencies should also be marked above the rectangles. The procedure to draw histograms is explained in following examples.

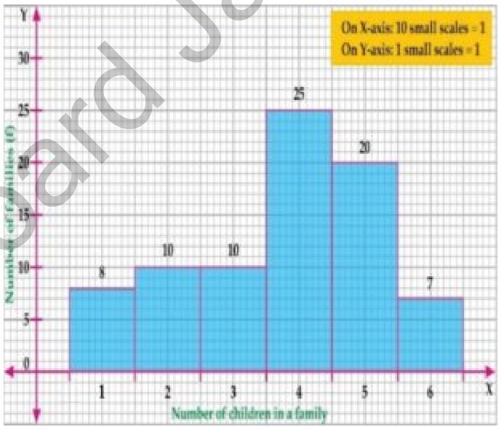
Example 1:

Draw histogram for the number of children per family in villages using the following data of 80 families.

Number of children in a family	1	2	3	4	5	6
Number of families	8	10	10	25	20	7

Solution:

Data represent a discrete frequency table. Writing distinct numbers: 1, 2, 3, 4, 5, 6 representing classes on X-axis with a fixed interval for all, then writing frequencies on Y-axis starting with "0" with suitable scale to include all frequencies up to 25. Drawing rectangles based at the distinct numbers with equal width and heights equal to frequencies, we get the following histogram. Axes are labeled and scales are defined.



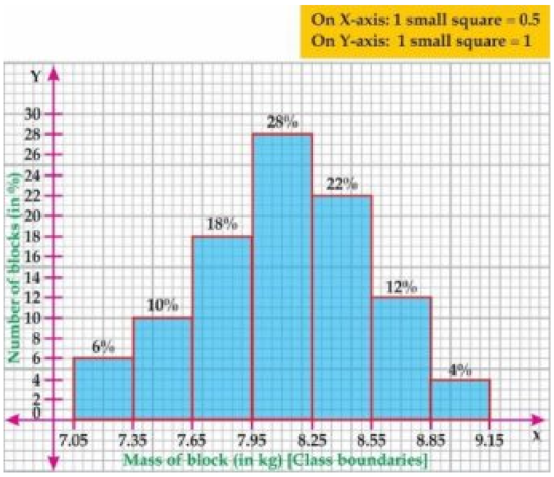
Example 2.

Draw percentage frequency histogram for the following distribution.

Mass of block (in kg)	7.1 - 7.3	7.4 - 76	7.7 - 7.9	8.0 - 8.2	8.3 - 8.5	8.6 - 8.8	8.9 - 9.1
Number of blocks	3	5	9	14	11	6	2

Solution:

Given is a continuous frequency table with continuous data up to one digit after the decimal, so $n = 50$, $m = 1$ and $d = 0.1$. The class boundaries are: 7.05 - 7.35, 7.35 - 7.65, 7.65 - 7.95, 7.95 - 8.25, 8.25 - 8.55, 8.55 - 8.85, 8.85 - 9.15. Writing these on X-axis and corresponding percentage frequencies: 6, 10, 18, 28, 22, 12, 4 on Y-axis with respective scales. Drawing adjacent rectangles based at class boundaries, of width $h = 0.3$ and heights equal to percentage frequencies, we get the required percentage frequency histogram.



22.1 (iii). Construct histograms with unequal class intervals:

For the case of **histograms with unequal class intervals**, the adjacent rectangles are constructed so that only their areas (not heights) are proportional to the frequencies. To do this, we define **adjusted heights** of rectangles to assure proportional areas and frequencies. If f^* and h^* are frequency and width of a class, respectively, then:

$$\text{Adjusted height of a rectangle}^* = \frac{f^*}{h^*} \qquad (1)$$

$$\text{So that, Area of a rectangle}^* = \text{width} \times \text{height} = h^* \times \frac{f^*}{h^*} = f^* \qquad (2)$$

From (1), we see that adjusted heights are not proportional to f^*, but areas in (2) are. This satisfies the requirement. The steps of drawing histograms with unequal class intervals are same as before, but we must write adjusted heights on Y-axis.

Example:

Construct histogram for the following data with unequal class intervals:

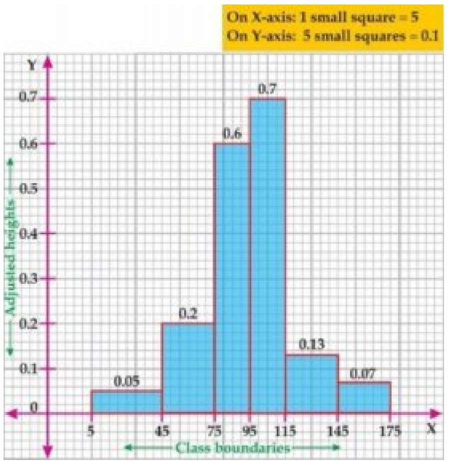
Class limits	10 - 40	50 - 70	80 - 90	100 - 110	120 - 140	150 - 170
Frequencies	2	6	12	14	4	2

Solution:

We can observe that the given data show a frequency table with unequal class intervals. We first compute class boundaries and adjusted heights. Here, $d = 10$.

Class limits	Class boundaries	Class intervals (h^*)	Frequencies (f^*)	Adjusted heights of rectangles (f^* / h^*)
10 – 40	5-45	40	2	$\frac{2}{40} = 0.05$
50 – 70	45-75	30	6	$\frac{6}{30} = 0.2$
80 – 90	75-95	20	12	0.6
100 – 110	95-115	20	14	0.7
120-140	115-145	30	4	≈ 0.13
150-170	145-175	30	2	≈ 0.07

Writing class boundaries on X-axis and adjusted heights on Y-axis, we draw rectangles for all classes with widths equal to unequal class intervals and heights equal to the adjusted as computed in above table. Thus, we get the required histogram with unequal intervals.



We can verify that areas of the rectangle are proportional, and particularly equal, to the frequencies here.

Area of rectangle-1 = $40 \times 0.05 = 2$.

Area of rectangle-2 = $30 \times 0.2 = 6$. Area of rectangle-3 = $20 \times 0.6 = 12$.

and so on.

22.1 (iv). Construct a frequency polygon:

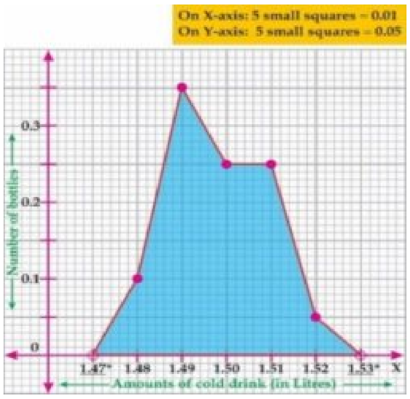
Frequency polygon is another way of graphical representation of a quantitative data grouped into frequency distribution. We often see such graphs in newspapers (especially in business section), in weather forecast news, and in cricket match stats (a commonly used abbreviation of statistics). The word **polygon** refers to a closed shape in a plane formed by connecting at least three line segments. For example, triangles, squares, pentagons, etc. Here, we will learn how this geometric concept is related with frequency distributions. The frequency polygons are used in advanced statistical analysis is to identify shape of the distribution.

To construct a frequency polygon, the points (x, y) are plotted, where the abscissas (the x-coordinates) are class marks and ordinates (the y-coordinates) are the class frequencies. Secondly, we add two dummy classes, one in the start and one in the end, with 0 frequencies, and mark corresponding points for these on X-axis. Finally, we join the scattered points on the graph by line segments to get the required closed shape, called **frequency polygon**, with base on X-axis and peaks showing frequencies. For clarity, filled marks (•) denote points of real classes and blank marks (○) denote points for dummy classes. It is obvious to label the axes and define suitable scales.

Example 1:

Draw a relative frequency polygon for the following data.

Amount of cold drink in liters	1.48	1.49	1.50	1.51	1.52
Numbers of bottles	2	7	5	5	1



Solution:

Given data show discrete frequency table. The distinct numbers: 1.48, 1.49, 1.50, 1.51, 1.52 are class marks (x). The frequencies (y) are: 2, 7, 5, 5, 1. First, we compute relative frequencies, which are: 0.1, 0.35, 0.25, 0.25 and 0.05. With two dummy classes: the plotted points are: (1.47*, 0), (1.48, 0.1), (1.49, 0.35), (1.50, 0.25), (1.51, 0.25), (1.52, 0.05) and (1.53*, 0). Joining points by line segments, we get the relative frequency polygon.

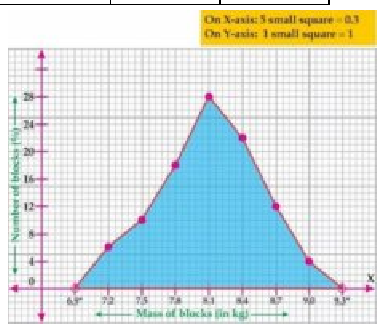
Example 2:

Draw percentage frequency polygon for the following distribution.

Mass of block (in kg)	7.1 - 7.3	7.4 - 7.6	7.7 - 7.9	8.0 - 8.2	8.3 – 8.5	8.6 – 8.8	8.9 – 9.1
Number of block	3	5	9	14	11	6	2

Solution:

Data show a continuous frequency distribution with equal class interval $h = 0.3$. The points to plot are (x, y), where x represent class marks: 7.2, 7.5, 7.8, 8.1, 8.4, 8.7, 9.0, and y are percentage frequencies: 6, 10, 18, 28, 22, 12, 4. Adding dummy class marks and the required percentage frequency polygon is plotted through (6.9*, 0), (7.2, 6), (7.5, 10), (7.8, 18), (8.1, 28), (8.4, 22), (8.7, 12), (9.0, 4) and (9.3*, 0) with suitable scales.



22.2. Cumulative frequency distribution:

A **cumulative frequency distribution** is a table listing upper class boundaries of classes together with the corresponding cumulative frequencies. We already know about the computation of cumulative frequencies. The cumulative frequencies computed by adding number of observations less than the UCBs of classes are called **"less than cumulative frequencies"**. These start with "0" and end at the total number of observations "n". We must add a dummy UCB in the start with cumulative frequency of 0.

"The greater than cumulative frequencies" are computed by summing all observations greater than the LCBs of classes. These start with "n" and end at "0". But we often use "less than cumulative frequencies" in practice. In the forthcoming discussion, we use the term **"cumulative frequency"** to refer the **"less than cumulative frequency"** for simplicity.

22.2 (i). Construct a cumulative frequency table:

Steps to construct cumulative frequency table of grouped quantitative data are:

1. Obtain upper class boundaries and cumulative frequencies.
2. Add a dummy upper class boundary in the start with 0 cumulative frequency.
3. Form a table to mention all UCBs with respective cumulative frequencies.

Example 1: Construct a cumulative frequency table using the following data.

Marks of Students	1	2	3	4	5
Number of Students	2	5	4	3	1

Solution: Given data shows a discrete frequency distribution.

Here, $d = 1$ and $n = 15$. Computing UCBs by adding $\frac{d}{2}$ in the class limits, and then finding cumulative frequencies we have:

Classes	UCBs	Frequencies	Cumulative Frequencies
1	1.5	2	2
2	2.5	5	7
3	3.5	4	11
4	4.5	3	14
5	5.5	1	15 = n

Adding a dummy UCB in start equal to 0.5* with cumulative frequency "0", the required cumulative frequency table is:

Marks of students (UCBs) Less than	0.5*	1.5	2.5	3.5	4.5	4.5
Number of students (Cumulative frequencies)	0	2	7	11	14	15

Example 2: Construct cumulative frequency table for electricity consumption data.

Electricity Consumption (kWh)	68-87	88-107	108-127	128-147	148-167	168-187	188-207
Number of days	10	13	15	10	4	6	2

Solution: Given data is a continuous frequency distribution, Here, $d = 1$ and $n = 60$. The UCBs are: 87.5. 107.5, 127.5, 147.5, 167.5, 187.5 and 207.5. Adding a dummy UCB of **67.5*** in start, we have the following cumulative frequency table.

Electricity Consumption Less than (kWh)	67.5*	87.5	107.5	127.5	147.5	167.5	187.5	207.5
Number of days	0	10	23	38	48	52	58	60

22.2 (ii). Draw a cumulative frequency polygon:

A cumulative frequency distribution is graphically represented by a **cumulative frequency polygon**, also called an **ogive**. We begin plotting the points (x, y) of the cumulative frequency table (x: UCBs and y: cumulative frequencies), then join these by line segments. Finally, we draw a perpendicular from the peak point on X-axis to get a closed shape, which is the required ogive or cumulative frequency polygon.

Example 1.

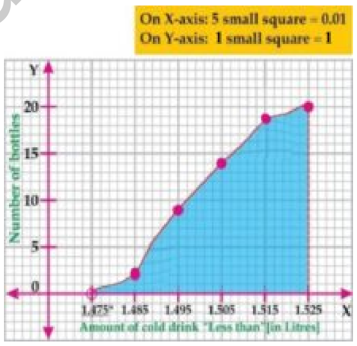
Draw an ogive (cumulative frequency polygon) for the following data.

Amount of cold drink (in liters)	1.48	1.49	1.50	1.51	1.52
Number of bottles	2	7	5	5	1

Solution:

Here, $n = 20$ and $d = 0.01$. By computing UCBs and cumulative frequencies, we get the cumulative frequency table, where 1.475* is a dummy starting UCB.

Amount of cold drink (in liters)	1.475*	1.485	1.495	1.505	1.515	1.525
Number of bottles	0	2	9	14	19	20



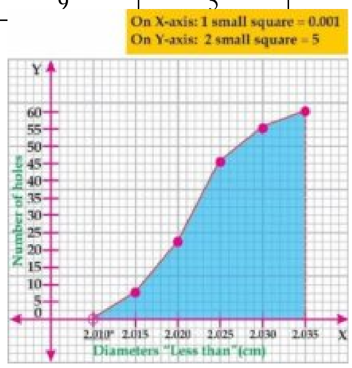
Now, we plot the points (1.475*, 0), (1.485, 2), (1.495, 9), (1.505, 14), (1.515, 19) and (1.525, 20) on graph, join these through line segments, and, finally draw a perpendicular from the last point towards X-axis to get the required cumulative frequency polygon or ogive. Axes are labeled clearly with proper scaling.

Example 2:

Draw ogive for the following distribution of diameters (in cm) of 60 holes bored in engine castings as measured in a certain study.

Diameters (cm)	2.011– 2.014	2.016 – 2.019	2.021 - 2.024	2.026 – 2.029	2.031 – 2.034
Number of holes	7	16	23	9	5

Solution:



Given is a continuous frequency table with $n = 60$ and $d = 0.002$. Using UCBs and cumulative frequencies, we get the cumulative frequency table with points: (2.010*, 0), (2.010, 7), (2.020, 23), (2.025, 46), (2.030, 55) and (2.035, 60). These points lead to the required ogive or the cumulative frequency polygon.

EXERCISE 22.2

1. Obtain percentage histogram, frequency polygon, cumulative frequency table and ogive for the following data.

Marks of Students	1	2	3	4	5
Number of Students	2	5	4	3	1

2. Construct histogram for the following data with unequal class intervals:

Class limits	0-39	40-49	50-79	80-99
Frequencies	6	8	12	4

3. Construct histogram, frequency polygon, cumulative frequency table and ogive.

Amount of money earned weekly	20-40	50-70	80-90	100-110	120-140	150-170
Number of people	2	6	12	14	4	2

4. Plot histogram, relative frequency polygon, and ogive for the following data.

Class limits	10.5-10.9	11.0-11.4	11.5-11.9	12.0-12.9	13.0-13.4
Frequencies	2	7	10	12	8

22.3. Measures of central tendency:

The ability of all observations in data to cluster around a **central point** is referred as the **central tendency**. A central point of the data is called a **measure of central tendency** or simply an **average**. A measure of central tendency represents the whole data by a single central point, and is a concise identification of whole data.

We usually talk about averages of numbers and categories frequently. For example, consider the following statements:

1. "The majority of students in a class secured Grade-B". Here, Grade B is an average.
2. "Ali spends 45 minutes in the GYM daily". Here, average time per day Ali spends in GYM is 45 minutes. Obviously, he may have spent more or less than 45 minutes.
3. "Average score of a batsman in T20", which is based on the scores of the batsman in all T20 matches he played. The average score is generalizes all individual scores.

22.3 (i). To calculate measures of central tendency:

We usually compute an average or measure central tendency of data by adding the observation and then dividing by the total numbers of observations, which is particularly the **arithmetic mean**. There are many more ways to compute averages. We discuss the following five types here:

(1) Arithmetic mean (2) Median (3) Mode (4) Geometric mean (5) Harmonic mean

These averages have some advantages and disadvantages when compared with each other. Also, the best use of an average depends on nature of data.

22.3 (i) (a) arithmetic mean by definition and using deviations from assumed mean

The **arithmetic mean** is based on the principle of equality among all observations. We mix all n observations by their sum, and then divide the sum in n equal parts.

If $x_1, x_2, x_3, \ldots, x_n$ are n numbers, then their arithmetic mean (denoted as $\bar{x}$) is:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} \quad \text{or} \quad \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad (1)$$

where "Σ" is an upper-cased Greek letter SIGMA used for summation.

Note that:

- Arithmetic mean is a unique number within the range of the data.
- If data are quantitative, then we directly find arithmetic mean. For nominal or ordinal, we may assign numeric codes or ranks, respectively.
- Arithmetic mean is not appropriate for nominal data.

Arithmetic mean by definition/ direct method:

The **direct method** or **by definition** refers to computing arithmetic mean directly from the given data without changing the location or scale of the observations.

In direct method/ by definition, we use following formulas:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum x}{n} \quad \text{(Ungrouped data)} \qquad \bar{x} = \frac{\sum_{i=1}^{n} x_i f_i}{\sum f_i} = \frac{\sum xf}{\sum f} \quad \text{(Grouped data)}$$

where x_i are individual observations in ungrouped data and class marks in grouped data, and f_i are corresponding class frequencies. For brevity, we can omit subscripts in the formulas.

Example 1: Find arithmetic mean of the following datasets by definition.

(i). 2, 3, 7, 5, 5, 13, 1, 7, 4, 8, 3, 4, 3 (ii). 7, 5, 74, 10 (iii). 18.92, 27.9, 34.7, 39.68

Solution:

(i). By definition or direct method, we have:

$$\bar{x} = \frac{\sum x}{n} = \frac{2+3+7+5+5+13+1+7+4+8+3+4+3}{13} = \frac{65}{13} = 5.$$

(ii). By definition, we have: $\bar{x} = \frac{\sum x}{n} = \frac{7+5+74+10}{4} = \frac{96}{4} = 24.$

(iii). $\bar{x} = \frac{\sum x}{n} = \frac{18.92+27.9+34.7+39.68}{4} = \frac{121.2}{4} = 30.3.$

Note that:

- The arithmetic mean is affected by **extreme values** (or **outliers**). In Example 1 (ii), $\bar{x} = 24$ is far away from all observations: 7, 5, 74 and 10 due to the outlier 74.

Example 2: The grades of five students who studied in group were: A+, B+, C, B, A. Identify the average grade using arithmetic mean.

Solution: The data are ordinal and can be ranked by numbers. We represent the grades: C, C+, B, B+, A, A+ by labels: 1, 2, 3, 4, 5, 6 respectively, then:

$$\bar{x} = \frac{\sum x}{n} = \frac{6+1+4+3+5}{5} = \frac{19}{5} = 3.8 \approx 4.$$ So, the average grade is B+.

Example 3:

Find arithmetic mean amount of cold drink by definition in the following data:

Amount of cold drink(in liters)	1.48	1.49	1.50	1.51	1.52
Number of bottles	2	7	5	5	1

Solution:

Data is grouped into discrete classes. The class marks (x) are: 1.48, 1.49, 1.50, 1.51, 1.52. The computations are shown in the table.

x	f	xf
1.48	2	$1.48\times 2=2.96$
1.49	7	10.43
1.50	5	7.5
1.51	5	7.55
1.52	1	1.52
	$\sum f=20$	$\sum xf=29.96$

Finally, $\bar{x}=\dfrac{\sum xf}{\sum f}=\dfrac{29.96}{20}=1.498$ liters.

The average amount of cold drink using arithmetic mean is 1.498 liters.

Example 4:

Find arithmetic mean of satisfaction of customers using definition.

Satisfaction level	Not at all	Not very	Not sure	Somewhat	Very
Number of customers	3	8	5	6	18

Solution:

We represent attributes of the ordinal variable: "Not at all, Not very, Not sure, Somewhat and Very" by ranks, say, 1, 2, 3, 4. 5, respectively. By definition, we have:

$$\bar{x}=\frac{\sum xf}{\sum f}=\frac{(1)(3)+(2)(8)+(3)(5)+(4)(6)+(5)(18)}{3+8+5+6+18}=\frac{148}{40}=3.7\approx 4.$$

So, average satisfaction level is 4^{th} rank, which is "Somewhat satisfied".

Example 5:

Find arithmetic mean of electricity consumption (in kWh) of a shop for 60 days:

Electricity consumption	68-87	88-107	108-127	128-147	148-167	168-187	188-207
Number of days	10	13	15	10	4	6	2

Solution:

Data show continuous frequency table. We find class marks and required sums in adjacent table.

By definition or direct method:

$\bar{x}=\dfrac{\sum xf}{\sum f}=\dfrac{7270}{60}=121.166.$

Arithmetic mean of electricity consumption of shop is 121.166 kWh.

Class limits	Class Marks (x)	Frequencies (f)	xf
68-87	77.5	10	775
88-107	97.5	13	1267.5
108-127	117.5	15	1762.5
128-147	137.5	10	1375
148-167	157.5	4	630
168-187	177.5	6	1065
188-207	197.5	2	395
Total	---	$\sum f=60$	7270

Note that:

Arithmetic mean of an ungrouped data and its discrete frequency table are always same. But, the arithmetic mean of a data from continuous frequency distribution is not equal to (but closer to) the mean of corresponding ungrouped data.

Arithmetic mean by using deviations/ indirect methods:

If x_i are observations and A is **assumed/provisional mean** (any number within or outside the range of data), then deviations about A are: $D_i = x_i - A$.

There are two indirect methods based on deviations: **shortcut** and **coding** methods, to compute arithmetic mean when dealing with larger data with higher values.

Shortcut method:

The arithmetic mean using shortcut method is computed as:

$$\bar{x} = A + \frac{\sum D}{n} \text{ (for ungrouped data) and } \bar{x} = A + \frac{\sum Df}{\sum f} \text{ (for grouped data)}$$

For ungrouped data, x_i are observations. For grouped data, x_i are class marks. It is better to take x_i corresponding to the highest frequency as "A" for grouped data.

Coding method:

The formulae for computing arithmetic mean using coding method are:

$$\bar{x} = A + h\frac{\sum u}{\sum n} \text{ (for ungrouped data) and } \bar{x} = A + h\frac{\sum uf}{\sum f} \text{ (for grouped data)}$$

where, $u_i = \frac{x_i - A}{h}$ or, simply $u = \frac{x - A}{h}$ is referred as the coded variable. For ungrouped data, x_i are observations, and for grouped data x_i are class marks.

Example 1:

The total marks of six students who studied in group for the examination are: 610, 640, 685, 680, 710, 580. Find arithmetic mean marks using indirect methods by taking (i). $A = 650$ and $h = 10$, (ii). $A = 680$ and $h = 20$.

Solution:

The shortcut method states: $\bar{x} = A + \frac{\sum D}{n}$

The coding method states: $\bar{x} = A + h\frac{\sum u}{n}$, where $u = \frac{x - A}{h}$.

The required calculations are shown in tables separately for both parts.

(i). $A = 650$ and $h = 10$.

x	$D = x - 650$	$u = \frac{x-650}{10}$
610	-40	-4
640	-10	-1
685	35	3.5
680	30	3
710	60	6
580	-70	-7
	$\sum D = 5$	$\sum u = 0.5$

By shortcut method, we have:

$$\bar{x} = 650 + \frac{5}{6} = 650.833.$$

By coding method, we have

$$\bar{x} = 650 + 10 \times \frac{0.5}{6} = 650 + \frac{5}{6} = 650.833.$$

(ii). $A = 680$ and $h = 20$.

x	$D = x - 680$	$u = \frac{x-680}{20}$
610	-70	-3.5
640	-40	2
685	5	0.25
680	0	0
710	30	1.5
580	-100	-5
	$\sum D = -175$	$\sum u = -8.75$

By shortcut method:

$$\bar{x} = 680 + \left(\frac{-175}{6}\right) = 680 - 29.1666$$

$\bar{x} = 650.833.$

By coding method:

$$\bar{x} = 680 + 20\left(\frac{-8.75}{6}\right)$$

$\bar{x} = 680 - 29.1666 = 650.833.$

The mean marks are 650.833 or 651 regardless of the choice of A and h.

Example 2: Find mean amount of cold drink (in liters) by shortcut and coding methods.

Amount of cold drink(in liters)	1.48	1.49	1.50	1.51	1.52
Number of bottles	2	7	5	5	1

Solution: Here, we use $A = 1.49$ and $h = 0.01$. The calculations are detailed in the table.

x	f	$D = x - 1.49$	$u = \frac{x-1.49}{0.01}$	Df	uf
1.48	2	-0.01	-1	-0.02	-2
1.49	7	0	0	0	0
1.50	5	0.01	1	0.05	5
1.51	5	0.02	2	0.1	10
1.52	1	0.03	3	0.03	3
	$\sum f = 20$			$\sum Df = 0.16$	$\sum uf = 16$

By shortcut method: $\bar{x} = A + \frac{\sum Df}{\sum f} = 1.49 + \frac{0.16}{20} = 1.498.$

By coding method: $\bar{x} = A + h \times \frac{\sum uf}{\sum f} = 1.49 + 0.01\left(\frac{16}{20}\right) = 1.498.$

We note that the result by both indirect methods and by definition match.

Example 3: Use shortcut and coding methods to find mean mass of 50 metal blocks.

Mass of block(in Kg)	7.1-7.3	7.4-7.6	7.7-7.9	8.0-8.2	8.3-8.5	8.6-8.8	8.9-9.1
Number of blocks	3	5	9	14	11	6	2

Solution: Taking $A = 8.1$ and $h = 0.1$, we proceed with the calculations in the table.

Class limits	Class marks (x)	f	$D = x - 8.1$	$u = \frac{x-8.1}{0.1}$	Df	uf
7.1-7.3	7.2	3	-0.9	-9	-2.7	-27
7.4-7.6	7.5	5	-0.6	-6	-3	-30
7.7-7.9	7.8	9	-0.3	-3	-2.7	-27
8.0-8.2	8.1	14	0	0	0	0
8.3-8.5	8.4	11	0.3	3	3.3	33
8.6-8.8	8.7	6	0.6	6	3.6	36
8.9-9.1	9.0	2	0.9	9	1.8	18
		$\sum f = 50$			$\sum Df = 0.3$	$\sum uf = 3$

By shortcut method: $\bar{x} = \text{A}+\frac{\sum Df}{\sum f} = 8.1+\frac{0.3}{50} = 8.106$.

By coding method: $\bar{x} = \text{A}+h\times\frac{\sum uf}{\sum f} = 8.1+0.1\times\left(\frac{3}{50}\right) = 8.106$.

So, the average/mean mass is 8.106Kg.

22.3(i) (b). Median, mode, geometric mean and harmonic mean

Besides, the arithmetic mean (based on equality), there are some other ways to compute measures of central tendency of a data. For example, in our electoral system, everyone has right to vote, but the elected representative is finally chosen on the basis of **majority**, as widely said "majority is the authority". Such a case represents **mode**, the most frequent observation. Similarly, the **median** focuses on the **middle most part** of a ranked data set. The **geometric and harmonic means** are mathematical in nature like arithmetic mean in contrast to median and mode which are descriptive in nature.

Median:

The **median** in an ordered data is the value dividing it into two equal parts. By computing median, we assume that the central point of the whole data arranged into ascending or descending order lies at middle position.

Note:

- In a data, median is always unique.
- We can compute median of quantitative data.
- For qualitative data, it is meaningful to find median of only the ordinal data.
- The median does not rely equally on all observations
- The median is not affected by extreme values (outliers) in the data.

For an ordered ungrouped data with "n" observations, median "m" is the middle-most part of data, and can be defined through the following two cases:

$$m = \left(\frac{n+1}{2}\right)^{th} \text{ observation, when n is odd} \qquad (1)$$

$$\text{Otherwise, } m = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th} \text{observations}\right] \text{, when n is even} \qquad (2)$$

For grouped data with discrete classes, by the help of cumulative frequencies, we choose the median observation using (1) and (2).

For grouped data with continuous classes, the median lies in the class containing $\left(\frac{n}{2}\right)^{th}$ observation, which is also called the median class (m-class). The formula is:

$$m = L + \frac{h}{f}\left(\frac{n}{2} - c\right) \qquad (3)$$

Where, L is the LCB of m-class, h is width of m-class, f is frequency of m-class and c is cumulative frequency of the class just before the m-class.

Example 1: Find median in the following:

(i). 2, 3, 7, 5, 5, 13, 1, 7, 4, 8, 3, 4, 3, (ii). 7, 5, 74, 10 (iii). 18.92, 27.9, 37.4, 39.68

Solution:

(i). Writing data is ascending order: 1, 2, 3, 3, 3, 4, 4, 5, 5, 7, 7, 8, 13.

Here, $n = 13$ (odd), So: $m = \left(\frac{13+1}{2}\right)^{th}$ observation $= 7^{th}$ observation $= 4$.

(ii). In ascending order, we have: 5, 7, 10, 74. Here, $n = 4$ (even), so:

$$m = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th} \text{observations}\right] = \frac{1}{2}\left[2^{nd} + 3^{rd} \text{ observations}\right] = \frac{1}{2}\left[7+10\right] = 8.5.$$

Here, n =4 (even). Data is already in ranked from.

$$m = \frac{1}{2}\left[2^{nd} + 3^{rd} \text{ observations}\right] = \frac{1}{2}\left[27.9+37.4\right] = 32.65.$$

Example 2: Identify the median grade of the following two groups of students:

(i). A+, B+, C, B, A (ii). C+, A, B, B, B+, C, B+, A, A+, C

Solution:

(i). The ranked data are: C, B, B+, A, A+. Also, $n = 5$ (odd), so:

$$m = \left(\frac{n+1}{2}\right)^{th} = \left(\frac{5+1}{2}\right)^{th} = 3^{\text{rd}} \text{ observation} = B+,$$ which is the median grade.

(ii). Arranging in ascending order, we have: C, C, C+, B, B, B+, B+, A, A, A+. Here, $n = 10$, the two middle observations are $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$, i.e. 5^{th} and 6^{th} observations. Which are: B and B+ grades (different). So, we conclude that the median grade is equally split between B and B+ grades.

Example 3:

What is the median satisfaction level in the following data of 40 customers?

Satisfaction level	Not at all	Not very	Not sure	Somewhat	Very
Number of customers	3	8	5	6	18

Solution:

Here, $n = 40$ (even). So, two middle observations are: 20^{th} and 21^{th}. These two lie in group "Somewhat". So, the median satisfaction level is "Somewhat satisfied".

Example 4: Find median amount of cold drink in the following data.

Amount of cold drink(in liters)	1.48	1.49	1.50	1.51	1.52
Number of bottles	2	7	5	5	1

Solution:

Here, $n = \sum f = 20$ (even).

So, $m = \frac{1}{2}\left[10^{th} + 11^{th} \text{ observations}\right]$.

Finding cumulative frequencies, we see that 10^{th} and 11^{th} observations lie in group with cumulative frequency 14.

x	f	Cumulative frequencies
1.48	2	2
1.49	7	9
1.50	5	14
1.51	5	19
1.52	1	20
	$\sum f = 20$	

$m = \frac{1}{2}[1.50 + 1.50] = \frac{1}{2}(3) = 1.50$ liters.

Example 5:

Find the median electricity consumption of a shop for 60 days using the data:

Electricity consumption	68-87	88-107	108-127	128-147	148-167	168-187	188-207
Number of days	10	13	15	10	4	6	2

Solution:

Here, $n = 60$, $h = 20$. Data are a continuous frequency table, so we need to compute class boundaries and cumulative frequencies as done in the table below.

Class limits	Class boundaries	f	Cumulative frequencies
68-87	67.5-87.5	10	10
88-107	87.5-107.5	13	23
108-127	107.5-127.5	15	38 (m-class)
128-147	127.5-147.5	10	48
148-167	147.5-167.5	4	52
168-187	167.5-187.5	6	58
188-207	187.5-207.5	2	60
		$n = 60$	

Median class (m-class) is one containing $\left(\frac{n}{2}\right)^{th}$ observation, i.e. 30^{th} observation. So, m-class is one with cumulative frequency of 38. The m-class is 108-127 (as highlighted in table). Using the formula:

$$m = L + \frac{h}{f}\left(\frac{n}{2} - c\right) = 107.5 + \frac{20}{15}\left(\frac{60}{2} - 23\right)$$

$$m = 107.5 + \frac{20}{15}(30 - 23) = 107.5 + \frac{20}{15} \times 7 = 107.5 + 9.333 = 116.833 \text{ kWh.}$$

Here, L = LCB of m-class = 107.5, h = class interval m-class = 20, f = frequency of m-class = 15 and c = Cumulative frequency of class just before the median class = 23.

Quartiles:

An ordered data may be divided into four equal parts to define **quartiles**. There are three quartiles within the range of data which divide data into 4 equal parts.

The **first (or lower) quartile** Q_1, divides ordered data into 25% to 75% ratio.

For ungrouped data: $Q_1 = \left(\frac{n+3}{4}\right)^{th}$ observation, if $\frac{n}{4}$ is not integer (1)

Otherwise, $Q_1 = \frac{1}{2}\left[\left(\frac{n}{4}\right)^{th} + \left(\frac{n}{4}+1\right)^{th} \text{ observations}\right]$, if $\frac{n}{4}$ is an integer (2)

For grouped data with discrete classes, we also use (1) and (2). For grouped data with continuous classes, we use: $Q_1 = L + \frac{h}{f}\left(\frac{n}{4} - c\right)$ (3)

where, L is LCB, h is width, f is frequency of Q_1-class, and c is cumulative frequency of a class just before Q_1-class. The Q_1-class is one containing the $\left(\frac{n}{4}\right)^{th}$ observation.

The **second (or middle) quartile** Q_2, divides ordered data into 50% – 50% ratio. It is same as the median of data.

The **third (or upper) quartile** Q_3 divides ordered data into 75% – 25% ratio.

For ungrouped data: $Q_3 = \left(\frac{3n+1}{4}\right)^{th}$ observation, if $\frac{3n}{4}$ is not integer (4)

Otherwise, $Q_3 = \frac{1}{2}\left[\left(\frac{3n}{4}\right)^{th} + \left(\frac{3n}{4}+1\right)^{th} \text{ observations}\right]$, if $\frac{3n}{4}$ is an integer (5)

For grouped data with discrete classes (4)-(5) are used with cumulative frequencies.

For grouped data with continuous classes: $Q_3 = L + \frac{h}{f}\left(\frac{3n}{4} \quad c\right)$ (6)

where, L is LCB, h is width, f is frequency of Q_3–class, which contains $\left(\frac{3n}{4}\right)^{th}$ observation, and c is the cumulative frequency of the class just before Q_3–class.

Example 1:

Find lower and upper quartiles in the following datasets.

(i). 1000, 1200, 1600, 1500, 1200 (ii). 32, 36, 36, 37, 39, 41, 45, 46

Solution

(i). The ordered data set is: 1000, 1200, 1200, 1500, 1600. Here, n = 5.

For $Q_1 : \frac{n}{4} = \frac{5}{4} = 1.25$ is not integer,

so: $Q_1 = \left(\frac{n+3}{4}\right)^{th} = 2^{nd}$ observation =1200.

For $Q_3 : \frac{3n}{4} = \frac{3\times 5}{4} = 3.75$ is not integer,

so: $Q_3 = \left(\frac{3n+1}{4}\right)^{th} = 4^{th}$ observation1500.

(ii): The data are already given in ordered form and $n = 8$.

For $Q_1 : \frac{n}{4} = \frac{8}{4} = 2$ is integer,

$$\text{so:} Q_1 = \frac{1}{2}\left[\left(\frac{n}{4}\right)^{th} + \left(\frac{n}{4}+1\right)^{th} \text{ observations}\right]$$

$$\Rightarrow Q_1 = \frac{1}{2}\left[2^{nd} + 3^{rd} \text{ observation}\right] = \frac{1}{2}[36+36] = 36.$$

For $Q_3 : \frac{3n}{4} = \frac{3\times 8}{4} = 6$ (integer),

$$\text{so:} Q_3 = \frac{1}{2}\left[6^{th} + 7^{th} \text{ observations}\right] = \frac{1}{2}[41+45] = 43.$$

Example 2:

Find lower and upper quartiles of the number of children in 80 families:

Number of children	1	2	3	4	5	6
Number of families	8	10	10	25	20	7

Solution:

Data are grouped with discrete classes. Here, $n = 80$.
Finding cumulative frequencies as in the following table.

x	f	Cumulative frequency
1	8	8
2	10	18
3	10	28 (Q_1-class)
4	25	53
5	20	73 (Q_3-class)
6	7	80

For $Q_1 : \frac{n}{4} = \frac{80}{4} = 20$ (an integer).

So, Q_1 is average of 20^{th} and 21^{st} observations.

These both lie in class with cumulative frequency 28, $Q_1 = \frac{1}{2}(3+3) = 3.$

For $Q_3 : \frac{3n}{4} = \frac{3\times 80}{4} = 60$.

60^{th} and 61^{st} observations lie in the class with cumulative frequency 73.

So, $Q_3 = \frac{1}{2}(5+5) = 5.$

Example 3:

Find quartile weights (in Kg) of 100 students reported below:

Weight (Kg)	70-74	75-79	80-84	85-89	90-94
Number of students	10	24	46	12	8

Solution:

Here, n =100, h = 5, and data are grouped with continuous classes. First we compute class boundaries and cumulative frequencies. The calculations are summarized in the following table.

Class limits	Class boundaries	f	Cumulative Frequencies
70-74	69.5-74.5	10	10
75-79	74.5-79.5	24	34 (Q_1-class)
80-84	79.5-84.5	46	80 (*m and* Q_3-class)
85-89	84.5-89.5	12	92
90-94	89.5-94.5	8	100

For $Q_1 : \frac{n}{4} = \frac{100}{4}$ =25. 25^{th} observation lies in class with cumulative frequency 34,

So, Q_1 – class is 75 – 79.

In this class: $Q_1 = L + \frac{h}{f}\left(\frac{n}{4} - c\right) = 74.5 + \frac{5}{24}(25 - 10) = 77.625$ Kg.

For $Q_2 = m : \frac{n}{2} = \frac{100}{2} = 50.$

50^{th} observation lies in the class with cumulative frequency 80.

So, m-class is 80-84.

$$m = L + \frac{h}{f}\left(\frac{n}{2} - c\right) = 79.5 + \frac{5}{46}(50 - 34) = 81.239 \text{ Kg.}$$

For $Q_3 : \frac{3n}{4} = \frac{3 \times 100}{4} = 75.$

75^{th} observation lies in the class with cumulative frequency of 80.

So, Q_3-class is same as m-class, which is: 80-84.

In this class:

$$Q_3 = L + \frac{h}{f}\left(\frac{3n}{4} - c\right) = 79.5 + \frac{5}{46}(75 - 34) = 83.956 \text{ Kg.}$$

Note that:

The quartiles are positional in nature, their positions do not change as long as number of observations remain same. For example, for an ordered data with $n = 10$:

$Q_1 = 3^{rd}$ observation, $Q_2 = m = \frac{1}{2}\left[5^{th} + 6^{th} \text{ observations}\right], Q_3 = 8^{th}$ observation

Any ordered dataset with 10 observations will follow these positions, as in:

a. 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 $\quad Q_1 = 5,\ m = 10,\ Q_3 = 15$

b. 10, 12, 12, 13, 17, 20, 20, 20, 21, 25 $\quad Q_1 = 12,\ m = 13.5,\ Q_3 = 20.$

c. 1, 1, 1, 3, 12, 12, 13, 17, 19, 30 $\quad Q_1 = 1,\ m = 12,\ Q_3 = 17.$

Mode:

The most frequent observation in a data is referred as **mode**. Mode relies on the principle "majority is the authority". Mode can be computed for both quantitative and qualitative data. Data with no mode is non-modal, one mode is uni-modal, two or more modes is multi-modal.

For ungrouped data, observation(s) which occur most frequently than others, if any, are referred as mode(s).

For grouped data with discrete classes, mode is the observation (class) with highest frequency. If two or more than two classes have tie for highest frequency, then data is multi-modal, and all observations with same highest frequency are modes.

For grouped data with continuous classes, a class with highest frequency is called modal class, and a mode lies in the modal class. The formula to compute mode (M) in modal-class (M-class) is: $\text{Mode} = M = L + \dfrac{(f_M - f_{M-1}) \times h}{2f_M - f_{M-1} - f_{M+1}}$

Where, L is LCB, h is width and f_M is frequency of M-class. f_{M-1} and f_{M+1} are frequencies of the classes just before and after M-class, respectively.

Example 1: Find mode in the following datasets:

(i). 75, 76, 80, 80, 82, 82, 82, 85

(ii). 13, 14, 15, 11, 16, 10, 19, 20, 18, 17

(iii). 1.49, 1.50, 1.51, 1.50, 1.48, 1.51

(iv). 1, 2, 2, 2, 5, 5, 5, 8, 9, 9, 9, 6, 1, 10

(v). C+, A, B, B, B+, C, B, A, A+, C (grades)

(vi). Good,Poor,Dull,Good,Fair,Fair (ratings)

Solution:

(i). Mode = 82 as 82 occurs 3 times, which is more than any other observation.

(ii). Data has no mode, it is non-modal data.

(iii). There are two modes: 1.50 and 1.51 as these equally occur more than others.

(iv). Data is tri-modal. There are three modes: 2, 5 and 9.

(v). Grade "B" occurs most frequently, so the modal grade is "B".

(vi). Data has two modes: Fair and Good as these are equally most frequent ratings.

Example 2. Find modal number of mobiles possessed by a family from data of 45 families.

Numbers of mobiles	0	1	2	3	4	5	6	7
Numbers of families	1	7	15	12	5	2	2	1

Solution: Data are grouped into discrete classes. Here, "15" is the highest frequency, and it corresponds to the class "2". Hence, the modal number of mobiles per family is 2.

Example 3: From the blood groups of 30 students given below, find the modal blood group.

Blood group	A+	B+	AB+	O+	A-	B-	AB-	O-
Number of student	6	5	3	7	3	1	2	3

Solution: The modal blood group is O+ as it is associated with the highest frequency "7".

Example 4: Compute modal satisfaction level of 40 customers from the following data.

Satisfaction level	Not at all	Not very	Not sure	Some what	Very
Number of customers	3	8	5	6	18

Solution: The modal satisfaction level is "Very satisfied" as it occurs most frequently.

Example 5: Find modal kWh electricity consumption of a shop for 60 days using the data:

Electricity consumption	68-87	88-107	108-127	128-147	148-167	168-187	188-207
Number of days	10	13	15	10	4	6	2

Solution: Data are grouped in continuous classes. The highest frequency is 15. The modal class is 108-127. The class boundaries are computed in table.

Class limits	Class boundaries	Frequency
68 - 87	67.5 - 87.5	10
88 - 107	87.5 - 107.5	13
108 - 127	107.5 - 127.5	15
128 - 147	127.5 - 147.5	10
148 – 167	147.5 - 167.5	4
168 - 187	167.5 - 187.5	6
188 - 207	188.5 - 207.5	2

$h = 20, L=107.5, f_M = 15, f_{M-1} = 13, f_{M+1} = 10$

$$\text{Mode} = M = L + \frac{(f_M - f_{M-1}) \times h}{2f_M - f_{M-1} - f_{M+1}}$$

$$M = 107.5 + \frac{(15-13)\times 20}{2(15)-13-10} = 107.5 + \frac{2\times 20}{30-23}$$

$$M = 107.5 + 5.71428 = 113.21428 \text{kWh}.$$

Note that:

The continuous frequency distribution in Example 5 was uni-modal as there was only one class with the highest frequency.

The following continuous frequency distribution representing diameters of a reel of wire using measured diameters at 24 places is bi-modal.

Diameters (mm)	2.10-2.13	2.14-2.17	2.18-2.21	2.22-2.25	2.26-2.29	2.30-2.33	2.34-2.37
Number of places	4	5	4	4	5	1	1

There are two modal classes: "2.14 – 2.17" and "2.26 – 2.29" corresponding to the highest frequency "5". Using formula in each class, we can get the respective modal diameters of 2.175mm and 2.263mm.

Geometric mean:

In **geometric mean**, all numbers are given **equal** importance. We mix the numbers by their product and then remove effect of mixing by finding n^{th} root of the product. It is meaningful to compute geometric mean (G.M.) of only quantitative data which are positive. If $x_1, x_2, x_3, ..., x_n$ are n numbers (all positive), then:

$$\text{G.M.} = (x_1.x_2...x_3)^{\frac{1}{n}} = \left[\prod(x)\right]^{\frac{1}{n}} \qquad (1)$$

Here, $\prod$ is upper-cased Greek letter pi for product. G.M., if exists, is unique.

For ungrouped data, formula (1) is referred as the **basic formula** to find G.M. But, when observations are bigger in magnitude, we use **logarithmic formula** obtained by taking logarithm on both sides of (1) and using properties, which is:

$$\text{G.M} = \text{antilog}\left[\frac{\sum_{i=1}^{n} \log x_i}{n}\right] = \text{antilog}\left[\frac{\sum \log x}{n}\right] \quad (2)$$

In (2), G.M. is the antilogarithm of the arithmetic mean of logarithms of data.
For grouped data, the **basic formula** is:

$$\text{G.M.} = \left[(x_1)^{f_1} \times (x_2)^{f_2} \times \ldots \times (x_n)^{f_n}\right]^{\frac{1}{\Sigma f}} = \left[\prod (x)^f\right]^{\frac{1}{\Sigma f}} \quad (3)$$

and, **logarithmic formula** is: $\text{G.M.} = \text{antilog}\left[\frac{\sum \text{f. } \log x}{\sum \text{f}}\right] \quad (4)$

In (3)-(4), $f_1, f_2, \ldots, f_n$ are class frequencies and $x_1, x_2, \ldots, x_n$ are class marks, all positive. None of the frequencies should be zero in (4).

Example 1: Find G.M. using basic and logarithmic formula. Which one is better and why?

(i). 2, 4, 2, 16 (ii). 1.48, 1.52, 1.47, 1.50 (iii). 1000, 1200, 1600, 1500, 1200

Solution:

(i). Using basic formula: $\text{G.M.} = (2\times 4\times 2\times 16)^{\frac{1}{4}} = (256)^{\frac{1}{4}} = 4.$

Using logarithmic formula: $\text{G.M.} = \text{antilog}\left[\frac{\log(2)+\log(4)+\log(2)+\log(16)}{4}\right]$

$$\text{G.M.} = \text{antilog}\left[\frac{0.3010+0.6021+0.3010+1.2041}{4}\right] = \text{antilog }(0.60205) = 3.9999 \approx 4.$$

Observations were small in magnitude, so basic formula is better.

(ii). Using basic formula: $\text{G.M.} = (1.48\times 1.52\times 1.47\times 1.50)^{\frac{1}{4}} = (4-9604)^{\frac{1}{4}} = 1.4924$

By logarithmic formula: $\text{G.M.} = \text{antilog}\left[\frac{\log(1.48)+\log(1.52)+\log(1.47)+\log(1.5)}{4}\right]$

$$\text{G.M.} = \text{antilog}\left[\frac{0.1703+0.1818+0.1673+0.1761}{4}\right] = \text{antilog }(0.173875) = 1.4924.$$

Again, basic formula is suitable as observation were smaller in magnitude.

(iii). Using basic formula:

$$\text{G.M.} = (1000\times 1200\times 1600\times 1500\times 1200)^{\frac{1}{5}} = (3456000000000000)^{\frac{1}{5}} = 1281.48 \approx 1281.5$$

Using logarithmic formula:

$$\text{G.M.} = \text{antilog}\left[\frac{\log(1000)+\log(1200)+\log(1600)+\log(1500)+\log(1200)}{5}\right]$$

$$\text{G.M.} = \text{antilog}\left[\frac{3+3.0792+3.2041+3.1761+3.0792}{5}\right] = \text{antilog}(3.10772) = 1281.504 \approx 1281.5$$

The logarithmic formula is better as logarithms made higher magnitudes smaller to use.

Example 2:

Find average amount of cold drink using geometric mean in the following data.

Amount of cold drink (liters)	1.48	1.49	1.50	1.51	1.52
Number of bottles	2	7	5	5	1

Solution:

Given data are grouped into discrete classes. Computations required in the basic and logarithmic formula are show in following table.

Classes (x)	f	$(x)^f$	log(x)	$f\times \log(x)$
1.48	2	2.1904	0.1703	0.3406
1.49	7	16.3044	0.1732	1.2124
1.50	5	7.5938	0.1761	0.8805
1.51	5	7.8503	0.1790	0.8950
1.52	1	1.52	0.1818	0.1818
---	$\sum f=20$	$\prod x^f=3236.0651$	---	$\sum f\log(x)=3.5103$

Using basic formula: $G.M. = \left[\prod (x)^f\right]^{\frac{1}{\sum f}} = (3236.0651)^{\frac{1}{20}} = 1.49796 \approx 1.498$ liters

Using logarithmic formula: $G.M. = \text{antilog}\left[\frac{\sum f\ \log x}{\sum f}\right] = \text{antilog}\left[\frac{3.5103}{20}\right]$

$G.M.=\text{antilog}\,(0.175515) = 1.4980\text{ L}$. The G.M. amount of cold drink is 1.498 liters.

Example 3:

Find G.M. of the electricity consumption data using logarithmic method.

Electricity consumption (kWh)	68-87	88-107	108-127	128-147	148-167	168-187	188-207
Number of days	10	13	15	10	4	6	2

Solution:

The computation required in the formula are carried art in the following table.

Class limits	f	Class marks (x)	Log (x)	$f\times\log(x)$
68-87	10	77.5	1.8893	18.8930
88-107	13	97.5	1.9890	25.8570
108-127	15	117.5	2.0700	31.0500
128-147	10	137.5	2.1383	21.3830
148-167	4	157.5	2.1973	8.7892
168-187	6	177.5	2.2492	13.4952
188-207	2	197.5	2.2956	4.5912
	$\sum f=60$			$\sum f\log x=124.0586$

So, $G.M. = \text{antilog}\left[\frac{\sum f\ \log x}{\sum f}\right] = \text{antilog}\left[\frac{124.0586}{60}\right] = \text{antilog}\,(2.0676) = 116.8422$ kWh.

Example 4:

Find average diameter of a rear of wire using G.M. Use both methods.

Diameter (mm)	2.10-2.13	2.14-2.17	2.18-2.21	2.22-2.25	2.26-2.29	2.30-2.33	2.34-2.37
Number of places	4	5	4	4	5	1	1

Solution:

Data are grouped into continuous classes. The computation follow in the table.

Class limits	Class marks (x)	f	x^f	$\log(x)$	$f \times \log x$
2.10-2.13	2.115	4	20.0097	0.3253	1.3012
2.14-2.17	2.155	5	46.4768	0.3334	1.6670
2.18-2.21	2.195	4	23.2134	0.3414	1.3656
2.22-2.25	2.235	4	24.9523	0.3493	1.3972
2.26-2.29	2.275	5	60.9406	0.3570	1.7850
2.30-2.33	2.315	1	2.315	0.3646	0.3646
2.34-2.37	2.355	1	2.355	0.3720	0.3720
		$\sum f = 24$	$\prod x^f = 178967745.4$	---	$\sum f \log x = 8.2526$

By basic formula: $G.M = \left[\prod x^f\right]^{\frac{1}{\sum f}} = (178967745.4)^{\frac{1}{24}} = 2.2073$ mm

By logarithmic formula: $G.M. = \text{antilog}\left[\dfrac{\sum f \log x}{\sum f}\right] = \text{antilog}\left[\dfrac{8.2526}{24}\right] = 2.2074$ mm

Harmonic mean:

The **harmonic mean** (H.M.) is the reciprocal of the arithmetic mean of reciprocals of data. In H.M., we mix all n non-zero numbers by the sum of their reciprocals and then remove the effect of mixing by dividing the same into n **equal** parts. Finally, the H.M. is the reciprocal of the result. The H.M is usually computed for quantitative data, and it is always unique. If $x_1, x_2, ..., x_n$ are n non-zero numbers, then:

For ungrouped data: H.M. = Reciprocal of "arithmetic mean of reciprocals".

$$H.M. = \text{Reciprocal of}\left[\frac{\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}}{n}\right] = \text{Reciprocal of}\left[\frac{\sum\left(\frac{1}{x}\right)}{n}\right] = \frac{n}{\sum\left(\frac{1}{x}\right)} \qquad (1)$$

For grouped data, if x's are non-zero class marks, then $H.M. = \dfrac{\sum f}{\sum\left(\frac{f}{x}\right)}$ (2)

Example 1:

Find H.M. of the speeds of a vehicle measured for each 10 km distance. The speeds are: 15 km/h, 30 km/h, 22km/h, 30 km/h and 45 km/h.

Solution:

$$H.M. = \frac{n}{\sum\left(\frac{1}{x}\right)} = \frac{5}{\frac{1}{15}+\frac{1}{30}+\frac{1}{22}+\frac{1}{30}+\frac{1}{45}} = \frac{5}{0.0667+0.0333+0.0454+0.0333+0.0222}$$

$H.M. = \dfrac{5}{0.2009} = 24.8880$. So, average speed of vehicle is 12.4719 km/h using H.M.

Example 2:

The amount of cold drink (in liters) is given for 20 bottles. Find H.M. amount.

Amount of cold drink(in liters)	1.48	1.49	1.50	1.51	1.52
Number of bottles	2	7	5	5	1

Solution:

Given data are grouped into discrete classes, so:

$$\text{H.M.} = \frac{\sum f}{\sum\left(\frac{f}{x}\right)} = \frac{2+7+5+5+1}{\frac{2}{1.48}+\frac{7}{1.49}+\frac{5}{1.50}+\frac{5}{1.51}+\frac{1}{1.52}}$$

$$\text{H.M.} = \frac{20}{1.3513+4.6980+3.3333+3.3112+0.6579} = \frac{20}{13.3517} = 1.4979 \text{ liters.}$$

Example 3:

Find harmonic mean mass (in Kg) of 50 blocks of metals given as::

Mass (Kg)	7.1-7.3	7.4-7.6	7.7-7.9	8.0-8.2	8.3-8.5	8.6-8.0	8.9-9.1
Numbers of blocks	3	5	9	14	11	6	2

Solution:

Data are grouped into continuous classes, so: $\text{H.M} = \frac{\sum f}{\sum\left(\frac{f}{x}\right)}$

Computing the required sums in the following table, we have:

Class limits	Class marks (x)	f	$\frac{f}{x}$
7.1-7.3	7.2	3	0.4167
7.4-7.6	7.5	5	0.6667
7.7-7.9	7.8	9	1.1538
8.0-8.2	8.1	14	1.7284
8.3-8.5	8.4	11	1.3095
8.6-8.8	8.7	6	0.6896
8.9-9.1	9.0	2	0.2222
Total	---	50	6.1869

$$\text{H.M} = \frac{50}{6.1869} = 8.0816 \text{ Kg.}$$

EXERCISE 22.3

1. Find A.M., G.M., H.M., median and mode in the following (wherever possible).
   - **a.** 3.2, 6, 10, 12, 12, -20, 25, 28, 30.8
   - **b.** 14, 12, 18, 19, 0, -19, -18, -12, -14
   - **c.** 6.5, 11, 12.3, 9, 8.1, 16, 18, 20.5, 25
   - **d.** 51, 55, 52, 54, 58, 60, 61, 62, 52, 57, 52, 64
   - **e.** A+, O-, AB+, O+, AB+, AB-, B+, AB+, O+, A- (blood groups)
   - **f.** North, South, East, West (directions)
   - **g.** B+, Fail, B+, A+, A-, C+, B+, A-, B-, Fail, A-, B+, C- (grades)

2. The daily earnings for ten workers in Rs. are: 188, 170, 172, 125, 115, 195, 181, 190, 195, 190. Find A.M. (by definition and deviations with A = 50, h = 10), G.M. (by definition and logarithmic method), H.M, median and mode.

3. Find average attitude for dogs using A.M. median and mode from 60 people data.

Attitude	Love dogs	Like dogs	No opinion	Dislike dogs	Hate dogs
Number of people	20	15	4	13	8

4. Find modal cause of death from the following data of mortality/month in a city.

Cause of death	T.B.	Diabetics	Malaria	Cholera	COVID	Cancer	B.P.	Heart Attack
Number of people	10	6	2	2	15	10	2	5

5. The sizes of shoe sold at a store on a 50% off price are listed. Calculate A.M., G.M., H.M., median, Q_1, Q_3 and modal shoe size sold that day.

Shoe size	5	5.5	6	6.5	7	7.5	8	8.5	9	9.5
Number of pairs sold	2	5	15	30	60	40	23	11	4	1

6. Daily wages (in Rs. 100) for thousand employees in a factory are given. Find A.M., G.M., H.M., median, quartiles and modal wages.

Daily wages (in Rs. 100)	22	24	26	28	30	32	34	36	38	40	42	44
Number of employees	3	13	43	102	175	220	204	139	69	25	6	1

7. The profits earned by a company for a period of last 50 days are summarized below. Find the A.M. profit using shortcut and coding methods with

(a). A = 9000, h = 2000 (b). A = 11000, h = 2000

Profits (Rs.)	4000-6000	6000-8000	8000-10000	10000-12000	12000-14000
Number of days	5	7	11	21	6

8. The marks obtained by students in a subject (out of 50) are given in the following grouped table. Find A.M., G.M. (using direct and logarithmic methods), H.M., median and mode.

Marks	25-29	30-34	35-39	40-44	45-49
Number of students	9	18	35	17	5

9. The following data show number of devices resulting in observed values in appropriate ranges. Find A.M., G.M., H.M., median, quartiles and mode.

Class limits	10.5-10.9	11.0-11.4	11.5-11.9	12.0-12.4	12.5-12.9
Frequencies	2	7	10	12	8

22.3(ii): Recognize properties of arithmetic mean:

Here, we discuss some important properties of A.M. with examples.

1. The sum of all deviations of observations in an ungrouped data about A.M. is zero.
2. A.M. of a constant dataset is that constant itself.
3. If A.M. of a set X is $\bar{x}$, then of Y = aX + b is $\bar{y} = a\bar{x} + b$, where a, b are real numbers.
4. If A.M. = G.M. = H.M., then all observations in data are same or constant.
5. For a non-constant data, A.M. > G.M. > H.M.
6. For a non-constant data: $(\text{A.M})(\text{H.M}) \approx (\text{G.M.})^2$.
7. If A.M. = Median, then the data are **symmetric**, otherwise **asymmetric**.
8. In a uni-modal symmetric data: A.M. = Median = Mode.
9. In a constant data: A.M. = G.M. = H.M. = Median = Mode.

Example 1:

Find the arithmetic mean of the following datasets:

(i). X={19, 19, 19, 19, 19} (ii). Z = 3Y + 7, where Y={3, 4, 6, 1, 6}

Solution:

(i). X= $\{19,19,19,19,19\}$ is constant with all values equal to 19, so: $\bar{x} = 19$.

(ii). First we find: $\bar{y} = \dfrac{3+4+6+1+6}{5} = \dfrac{20}{5} = 4$

Now, as Z = 3Y + 7, so: $\bar{z} = 3\bar{y} + 7 = 3(4) + 7 = 19$.

We can verify that, when Z = 3Y + 7 $= \{16,19,25,10,25\}$, then $\bar{z} = 19$ is correct.

Example 2:

Show that sum of all deviations in $\{3,4,6,1,6\}$ about its A.M. is zero.

Proof:

Let $X = \{3,4,6,1,6\}$. Finding $\bar{x}$ first, which is: $\bar{x} = \dfrac{\sum x}{n} = \dfrac{20}{5} = 4$.

Now, the deviations about $\bar{x}$ are: (3-4), (4-4), (6-4), (1-4), and (6-4), or simply: -1, 0, 2, -3 and 2. Now, we observe the sum of all deviations about $\bar{x}$:

$\sum(X - \bar{x}) = (-1) + (0) + (2) + (-3) + (2) = 0. \Rightarrow \sum(X - \bar{x})$ 0. Hence shown.

Example 3:

Verify that A.M. = G.M. = H.M. = Median = Mode for $\{19,19,19,19,19\}$.

Solution:

Finding all averages for the given constant dataset. Here, n = 5 (odd).

A.M.= $\dfrac{19+19+19+19+19}{5} = 19$ (i)

G.M.= $(19\times19\times19\times19\times19)^{1/5} = 19^{5\times\frac{1}{5}} = 19$. (ii)

H.M.= $\dfrac{5}{\frac{1}{19}+\frac{1}{19}+\frac{1}{19}+\frac{1}{19}+\frac{1}{19}} = \dfrac{5}{\frac{5}{19}} = 19$. (iii)

Median = $\left(\dfrac{5+1}{2}\right)^{th}$ observation = 3^{rd} observation = 19. (iv)

Mode = 19 (Most frequent observation). (v)

From equations (i)-(v), we have: A.M. = G.M. = H.M. = Median = Mode. Hence proved

Example 4:

Use A.M. and median to see if the following data are symmetric/ asymmetric.

(i). 4, 5, 6, 6, 6, 7, 7, 7, 7, 7, 7, 8, 8, 8, 9, 10 (ii). 4, 8, 13, 14, 19, 20, 23

Solution:

(i). A.M.= $\dfrac{4+5+6+6+6+7+7+7+7+7+7+8+8+8+9+10}{16} = \dfrac{112}{16} = 7$

Median = $\dfrac{1}{2}\left[8^{th} + 9^{th} \text{ observation}\right] = \dfrac{1}{2}[7+7] = 7$

As A.M = Median, so data are symmetric.

(ii). $\text{A.M} = \dfrac{4+8+13+14+19+20+23}{7} = 14.42$

$\text{Median} = \left[\dfrac{7+1}{2}\right]^{th}$ observation $= 4^{th}$ observation $= 14$

As A.M. ≠ Median, so data are asymmetric.

Example 5:

For {4,5,6,6,6,7,7,8}, verify (i). A.M. > G.M. > H.M. and
(ii). $(\text{G.M})^2 \approx (\text{A.M})(\text{H.M})$.

Solution:

$\text{A.M.} = \dfrac{4+5+6+6+6+7+7+8}{8} = 6.125$, $\text{G.M.} = (4\times5\times6\times6\times6\times7\times7\times8)^{1/8} = 6.006$

$\text{H.M.} = \dfrac{8}{\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{7}+\frac{1}{7}+\frac{1}{8}} = 5.879$.

As 6.125 > 6.006 > 5.879, so: A.M. > G.M. > H.M. Hence verified (i)

As $(\text{G.M.})^2 = 36.072$ and (A.M.) (H.M.)=36.008;

So, $(\text{G.M.})^2 \approx (\text{A.M.})(\text{H.M.})$. Hence verified (ii).

22.3 (iii): Calculate weighted mean and moving averages

(a). Weighted mean:

When some observations are more important than others in a data, then we cannot give equal weight (relative importance) to all for computing the mean. The mean which is computing by using relative importance / weights of the observations is called **weighted mean**.

If $x_1, x_2, ..., x_n$ are numbers associated with observations in a data and $w_1, w_2, ..., w_n$ are corresponding weights, then weighted A.M., G.M. and H.M. are:

$$\bar{x}_w = \frac{\sum_{i=1}^{n} w_i x_i}{\sum w_i} = \frac{\sum wx}{\sum w}, \qquad G_w = \text{antilog}\left[\frac{\sum w \log x}{\sum w}\right] \quad \text{and} \quad H_w = \frac{\sum w}{\sum\left(\frac{w}{x}\right)}$$

Example 1:

The marks of a student in a social sciences diploma course are given for each subject taught with relative weights. Find weighed mean marks:

Subject	English	French	History	Science	Mathematics
Marks	73	82	60	62	57
Weights	4	3	2	1	1

Solution:

The formulas are: $\bar{x}_w = \dfrac{\sum xw}{\sum w}$, $G_w = \text{antilog}\left[\dfrac{1}{\sum w}\left(\sum w \log x\right)\right]$ and $H_w = \dfrac{\sum w}{\sum\left(\frac{w}{x}\right)}$.

We compute the required sums in following table:

x	w	$x.w$	$\frac{w}{x}$	log x	w. log x
73	4	292	0.0548	1.8633	7.4532
82	3	246	0.0366	1.9138	5.7414
60	2	120	0.0333	1.7782	3.5564
62	1	62	0.0161	1.7924	1.7924
57	1	57	0.0175	1.7559	1.7559
Sum	11	777	0.1538	---	20.2993

$$\bar{x}_w = \frac{777}{11} = 70.6 \text{ marks.}$$

$$G_w = \text{anti}\log\left[\frac{1}{11}\times 20.2993\right]$$

$$G_w = \text{antilog}(1.8454) = 70.04 \text{ marks.}$$

$$H_w = \frac{11}{0.1583} = 69.5 \text{ marks.}$$

Note that:

In Example 1, the unweighted means are: $\bar{x} = 66.8$, G.M.=66.17 and H.M.=65.58 which are lower than the weighted means. The weighted mean marks best describe the average performance of student by giving more share to important courses.

(b). Moving Averages:

When we want to examine average behavior of a variable of interest over fixed intervals of time (in days/months/years/etc) then it is important to track the change in average as time proceeds for specific periods. In this case, the average of specific periods/ intervals of time keep changing/ moving as time increases, and are referred as **moving averages**.

For the case of an odd-period moving averages, we place the moving averages at the middle cells of specific period only and rest of the cells are left blank. The placement of moving averages should be done at the already existing cells.

For the case of an even-period moving averages, we place the average of two middle cell. This way placement can be done at already existing cells.

The process is illustrated using following examples.

Example 1:

The data summarizes monthly price of an item in Rs./month for the last year.

Month	Jan.	Feb.	Mar.	Apr.	May.	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
Price (Rs../month)	180	87	91	102	108	139	150	200	220	307	289	330

Compute (i). 3-months, (ii). 5-months and (iii). 4-months moving averages.

Solution:

(i). First we compute 3-months moving averages by considering 3 consecutive months starting from: Jan.-Mar., Feb.-Apr., Mar.-May., and so on until the last 3-months period: Oct.-Dec. These are placed at middle month of the 3-months period. For example, the moving average of Jan.-Mar. is placed beside Feb.

The 3-months moving averages are summarized in the adjacent table.

Month	Price	3- months moving averages
Jan.	180	------------
Feb.	87	(180+87+91)/3=119.33
Mar.	91	93.33
Apr.	102	100.33
May.	108	166.33
Jun.	139	132.33
Jul.	150	163
Aug.	200	190
Sep.	220	242.33
Oct.	307	272
Nov.	289	308.66
Dec.	330	------------

(ii). For 5- months moving averages, we consider 5 consecutive months each time starting from Jan.-May, Feb.-June, and so on up to Aug.-Dec.

Moving averages are shown in the adjacent table.

We can see that there are total 8 such possibilities. The first and last two cells are left blank.

The 5-months moving averages are positioned at the middle months.

Month	Price	5- months moving averages
Jan.	180	------------
Feb.	87	------------
Mar.	91	(180+87+91+102+108)/5=113.6
Apr.	102	105.4
May.	108	118
Jun.	139	139.8
Jul.	150	163.4
Aug.	200	203.2
Sep.	220	233.2
Oct.	307	269.2
Nov.	289	------------
Dec.	330	------------

(iii). For 4- months moving averages, we consider groups of 4 consecutive months each time starting from Jan.-Apr., Feb.-May, … , Sep.-Dec.

Initially the averages are placed mid-way between the 4 months, then each pair of two 4-months averages are averaged further to get the required 4-months centered moving averages.

The results are summarized in the following table. We have to add two columns for even-period group averages always.

Months	Prices	4- months moving averages(Initial)	4- month moving averages(centered)
Jan.	180		------------
Feb.	87		------------
		115	
Mar.	91		106
		97	
Apr.	102		103.5
		110	
May.	108		117.375
		124.75	
Jun.	139		137
		149.25	
Jul.	150		163.25
		177.25	
Aug.	200		198.25
		219.25	
Sep.	220		236.625
		254	
Oct.	307		270.25
		286.5	
Nov.	289		------------
Dec.	330		------------

Note that:

The average price (arithmetic mean) of the 12 months for the year equals Rs. 183.58 per month. We can see that the prices were not constant, and the 3, 5, 4- months moving averages better summarize the variations in respective periods over the year.

22.3 (iv): Estimate medians, quartiles and mode, graphically.

Graphically, to locate and estimate median and quartiles, we use ogive (cumulative frequency polygon) of the data. The median, lower quartile (Q_1) and upper quartile (Q_3) in an ogive are simply the x-coordinates of the points on ogive whose y-coordinates (cumulative frequencies) are $\frac{n}{2}$, $\frac{n}{4}$ and $\frac{3n}{4}$, respectively.

To locate and estimate mode (if it exists) graphically, we use the histogram of data. The modal class is one for which the rectangle has highest height, and in this class, the mode is the x-coordinate of the point of intersection of two line segments. First segment is drawn by joining left peaks of the rectangles corresponding to the modal class and class after it. The second line segment is drawn by joining right peaks of the rectangles corresponding to the modal class and class before it. The perpendicular from point of intersection to the x-axis hits it at the location of mode. This procedure is used for a grouped data with continuous classes.

Example 1:

Locate and estimate median, quartiles and modal weights (in Kg) graphically.

Weights (in Kg)	70-74	75-79	80-84	85-89	90-94
Number of students	10	24	46	12	8

Solution:

We have n=100 and, and given data are grouped with continuous classes. First, computing the class boundaries and cumulative frequencies in the adjacent table to be able to draw ogive and histogram.

Class limits	Class boundaries	f	Cumulative frequencies
70-74	69.5-74.5	10	10
75-79	74.5-79.5	24	34
80-84	79.5-84.5	46	80
85-89	84.5-89.5	12	92
90-94	89.5-94.5	8	100

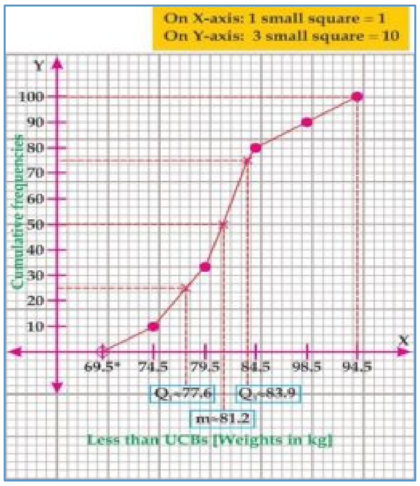
For median and quartiles, we identify $\frac{n}{2} = 50$, $\frac{n}{4} = 25$ and $\frac{3n}{4} = 75$ on y-axis.

Then, drawing perpendiculars through these to ogive, and then from ogive to x-axis lead to median, Q_1 and Q_3, respectively. We read values as:

Median ≈ 81.2,

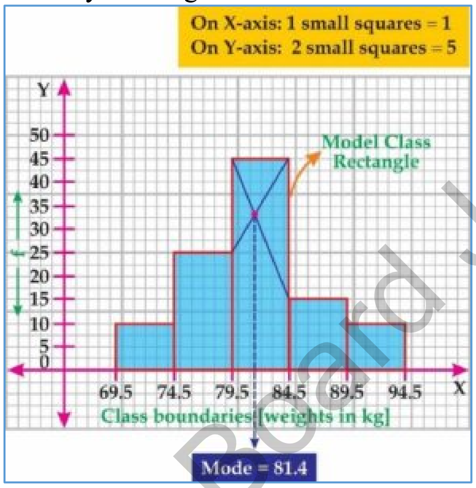
$Q_1 \approx 77.6$,

$Q_3 \approx 83.9$.



For mode, we first draw the histogram, as shown here. The modal class corresponds to the highest height rectangle as highlighted in the figure.

Joining left peaks of the rectangles of modal class and class after it, and then right peaks of the rectangles of the modal class and class before it to get two line segments. Now, the perpendicular from point of intersection to the x-axis hits at nearly 81.4 where mode is located. The modal weight is approximately 81.4 Kg.



EXERCISE 22.4

1. If W = {148, 145, 160, 157, 156, 160, 160, 165}, X = {-2, -2, -2, -2}, then:
   a. Show that sum of all deviations of W about its A.M. is zero.
   b. Show that A.M. = G.M. = H.M. = Median = Mode for X.
   c. Compute A.M. of Y = 3X.
   d. Compute A.M. of Z = 3W-11
   e. Show that H.M. < G.M. < A.M. < Median < Mode for W.
2. Check if X = {7,9,3,3,3,4,1,3,2,2} and Y = {2,1,4,4,4,6,6,5,7,1} are symmetric?
3. The required weights (Kg) and price (Rs./Kg) of monthly items required by a family are given. Find weighted mean prices using A.M., G.M. and H.M.

Item	A	B	C	D	E
Price (Rs./Kg)	300	200	600	250	650
Required weight (Kg)	25	5	4	8	3

4. Of 50 bricks bought, 21 bricks have mean mass of 24.2Kg, 29 bricks have mean mass of 23.6Kg, Find weighted mean mass of the 50 bricks.
5. Calculate 2, 3 and 4-days moving average from the following data of number of deaths monitored for a peak week in a province due to nCOVID'19.

Days	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.	Sun.
Number of deaths	102	130	158	188	196	259	310

6. Calculate 4 and 5 years moving averages of sales (in million Rs.) over 11 years.

Year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
Sales	102	130	158	188	196	259	310	188	196	259	310

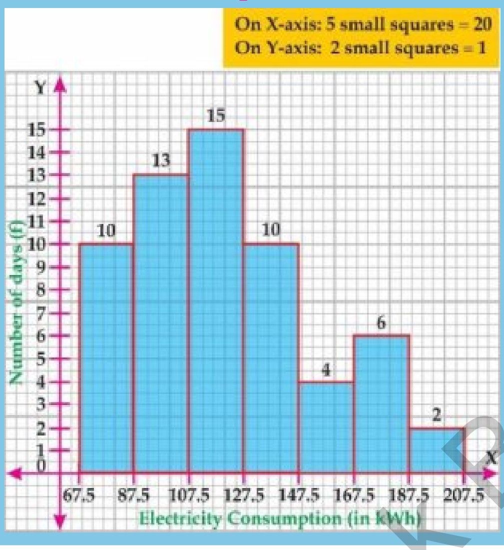
7. Locate and estimate median, quartiles and mode graphically in the data.

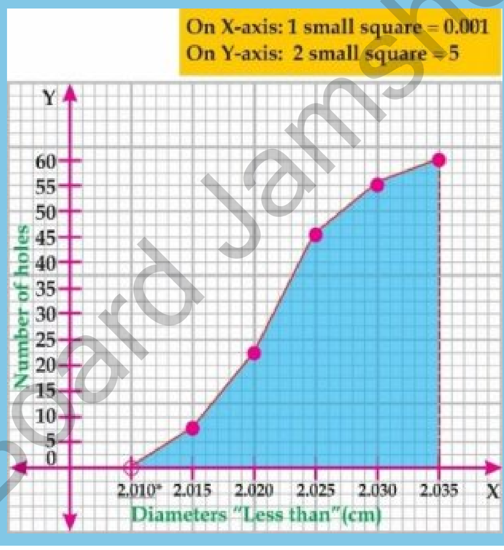
Overtime (hrs/week)	25-29	30-34	35-39	40-44	45-49	50-54	55-59
Number of days	5	4	7	11	12	8	1

8. Locate and estimate the indicated measures in the following.

a. Mode

b. Median and quartiles





22.4. Measures of Dispersion

The measures of central tendency only focus on the average or central behavior of data without focusing on the variations and consistency of data. When observations are much away from the average, then the measures of central tendency alone are not sufficient to have a thorough understanding of the behavior and properties of a data. **For example:**

The table below shows the marks obtained by two students in five subjects.

Subjects	I	II	III	IV	V
Marks of student-A	90	30	85	50	25
Marks of student-B	63	50	60	55	52

If we find average marks obtained by each student using A.M., then we observe that the A.M. marks of both students are 56, but is it sufficient to compare the performance of both students? The answer is No. Student-A marks have higher fluctuations about the mean, whereas Student-B marks are closer to the mean marks.

The performance of Student-B is more stable/ consistent/ reliable than of Student-A due to lesser fluctuations/ variations/ scatter/ dispersions about the mean.

We can also observe average marks of students using all types we learnt below:

Average	A.M.	Median	Mode	G.M.	H.M.
Marks of student A	56	50	None	49 (Approximately)	43 (Approximately)
Marks of student B	56	55	None	56 (Approximately)	56 (Approximately)

Due to variation in marks, the averages for Student-A are also much different than each other. The averages of the marks of Student-B are closer to each other.

Due to stable and consistent performance, we are in a better position to predict the marks of Student-B in a forthcoming subject as compared to Student-A.

The knowledge of variation in data matters a lot in decision making and forecasting. Therefore, it is important to compute the degree of variation (or dispersion) in a data besides computing the average to have a better understanding of data. The **measures of dispersion** are indicators of the extent to which the data are spread-out/scattered about the central point or average. A **measure of dispersion** indicates the average variation in a data. It is mostly used to compare two or more data sets from the view-point of stability/ consistency / reliability. Higher is the variation/ dispersion in a data, lower is the stability/ consistency/ reliability.

When we are interested in variation/dispersion in only one data set, we compute the measures of dispersion in the units of data, and are thus called **absolute measures of dispersion**. The variation of two or more data sets with same nature/units, number of observations and equal/closer averages can also be compared using absolute measures of dispersion.

For two or more data sets with different nature/units, number of observations or different averages, we use **relative measures of dispersion**, which are not in the units of data but show the relative behavior of the variation.

22.4(i) Define, identify and measure range, calculate variance, mean deviation and standard deviation:

We discuss some important measures of dispersion here.

(a) Range: **Range** of a data refers to the difference between the extreme observations.

For ungrouped data, range (R) is the difference between the largest observation (x_L) and smallest observation (x_S), i.e. $R = x_L - x_S$ (1)

For grouped data, R is the difference between UCB of the final group (UCB_F) and the LCB of the initial group (LCB_I), i.e. $R = UCB_F - LCB_I$ (2)

Formulas (1)-(2) calculate **absolute range**. For **relative range** we divide by $x_L + x_S$ and $UCB_F + LCB_I$ in equations (1) and (2), respectively for comparison.

Note that:

- Range does not properly measure variation of data with outliers.
- Range uses extremes in data only without using frequencies and other parameters.

Example 1: Find range of the data sets: W = {27.90, 34.70, 54.40, 18.92, 47.60, 39.68}, X = {3, 8, 10, 7, 5, 14, 2, 12, 8}, Y = {8, 8, 8, 8, 8} and Z = {1, 3, 2, 4, 3, 2, 43}. Also interpret the results.

Solution: We know that for an ungrouped data: $R = x_L - x_S$ .So,

$$R_W = 54.40 - 18.92 = 35.48, R_X = 14 - 2 = 12, R_Y = 8 - 8 = 0 \text{ and } R_Z = 43 - 1 = 42.$$

We can interpret range for dataset, say W, as: "All observations in dataset W lie within a distance of 35.48 between the extreme values 18.92 and 54.40.

The observations in data W and X are well-spread around the extreme values, so range meaningfully represent the variation. The range of set Y is zero as all observations were same and there is no variation. The majority of observations in set Z lie within 1 and 4, but due to an outlier 43, the range equal to 42 is misleading.

Example 2:

Compare the variation of marks of two students in five subjects as given below using range, and interpret the results. Which student performs more consistently?

Marks of student A	90	30	85	50	25
Marks of student B	63	50	60	55	52

Solution:

Data for both students are ungrouped. The absolute range in marks are: $R_A = 90 - 25 = 65$ marks and $R_B = 63 - 50 = 13$ marks. Here, $x_L + x_S$ is same for both students, so relative range is not useful here. The performance of Student-A show higher variation as compared to Student-B on the basis of absolute range. So, Student-B performs more consistently than Student-A.

Example 3:

Using range compare variation of price and life of five similar rating batteries manufactured by two different companies. Interpret the results.

Price (in thousand Rs.)	8	13	18	23	30
Life (in years)	1.3	1.5	1.8	2.5	3.5

Solution:

The data of price (P) and life (L) of batteries are ungrouped, and different in nature and units. Also, $x_L + x_S$ are different for both datasets. So we use the relative range to compare variation.

Relative $R_P = \frac{P_L - P_S}{P_L + P_S} = \frac{30-8}{30+8} = \frac{22}{38} = 0.5789$. Relative $R_L = \frac{L_L - L_S}{L_L + L_S} = \frac{3.5 - 1.3}{3.5+1.3} = \frac{2.2}{4.8} = 0.4583$. The relative variation in price of batteries manufactured by five different companies is higher in comparison to the relative variation in lives of batteries.

Example 4:

Compare the range of number of mobiles possessed by a family for Karachi and Moro using the grouped data of some randomly selected families in the cities. Is it sufficient to compare the variation on basis of range?

Number of mobiles	0	1	2	3	4	5	6	7
Number of families (Karachi)	1	7	15	12	5	2	2	1
Number of families (Moro)	4	13	8	5	2	1	0	1

Solution:

The data are grouped into discrete classes with spacing between the classes d=1. So, the range is: $R = UCB_F - LCB_I = (7+0.5)-(0-0.5) = 7.5 + 0.5 = 8$ mobiles. The range of number of mobiles possessed by a family in Karachi and Moro is same, i.e. 8 mobiles. But, range does not consider number of families in the cities (frequencies). The variation between the number of mobiles possessed by an average family in Karachi and Moro are surely different, but this cannot be reflected by the range.

Example 5:

Find the range of electricity consumption (in kWh) of a shop using the data:

Electricity consumption	68-87	88-107	108-127	128-147	148-167	168-187	188-207
Number of days	10	13	15	10	4	6	2

Solution:

The data are grouped into continuous classes with spacing between the classes $d = 1$. So,

$R = UCB_F - LCB_I = \left[207 + \frac{1}{2}\right] - \left[68 - \frac{1}{2}\right] = 207.5 - 67.5 = 140\,\text{kWh}$. Here, R does not use the frequencies of classes, so can be misleading.

(b) Variance, mean deviation and standard deviation

To better calculate the amount of dispersion in data, we use all observations instead of only the extreme ones along with their frequencies in the concepts of **variance, mean deviation and standard deviation**. The deviation of an observation x_i about the mean (usually A.M.) is defined as:

Deviation about mean = $D_i = x_i - \bar{x}$ (1)

Deviations can be negative, positive or zero. But, to calculate total of all deviations, simply summing them will not be useful as the result will always be zero.

To overcome this, we must avoid negative deviations, and this is done in two ways. We can use the squared deviations or the absolute deviations, defined as:

Squared deviation about mean = $D_i^2 = \left(x_i - \bar{x}\right)^2$ (2)

Absolute deviation about mean = $|D_i| = |x_i - \bar{x}|$ (3)

The squared deviations are always are non-negative. The absolute value of a real number is always non-negative, for example: $|3| = 3, |-22| = 22, |0| = 0, |-1.5| = 1.5$. These two ways lead us to define more appropriate measures of dispersions.

The **variance** (Var or s^2) is the mean of squared deviations of observations about their A.M. The **mean (or absolute) deviation** ($M.D.$) is the mean of absolute deviations of observations about their A.M. Because the dispersion is usually meaningful in original units of data, and the variance finds it in squared units, so we find its square root to define the standard deviation. The **standard deviation** ($S.D.$ or s) is the positive square root of variance.

For ungrouped data with observations: $x_1, x_2, x_3, \ldots, x_n$ with their A.M. $= \bar{x}$:

$$Var = s^2 = \frac{\sum (x - \bar{x})^2}{n} \quad (4) \qquad M.D. = \frac{\sum |x - \bar{x}|}{n} \quad (5)$$

$$S.D. = s = \sqrt{s^2} = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \quad (6) \qquad \text{where } \bar{x} = \frac{\sum x}{n}$$

For grouped data, if x_i are class marks and f_i are class frequencies, then:

$$Var = s^2 = \frac{\sum (x - \bar{x})^2 f}{\sum f} \quad (7) \qquad M.D. = \frac{\sum |x - \bar{x}| f}{\sum f} \quad (8)$$

$$S.D. = s = \sqrt{s^2} = \sqrt{\frac{\sum (x - \bar{x})^2 f}{\sum f}} \quad (9) \qquad \text{where } \bar{x} = \frac{\sum xf}{\sum f}$$

Formulae (4)-(9) calculate **absolute measures,** the **relative measures** are computed by dividing $\bar{x}$ in the units of data for comparison.

Example 1:

Find variance, mean deviation and standard deviation of the marks of students A and B. Which student performs more consistently?

Marks of student A	90	30	85	50	25
Marks of student B	63	50	60	55	52

Solution:

Marks of both students show ungrouped data. First, we compute mean marks.

For student-A (x) and student-B (y), we have:

$$\bar{x} = \frac{\sum x}{n} = \frac{280}{5} = 56 \quad \text{and} \quad \bar{y} = \frac{\sum y}{n} = \frac{280}{5} = 56$$

As means are same, so we use the absolute variation formulas variance, mean deviation and standard deviation. Computations are shown in the following tables.

x	$x-\bar{x}$	$(x-\bar{x})^2$	$\lvert x-\bar{x}\rvert$
90	34	1156	34
30	-26	676	26
85	29	841	29
50	-6	36	6
25	-31	961	31
Sum=280	0	3670	126

$$Var_A = \frac{\sum (x-\bar{x})^2}{n} = \frac{3670}{5} = 734.$$

$$(M.D.)_A = \frac{\sum \lvert x-\bar{x}\rvert}{n} = \frac{126}{5} = 25.2.$$

$$(S.D.)_A = \sqrt{734} = 27.0924.$$

y	$y-\bar{y}$	$(y-\bar{y})^2$	$\lvert y-\bar{y}\rvert$
63	7	49	7
50	-6	36	6
60	4	16	4
55	-1	1	1
52	-4	16	4
Sum=280	0	118	22

$$Var_B = \frac{\sum (y-\bar{y})^2}{n} = \frac{118}{5} = 23.6.$$

$$(M.D.)_B = \frac{\sum \lvert y-\bar{y}\rvert}{n} = \frac{22}{5} = 4.4.$$

$$(S.D.)_B = \sqrt{23.6} = 4.8578.$$

We observe that the variation in marks of student A is much higher than that in the marks of student B by all formulae, so performance of student B is more consistent.

Example 2: Compare the variation using standard deviation of price and life of five similar rating batteries manufactured by different companies. Also interpret the results.

Price (in thousand Rs.)	8	13	18	23	30
Life (in years)	1.3	1.5	1.8	2.5	3.5

Solution: Data of price (P) and life (L) are ungrouped and different in nature and units. First, we compute means.

$$\bar{x}_P = \frac{\sum x_P}{n} = \frac{92}{5} = 18.4(1000\text{ Rs}) \text{ and } \bar{x}_L = \frac{\sum x_L}{n} = \frac{10.6}{5} = 2.12 \text{ years.}$$

As means are different, we use the relative standard deviation formula for comparison. Computations are shown in the following tables for P and L.

Price (P)	$x_P - \bar{x}_P$	$(x_P - \bar{x}_P)^2$
8	-10.4	108.16
13	-5.4	29.16
18	-0.4	0.16
23	4.6`	21.16
30	11.6	134.56
Sum=92	0	293.2

$$s_P = \sqrt{\frac{\sum (x_P - \bar{x}_P)^2}{n}} = \sqrt{\frac{293.2}{5}} = 7.658(1000\text{ Rs})$$

$$\text{Relative } s_P = \frac{s_P}{\bar{x}_P} = \frac{7.658}{18.4} = 0.416.$$

Life (L)	$x_L - \bar{x}_L$	$(x_L - \bar{x}_L)^2$
1.3	-0.82	1.69
1.5	-0.62	2.25
1.8	-0.32	3.24
2.5	0.38	6.25
3.5	1.38	12.25
Sum=10.6	0	25.68

$$s_L = \sqrt{\frac{\sum (x_L - \bar{x}_L)^2}{n}} = \sqrt{\frac{25.68}{5}} = 2.266 \text{ years}$$

$$\text{Relative } s_L = \frac{s_L}{\bar{x}_L} = \frac{2.266}{2.12} = 1.069.$$

The relative standard deviation in life of batteries is higher than price. This means that data of price of batteries is more consistent than the lives of batteries. We note that if relative formula was not used then conclusion would be different and wrong.

Example 3: Find variance, mean deviation and standard deviation of amount of cold drink.

Amount of cold drink (in liters)	1.48	1.49	1.50	1.51	1.52
Number of bottles	2	7	5	5	1

Solution: Data are grouped with discrete classes. Calculations are given in table below.

Amounts (x)	Number of bottles (f)	xf	$(x-\bar{x})$	$(x-\bar{x})^2$	$\lvert x-\bar{x}\rvert$	$(x-\bar{x})^2 f$	$\lvert x-\bar{x}\rvert f$
1.48	2	2.96	-0.018	0.0003240	0.018	0.000648	0.036
1.49	7	10.43	-0.008	0.0000640	0.008	0.000448	0.056
1.5	5	7.5	0.002	0.0000040	0.002	0.00002	0.01
1.51	5	7.55	0.012	0.0001440	0.012	0.00072	0.06
1.52	1	1.52	0.022	0.0004840	0.022	0.000484	0.022
Sum	20	29.96	--	--	--	0.00232	0.184

$$\bar{x}=\frac{\sum xf}{\sum f}=\frac{29.96}{20}=1.498\text{ liters.}\qquad Var=s^2=\frac{\sum(x-\bar{x})f}{\sum f}=\frac{0.00232}{20}=0.000116\,(\text{liters})^2.$$

$$S.D.=s=\sqrt{0.000116}=0.01078\text{ liters.}\qquad \text{M.D.}=\frac{\sum|x-\bar{x}|f}{\sum f}=\frac{0.184}{20}=0.0092\text{ liters.}$$

The variation in amounts of cold drink is much smaller, or negligible.

Example 4: Find mean deviation and standard deviation of electricity consumption data.

Electricity consumption (kWh)	68-87	88-107	108-127	128-147	148-167	168-187	188-207
Number of days	10	13	15	10	4	6	2

Solution: The calculations are summarized in the table below, where x show class marks.

Class limits	Class marks (x)	f	xf	$(x-\bar{x})$	$(x-\bar{x})^2$	$\lvert x-\bar{x}\rvert$	$(x-\bar{x})^2 f$	$\lvert x-\bar{x}\rvert f$
68-87	77.5	10	775	-44.167	1950.724	44.167	19507.24	441.67
88-107	97.5	13	1267.5	-24.167	584.0439	24.167	7592.571	314.171
108-127	117.5	15	1762.5	-4.167	17.36389	4.167	260.4583	62.505
128-147	137.5	10	1375	15.833	250.6839	15.833	2506.839	158.33
148-167	157.5	4	630	35.833	1284.004	35.833	5136.016	143.332
168-187	177.5	6	1065	55.833	3117.324	55.833	18703.94	334.998
188-207	197.5	2	395	75.833	5750.644	75.833	11501.29	151.666
Sum	--	60	7270	--	--	--	65208.35	1606.672

First we need mean consumption, which is: $\bar{x}=\frac{\sum xf}{\sum f}=\frac{7270}{60}=121.667\text{kWh.}$

Now, using the required sums, the mean and standard deviations are:

$$\text{M.D.}=\frac{\sum|x-\bar{x}|f}{\sum f}=\frac{1606.672}{60}=26.778\text{kWh.}\quad S.D.=s=\sqrt{\frac{\sum(x-\bar{x})^2 f}{\sum f}}=\sqrt{\frac{65208.35}{60}}=32.967\text{kWh.}$$

Example 5:

Some randomly selected families in the Karachi and Moro, when asked about number of mobiles possessed by them, resulted in an average mobiles/family of 5.773 and 6.133 with standard deviations 1.412 and 3.408, respectively. Compare which city shows more consistent use of mobiles per family?

Solution:

The average mobiles/family for Karachi (K) and Moro (M) are different as: $\bar{x}_K=5.773\text{mibiles/family}$ and $\bar{x}_M=6.133\text{mibiles/family}$.

The standard deviations are given as:

s_K=1.412 mobiles/family and $s_M=3.408$ mobiles/family.

As mean mobiles/family differ for both cities, we use relative measures:

$$\text{Relative}\,s_K=\frac{s_K}{\bar{x}_K}=\frac{1.412}{5.733}=0.246\text{ and Relative }s_M=\frac{s_M}{\bar{x}_M}=\frac{3.408}{6.133}=0.555.$$

By observing relative standard variations of both cities, we conclude that the relative variation for number of mobiles possessed by a family in Karachi is smaller than that for Moro. So, average use of mobiles/ family is higher and more consistent for Karachi than Moro.

EXERCISE 22.5

1. Find range, variance, mean deviation and standard deviation of number of absentees in a class for last seven days: 3, 5, 3, 2, 4, 1, 8.
2. Find range, mean deviation, variance and standard deviations of scores of two batsmen in 6 innings. Who is more consistent player?

Batsman-A	12	15	0	185	7	19
Batsman-B	47	12	76	48	4	51

3. Compare variation using range and standard deviation of income and expenditure of six families. Also interpret the results.

Income (in thousand Rs.)	10	20	30	40	50	60
Expenditure (in thousand Rs.)	7	21	23	34	36	53

4. Goals scored by teams A and B in a football season are given. Use relative standard and mean deviations to find which team performed more consistently?

Goals scored	0	1	2	3	4
Number of games A played	27	9	8	5	4
Number of games B played	17	9	6	5	3

5. Find range, variance, standard deviation and mean deviation in mass of 50 blocks of metal as distributed in the following data:

Mass of block(in Kg)	7.1-7.3	7.4-7.6	7.7-7.9	8.0-8.2	8.3-8.5	8.6-8.8	8.9-9.1
Number of blocks	3	5	9	14	11	6	2

Review Exercise 22

1. Tick the correct answer.

(i). Arithmetic mean is based on the use of _____ observations of data.
(a). Middle (b). Extreme (c). All (d). None

(ii). The ungrouped data must be ordered first to find _________ .
(a). A.M. (b). Mode (c). Median (d). Range

(iii). The number of classes in a continuous frequency table lies between 5 and ___.
(a). 10 (b). 15 (c). 20 (d). 25

(iv). Mode of grouped data is obtained graphically using ________.
(a). Histogram (b). Polygon (c). Ogive (d). Bar chart

(v). If data has outliers, then _______ is misleading.
(a). A.M. (b). Range (c). Median (d). Mode

(vi). The A.M. of {0, 90, k, 10, 100} is 40, then k = ____ .
(a). 0 (b). 90 (c). 10 (d). 100

(vii). Median and quartiles are _________ in nature.
(a). Mathematical (b). Positional (c). Logical (d). None of these

(viii). Upper quartile divides data in _________ ratio.
(a). 50%–50% (b). 25%–25% (c). 75%–25% (d). 40%–60%

(ix). If data contain a number equal to 0, then ____ cannot be computed.
(a). A.M. (b). G.M. (c). H.M. (d). Median

(x). If all numbers in data are equal, then:
(a). A.M.=G.M.=H.M. (b). Range = 0 (c). S.D. = 0 (d). All of these

SUMMARY

- Quantitative data refers to numbers, and are further categorized into discrete (only integers) and continuous data (any real numbers).
- Qualitative data refers to attributes of a variable, and are further categorized into nominal (without ordering) and ordinal (with ordering) data.
- To get meaningful information, data are grouped into frequency distribution.
- Frequency distribution lists classes and respective frequencies.
- Classes defined by a single point are discrete, whereas by a range of numbers are continuous.
- Observations are distributed into classes using tally or list entries method.
- Number of classes can be obtained using Sturges' (1926) rule.
- The range is the difference between highest and lowest observations in a data.
- The number of observations that can fall in a class is class interval or width.
- Class limits are starting and ending points of a class.
- Class boundaries are obtained by averaging any two consecutive class limits.
- The spacing between classes is the distance between any two adjacent classes.
- Relative and percentage frequencies lie from 0 to 1 and 0 to 100, respectively.
- Cumulative frequencies are the sum of all frequencies upto a particular class.
- Histogram is a graph of adjacent rectangles with bases at class boundaries and heights proportional to class frequencies.
- Frequency polygon is formed by connecting points with line segments whose x-coordinates are class marks and y-coordinates are frequencies.
- Ogive or cumulative frequency polygon is obtained by joining the points with coordinates (class boundaries, cumulative frequencies).
- A measure of central tendency is capacity of data to cluster about a central point.
- A.M., G.M. and H.M. are based on the equality among all observations.
- Median is the middle most part of a ranked dataset.
- Mode is based on the principle of majority, and is the most frequent observation.
- A.M. $\geq$ G. M. $\geq$ H.M.
- If A.M. = median, then data are symmetric, otherwise asymmetric.
- Mode can be located and estimated graphically using histogram.
- Median and quartiles can be located and estimated graphically using ogive.
- A measure of dispersion refers to the amount of variation / scatter in a data.
- The higher the variation/ dispersion in a data, lesser is the stability/ consistency.
- Range measures variation in data by difference of two extreme observations.
- The sum of all deviations about mean in an ungrouped data is zero.
- Variance is the A.M. of squared deviations of data about A.M.
- Mean/absolute deviation is the A.M. of squared deviations of data about A.M.

PYTHAGORAS THEOREM

Unit 23

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorem along with itscorollaries and apply them to solve allied problems.
  - In a right-angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides. (Pythagoras'Theorem).
  - If the square of one side of a triangle is equal to thesum of the squares of the other two sides then the triangle is a right angled triangle, (converse to Pythagoras' Theorem).

Introduction:

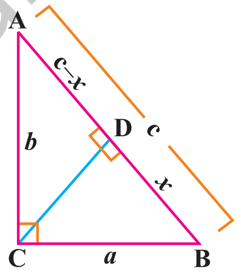
Pythagoras, a Greek philosopher and mathematician was born around 570BC. He discovered a very important relationship between the sides of right angled triangle. The Pythagorean theorem or Pythagoras theorem is a fundamental relation in Ecludian geometry among the three sides of a right angled triangle. He developed this relationship in the form of theorem called Pythagoras theorem after his name. The theorem can be proved by various methods. Here we shall prove it by using the concept of similar triangles. We shall state and prove its converse also and then apply them to solve different problem of daily life.

23.1.1 Pythagoras Theorem:

Theorem 23.1.

In a right angled triangle, the square of the length of hypotenuse is equal to sum of the squares of the length of the other two sides.

Given: ΔABC is a right angled triangle, having right angle at C. The measures of sides $\overline{AB}, \overline{AC}$ and $\overline{BC}$ are c, b and a respectively.



To prove: $c^2 = a^2 + b^2$

Construction:

Draw an altitude of vertex C to side $\overline{AB}$. say $\overline{BD} = x$

Proof:

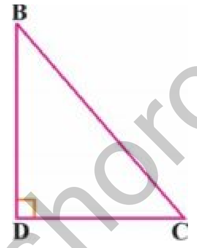
Statements	Reason
In $\Delta ABC \leftrightarrow \Delta CBD$	
$\angle ACB \cong \angle CDB$	$\angle CDB$ is right angle (Construction)
$\angle B \cong \angle B$	Common angle
and $\angle BAC \cong \angle BCD$	Complements of $\angle B$
$\therefore$ $\Delta ACB \quad \Delta CBD$	By definition of similar Δs.
Hence $\frac{c}{a} = \frac{a}{x}$	Corresponding sides of similar triangles
$cx = a^2$ ...(i)	
Again in $\Delta ACB \leftrightarrow \Delta ADC$	
$\angle ACB \cong \angle ADC$	$\angle ADC$ is right angle (Construction)
$\angle A \cong \angle A$	Common angles
and $\angle CBA \cong \angle DCA$	Complements of $\angle A$
$\therefore$ $\Delta ACB \quad \Delta ADC$	
Hence $\frac{c}{b} = \frac{b}{c-x}$	Corresponding sides of similar triangles
$c(c-x) = b^2$	
or $c^2 - cx = b^2$...(ii)	
Adding equation (i) and (ii) we get,	
$cx + c^2 - cx = a^2 + b^2$	
$c^2 = a^2 + b^2$	

Q.E.D

Corollary: In a right angled triangle *ABC*, right angle at *B*.

i. $\left(m\overline{BC}\right)^2 = \left(mAC\right)^2 - \left(mAB\right)^2$

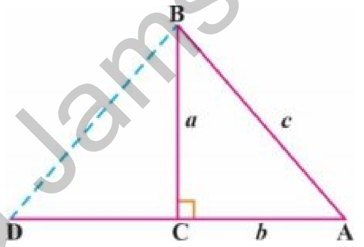
ii. $\left(m\overline{AB}\right)^2 = \left(mAC\right)^2 - \left(mBC\right)^2$



Theorem 23.2

(Converse of Pythagoras theorem)

If the square of one side of triangle is equal to the sum of the squares of the other two sides, then the triangle is a right angled triangle.



Given:

In a $\Delta ABC, m\overline{BC} = a, m\overline{AC} = b$ and $m\overline{AB} = c$

such that $a^2 + b^2 = c^2$

To prove:

ΔABC is a right angled triangle.

Construction:

Draw $\overline{CD}$ perpendicular to $\overline{BC}$ such that $\overline{CD} \cong \overline{CA}$. Join the points *B* and *D*.

Proof:

Statements	Reason
ΔDCB is a right angled triangle	Construction
$\therefore \quad \left(m\overline{BD}\right)^2 = a^2 + \left(m\overline{DC}\right)^2$	Pythagoras theorem
$\Rightarrow \quad \left(m\overline{BD}\right)^2 = a^2 + b^2$	
But $\quad a^2 + b^2 = c^2$	Given
$\therefore \quad \left(m\overline{BD}\right)^2 = c^2$	Transitive property of equality
$\therefore \quad m\overline{BD} = c$	By taking square root of both sides
Now in	
$\Delta DCB \leftrightarrow \Delta ACB$	
$\overline{CD} \cong \overline{CA}$	Construction
$\overline{BC} \cong \overline{BC}$	Common
$\overline{BD} \cong \overline{AB}$	Each side equal to C (Proved above).
$\therefore \quad \Delta DCB \cong \Delta ACB$	S.S.S ≅ S.S.S
$m\angle DCB = m\angle ACB$	Corresponding angles of congruent triangles
But $\quad m\angle DCB = 90^\circ$	Construction
$\therefore \quad m\angle ACB = 90^\circ$	Transitive property of equality.
Hence ABC is a right angled triangle.	$\therefore \quad m\angle ACB = 90^\circ$

Q.E.D

Corollary:

Let a, b and c be the sides of a triangle. Such that the side c be the longest side. Then,

i. If $a^2+b^2=c^2$ then the triangle is right angled triangle.

ii. If $a^2+b^2>c^2$ then the triangle is acute angled triangle.

iii. If $a^2+b^2<c^2$ then the triangle is obtuse angled triangle.

Example 1:

Measures of the sides of a triangle are given decide which of these represent right triangles.

i. $a=5\text{cm},\ b=12\text{cm}$ and $c=13\text{cm}$

ii. $a=6\text{cm},\ b=7\text{cm}$ and $c=8\text{cm}$

iii. $a=9\text{cm},\ b=12\text{cm}$ and $c=15\text{cm}$

Solution:

i. Since $(13)^2=(5)^2+(12)^2$

$$169=25+144$$
$$169=169$$

By the converse of Pythagoras theorem, the a,b and c are the sides of right triangle.

ii. Since $(8)^2\neq(6)^2+(7)^2$

i.e. $64\neq 36+49$

or $64\neq 85$

By the converse of Pythagoras theorem, the a,b and c are not the sides of right triangle.

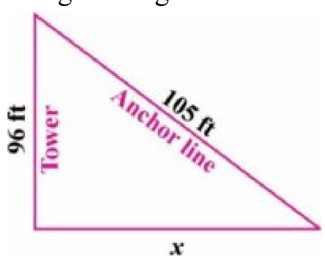
iii. Since $(15)^2=(12)^2+(9)^2$

$$225=144+81$$
$$225=225$$

By the converse of Pythagoras theorem, the a,b and c are the sides of right triangle.

Example 2:

An anchor line for a tower needs to be placed. The tower is 96fts. The anchor line is 105fts long. How far from the tower can it be placed?



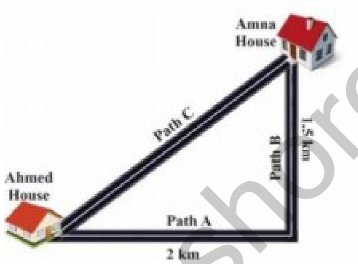
Solution:

Let the anchor line be placed at x ft from the tower. By Pythagoras theorem.

$$(105)^2=(96)^2+(x)^2$$
$$\Rightarrow \quad 11025=9216+x^2$$
$$\Rightarrow \quad x^2=1809$$
$$x=\sqrt{1809}=42.53\text{ft}$$

Hence the required distance is 42.53 ft approx.

Example 3:



There are two streets that one can choose to go from Amna's house to Ahmed's house. One way is to take C path, and the other way requires to take A path of 2 km and then B path of 1.5 km. How much shorter is the direct path along C path as shown in the figure.

Solution:

Let x be the length of path C.

By Pythagoras theorem.

$$x^2 = (2)^2 + (1.5)^2$$

$$\Rightarrow \quad x = \sqrt{(2)^2 + (1.5)^2} = \sqrt{6.25} = 2.5\text{km}$$

By using the alternative way, he has to cover the distance $= 2 + 1.5 = 3.5\text{km}$

The difference between these two paths $= 3.5 - 2.5 = 1\text{km}$

EXERCISE 23

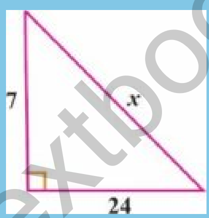
1. Following are the length of sides of triangle, verify that the triangle is right triangle.

i. $a = 16 \quad b = 30 \quad c = 34$ ii. $a = 6 \quad b = 8 \quad c = 10$

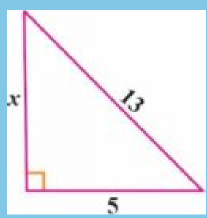
iii. $a = 15 \quad b = 20 \quad c = 25$ iv. $a = 13 \quad b = \sqrt{56} \quad c = 15$

2. Find unknown values in each of the following figures.

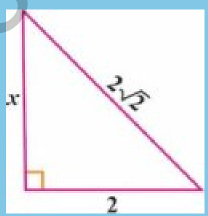
i.



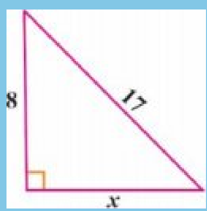
ii.



iii.



iv.

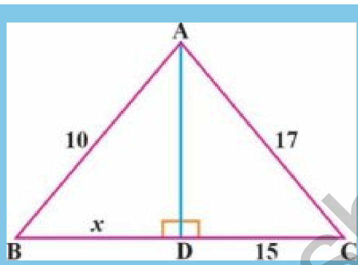


3. The three sides of a triangle are of measure 9.5cm, 7.5cm and $x$cm. For what value of x will the side represent right triangle?

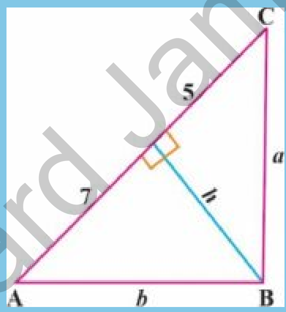
4. ABC is an isosceles triangle with $m\overline{AB} = m\overline{AC} = 13\text{cm}$ and $m\overline{BC} = 10\text{cm}$. Calculate the perpendicular distance from A to $\overline{BC}$.

5. The foot of a ladder is placed 6 feet from a wall. If the top of the ladder rests 8 feet up on the wall. How long is the ladder?

6. Find the value of x in the adjacent figure.



7. In the ΔABC as $\angle ADB$ is right angle as shown in the adjacent figure. Find the lengths a, h and b if $m\overline{CD} = 5$ units and $m\overline{AD} = 7$ units.



8. The sides of a rectangular swimming pool are 50m and 30m. What is the length between the opposite corners?
9. The length of each side of an equilateral triangle is 8 units. Find the length of any one altitude.
10. The sides of a triangle have lengths $x, x+4$ and 20. If the length of the longest side is 20. What values of x make the right triangle?
11. A manager goes 18m due east and then 24m due north. Find the distance of his current position from the starting point.
12. In the rectangle ABCD, $m\overline{BC} + m\overline{CD} = 17\text{cm}$ and $m\overline{BD} + m\overline{AC} = 26\text{cm}$ calculate the length and breadth of the rectangle.

REVIEW EXERCISE 23

1. Encircle the correct option

i. Diagonal of a rectangle measures 6.5cm. If its width is 2.5cm, its length is
   (a) 9cm (b) 4cm
   (c) 6cm (d) 3cm

ii. Which of the following are the sides of a right angled triangle?
   (a) 3, 4, 5 (b) 2, 3, 4
   (c) 5, 6, 7 (d) 4, 5, 6

iii. In a right angled triangle the greatest angle is
   (a) 100^o (b) 90^o
   (c) 80^o (d) 110^o

iv. In a right angled triangle hypotenuse is opposite side to

(a) Acute angle (b) Right angle

(c) Obtuse angle (d) None

v. If *a*, *b*, *c* are sides of right angled triangle, with *c* is the larger side, then

(a) $c^2 = a^2 + b^2$ (b) $b^2 = c^2 + a^2$

(c) $a^2 = b^2 + c^2$ (d) $c^2 = a^2 - b^2$

vi. If 5cm and 12cm are two sides of a right angled triangle. Then hypotenuse is

(a) 16 (b) 15

(c) 14 (d) 13

vii. If hypotenuse of an isosceles right-angled triangle is $3\sqrt{2}$cm, then each of other side is of length

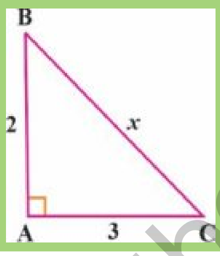
(a) 2cm (b) 5cm

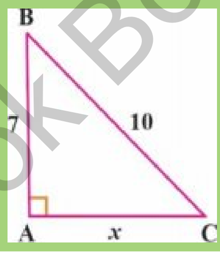
(c) 3cm (d) 1cm

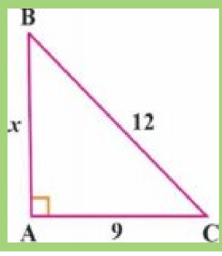
2. Explain the Pythagoras theorem.

3. A ladder 25m long rests against a vertical wall. The foot of the ladder is 7m away from the base of the wall, how high up the wall will the ladder reach.

4. Find the unknown values in each of the following figures

i)


ii)


iii)


SUMMARY

- In a right angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides.
- If the square of the one side of a triangle is equal to the sum of the square of the other two sides then the triangle is a right angled triangle.

RATIO AND PROPORTION

Unit 24

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with theircorollaries and apply them to solve allied problems.
    - A line parallel to one side of a triangle, intersectingthe other two sides, divides them proportionally.
    - If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the third side.
    - The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the sides containing the angle.
    - If two triangles are similar, the measures of their corresponding sides are proportional.

24.1 Ratio and proportion

In this chapter, we will study the theorems related to the ratio and proportion of sides of a triangle along with the similarity of triangles. So, first of all we have to recall the concepts of ratio, proportion and similarity.

Ratio:

Ratio is the comparison of two quantities of same kind with same units. The ratio of a and b is written as $a:b$ or $\frac{a}{b}$ where a is called antecedent and b is called consequent.

For example the ratio of 25 litres and 5 litres is $25:5$ or 5:1.

Proportion:

Equality of two ratios is called proportion.

If two ratios $a:b$ and $c:d$ are equal then we write it as $a:b=c:d$ or $a:b::c:d$ and call it as proportion.

In the proportion $a:b=c:d$, a and d are called extremes, whereas b and c are called means.

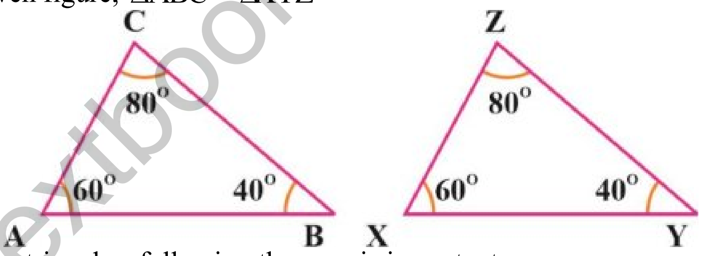
In a proportion, the product of means is always equal to the product of extremes.

Similar Triangles:

Two triangles ABC and PQR are called similar triangles if their corresponding angles are congruent.

Symbolically, we write it as $\Delta ABC \sim \Delta PQR$

In the given figure, $\Delta ABC \sim \Delta XYZ$

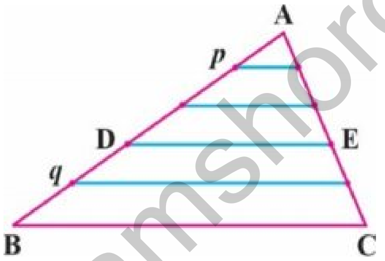


For similar triangles, following theorem is important

"If two triangles are similar then their corresponding sides are proportional". We will prove this theorem in last.

Theorem 24.1

A line parallel to one side of a triangle and intersecting the other two sides, divides them proportionally.



Given: In $\Delta ABC, \overline{DE} \parallel \overline{BC}$ and $\overline{DE}$ intersects the other sides $\overline{AB}$ and $\overline{AC}$ at points D and E respectively.

To prove: $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EC}}$

Construction:

Select such a unit of length so that $m\overline{AD} = p$ units, and $m\overline{BD} = q$ units where p and q are natural numbers.

Divide $\overline{AD}$ into p congruent segments and $\overline{BD}$ into q congruent segments.

So, $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{p}{q}$ .From the points of division, draw lines parallel to $\overline{BC}$.

Proof:

Statements	Reasons
$\overline{AD}$ is divided into p congruent segments by the parallel lines.	Construction
$\overline{AE}$ is also divided into p congruent segments by the same parallel lines.	Parallel lines make equal numbers of congruent intercepts on each transversal.
Similarly, $\overline{EC}$ is also divided into q congruent segments.	$\because$ $\overline{BD}$ is divided into q congruent segments by the parallel lines.
Now, $\dfrac{m\overline{AE}}{m\overline{EC}} = \dfrac{p}{q}$	$\because$ $\overline{AE}$ and $\overline{EC}$ are divided into p and q congruent segments respectively (Proved above)
But $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{p}{q}$	By construction
So $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EC}}$	By transitive property of equality

Q.E.D

Corollary 1:

From the figure of above theorem, it can be proved that $\dfrac{m\overline{AB}}{m\overline{DB}} = \dfrac{m\overline{AC}}{m\overline{EC}}$.

Corollary 2:

From the figure of the above theorem, it can be proved that $\frac{m\overline{AD}}{m\overline{AB}} = \frac{m\overline{AE}}{m\overline{AC}}$.

Corollary 3:

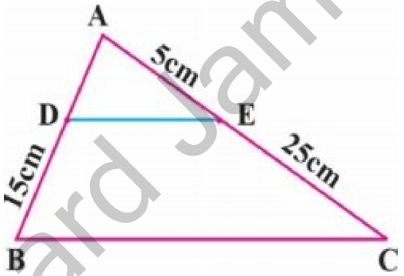
If a line is parallel to the base of an isosceles triangle and intersects other two congruent sides then the corresponding intercepts on the sides will be congruent.

Example:

In the adjacent figure, $\overline{DE} \parallel \overline{BC}$ in ΔABC.

Find $m\overline{AD}$ if $m\overline{AE} = 5cm$,

$m\overline{BD} = 15cm$ and $m\overline{CE} = 25cm$.



Solution:

$\because \quad \overline{DE} \parallel \overline{BC}$

$\therefore \quad \frac{m\overline{AD}}{m\overline{BD}} = \frac{m\overline{AE}}{m\overline{EC}}$

i.e. $\quad \frac{m\overline{AD}}{15} = \frac{5}{25}$

$\Rightarrow \quad m\overline{AD} = \frac{15\times 5}{25}$

$\Rightarrow \quad m\overline{AD} = 3cm$

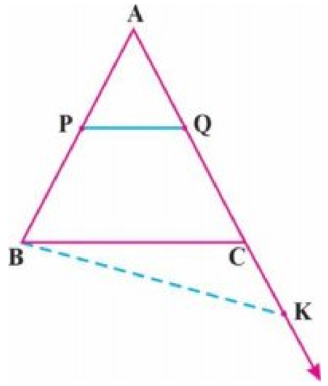
Theorem 24.2

If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the third side.

Given:

In ΔABC, $\overline{PQ}$ cuts $\overline{AB}$ and $\overline{AC}$ at points P and Q respectively. such that

$$\frac{m\overline{AP}}{m\overline{PB}} = \frac{m\overline{AQ}}{m\overline{QC}}$$



To prove:

$\overline{PQ} \parallel \overline{BC}$

Construction:

If $\overline{PQ}$ is not parallel to $\overline{BC}$, draw $\overline{BK}$ meeting $\overrightarrow{AC}$ at point K other than C such that $\overline{PQ} \parallel \overline{BK}$

Proof:

Statements	Reasons
In ΔABK	
$\overline{PQ} \parallel \overline{BK}$	Construction
$\dfrac{m\overline{AP}}{m\overline{PB}} = \dfrac{m\overline{AQ}}{m\overline{QK}}$	Line parallel to one side of triangle divides other sides proportionally
But	
$\dfrac{m\overline{AP}}{m\overline{PB}} = \dfrac{m\overline{AQ}}{m\overline{QC}}$	Given
So	
$\dfrac{m\overline{AQ}}{m\overline{QK}} = \dfrac{m\overline{AQ}}{m\overline{QC}}$	Transitive property of equality
$\Rightarrow \quad m\overline{QK} = m\overline{QC}$	If antecedents are equal then consequents are also equal in equal ratios.
i.e. $\quad \overline{QK} \cong \overline{QC}$	By definition of congruent segments.
This is possible only when K coincides with C.	Q is common point in both
Hence $\overline{PQ} \parallel \overline{BC}$	Our assumption is wrong

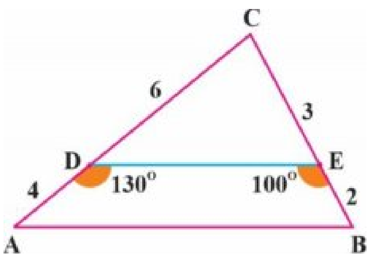
Q.E.D

Corollary:

The line segment joining the mid points of two sides of a triangle is parallel to the third side.

Example:

In adjacent figure, $\overline{DE}$ cuts two sides $\overline{AC}$ and $\overline{BC}$ of ΔABC at points D and E respectively. Find the base angles A and B if $m\angle ADE = 130°$ and $m\angle BED = 100°$ where lengths of sides are given as shown in the figure.



Solution:

$\because \quad \angle ADE$ and $\angle CDE$ are supplementary

$\therefore \quad m\angle CDE = 50° \left(\quad m\angle ADE = 130° \right)$

Similarly

$m\angle DEC = 80°$

$\because \quad \dfrac{4}{6} = \dfrac{2}{3}$ i.e. $\overline{DE}$ cuts two sides $\overline{AC}$ and $\overline{BC}$ proportionally

$\therefore \quad \overline{DE} \parallel \overline{AB}$

So, $m\angle A = m\angle CDE$, because corresponding angles of parallel lines are equal

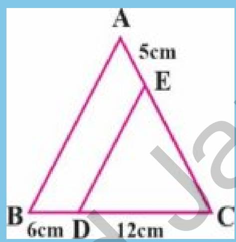
i.e. $m\angle A = 50°$

Similarly

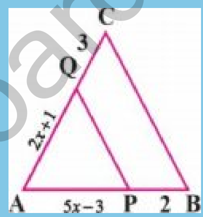
$m\angle B = m\angle DEC = 80°$

EXERCISE 24.1

1. In ΔABC, $\overline{DE} \parallel \overline{AB}$. Find $m\overline{CE}$ if $m\overline{BD} = 6cm$, $m\overline{DC} = 12cm$ and $m\overline{AE} = 5cm$.



2. In ΔABC, $\overline{PQ} \parallel \overline{BC}$. Find x if $m\overline{AP} = 5x-3$, $m\overline{PB} = 2$, $m\overline{AQ} = 2x+1$ and $m\overline{QC} = 3$



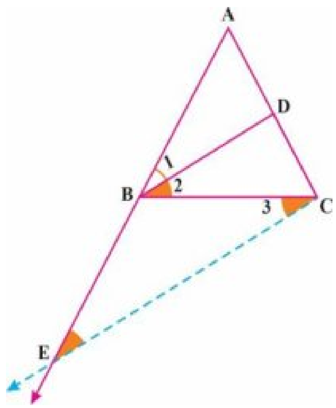
3. Prove that the line drawn parallel to the parallel sides of a trapezium divides the non-parallel sides proportionally.

4. Prove that the line segment drawn through the mid point of a side of a triangle and parallel to another side bisects the third side.

5. Prove that the line which divides the non-parallel sides of a trapezium proportionally is parallel to the third side.

Theorem 24.3

The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the sides containing the angle.



Given:

$\overline{BD}$ is the bisector of $\angle ABC$ of ΔABC.

i.e $<1 \cong <2$

To prove:

$$\frac{m\overline{AD}}{m\overline{DC}} = \frac{m\overline{AB}}{m\overline{BC}}$$

Construction:

Draw $\overrightarrow{CE}$ parallel to $\overline{BD}$ meeting $\overrightarrow{AB}$ at point E

Proof:

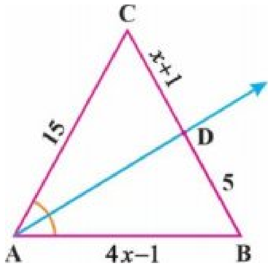
Statements	Reasons
$\because$ $\overline{EC} \parallel \overline{BD}$	Construction
$\therefore$ $m\angle E = m\angle 1$ ...(i)	Corresponding angles of parallel lines.
Also $m\angle 2 = m\angle 3$	Alternate angles of parallel lines
But $m\angle 1 = m\angle 2$	Given
So $m\angle 1 = m\angle 3$	Transitive property
$m\angle E = m\angle 3$	Using eq (i)
In ΔBCE, $\overline{BC} \cong \overline{BE}$	Sides opposite to congruent angles of a triangle are congruent
In ΔACE,	
$\because$ $\overline{BD} \parallel \overline{EC}$	Construction
$\therefore$ $\dfrac{m\overline{AD}}{m\overline{DC}} = \dfrac{m\overline{AB}}{m\overline{BE}}$	Line parallel to one side of triangle divides other sides proportionally
Or $\dfrac{m\overline{AD}}{m\overline{DC}} = \dfrac{m\overline{AB}}{m\overline{BC}}$	$\overline{BC} \cong \overline{BE}$ (Proved above)

Q.E.D

Corollary:

If the internal bisector of an angle of a triangle bisects the opposite side, the triangle is isosceles.

Example: In adjacent figure, $\overrightarrow{AD}$ is the bisector of $\angle A$ of ΔABC. Find x if $m\overline{AC} = 15$ cm, $m\overline{AB} = 4x - 1$ cm, $m\overline{CD} = x + 1$ cm and $m\overline{BD} = 5$cm. Also specify the type of triangle where $x \in N$.



Solution:

$\because$ $\overrightarrow{AD}$ is the bisector of $\angle A$

$\therefore$ $\dfrac{m\overline{CD}}{m\overline{DB}} = \dfrac{m\overline{AC}}{m\overline{AB}}$

i.e. $\dfrac{x+1}{5} = \dfrac{15}{4x-1}$

$\Rightarrow$ $4x^2 - x + 4x - 1 = 75$

$\Rightarrow$ $4x^2 + 3x - 76 = 0$

$\Rightarrow$ $4x^2 + 19x - 16x - 76 = 0$

$\Rightarrow$ $x(4x+19) - 4(4x+19) = 0$

$\Rightarrow$ $(x-4)(4x+19) = 0$

$\Rightarrow$ $x - 4 = 0$ or $4x + 19 = 0$

$\Rightarrow \quad x = 4$ | $\Rightarrow \quad x = \frac{-19}{4}$

$\because \quad \frac{-19}{4} \notin N$

$\therefore$ we neglect $\frac{-19}{4}$

Hence $x = 4$

Now $\quad m\overline{AB} = 4x - 1$

$= 4 \times 4 - 1$

$= 15$

$m\overline{AB} = m\overline{AC} = 15$ cm

and $\quad m\overline{BC} = x + 1 + 5 = 10$ cm

$\therefore \quad \Delta ABC$ is an isosceles triangle.

Theorem 24.4:

If two triangles are similar, the measures of their corresponding sides are proportional.

Given:

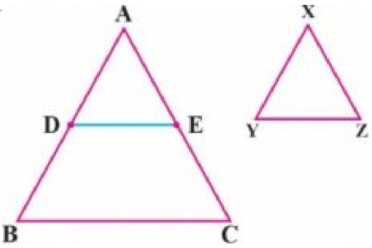
ΔABC and ΔXYZ are similar triangles.

i.e. In $\Delta ABC \leftrightarrow \Delta XYZ$

$\angle A \cong \angle X$

$\angle B \cong \angle Y$

and $\angle C \cong \angle Z$



To prove:

$$\frac{m\overline{AB}}{m\overline{XY}} = \frac{m\overline{BC}}{m\overline{YZ}} = \frac{m\overline{AC}}{m\overline{XZ}}$$

Construction:

From $\overline{AB}$, cut off $\overline{AD} \cong \overline{XY}$ and from $\overline{AC}$, cut off $\overline{AE} \cong \overline{XZ}$. Draw $\overline{DE}$.

Proof:

Statements	Reasons
In $\Delta ADE \leftrightarrow \Delta XYZ$	
$\overline{AD} \cong \overline{XY}$	Construction
$\angle A \cong \angle X$	Given
$\overline{AE} \cong \overline{XZ}$	Construction
$\Delta ADE \cong \Delta XYZ$	By S.A.S postulate
$\angle ADE \cong \angle Y$	Corresponding angles of congruent triangles
But $\angle B \cong \angle Y$	Given
So $\angle ADE \cong \angle B$	Transitive property
Hence	

Statements	Reasons
$\overline{DE} \parallel \overline{BC}$	Corresponding angles $\angle ADE$ and $\angle B$ are congruent
$\dfrac{m\overline{AB}}{m\overline{AD}} = \dfrac{m\overline{AC}}{m\overline{AE}}$	By Theorem 1 (Corollary)
Or $\dfrac{m\overline{AB}}{m\overline{XY}} = \dfrac{m\overline{AC}}{m\overline{XZ}}$ ...(i)	$m\overline{AD} = m\overline{XY}$ and $m\overline{AE} = m\overline{XZ}$
Similarly $\dfrac{m\overline{AB}}{m\overline{XY}} = \dfrac{m\overline{BC}}{m\overline{YZ}}$ ...(ii)	By the above process
So $\dfrac{m\overline{AB}}{m\overline{XY}} = \dfrac{m\overline{BC}}{m\overline{YZ}} = \dfrac{m\overline{AC}}{m\overline{XZ}}$	From (i) and (ii)

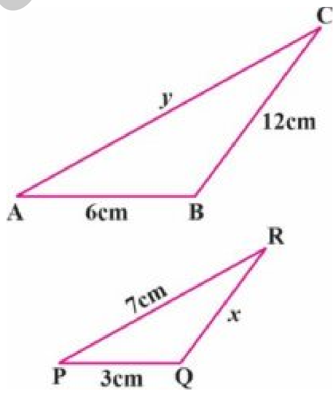
Q.E.D

Corollary:

In a correspondence of two triangles, if two angles of a triangle are congruent to the corresponding two angles of other triangle then their corresponding sides are proportional.

Example 1:

In the given figure, ΔABC and ΔPQR are similar. Find the values of x and y if $m\overline{AB} = 6\text{cm}$, $m\overline{BC} = 12\text{cm}$, $m\,\overline{PQ} = 3\text{cm}$, and $m\overline{PR} = 7\text{cm}$



Solution:

ΔABC and ΔPQR are similar

$\therefore$ Their corresponding sides are equal

i.e. $\dfrac{m\overline{AB}}{m\overline{PQ}} = \dfrac{m\overline{BC}}{m\overline{QR}} = \dfrac{m\overline{AC}}{m\overline{PR}}$

$\Rightarrow \dfrac{6}{3} = \dfrac{12}{x} = \dfrac{y}{7}$ (Using the given measures)

$\Rightarrow \dfrac{6}{3} = \dfrac{12}{x}$ and $\dfrac{6}{3} = \dfrac{y}{7}$

or $2 = \dfrac{12}{x}$ or $2 = \dfrac{y}{7}$

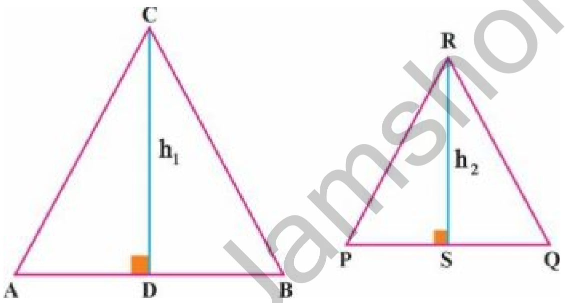
or $2x = 12$ or $14 = y$

$\Rightarrow x = 6$ cm or $y = 14$ cm

So the values of x and y are 6cm and 14cm respectively.

Example 2

In the given figure, ΔABC and ΔPQR are similar and $\frac{m\overline{AB}}{m\overline{PQ}} = \frac{x}{y}$ where h_1 and h_2 are the altitudes of the given triangles.



Prove that:

$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{x^2}{y^2}$$

Proof:

we have $\frac{m\overline{AB}}{m\overline{PQ}} = \frac{x}{y}$

$\because$ ΔABC and ΔPQR are similar

$\therefore$ $\frac{m\overline{AC}}{m\overline{PR}} = \frac{m\overline{BC}}{m\overline{QR}} = \frac{m\overline{AB}}{m\overline{PQ}} = \frac{x}{y}$ ... (i)

In $\Delta ADC \leftrightarrow \Delta PSR$

$m\angle A = m\angle P$ (Given)

$\because$ and $m\angle D = m\angle S = 90^\circ$

$\therefore$ $\Delta ADC \sim \Delta PSR$

Hence $\frac{h_1}{h_2} = \frac{m\overline{AC}}{m\overline{PR}} = \frac{x}{y}$ (using eq: i)

Now

$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{\frac{1}{2}\left(m\overline{AB}\right)h_1}{\frac{1}{2}\left(m\overline{PQ}\right)h_2}$$

$$= \frac{m\overline{AB}}{m\overline{PQ}} \times \frac{h_1}{h_2}$$

$$= \frac{x}{y} \times \frac{x}{y} \quad \left(\frac{m\overline{AB}}{m\overline{PQ}} = \frac{x}{y} \text{ and } \frac{h_1}{h_2} = \frac{x}{y}\right)$$

$$= \frac{x^2}{y^2}$$

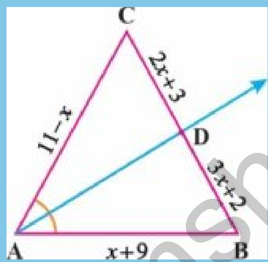
Hence proved.

From above example, we conclude that

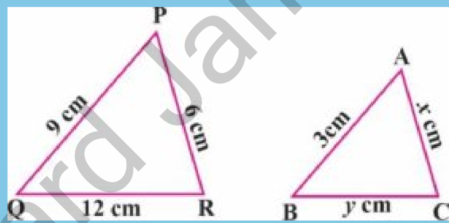
The ratio of areas of two similar triangles is equal to the square of the ratio of any two corresponding sides.

EXERCISE 24.2

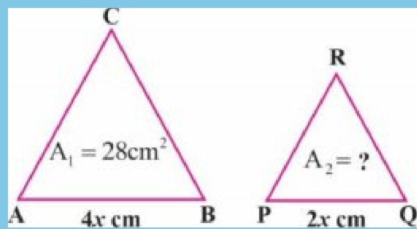
1. In the adjacent figure, $\overrightarrow{AD}$ is the bisector of $\angle A$ of ΔABC. Find the value of x if $m\overline{AB} = x+9, m\overline{AC} = 11-x, m\overline{CD} = 2x+3$ and $m\overline{BD} = 3x+2$. Also specify the type of triangle.



2. In the adjacent figure, ΔPQR and ΔABC are similar. Find the values of x and y if lengths of sides are indicated in the figure.



3. Let A_1 and A_2 be the areas of two similar triangles ΔABC and ΔPQR respectively as shown in the figure. Find A_2 if $A_1 = 28\text{cm}^2, m\overline{AB} = 4x\text{cm}$ and $m\overline{PQ} = 2x\text{cm}$



4. Ratio of corresponding sides of two similar triangles is $2 : x-5$ and the ratio of their areas is $1:9$. Find the value of x.
5. Prove that two right triangles have their sides proportional if an acute angle of the one is congruent to an acute angle of the other.
6. In a right triangle the perpendicular drawn from the right angle to the hypotenuse divides the triangle into two triangles. Prove that each of these triangles is similar to the original one.

REVIEW EXERCISE 24

1. **Tick the correct option.**

i. In a proportion, the product of means is equal to __________ of extremes.
(a) sum (b) difference (c) quotient (d) product

ii. __________triangles are always similar.
(a) right (b) scalene (c) acute angled (d) equilateral

iii. The symbol of similarity of triangles is __________.
(a) $=$ (b) $\cong$ (c) $\sim$ (d) $\leftrightarrow$

iv. If a line parallel to base of a triangle and divides one side in 2:3 then it will divide other side in __________.

(a) 3:2 (b) 2:3 (c) 2:6 (d) 5:3

v. If a line intersects two sides of a triangle in same ratio then it is ________ to other side.

(a) parallel (b) non-parallel (c) coincident (d) all of these

vi. In ΔABC, the bisector of $\angle A$ divides $\overline{BC}$ in ratio _________if $m\overline{AB} = 6cm$ and $m\overline{AC} = 8cm$.

(a) 5:8 (b) 3:4 (c) 1:1 (d) 5:7

vii. The bisector of an angle of equilateral triangle divides the opposite side in

(a) 2:3 (b) 3:2 (c) 1:1 (d) 5:2

viii. The corresponding sides of two similar triangles are__________.

(a) equal (b) un- equal (c) proportional (d) None of these

ix. If the ratio of two corresponding sides of similar triangle is 5:7 then ratio of their areas is equal to __________.

(a) 5:7 (b) 7:5 (c) 25:7 (d) 25:49

x. If the ratio of areas of two similar triangles is 36:121 then the ratio of its corresponding sides will be __________.

(a) 6:10 (b) 6:11 (c) 11:6 (d) 10:6

xi. If the ratio of corresponding sides of similar triangles is 2:x and that of areas is 4:9 then $x =$ _______.

(a) 3 (b) -3 (c) both a and b (d) none of these

xii. Two equilateral triangles are also __________.

(a) congruent (b) similar (c) proportional (d) equivalent

SUMMARY

- Ratio is the comparison of two similar quantities.
- In ratio $a:b$, a is called antecedent and b is called consequent.
- Proportion is the equality of two ratios.
- In proportion, the product of means is equal to the product of extremes.
- Two triangles are similar if they are equiangular.
- If two triangles are similar then their corresponding sides are proportional.
- A line parallel to one side of a triangle and intersecting the other two sides, divides them proportionally.
- If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the third side.
- The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the sides containing the angle.

CHORDS OF A CIRCLE

Unit 25

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

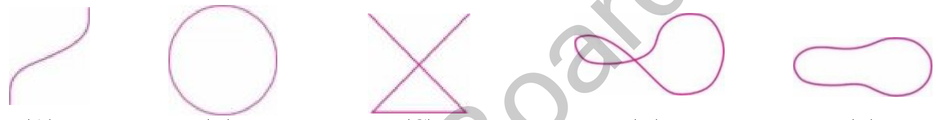
- Understand the following theorems along with their corollaries and apply them to solve allied problems.
  - One and only one circle can pass through three non-collinear points.
  - A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.
  - Perpendicular from the centre of a circle to a chord bisects it.
  - If two chords of a circle are congruent then they will be equidistant from the centre.
  - Two chords of a circle which are equidistant from the centre are congruent.

Introduction

In previous classes and units, we studied detailed geometry involving triangles and quadrilaterals which are all formed by line segments. Line segments do not have bends. Here, we focus on theorems with proofs and allied examples related with circles.

Circles play an important role in understanding the shapes of real world objects. Without circles, we will be unable to understand movement of a vehicle around a curve on the road, the starting developments about the orbits of planets, movements of electrons in an atom, shapes of cyclones, etc. Circles form a basis to study advanced shapes: ellipses, spheres, cylinders, and cones which are used in trunks, trees, water drops, wires, pipes, balloons, pies, wheels, ball-bearings, etc.

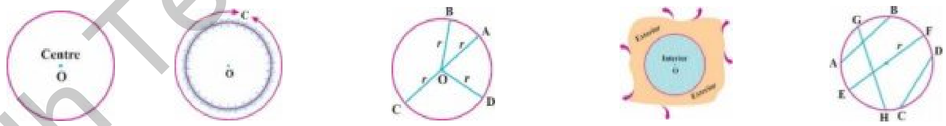
The path followed by a moving point is termed as **curve**. An **open curve** has different starting and ending points. A **closed curve** starts and ends at the same point (or) which has no starting and ending points. A curve which does not cross itself is a **simple curve**, otherwise **non-simple**. A curve which is simple as well as closed is known as a **simple closed curve**. For examples, refer to the curves in Figure (i).



(A) Simple, open | (B) Simple, closed | (C) Non-simple, open | (D) Non-simple, closed | (E) Simple, closed

Figure (i). Open, closed, simple and non-simple curves

A **circle** is a simple closed curve and all its points are at the same distance from a fixed point, called its **centre**. The length of the boundary of a circle is called its **circumference**. A line segment joining centre of a circle and any point on its boundary is a **radial segment**, and its length is called the **radius** of the circle. The points lying inside the boundary of a circle form its **interior**, and those outside form its **exterior**. A line segment joining any two points of circle is a **chord** of the circle. A chord passing through the centre of circle is called a **diameter** or a **central chord**, and its length is highest of all the lengths of chords of a circle. These terms are explained in Figure (ii).



(A) Circle centred at O (O)

(B) Circumference "C" of O

(C) Radius r and radial segments $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$

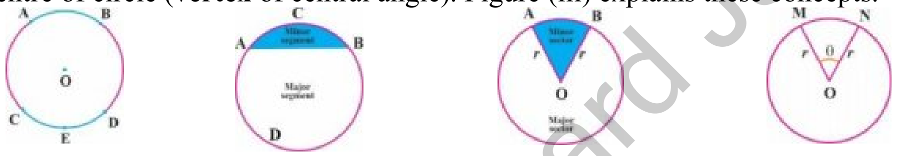
(D) Interior $(\equiv)$ and exterior of O

(E) Chords $\overline{AB}, \overline{CD}, \overline{EF}, \overline{GH}$, diameter $\overline{EF}\left(m\overline{EF} = 2r\right)$

Figure (ii)

If r is radius of a circle, then its diameter, circumference and **circular area** are equal to: $2r, 2\pi r$ and πr^2, where π (Greek letter PIE) is an irrational number. π is ratio of circumference of a circle to its diameter. It is approximated by $\frac{22}{7}$, but is not equal to it. In real, $\pi = 3.141592653$. We see that $\frac{22}{7} = 3.142857142$ matches with only first two decimal places of π.

A portion of the circumference of a circle is an **arc** of the circle. A chord chops the interior of a circle into exactly two parts or segments. The diameter divides the circle into two equal segments. The **segments** are bounded by an arc and a respective chords of a circle. For a particular chord, a segment with larger portion of the interior of circle is **major segment** and the other one is **minor segment**. The corresponding arcs are referred as **major** and **minor arcs**. A portion in interior of a circle confined between two radii (plural of radius) and the intercepted arc in-between is a **sector** of the circle. This leads to the **minor** and **major sectors** for an arc and chosen radii. The word **subtend** means "holds under", is usually used for angles under a chord or an arc at a point in a circle. The **central angle** is an angle subtended by an arc with the centre of circle (**vertex** of central angle). Figure (iii) explains these concepts.



(A). Arcs AB, CED

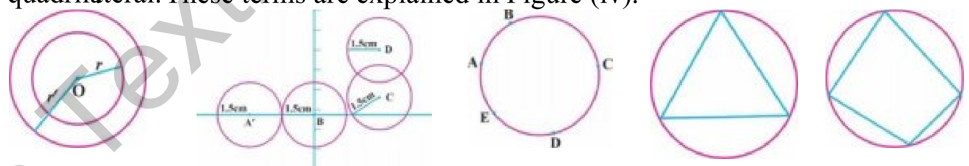
(B). Minor arc ACB, major arc ADB, minor and major segments

(C). Sector bounded by AB and $\overline{OA}, \overline{OB}$.

(D). Central angle $\angle MON = \theta$. Vertex O, arms $\overline{OM}, \overline{ON}$.

Figure (iii)

The **location and size** of a circle are determined by its centre and radius, respectively. Two circles are **congruent** if their radii are equal. Two circles with same centres are **concentric**. Points lying on the same line are **collinear**, else **non-collinear**. Points lying on the boundary of a circle are **concyclic**. A circle through vertices of a triangle is a **circumscribed circle** (or) **circumcircle**. A quadrilateral whose all vertices lie on a single circle is called cyclic quadrilateral. These terms are explained in Figure (iv).



(A). Concentric circles

(B). Congruent circles

(C). Concyclic points

(D). Circumcircle

(E). Cyclic quadrilateral

Figure (iv)

25.1 Chords of a Circle:

Theorem 25.1: One and only one circle can pass through three non-collinear points.

Given:

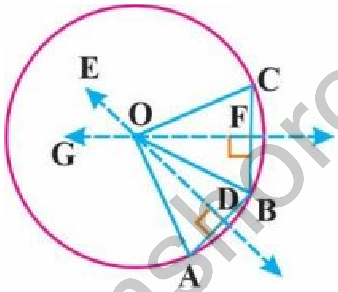
Three non-collinear points, say A, B and C.

To prove:

One and only one circle can pass through A,B and C.

Construction:

Draw line segments $\overline{AB}$ and $\overline{BC}$. Draw right bisectors $\overleftrightarrow{ED}$ and $\overleftrightarrow{GF}$ of $\overline{AB}$ and $\overline{BC}$, respectively. $\overleftrightarrow{ED}$ and $\overleftrightarrow{GF}$ intersect at a point, say O. Draw $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$.



Proof:

Statements	Reasons
All points on $\overleftrightarrow{ED}$ are equidistant from A and B, so $m\overline{OA} = m\overline{OB}$ (i)	$\overleftrightarrow{ED}$ is the right bisector of $\overline{AB}$, and O is a point on $\overleftrightarrow{ED}$
All points on $\overleftrightarrow{GF}$ are equidistant from B and C, so $m\overline{OB} = m\overline{OC}$ (ii)	$\overleftrightarrow{GF}$ is the right bisector of $\overline{BC}$, and $\overleftrightarrow{GF}$ passes through O.
O is the unique point of intersection of $\overleftrightarrow{ED}$ and $\overleftrightarrow{GF}$. (iii)	$\overleftrightarrow{ED}$ and $\overleftrightarrow{GF}$ are non-parallel lines.
The point O is equidistant from A, B and C, i.e.	From (i) and (ii).
$m\overline{OA} = m\overline{OB} = m\overline{OC} = r$, say (iv)	Transitive property.
The circle with centre only at O and radius r passes through A, B and C. (v)	$\overline{OA}, \overline{OB}$ and $\overline{OC}$ are radial segments, and by (iii).
A, B and C are non-collinear. (vi)	Given
Therefore, there exists one and only one circle centered at O and with radius r passing through non-collinear points A,B and C.	From (iii), (iv), (v) and (vi).

Q.E.D

Corollary:

Any three distinct points on a circle are non-collinear.

Note 1:

Theorem 25.1 demonstrates the existence and uniqueness of the circle which can be drawn through any three non-collinear points.

Note 2:

Any number of distinct points more than three on a circle are non-collinear.

Example 1:

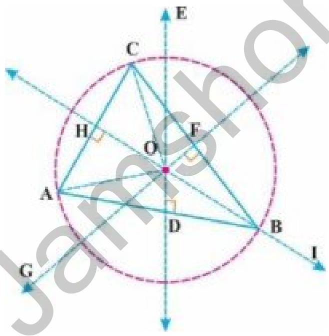
Prove that one and only one circle can pass through three vertices of a triangle.

Given:

ΔABC with vertices A, B and C.

To prove:

One and only one circle can pass through vertices of ΔABC.



Construction:

Draw right bisectors: $\overleftrightarrow{DE}$, $\overleftrightarrow{FG}$ and $\overleftrightarrow{HI}$ of sides of ΔABC : $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$, respectively, intersecting at the point O, say. Draw $\overline{OA}, \overline{OB}$ and $\overline{OC}$.

Proof:

Statements	Reasons
O is the unique point of intersection of $\overleftrightarrow{DE}, \overleftrightarrow{FG}$ and $\overleftrightarrow{HI}$.	$\overleftrightarrow{DE}$, $\overleftrightarrow{FG}$ and $\overleftrightarrow{HI}$ are non-parallel lines, and O lies on all of these.
The point O is circumcenter of ΔABC.	By definition of triangle.
The point O is equidistant from vertices of ΔABC, i.e. $m\overline{OA} = m\overline{OB} = m\overline{OC} = r$	By definition of circumcentre.
The circle with centre O and radius r passes through vertices of ΔABC.	$\overline{OA}, \overline{OB}$ and $\overline{OC}$ are radial segments.

Q.E.D

Example 2:

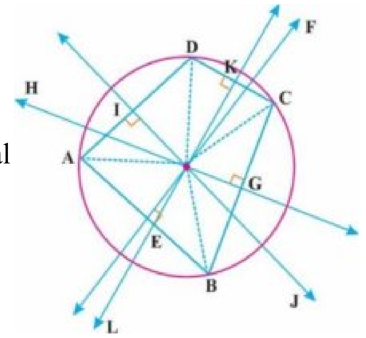
Show that one and only one circle can pass through vertices of a quadrilateral.

Given:

A quadrilateral ABCD.

To prove:

Only one circle passes through vertices of quadrilateral ABCD.



Construction:

Draw $\perp$ bisectors $\overleftrightarrow{EF}, \overleftrightarrow{GH}, \overleftrightarrow{IJ}$ and $\overleftrightarrow{KL}$ on $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{AD}$. These all meet at point O. Construct $\overline{OA}, \overline{OB}, \overline{OC}$ and $\overline{OD}$.

Proof:

Statements	Reasons
All points on EF are equidistant from A and B.	EF is $\perp$ bisector of $\overline{AB}$.
$\therefore \quad \overline{OA} \cong \overline{OB}$ (i)	O lies on $\overleftrightarrow{EF}$.
Similarly, $\overline{OB} \cong \overline{OC}$ (ii) $\overline{OC} \cong \overline{OD}$ (iii) $\overline{OD} \cong \overline{OA}$ (iv)	Using similar reasoning for $\overleftrightarrow{GH}$ and $\overleftrightarrow{BC}$, $\overleftrightarrow{IJ}$ and $\overleftrightarrow{CD}$, $\overleftrightarrow{KL}$ and $\overleftrightarrow{AD}$, as above.
O is the unique point of intersection of $\overleftrightarrow{EF}, \overleftrightarrow{GH}, \overleftrightarrow{IJ}$ and $\overleftrightarrow{KL}$	O lies on all these lines.
O is equidistant from vertices A, B, C and D of the quadrilateral so: $m\overline{OA} = m\overline{OB} = m\overline{OC} = m\overline{OD} = r$	From (i) to (iv).
The unique circle with centre O and radius r passes through vertices A, B, C and D of quadrilateral.	$\overline{OA}, \overline{OB}, \overline{OC}$ and $\overline{OD}$ form radial segments.

Q.E.D

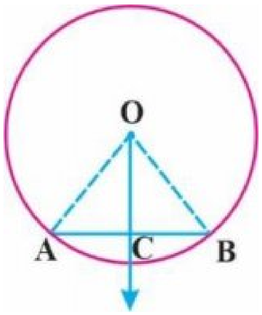
EXERCISE 25.1

1. Can a circle pass through three collinear points? Explain with reason.
2. Can you draw a circle from any four non-collinear points? Explain with reasons.
3. Show that one and only one circle can pass through the vertices of a square.
4. Show that only one circle can pass through the vertices of a regular pentagon.
5. Three villages are situated in a way that B is east of A at a distance of 6km and C is north of B at a distance of 8km. Determine the location of mosque so that all have to walk same distance from each village by using ruler, compass and divider. How much distance each villager has to walk down?

Theorem 25.2: A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.

Given:

A circle with centre O and a chord $\overline{AB}$ which does not pass through O, i.e. not a diameter. A straight line OC bisects $\overline{AB}$ at C, i.e. $m\overline{AC} = m\overline{BC}$.



To prove: $OC \perp \overline{AB}$

Construction:

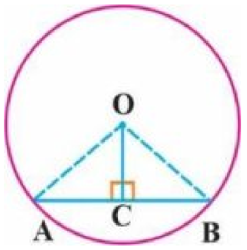
Draw $\overline{OA}$ and $\overline{OB}$.

Proof:

Statements			Reasons
In	$\Delta OCA \leftrightarrow \Delta OCB$		
	$m\overline{OA} = m\overline{OB}$		Radius of same circle.
	$m\overline{AC} = m\overline{BC}$		Given.
	$m\overline{OC} = m\overline{OC}$		Common side.
	$\Delta OCA \cong \Delta OCB$		S.S.S $\cong$ S.S.S.
or	$m\angle OCA = m\angle OCB$	(i)	Corresponding angles of congruent Δs.
	$m\angle OCA + m\angle OCB = 180^\circ$	(ii)	Supplement postulate.
	$m\angle OCA = 90^\circ = m\angle OCB$		From (i) and (ii).
Thus,	$OC \perp \overline{AB}$		By definition of perpendicular.

Q.E.D

Theorem 25.3: Perpendicular from the centre of a circle to a chord bisects it.



Given: A circle with centre O and its chord $\overline{AB}$. The perpendicular $\overline{OC}$ from O meets $\overline{AB}$ at C, i.e. $\overline{OC} \perp \overline{AB}$, so $\angle OCA$ and $\angle OCB$ are right angles.

To prove: $\overline{OC}$ bisects $\overline{AB}$ at C.

Construction: Draw $\overline{OA}$ and $\overline{OB}$.

Proof:

Statements		Reasons
In	$\Delta AOC \leftrightarrow \Delta BOC$	
	$m\angle OCA = 90^\circ = m\angle OCB$	Given.
	$\overline{OA} \cong \overline{OB}$	Radial segments of same circle.
	$\overline{OC} \cong \overline{OC}$	Common side.
So,	$\Delta AOC \cong \Delta BOC$	H.S $\cong$ H.S.
$\therefore$	$\overline{AC} \cong \overline{BC}$	Corresponding sides of congruent Δs.
$\therefore$	$\overline{OC}$ bisects the chord $\overline{AB}$.	C is the midpoint of $\overline{AB}$.

Q.E.D

Corollary 1:

Perpendicular bisector of the chord of a circle passes through the centre of the circle.

Corollary 2:

The shortest distance from a point on chord and the centre of the circle is from its midpoint.

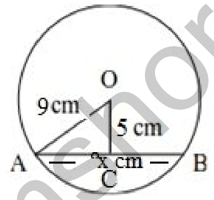
Note:

Theorems 25.2 and 25.3 highlight relationship between a chord of a circle and a line segment dividing it equally through the centre of circle. With these and the Pythagoras' theorem, we can find length of a chord of a circle.

Example 1:

Calculate length of a chord which stands at a perpendicular distance of 5cm from the centre of a circle with radius 9cm.

Solution:



Consider a circle with centre O and its chord AB. The perpendicular from centre O meets chord AB at a point C, as shown in the adjacent figure. Using Pythagoras' theorem in ΔOCA, we have:

$\left(m\overline{OA}\right)^2 = \left(m\overline{OC}\right)^2 + \left(m\overline{AC}\right)^2$ (or) $9^2 = 5^2 + \left(m\overline{AC}\right)^2$

$\therefore\ m\overline{AC} = \sqrt{81-25} = \sqrt{56} = 2\sqrt{14}$ cm.

Finally,

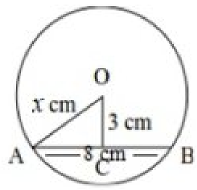
$$\text{Length of chord} = \overline{AB} = 2\left(m\overline{AC}\right) \quad (\text{As } \overline{OC} \text{ bisects } \overline{AB})$$

$$= 2\left(2\sqrt{14}\right) = 4\sqrt{14} = 14.967\text{cm}$$

Example 2:

If length of a chord in a circle is 8cm, and perpendicular distance from centre of circle to the chord is 3cm, what is radius of that circle?

Solution:



Consider a circle with centre O and its chord AB. The perpendicular from centre O meets chord AB of length 8 cm at a point C, as shown in the adjacent figure. Using Pythagoras' theorem in ΔOCA, we have:

$\left(m\overline{OA}\right)^2 = \left(m\overline{OC}\right)^2 + \left(m\overline{AC}\right)^2$

(or) $x^2 = 3^2 + 4^2$ ($\overline{OC}$ bisects $\overline{AB}$)

$\therefore\ x^2 = 25 \ \Rightarrow x = 5$, ignoring negative sign.

Thus, Radius $= m\overline{OA} = x = 5$ cm.

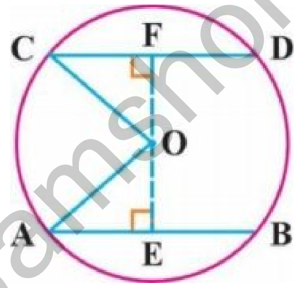
EXERCISE 25.2

1. Show that the diameters of a circle bisect each other.
2. Show that angle subtended by the centre of a circle and midpoint of a chord is right angle.
3. If length of the chord is 8cm. Its perpendicular distance from the centre of circle is 3cm, then find circumference and area of the circle.
4. Calculate length of a chord which stands at a perpendicular distance of k from the centre of a circle with radius r when:
   a. k=4cm, $r = 9$cm b. k=3cm, $r = 6$cm
5. What will be radius of a circle in which the distance of a chord of length 10cm from the centre of circle is 3cm?

Theorem 25.4: If two chords of a circle are congruent then they will be equidistant from the centre.

Given:

A circle with centre O. Two congruent chords: $\overline{AB}$ and $\overline{CD}$ of circle, where $\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$.



To prove:

$\overline{OE} \cong \overline{OF}$

Construction:

Draw $\overline{OA}$ and $\overline{OC}$.

Proof:

Statements	Reasons
$m\overline{AE} = \frac{1}{2} m\overline{AB}$ (i)	Perpendicular $\overline{OE}$ bisects the chord $\overline{AB}$.
$m\overline{CF} = \frac{1}{2} m\overline{CD}$ (ii)	Perpendicular $\overline{OF}$ bisects the chord $\overline{CD}$.
$m\overline{AB} = m\overline{CD}$ (iii)	$\overline{AB} \cong \overline{CD}$ (Given).
$\therefore$ $\overline{AE} \cong \overline{CF}$	Using (i), (ii) and (iii).
In right $\Delta AEO \leftrightarrow \Delta CFO$	
$\overline{OA} \cong \overline{OC}$	Radial segments of same circle.
$\overline{AE} \cong \overline{CF}$	Proved above.
$\therefore$ $\Delta AEO \cong \Delta CFO$	H.S. $\cong$ H.S.
$\Rightarrow$ $\overline{OE} \cong \overline{OF}$	Corresponding sides of congruent Δs.

Q.E.D

Corollary:

Two congruent chords of a circle subtend equal angles at the centre.

Note 1:

Theorem 25.4 explains the association between congruence of two chords and their distances from the centre of the circle.

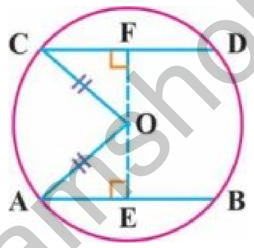
Note 2:

If two chords are not congruent, then they are not equidistant from the centre.

Theorem 25.5: Two chords of a circle which are equidistant from the centre are congruent.

Given:

A circle with centre O. Two chords $\overline{AB}$ and $\overline{CD}$ of circle, which are equidistant from O. i.e. $\overline{OE} \cong \overline{OF}$. This also means that $\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$.



To prove:

$\overline{AB} \cong \overline{CD}$

Construction:

Draw $\overline{OA}$ and $\overline{OC}$.

Proof:

Statements	Reasons
In right $\Delta AEO \leftrightarrow \Delta CFO$	
$\overline{OA} \cong \overline{OC}$	Radial segments of same circle.
$\because$ $\overline{OE} \cong \overline{OF}$	Given.
$\therefore$ $\Delta AEO \cong \Delta CFO$	H.S. ≅ H.S.
So, $m\overline{AE} = m\overline{CF}$ (i)	Corresponding sides of congruent Δs.
$m\overline{AE} = \frac{1}{2} m\overline{AB}$ (ii)	Perpendicular $\overline{OE}$ bisects the chord $\overline{AB}$.
$m\overline{CF} = \frac{1}{2} m\overline{CD}$ (iii)	Perpendicular $\overline{OF}$ bisects the chord $\overline{CD}$.
$\because$ $\frac{1}{2} m\overline{AB} = \frac{1}{2} m\overline{CD}$	Using (i), (ii), (iii)
$\therefore$ $\overline{AB} \cong \overline{CD}$	

Q.E.D

Corollary:

If two chords subtend equal angles at the centre, then they are congruent.

Note 1:

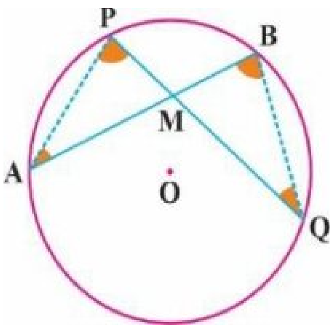
Theorems 25.5 is converse of Theorem 25.4

Note 2:

If two chords are not equidistant from the centre, they are not congruent.

Example 1:

Show that if two chords of a circle intersect each other, then the product of their lengths of segments are equal.



Given: Two chords $\overline{AB}$ and $\overline{PQ}$ of a circle centered at O intersecting each other at M.

To prove: $(m\overline{AM}) \times (m\overline{MB}) = (m\overline{PM}) \times (m\overline{QM})$

Construction: Draw $\overline{AP}$ and $\overline{BQ}$.

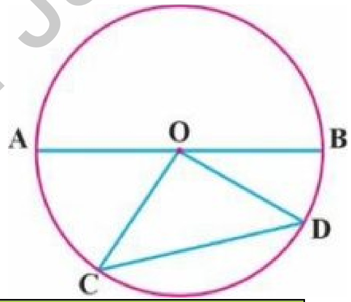
Proof:

Statements		Reasons
$\angle PAM \cong \angle BQM$	(i)	Angles of same arc PB.
$\angle APM \cong \angle QBM$	(ii)	Angles of same arc AQ.
$\Delta APM \sim \Delta QBM$		From (i) and (ii): A.A~ A.A.
$\dfrac{m\overline{AM}}{m\overline{QM}} = \dfrac{m\overline{PM}}{m\overline{BM}}$	(iii)	Corresponding sides of similar Δs.
$(m\overline{AM}) \times (m\overline{BM}) = (m\overline{PM}) \times (m\overline{QM})$		Cross multiplication in (iii)

Q.E.D

Example 2:

Show that the largest chord in a circle is the diameter.



Given: A circle with centre O and its diameter $\overline{AB}$. $\overline{CD}$ is any other chord of circle except $\overline{AB}$.

To prove: $m\overline{AB} > m\overline{CD}$

Construction: Draw $\overline{OC}$ and $\overline{OD}$ to complete ΔOCD.

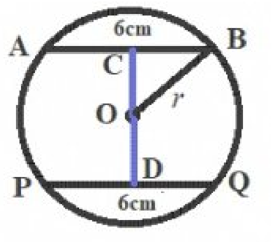
Proof:

Statements		Reasons
In ΔOCD,		
$m\overline{OC} + m\overline{OD} > m\overline{CD}$	(i)	Sum of two sides of a Δ is greater than the third.
$m\overline{OC} + m\overline{OD} = m\overline{AB}$	(ii)	$\overline{OC}$ and $\overline{OD}$ are radial segments, and by definition of diameter.
$m\overline{AB} > m\overline{CD}$	(iii)	Using (i) and (ii).
The length of chord $\overline{AB}$, which is also the diameter is larger than length of any other chord $\overline{CD}$ of the circle.		From (iii), and for any chord $\overline{CD}$ except diameter.

Q.E.D

Example 3:

Find distance between two congruent chords of lengths 6cm each in circle whose circumference is 10π cm.



Solution:

Two congruent chords $\overline{AB}$ and $\overline{PQ}$ of length 6 cm of a circle centred at O with radius r are shown in the adjacent figure. We need $m\overline{CD}$.

Radius of the circle: $m\overline{OB} = r = \dfrac{C}{2\pi} = \dfrac{10\pi}{2\pi} = 5\text{cm}.$

Using Pythagoras' theorem in right ΔOCB, we have:

$$\left(m\overline{OB}\right)^2 = \left(m\overline{OC}\right)^2 + \left(m\overline{CB}\right)^2$$

$$5^2 = \left(m\overline{OC}\right)^2 + \left(\frac{1}{2}m\overline{AB}\right)^2$$ (Perpendicular $\overline{OC}$ from centre O bisects the chord $\overline{AB}$)

$$5^2 = \left(m\overline{OC}\right)^2 + 3^2 \qquad \Rightarrow m\overline{OC} = \sqrt{25-9} = \sqrt{16} = 4cm.$$

Because, two congruent chords are equidistant from the centre of a circle, so:

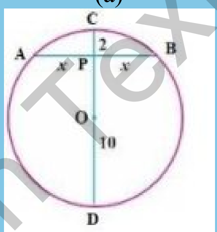
$m\overline{OD} = m\overline{OC} = 4\text{cm}.$

Finally, $m\overline{CD} = m\overline{OC} + m\overline{OD} = 4 + 4 = 8\text{cm}.$

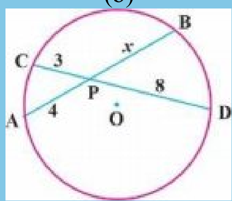
EXERCISE 25.3

1. Prove that two congruent chords of two congruent circles are equidistant from their centres.
2. Show that two chords of congruent circles equidistant from the centres are congruent.
3. Find distance between two congruent chords of length α in a circle of radius r for the following given values of α and r:
   (a). $\alpha = 7\text{cm}$ and $r = 6\text{cm}$ (b). $\alpha = 5\text{cm}$ and $r = 4\text{cm}$
   (c). $\alpha = 2\text{cm}$ and $r = 4\text{cm}$ (d). $\alpha = 4\text{cm}$ and $r = 9\text{cm}$
4. Find distance between two parallel chords of length α and β in a circle of radius r for the following given values of α, β and r:
   a. $\alpha = 6\text{cm}, \beta = 8\text{cm}, r = 5\text{cm}$ b. $\alpha = 3\text{cm}, \beta = 6\text{cm}, r = 14\text{cm}$
5. If two chords $\overline{AB}$ and $\overline{CD}$ of a circle with centre O intersect at P, find x in the following.

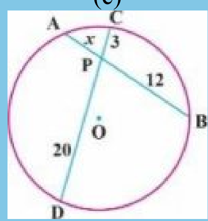
(a)



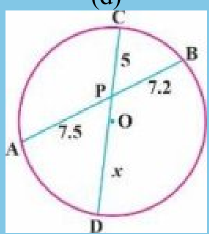
(b)



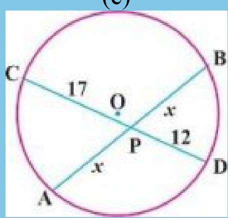
(c)



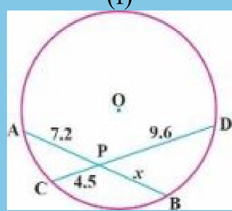
(d)



(e)



(f)



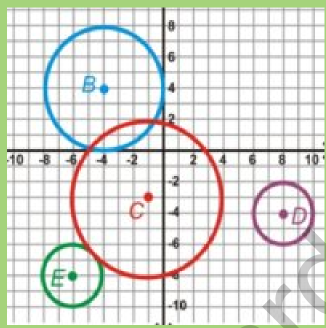
Review Exercise 25

1. Tick the correct option.

i. All circles are always _____ .

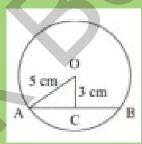
(a) congruent (b) similar
(c) tangent (d) none of these

ii. In the following figure, circles centred at D and E are ________ .



(a) congruent (b) similar
(c) both (a) and (b) (d) none of these

iii. In the given figure, the length of the chord $\overline{AB}$ is _______ .



(a) 4cm (b) 6cm (c) 8cm (d) 15cm

iv. One and only one circle passes through three _______ points.

(a) collinear (b) non-collinear (c) disjoint (d) none of these

v. The hypothesis of the enunciation "If two chords of a circle are congruent, then they are equidistant from the centre." is:

(a) two chords of a circle are equidistant from the centre.
(b) two chords of a circle are congruent.
(c) a circle has two chords.
(d) the centre of circle is equidistant from chords.

vi. The conclusion of the enunciation "If two chords of a circle are congruent, then they are equidistant from the centre." is:

(a) two chords of the circle are equidistant from the centre.
(b) two chords of a circle are congruent.
(c) a circle has two chords.
(d) the centre of circle is equidistant from chords.

vii. The hypothesis of the enunciation "One and only one circle can pass through three non-collinear points" is:

(a) three points are non-collinear.
(b) one and only one circle passes through three points.

(c) two circles pass through three points.
(d) three points are collinear.

viii. The conclusion of the enunciation "One and only one circle can pass through three non-collinear points" is:
(a) three points are non-collinear.
(b) one and only one circle passes through three points.
(c) two circles pass through three points.
(d) three points are collinear.

ix. A circle is an example of a __________ curve.
(a) simple and closed (b) simple and open
(c) non-simple and closed (d) non-simple and open

SUMMARY

- A circle is a simple closed curve.
- All points of a circle are equidistant from a fixed point. The fixed point is the centre and constant distance is radius of circle.
- A radial segment in a circle is a line segment from its centre to any point on it.
- A chord is a line segment joining any two points of a circle. A chord passing through the centre is known as diameter or central chord of the circle.
- For a circle of radius r, its diameter, circumference and circular area are equal to: $2r$, $2\pi r$ and πr^2, respectively.
- A portion of circumference of a circle is called an arc. These are further categorized as minor and major arcs.
- Sectors are portions of circle bounded by two radii and the intercepted arc. These can be further categorized as minor and major sectors.
- Segments are portions of circle bounded between a chord and the intercepted arc. These are further divided into major and minor sectors.
- All circles are similar to each other.
- Two circles are congruent if their radii are equal.
- Circles having same centre are concentric.
- Points lying on a circle are concyclic.
- One and only one circle can pass through three non-collinear points.
- A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.
- Perpendicular from the centre of a circle to a chord bisects it.
- If two chords of a circle are congruent then they will be equidistant from the centre.
- Two chords of a circle which are equidistant from the centre are congruent.

TANGENTS OF A CIRCLE

Unit 26

Unit No. 26
Tangents Of A Circle

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with their corollaries and apply them to solve allied problems.
  - If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.
  - The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.
  - The two tangents drawn to a circle from a point outside it are equal in length.
  - If two circles touch externally or internally, the distance between their centres is, respectively, equal to the sum or difference of their radii.

Introduction

The theorems and proofs related to **tangent(s)** to a circle are discussed here. The concept of tangent is also important in defining derivatives in a branch of Mathematics called **Calculus.** A straight line may or may not intersect a circle. If a line touches a circle at only one point, then it is a **tangent to the circle**. The common point between a tangent and the circle is the **point of contact** (or) **point of tangency**. A line which intersects the circle at two points is a **secant to the circle**. There are two points of contact between a secant and the circle. In Figure (i), lines l_1 and l_2 do not touch the circle. The lines m_1 and m_2 are tangents at P and Q, respectively. The lines n_1 and n_2 are secants to the circle at points A, B and C, D, respectively. A line segment from the point of tangency to any other point of the tangent is a **tangent segment**. In Figure (ii), $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$ are tangents to the circle, whereas $\overline{PQ}$ and $\overline{AB}$ are tangent segments, respectively. $m\overline{AB}$ is **length of tangent segment** $\overline{AB}$. For a point P lying outside the circle, only two tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$ can be drawn, as shown in Figure (iii).

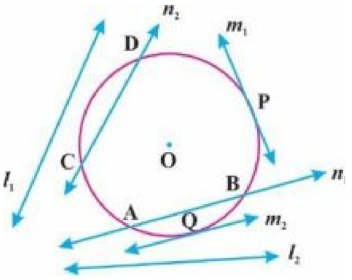


Figure (i).

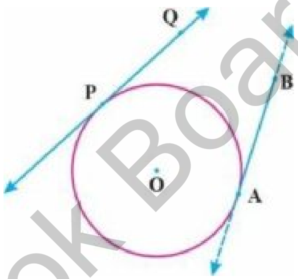


Figure (ii).

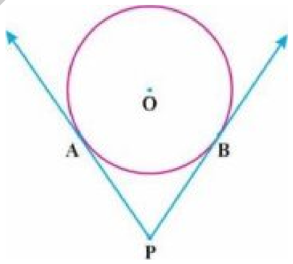


Figure (iii).

Circles touching each other internally or externally share the common point of tangency. The point *P* is the **common point of tangency** of circles with centres O, M and N in Figure (iv). Circles with centres O and M touch internally, whereas circles with centres O and N (also M and N) touch externally. Two circles not touching each other can have common tangents but different points of contact. A common tangent between two circles not touching each other is an **internal (or transverse) common tangent** if it intersects the segment joining their centres, otherwise an **external (or direct) common tangent**. In Figure (v), line *m* is an external (or direct) common tangent, whereas line *l* is an internal (or transverse) common tangent to the circles.

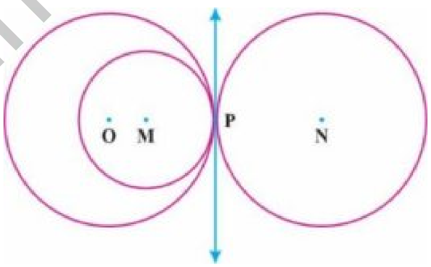


Figure (iv).

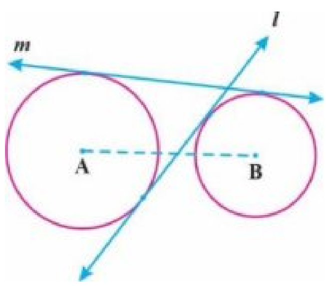


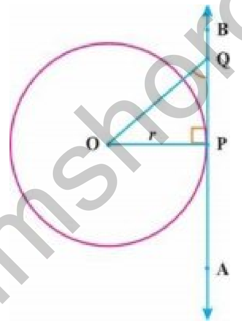
Figure (v).

26.1 Tangent(s) to a Circle:

Theorem 26.1: If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.

Given:

A circle with centre O and radius r. $\overleftrightarrow{AB}$ is drawn perpendicular on radial segment $\overline{OP}$, i.e. $\overleftrightarrow{AB} \perp \overline{OP}$ at its outer end point P.



To prove:

$\overleftrightarrow{AB}$ is tangent to given circle at P only.

Construction:

Draw $\overline{OQ}$, where Q is any other point on AB except P.

Proof:

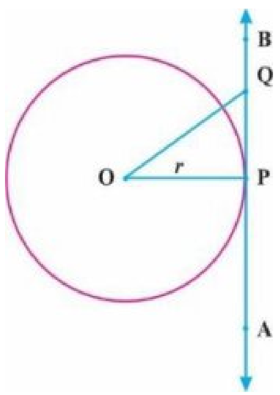
Statements	Reasons
In right ΔOPQ, $m\angle OPQ = 90°$	$\overleftrightarrow{AB} \perp \overline{OP}$ at P (given)
But, $m\angle OQP < 90°$	Except right angle, other angles are acute in right Δ.
So, $m\overline{OQ} > m\overline{OP}$	Greater angle has greater side opposite.
Hence, Q lies outside the circle.	$m\overline{OQ} > r$
All points on $\overleftrightarrow{AB}$ except P lie outside the circle.	Q is any other point except P (construction),
Only point of contact of circle and $\overleftrightarrow{AB}$ is P. So, $\overleftrightarrow{AB}$ is tangent to circle at P.	By definition of tangent.

Q.E.D.

Theorem 26.2: The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.

Given:

A circle with centre O, radius r and a radial segment $\overline{OP}$.
A tangent $\overleftrightarrow{AB}$ at P.



To prove:

$\overline{OP} \perp \overleftrightarrow{AB}$

Construction:

Draw $\overline{OQ}$, where Q is any point on $\overleftrightarrow{AB}$ other than P.

Proof:

Statements	Reasons
Q lies outside the circle.	Q is any other point on $\overleftrightarrow{AB}$ except P.
So, $m\overline{OQ} > m\overline{OP} = r$ (i)	$\overline{OP}$ is radial segment.
Distance from the centre of all points on $\overleftrightarrow{AB}$ except P exceed r. So, all points except P lie outside the circle.	$m\overline{OP} = r$, and Q is any point on $\overleftrightarrow{AB}$ except P. (construction).
The shortest line segment from O to any point on $\overleftrightarrow{AB}$ is $\overline{OP}$.	From (i) for any Q on $\overleftrightarrow{AB}$ except P.
So, $m\angle OPQ = 90°$ and, $\overline{OP} \perp \overleftrightarrow{AB}$	All other angles are acute.

Q.E.D.

Note 1:

Theorems 26.1 and 26.2 highlight relationship between a tangent to a circle and the corresponding radial segment at the point of tangency.

Note 2:

Theorem 26.1 is a converse of Theorem 26.2, and vice-versa.

Corollary 1:

The line drawn at right angle to a tangent of a circle at its point of tangency passes through the centre of the circle.

Corollary 2:

The triangle formed by centre of the circle, point of tangency and any other point on the tangent is a right-angled triangle.

Corollary 3:

Only one tangent can be drawn to a circle at a given point on its boundary.

Example 1:

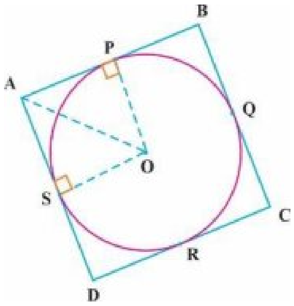
Show that the parallelogram circumscribing a circle is a rhombus.

Given:

$\|^{m}$ ABCD circumscribing a circle so that: $m\overline{AB} = m\overline{CD}$ and $m\overline{AD} = m\overline{BC}$.

To prove:

$\|^{m}$ $ABCD$ is a rhombus, i.e. $m\overline{AB} = m\overline{BC} = m\overline{CD} = m\overline{AD}$.



Construction:

$\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{AD}$ are tangent to circle at P, Q, R and S, respectively. Draw $\overline{OS}, \overline{OA}$ and $\overline{OP}$.

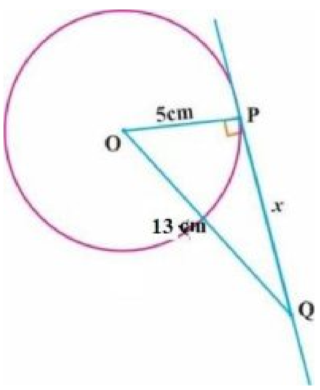
Proof:

Statements	Reasons
In $\Delta OSA \leftrightarrow OPA$	
$\overline{OS} \cong \overline{OP}$	Radial segments of same circle.
$m\angle OSA = 90^\circ = m\angle OPA$	Tangent perpendicular to radial segment
$\overline{OA} \cong \overline{OA}$	Common side of triangles.
$\Rightarrow \quad \Delta OSA \cong \Delta OPA$	$H.S \cong H.S$
and $m\overline{AS} = m\overline{AP} \quad (i)$	Corresponding sides of congruent Δs.
$\Delta OPB \cong \Delta OQB$	By same process
Similarly, $m\overline{BQ} = m\overline{BP} \quad (ii)$	Corresponding sides of congruent Δs.
$\Delta OQC \cong \Delta ORC$	By same process
$m\overline{QC} = m\overline{RC} \quad (iii)$	Corresponding sides of congruent Δs.
$\Delta ORD \cong \Delta OSD$	By same process
$m\overline{DS} = m\overline{DR} \quad (iv)$	Corresponding sides of congruent Δs.
Now, $m\overline{AS} + m\overline{BQ} + m\overline{QC} + m\overline{DS}$ $= m\overline{AP} + m\overline{BP} + m\overline{RC} + m\overline{DR}$	Adding (i)-(iv)
or $\left(m\overline{AS} + m\overline{DS}\right) + \left(m\overline{BQ} + m\overline{QC}\right)$ $= \left(m\overline{AP} + m\overline{BP}\right) + \left(m\overline{RC} + m\overline{DR}\right)$	Re-arranging
or $m\overline{AD} + m\overline{BC} = m\overline{AB} + m\overline{CD} \quad (v)$	From figure
$2m\overline{BC} = 2m\overline{AB} \Rightarrow m\overline{AB} = m\overline{BC} \quad (vi)$	By definition of parallelogram (given)
$m\overline{AB} = m\overline{BC} = m\overline{CD} = m\overline{AD}$	Using (vi) and the given.
So, $\|^{m}$ ABCD is a rhombus.	

Q.E.D.

Example 2: Find the length of the tangent segment to a circle of radius 5cm from a point 13cm away from the centre of the circle.

Solution: Let centre of the circle is O. The point of tangency is P. Let Q be the point on tangent at a distance of 13cm from the centre as shown in the figure. From Figure, we have:



$m\overline{OP} = r = 5cm,\ m\overline{OQ} = 13cm$ and $m\overline{PQ} = x = ?$

Using Pythagoras' theorem:

$$\left(m\overline{OQ}\right)^2 = \left(m\overline{OP}\right)^2 + \left(m\overline{PQ}\right)^2$$

$$(13)^2 = (5)^2 + (x)^2$$

$$169 = 25 + x^2 \quad or \quad x^2 = 169 - 25 = 144$$

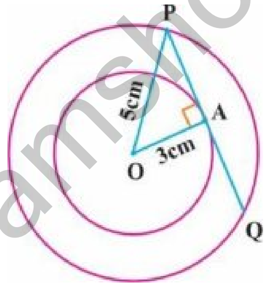
$$m\overline{PQ} = x = \sqrt{144} = 12cm.$$

Example 3:

Two concentric circles have radii 5cm and 3cm, respectively. Find the length of the tangent segment to smaller circle from the points on larger circle. Also find the length of chord of the larger circle which touches smaller circle.

Solution:

Two concentric circle of given radii are shown in the figure with common centre O. $m\overline{OA} = 3\text{cm}$ and $m\overline{OP} = 5\text{cm}$.



We first need to find $m\overline{AP}$ and $m\overline{AQ}$, i.e. length of tangent segment to smaller circle from points P and Q of larger circle.

Using Pythagoras' theorem, we have:

$$\left(m\overline{OP}\right)^2 = \left(m\overline{OA}\right)^2 + \left(m\overline{AP}\right)^2$$

$$25 = 9 + \left(m\overline{AP}\right)^2$$

$$\Rightarrow \quad \left(m\overline{AP}\right)^2 = 16 \quad or \quad m\overline{AP} = 4\text{cm}$$

But, the radial segment $\overline{OA}$ bisects the chord $\overline{PQ}$, so:

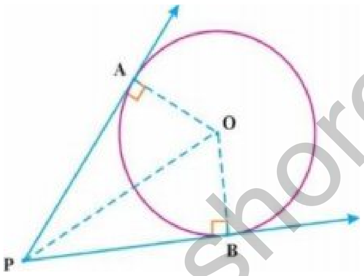
$$m\overline{AP} = m\overline{AQ} \quad \Rightarrow \quad m\overline{AQ} = 4cm$$

Finally, we need length of chord $\overline{PQ}$ of larger circle which touches smaller circle, which is: $m\overline{PQ} = m\overline{AP} + m\overline{AQ} = 8cm.$

EXERCISE 26.1

1. Show that a triangle circumscribing a circle is an equilateral triangle.
2. Show that a rectangle circumscribing a circle must be a square.
3. The diameters of two concentric circles are 10cm and 5cm, respectively. Find length of the tangent segment to the smaller circle to a point on it touching outer circle. Also find length of chord of outer circle which touches inner circle.
4. The length of tangent segment from a point at a distance of 5cm from centre of the circle is 4cm. Find diameter, circumference and area of the circle?
5. Find length of the tangent segment to a circle of radius 7cm from a point at a distance of 25cm from the centre of circle.
6. How far from centre of the circle of radius 3cm, a tangent segment of length 10cm can be drawn?

Theorem 26.3: The two tangents, drawn to a circle from a point outside it, are equal in length.



Given: $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are tangents to the circle at A and B from a point P outside it.

To prove: $\overline{PA} \cong \overline{PB}$.

Construction: Draw $\overline{OA}, \overline{OB}$ and $\overline{OP}$.

Proof:

Statements	Reasons
$\overline{OA} \perp \overrightarrow{PA}$ and $m\angle OAP = 90°$	Tangent is perpendicular to radial segment
Similarly, $\overline{OB} \perp \overrightarrow{PB}$ and $m\angle OBP = 90°$	Tangent is perpendicular to radial segment
In right triangles $\Delta OAP \leftrightarrow \Delta OBP$	
$\overline{OP} \cong \overline{OP}$	Common side
$\overline{OA} \cong \overline{OB}$	Radial segments of same circles
$\therefore$ $\Delta OAP \cong \Delta OBP$	H.S $\cong$ H.S
$\therefore$ $\overline{PA} \cong \overline{PB}$	Corresponding sides of congruent Δs .

Q.E.D.

Note 1:

Theorem 26.3 demonstrates that the two tangents to a circle intersecting at a point outside the circle must be equal in length.

Corollary 1:

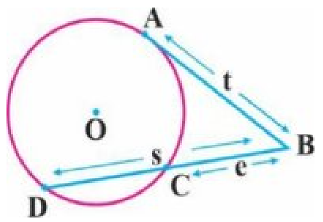
The two tangents drawn to a circle from an external point subtend congruent angles at the centre.

Corollary 2:

Two parallel tangents to a circle do not intersect at the same point outside it.

Note 2:

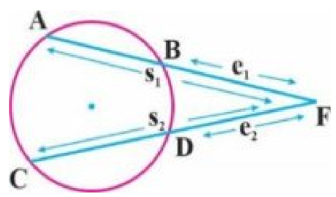
If a tangent and a secant to a circle intersect outside a circle as shown in adjacent figure, then the square of the length of the tangent segment equals the product of the lengths of the secant segment and its external portion, i.e.



$\left(m\overline{AB}\right)^2 = \left(m\overline{DB}\right) \cdot \left(m\overline{BC}\right)$ or $t^2 = s \cdot e$

Note 3:

If two secants intersect outside a circle as shown in the adjacent figure, then product of the lengths of one secant segment and its external portion equals the product of the lengths of other secant segment and its external portion, i.e.



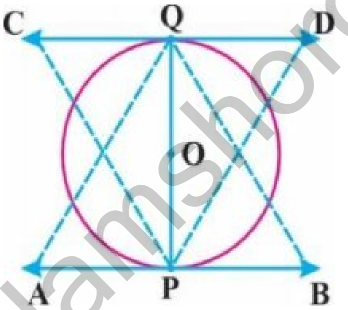
$\left(m\overline{AF}\right) \times \left(m\overline{BF}\right) = \left(m\overline{CF}\right) \times \left(m\overline{DF}\right)$ or $s_1 e_1 = s_2 e_2$

Example 1:

Prove that the tangents drawn at the ends of a diameter of a circle are parallel.

Given:

A circle centred at O. AB and CD are tangent at end points P and Q of the diameter, respectively.



To prove:

$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

Construction:

Draw $\overline{CP}, \overline{DP}, \overline{AQ}$ and $\overline{BQ}$.

Proof:

Statements	Reasons
$m\angle APO = 90^\circ = m\angle BPO$	$\overleftrightarrow{AB} \perp \overline{OP}$
$m\angle CQO = 90^\circ = m\angle DQO$	$\overleftrightarrow{CD} \perp \overline{OQ}$
$m\angle CPQ = m\angle BQP$ (i)	Alternating angles
and $m\angle AQP = m\angle QPD$ (ii)	Alternating angles
$\therefore$ $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	Same interior alternating angles, (i), (ii)

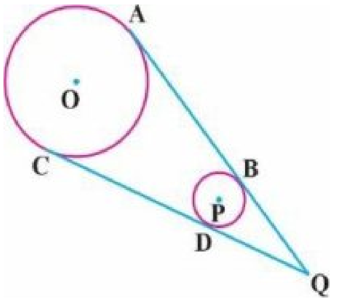
Q.E.D.

Example 2:

Prove that the two direct common tangents to two circles are equal in length.

Given:

$\overline{AB}$ and $\overline{CD}$ are directcommon tangents to circles with centres O and P.



To prove:

$m\overline{AB} = m\overline{CD}$

Construction:

Extend $\overline{AB}$ and $\overline{CD}$ to intersect at a point, say Q.

Proof:

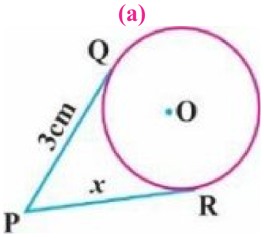
Statements	Reasons
$m\overline{AQ} = m\overline{CQ}$ (i)	Tangents drawn from point Q outside ⊙O.
$m\overline{BQ} = m\overline{DQ}$ (ii)	Tangents drawn from point Q outside ⊙P.
$m\overline{AQ} - m\overline{BQ} = m\overline{CQ} - m\overline{DQ}$	Subtracting (ii) from (i).
$\therefore$ $m\overline{AB} = m\overline{CD}$	Using figure.

Q.E.D.

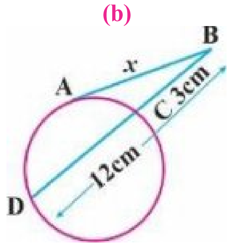
Example 3:

Find unknown length x in the following figures:

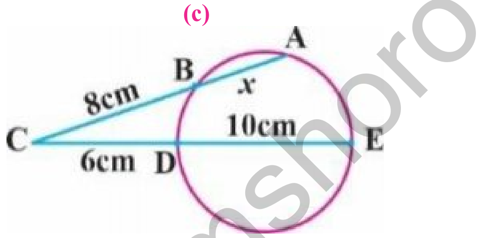
(a)



(b)



(c)



Solution:

(a). Tangents $\overline{PQ}$ and $\overline{PR}$ meet at point P outside the circle, so they must be equal in length, i.e. $m\overline{PQ} = m\overline{PR}$. Therefor, $x = 3\text{cm}$.

(b). The tangent $\overline{AB}$ meets the secant DC at point B outside the circle, so we have:
$\left(m\overline{AB}\right)^2 = \left(m\overline{DB}\right)\cdot\left(m\overline{BC}\right)$. (or) $x^2 = 12 \times 3 \Rightarrow x^2 = 36 \Rightarrow x = 6cm.$

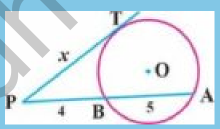
(c). Two secants AB and ED of circle intersect at C outside it, so we have:
$\left(m\overline{AC}\right)\times\left(m\overline{BC}\right) = \left(m\overline{EC}\right)\times\left(m\overline{CD}\right)$ (or) $(x+8)\times 8 = (10+6)\times 6 \Rightarrow 8x + 64 = 96$

Finally, $8x = 96 - 64 \Rightarrow 8x = 32 \Rightarrow x = 4\text{cm}.$

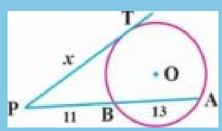
EXERCISE 26.2

1. Show that the tangents drawn at the ends of a chord in a circle make equal angles with the chord.
2. Show that if two tangents to a circle are parallel, then the points of tangency are the end points of a diameter of the circle.
3. Show that if two tangents to a circle are not parallel, then the points of tangency are end points of a chord of the circle.
4. Find unknown x in the following.

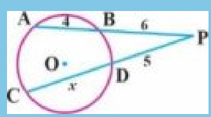
(a)



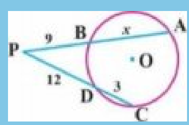
(b)



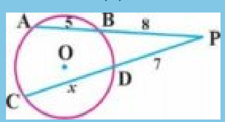
(c)



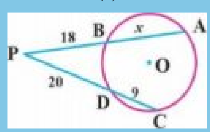
(d)



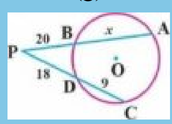
(e)



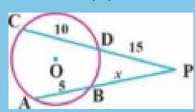
(f)



(g)

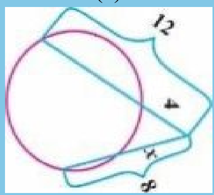


(h)

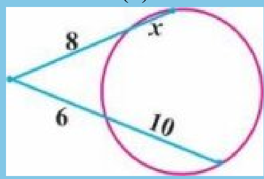


5. Find unknown x in the following.

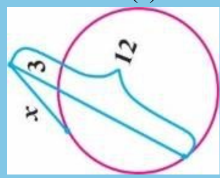
(a)



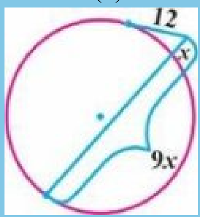
(b)



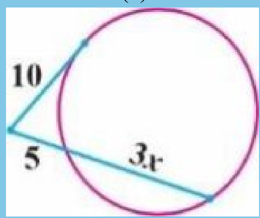
(c)



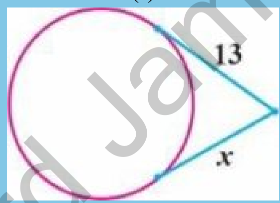
(d)



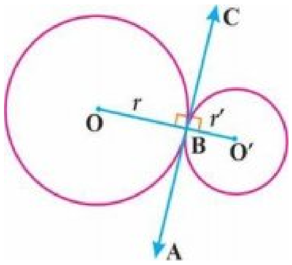
(e)



(f)



Theorem 26.4: Case (A): If two circles touch externally, the distance between their centres is equal to the sum of their radii.



Given: Circles with centres O and O' having radii r and r', respectively, touching each other externally at point B.

To prove: $m\overline{OO'} = r + r'$

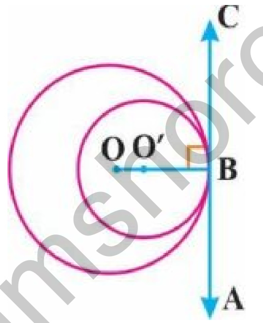
Construction: Draw a common tangent $\overleftrightarrow{AC}$ through B of both circles.

Proof:

Statements	Reasons
$\overline{OB} \perp \overleftrightarrow{AC}$ so $m\angle OBC = 90^\circ$ (i)	Tangent is perpendicular to radial segment.
$\overline{O'B} \perp \overleftrightarrow{AC}$ so $m\angle O'BC = 90^\circ$ (ii)	Tangent is perpendicular to radial segment.
$m\angle OBC + m\angle O'BC = 180^\circ$ (iii)	Adding (i) and (ii).
O, B and O′ are collinear, and B lies between O and O′.	From (i), (ii) and (iii)
$\therefore \quad m\overline{OB} + m\overline{O'B} = m\overline{OO'}$	By definition.
or $\quad m\overline{OO'} = r + r'$	$m\overline{OB} = r$ and $m\overline{O'B} = r'$.

Q.E.D.

Theorem 26.4: Case (B): If two circles touch internally, the distance between their centres is equal to the difference of their radii.



Given:

Circles with centres O and O' with radii r and r', respectively, touching each other internally at B.

To prove:

$m\overline{OO'} = r - r'$

Construction:

Draw a common tangent $\overleftrightarrow{AC}$ through B of both circles.

Proof:

Statements	Reasons
$\overline{OB} \perp \overleftrightarrow{AC}$ so $m\angle OBC = 90^\circ$ (i)	Tangent is perpendicular to radial segment
$\overline{O'B} \perp \overleftrightarrow{AC}$ so $m\angle O'BC = 90^\circ$ (ii)	Tangent is perpendicular to radial segment
$m\angle OBC = m\angle O'BC = 90^\circ$ (iii)	From (i) and (ii).
O, O′ and B are collinear, and O′ lies between O and B,	From (i), (ii) and (iii).
$\therefore$ $m\overline{OO'} + m\overline{O'B} = m\overline{OB}$ (iv)	By definition.
or $m\overline{OO'} = m\overline{OB} - m\overline{O'B} = r - r'$	From (iv), and $m\overline{OB} = r$, $m\overline{O'B} = r'$.

Q.E.D.

Corollary 1:

If distance between centres of two circles is sum (difference) of their radii, then circles touch externally (internally).

Corollary 2:

If the distance between centres of two circles is not equal to the sum or difference of their radii, then circles do not touch each other.

Example 1.

If three circles touch in pair externally, then the perimeter of a triangle formed by joining their centres is equal to twice the sum of their radii.

Given:

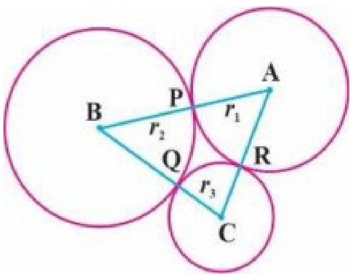
Three circles with centres A, B, C and radii r_1, r_2, r_3, respectively, touch in pairs externally at points P, Q and R.

To prove:

Perimeter of $\Delta ABC = 2(r_1 + r_2 + r_3)$.

Construction:

Construct $\overline{AB}, \overline{BC}$ and $\overline{AC}$ through P, Q and R, respectively, to form ΔABC.



Proof:

Statements	Reasons
$m\overline{AB} = m\overline{AP} + m\overline{BP}$ (i)	P lies between A and B.
$m\overline{BC} = m\overline{BQ} + m\overline{CQ}$ (ii)	Q lies between B and C.
$m\overline{AC} = m\overline{AR} + m\overline{CR}$ (iii)	R lies between A and C.
$m\overline{AB} + m\overline{BC} + m\overline{AC}$ $= m\overline{AP} + m\overline{BP} + m\overline{BQ} + m\overline{CQ} + m\overline{AR} + m\overline{CR}$	Adding (i), (ii) and (iii)
$m\overline{AB} + m\overline{BC} + m\overline{AC}$ $= \left(m\overline{AP} + m\overline{AR}\right) + \left(m\overline{BP} + m\overline{BQ}\right) + \left(m\overline{CQ} + m\overline{CR}\right)$	Re-arranging
$m\overline{AB} + m\overline{BC} + m\overline{AC} = \left(r_1 + r_1\right) + \left(r_2 + r_2\right) + \left(r_3 + r_3\right)$ $= 2r_1 + 2r_2 + 2r_3 = 2\left(r_1 + r_2 + r_3\right)$	$m\overline{AP} = m\overline{AR} = r_1$ $m\overline{BP} = m\overline{BQ} = r_2$ $m\overline{CQ} = m\overline{CR} = r_3$
So, Perimeter of $\Delta ABC = 2\left(r_1 + r_2 + r_3\right)$	

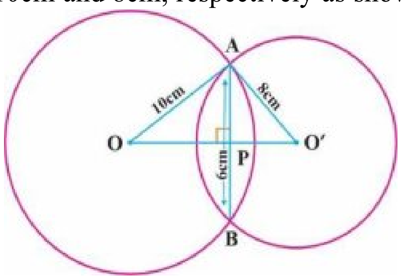
Q.E.D.

Example 2:

The radii of two intersecting circles are 10cm and 8cm. If the length of their common chord is 6cm, what is the distance between their centres?

Solution:

Consider circles with centres O and O' with radii 10cm and 8cm, respectively as shown in the figure. From figure, we have:



$m\overline{OA} = 10\text{cm}$, $m\overline{O'A} = 8\text{cm}$, $m\overline{AB} = 6\text{cm}$, $m\overline{OO'} = ?$

P bisects $\overline{AB}$, so $m\overline{AP} = \frac{1}{2} m\overline{AB} = \frac{6}{2} = 3\text{cm}$.

In ΔAPO, using Pythagoras' theorem we have:

$10^2 = 3^2 + \left(m\overline{OP}\right)^2 \Rightarrow m\overline{OP} = \sqrt{100 - 9} = \sqrt{91}\text{cm}$.

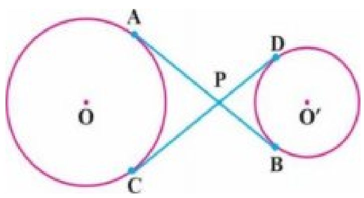
In $\Delta APO'$, using Pythagoras' theorem we have:

$8^2 = 3^2 + \left(m\overline{O'P}\right)^2 \Rightarrow m\overline{O'P} = \sqrt{64 - 9} = \sqrt{55}\text{cm}$

Finally, $m\overline{OO'} = m\overline{OP} + m\overline{O'P} = \sqrt{91} + \sqrt{55} = 16.955\text{cm}$ (approximately).

Example 3:

The two transverse common tangents to the two circles are equal in length.



Given: $\overline{AB}$ and $\overline{CD}$ are two internal (transverse) common tangents to circles with centres O and O', respectively.

To prove: $m\overline{AB} = m\overline{CD}$

Construction: Tangents $\overline{AB}$ and $\overline{CD}$ intersect at point P.

Proof:

Statements	Reasons
$m\overline{AP} = m\overline{CP}$ (i)	Tangents from a point outside circle.
$m\overline{BP} = m\overline{DP}$ (ii)	Tangents from a point outside circle.
$m\overline{AP} + m\overline{BP} = m\overline{CP} + m\overline{DP}$ (iii)	Adding (i) and (ii)
$\Rightarrow$ $m\overline{AB} = m\overline{CD}$	From (iii) and figure.

Q.E.D.

Example 4:

Two circles touch each other externally, and the length of segment joining their centres is 7cm. If radius of one is 3cm, what is area of other circle?

Solution:

Two circles touch externally, so:

Length of segment joining centres $= r_1 + r_2$

$$7 = 3 + r_2 \quad \Rightarrow \quad r_2 = 7 - 3 = 4cm.$$

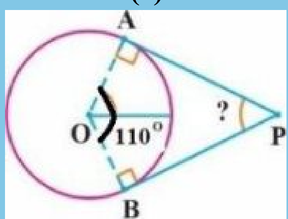
Now, area of second circle $= \pi r_2^2 = \pi \cdot (4)^2 = 16\pi \text{cm}^2$

$$= 16\left(\frac{22}{7}\right)\text{cm}^2 = 50.286\text{cm}^2 \text{ (approximately)}.$$

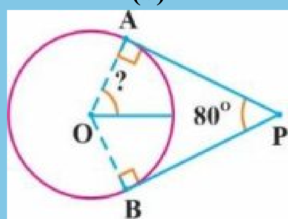
EXERCISE 26.3

1. If length of the segment joining centres of two circles is the sum of their radii, show that the circles touch externally.
2. If perimeter of the triangle with vertices at the centres of three circles is equal to the sum of their diameters, show that the three circles touch in pairs externally.
3. If length of segment joining two congruent circles touching externally is 12cm, find their radii and circumferences.
4. If $\overline{PA}$ and $\overline{PB}$ are tangents to the given circle from a point P outside as indicated in the figures. Find the unknown angle.

(a)



(b)



5. Show that if two circles touch externally, then the point of contact lies on the segment joining their centres.

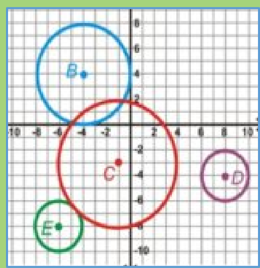
6. If two chords in a circle of radius 2.5cm are 3.9cm apart, and length of a chord is 1.4cm, then find the length of the other chord.
7. If three circles touch in pairs externally and radii of two circles are 4cm and 5cm, then find the radius of the third circle if the triangle formed through their centres has perimeter of 35cm.

Review Exercise 26

1. Tick the correct answer.

i. If a point is outside the circle then from this point we can draw ____ tangent(s) to the circle.
   (a) one (b) two (c) three (d) none

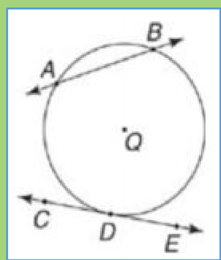
ii. Angle between the radial segment and tangent at its outer end point is ____ .
   (a) 45^0 (b) 60^0 (c) 90^0 (d) 120^0



iii. In the adjacent figure, circles with centres E and C __ .
   (a) touch internally (b) touch externally
   (c) do not touch (d) are congruent

iv. In the adjacent figure, circles with centres B and C have ______ point(s) of contact.
   (a) no (b) one
   (c) two (d) none of these

v. In adjacent figure, circles with centres E and C have ______ point(s) of contact.
   (a) no (b) one
   (c) two (d) none of these



vi. In the adjacent figure, $\overleftrightarrow{AB}$ is ________ .
   (a) tangent (b) secant
   (c) chord (d) none of these

vii. In the adjacent figure, $\overleftrightarrow{CDE}$ is _______ .
   (a) tangent (b) secant
   (c) chord (d) none of these

viii. In the adjacent figure, the point of tangency is __ .
   (a) A (b) B (c) C (d) D

ix. Tangents drawn at end points of a diameter of a circle are ______ to each other
   (a) parallel (b) perpendicular (c) intersecting (d) both (b) and (c)

x. The maximum number of common tangents between two circles touching internally is ____ .
   (a) 0 (b) 1 (c) 2 (d) 3

xi. The maximum number of common tangents between two circles touching externally is ____ .
   (a) 0 (b) 1 (c) 2 (d) 3

SUMMARY

- A straight line touching a circle at only one point is a tangent to the circle, and the common point is called point of tangency or point of contact.
- A straight line intersecting a circle at two points is a secant to the circle. There are two points of contact between a secant and the circle.
- A tangent segment is a line segment from the point of tangency to any other point on the tangent.
- The triangle formed by joining the centre of circle, the point of tangency and another point on tangent line is always a right angled triangle.
- When two circles touch internally or externally, they share a common tangent between them and the common point of contact.
- Two circles not touching each other can also have common tangents: external (direct) and internal (transverse) common tangents.
- The lengths of two direct common tangents to two circles are equal.
- The lengths of two transverse common tangents to two circles are equal.
- Only one tangent can be drawn to a circle at a point on it.
- If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.
- The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.
- The two tangents drawn to a circle from a point outside it are equal in length.
- If two circles touch externally, the distance between their centres is equal to the sum of their radii.
- If two circles touch internally, the distance between their centres is equal to the difference of their radii.

CHORDS AND ARCS

Unit 27

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with their corollaries and apply them to solve allied problems.
  - If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.
  - If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor,major, semi-circular) are congruent.
  - Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
  - If the angles subtended by the two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.

Introduction:

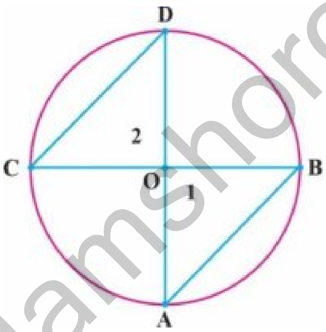
In this unit, we discuss the theorems related to chords and arcs of a circle.

Theorem 27. 1

If two arcs of a circle (or congruent circles) are congruent then the corresponding chords arc equal.

We shall prove the theorem:

i. For one circle
ii. For two congruent circles



i. For one circle

Given: A circle with center O whose $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ arc congruent arcs i.e. $\overset{\frown}{AB} \cong \overset{\frown}{CD}$.

$\overline{AB}$ and $\overline{CD}$ are the corresponding chords of the given congruent arcs.

To prove: $\overline{AB} \cong \overline{CD}$

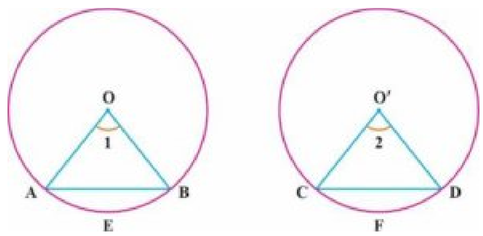
Construction: Join the points O with A,B,C and D.

Proof:

Statement	Reason
In $\Delta OAB \leftrightarrow OCD$	
$\overline{OA} \cong \overline{OC}$	Radii of same circle
$\overline{OB} \cong \overline{OD}$	Radii of same circle
$m\angle 1 = m\angle 2$	Central angles of two congruent arcs
$\Delta OAB \cong OCD$	S.A.S postulate
$\overline{AB} \cong \overline{CD}$	Corresponding sides of congruent triangles

Q.E.D

ii. For two congruent circles



Given: Two congruent circles with centers O and O′ respectively. $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are congruent arcs of these circles where $\overline{AB}$ and $\overline{CD}$ are the corresponding chords.

To prove: $\overline{AB} \cong \overline{CD}$

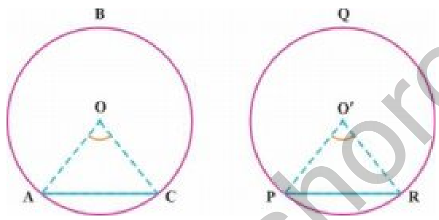
Construction: Join O to A and B. Join O′ to C and D.

Proof:

Statement	Reason
In $\Delta OAB \leftrightarrow \Delta O'CD$	
$\overline{OA} \cong \overline{O'C}$	Radii of same circle
$\overline{OB} \cong \overline{O'D}$	Radii of same circle
$m\angle 1 = m\angle 2$	Central angles of two congruent arcs
$\therefore$ $\Delta OAB \cong \Delta O'CD$	S.A.S postulate
$\therefore$ $\overline{AB} \cong \overline{CD}$	Corresponding sides of two congruent triangle

Theorem 27.2 (Converse of theorem 1)

If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major, semi-circle) are congruent.



Given: Two congruent circles with center O and O′ respectively having two congruent arcs i.e., $\overline{AC} \cong \overline{PR}$.

To prove: $\overset{\frown}{AC} \cong \overset{\frown}{PR}$

Construction: Join O with A and C, O′ with P and R.

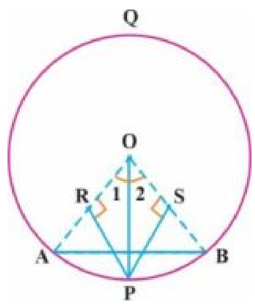
Proof:

Statement	Reason
In $\Delta AOC \leftrightarrow \Delta PO'R$	
$\overline{OA} \cong \overline{O'P}$	Radii of congruent circles
$\overline{OC} \cong \overline{O'R}$	Radii of congruent circles
$\overline{AC} \cong m\overline{PR}$	Given
$\Delta OAC \cong \Delta PO'R$	S.S.S ≅ S.S.S
$m\angle AOC = m\angle PO'R$ (i)	Corresponding angles of congruent Δs.
Hence $\overset{\frown}{AC} \cong \overset{\frown}{PR}$	From eq: (i)

Q.E.D

Example 1:

A point P on the circumference of a circle is equidistant from the radial segments $\overline{OA}$ and $\overline{OB}$. Prove that $m\overset{\frown}{AP} = m\overset{\frown}{BP}$



Solution:

Given: $\overline{AB}$ is the chord of a circle with center O. Point P on the circumference of the circle is equidistant from the radial segment $\overline{OA}$ and $\overline{OB}$. i.e. $m\overline{PR} = m\overline{PS}$

To prove: $m\overset{\frown}{AP} = m\overset{\frown}{BP}$

Construction: Join O with P. Write $m\angle 1$ and $m\angle 2$ as shown in the figure.

Proof:

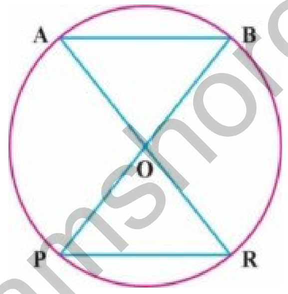
Statement	Reason
In right angle $\Delta OPR \leftrightarrow \Delta OPS$	
$m\overline{OP} = m\overline{OP}$	Common
$m\overline{PR} = m\overline{PS}$	
$\therefore$ $\Delta OPR \cong \Delta OPS$	H.S ≅ H.S
So $m\angle 1 = m\angle 2$ (i)	Corresponding angles of congruent Δs
$\Rightarrow$ $\overset{\frown}{AP} \cong \overset{\frown}{BP}$	From (i)
Hence $m\overset{\frown}{AP} = m\overset{\frown}{BP}$	By definition of congruent arcs

Theorem 27.3

Equal chords of a circle (or of congruent circles) subtend equal angles at the center (at the corresponding centers).

We shall prove theorem:

i) For one circle ii) For two circles



i) For one circle

Given: Circle with centre O, having two congruent chords i.e., $\overline{AB} \cong \overline{PR}$.

To prove: $m\angle AOB = m\angle POR$

Construction: Join O with A and B and also O with P and R.

Proof:

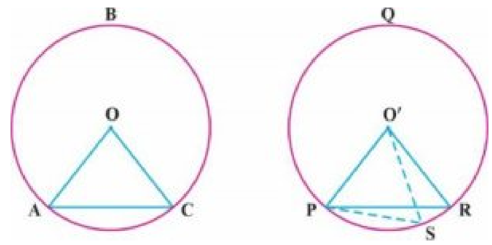
Statement	Reason
$\overline{AB} \cong \overline{PR}$ (i)	Given
$\overline{OB} \cong \overline{OR}$ (ii)	$\overline{OB}, \overline{OR}, \overline{OA}$ and $\overline{OP}$ are radii of same circle
and $\overline{OA} \cong \overline{OP}$ (iii)	
$\Delta AOB \cong \Delta POR$	S.S.S ≅ S.S.S
$m\angle AOB = m\angle POB$	Corresponding angles of congruent triangles

Q.E.D

ii) For two congruent circles

Given:

Two congruent circles with center O and O' respectively so that $m\overline{AC} = m\overline{PR}$. Where $\overline{AC}$ and $\overline{PR}$ are the chords of circles.



To prove:

$m\angle AOC = m\angle PO'R$

Construction:

Take a point S on arc $\overset{\frown}{PR}$ and join S to P and O' such that $\angle AOC \cong \angle PO'S$.

Proof:

Statement	Reason
Let $m\angle AOC = m\angle PO'R$	
But $m\angle AOC \cong m\angle PO'S$	Construction
$\therefore$ $\overset{\frown}{AC} \cong \overset{\frown}{PS}$ (i)	Arcs subtended by equal central angles of congruent circles
$\therefore$ $m\overline{AC} = m\overline{PS}$ (ii)	From theorem (1)
But $m\overline{AC} = m\overline{PR}$ (iii)	Given
$\therefore$ $m\overline{PR} = m\overline{PS}$	Using (ii) and (iii)
Which is only possible, if R concides with S.	Because P is common point
$\Rightarrow$ $m\angle AOC = m\angle POR$	Our supposition is working

Corollary 1:

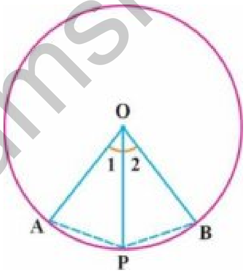
In congruent circles or in same circle, if central angles are equal then corresponding sectors are equal.

Corollary 2:

In congruent circles or in same circle, arcs will subtend unequal central angles for unequal chords.

Example 1:

Prove that the internal bisector of a central angle in a circle bisects the corresponding arc.



Given: A circle with center O. $\overline{OP}$ is an internal bisector of central angle AOB. i.e. $\angle 1 \cong \angle 2$

To prove: $\widehat{AP} \cong \widehat{PB}$

Construction: Draw $\overline{AP}$ and $\overline{BP}$, then write $\angle 1$ and $\angle 2$ are corresponding angles of $\widehat{AP}$ and $\widehat{BP}$ respectively.

Proof:

Statement	Reason
In $\Delta OAP \leftrightarrow \Delta OBP$	
$m\overline{OA} = m\overline{OB}$	Radii of same circle
$m\angle 1 = m\angle 2$	Given
and $m\overline{OP} = m\overline{OP}$	Common
$\therefore$ $\Delta OAP \cong \Delta OBP$	S.A.S postulate
Hence $\overline{AP} \cong \overline{BP}$	Corresponding sides of congruent Δs
$\Rightarrow$ $\widehat{AP} \cong \widehat{BP}$	Arcs corresponding to equal chords in a circle.

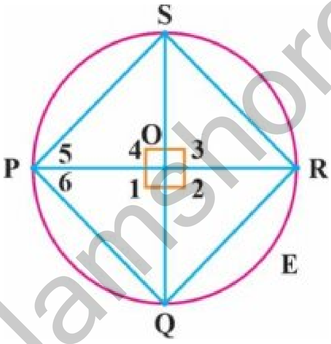
Q.E.D

Example 2:

Prove that in a circle, if two diameters are perpendicular to each other then the lines joining their ends form a square.

Given:

Given $\overline{PR}$ and $\overline{QS}$ are two perpendicular diameters of a circle with center O. So PQRS is a quadrilateral. Naming $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5$ and $\angle 6$ as shown in figure.



To prove:

PQRS is a square.

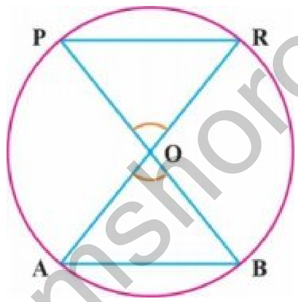
Proof:

Statement	Reason
$\overline{PR}$ and $\overline{QS}$ are two perpendicular diameters of a circle with center O.	Given
$m\angle 1 = m\angle 2 = m\angle 3 = m\angle 4 = 90^\circ$	Diameters are perpendicular to each other
$m\widehat{PQ} = m\widehat{QR} = m\widehat{RS} = m\widehat{SP}$ (i)	Arcs opposite to the equal central angles in a circle
$m\overline{PQ} = m\overline{QR} = m\overline{RS} = m\overline{SP}$ (ii)	Chords corresponding to equal arcs.
In right Δ POS,	
$m\overline{PO} = m\overline{OS}$ (iii)	$\overline{PO}$ and $\overline{OS}$ are radii of circle
ΔPOS is right isosceles, triangle	From (iii)
$m\angle 5 = 45^\circ$ (iv)	ΔPOS is right isosceles, triangle
Similarly, ΔPOQ is right isosceles triangle	By the above process
$m\angle 6 = 45^\circ$ (v)	
Moreover	
$m\angle P = m\angle 5 + m\angle 6$	Angle addition postulate
$m\angle P = 45^\circ + 45^\circ$	By using (iv) and (v)
$m\angle P = 90^\circ$ (vi)	
Similarly, $m\angle Q = m\angle R = m\angle S = 90^\circ$ (vii)	By the above process
PQRS is a square	Using eq (ii), (vi) and (vii)

Q.E.D

Theorem 27.4

If the angles subtended by two chords of a circle (or congruent circles) at the centres (corresponding centres) are equal, the chord are equal.



(a) For one circle

Given:

Circle with centre O, having chords $\overline{PR}$ and $\overline{AB}$. The angles substended by both arcs are congruent i.e. $\angle POR \cong \angle AOB$.

To prove: $m\overline{AB} \cong m\overline{PR}$

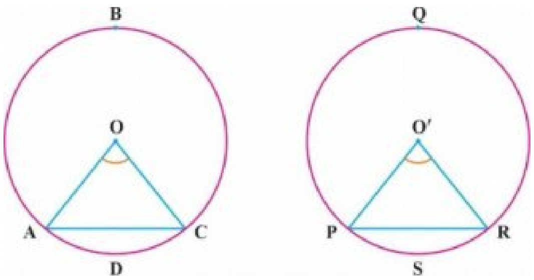
Construction: Join O with P and R, and also with A and B.

Proof:

Statement	Reason
$\overline{OR} \cong \overline{OB}$ (i)	Radius of same circle
and $\overline{OP} \cong \overline{OA}$ (ii)	Radii of same circle
$\angle POR \cong \angle AOB$ (iii)	Given
$\Delta PQR \cong \Delta AOB$	$S.A.S \cong S.A.S$ (Postulate)
$\overline{AB} \cong \overline{PR}$	Corresponding side of congruent triangles

Q.E.D

(b) For two congruent circles



Given:

Two congruent circles with center O and O′ with $\overline{AC}$ and $\overline{PR}$ are chords. Subtending equal angles at centre i.e. $\angle AOC \cong \angle PO'R$.

To prove: $m\overline{AC} = m\overline{PR}$

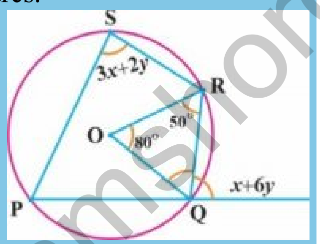
Proof:

Statement	Reason
In $\Delta OAC \leftrightarrow \Delta O'PR$	
$\overline{OA} \cong \overline{O'P}$	Radii of congruent circles
$\angle AOC \cong \angle PO'R$	Given
$\overline{OC} \cong \overline{O'R}$	Radii of congruent circle
$\therefore$ $\Delta OAC \cong \Delta O'PR$	S.A.S postulate
Hence $m\overline{AC} = m\overline{PR}$	Corresponding sides of congruent Δs

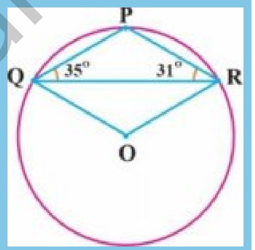
Q.E.D

EXERCISE 27.1

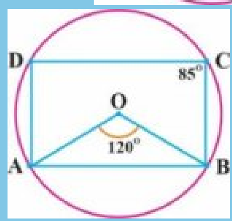
1. In a circle prove that the arcs between two parallel and equal chords or equal.
2. Prove that in equal circles, equal central angles have equal arcs.
3. In the following figure, O is the centre of the circle. Find the value of x and y.



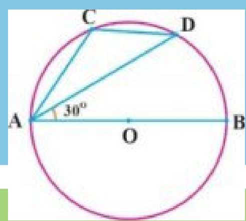
4. In the following figure $m\angle PQR = 35°$ and $m\angle PRQ = 31°$. Find $m\angle QPR$ and $m\angle OQR$.



5. In the figure O is the center of circle, given that $m\angle AOB = 120°$ and $m\angle BCD = 85°$. Find $m\angle OAD$



6. In the following figure, $\overline{AB}$ is diameter of the circle given that $m\angle DAB = 30°$ Find $m\angle ACD$



REVIEW EXERCISE 27

1. **Multiple Choice Question:**
   Tick the correct option.

i. If a chord of a circle subtends, a central angle of $60°$. Then the chord and the radial segment are _____.

(a) Parallel (b) Perpendicular (c) Congruent (d) Incongruent

ii. An arc subtends a central angle of $45°$ then the corresponding chords will subtend a central angle of ?

(a) $15°$ (b) $30°$ (c) $45°$ (d) $60°$

iii. A pair of chords of a circle subtending two congruent central angle are.

(a) Perpemdicular (b) Non congruent (c) Congruent (d) None of these

iv. The arcs opposite to congruent central angles of a circle are always.

(a) Parallel (b) Congruent (c) Perpendicular (d) None of these

v. A 6cm long chord subtends a central angle of 60°. The radial segment of this circle is

(a) 4cm (b) 6cm (c) 5cm (d) 8cm

vi. The chord length of a circle subtending a central angle of 180° is always

(a) equal to the radial segment
(b) less than radial segment
(c) double of radial segment
(d) half of the radial segment.

viii. The length of a chord and the radial segment of a circle are congruent, the central angle made by the chord will be:

(a) 60° (b) 75° (c) 90° (d) 45°

ix. Out of two congruent arcs of a circle, if one arc makes a central angle of 30° then the other arc will subtend the central angle:

(a) 60° (b) 90° (c) 75° (d) 30°

x. Diameter divides the circle into ______ parts.

(a) Two (b) Three (c) Four (d) All of these

SUMMARY

- The circles are congruent of their radii are equal.
- Equal chords of circle subtend equal angles at the centre.
- The straight line joining any two points of the circumference is called a chord of the circle.
- The portion of a circle bounded by an arc and a chord is known as the segment of a circle.
- The boundary traced by a moving point in a circle is called its circumference.
- An portion of the circumference of a circle is called an arc of the circle.
- A straight line, drawn from the centre of a circle bisecting a chord is perpendicular to the chord
- A perpendicular drawn from the centre of a circle on a chord, it bisect the chord.
- If two arcs of a circle(or of congruent circles) are congruent, then the corresponding chords are equal.
- If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major, semi circular) are congruent.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
- If the angle subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.

ANGLES IN A SEGMENT OF A CIRCLE

Unit 28

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with theircorollaries and apply them to solve allied problems.
- The measure of a central angle of a minor arc of a circle is double that of the angle subtended by the corresponding major arc.
- Any two angles in the same segments of a circleare equal.
- The angle
  - In a semi-circle is a right angle,
  - In a segment greater than the semi-circle is less than a right angle, (i.e., an acute angle)
  - In a segment less than a semi-circle is greater than a right angle, (i.e., an obtuse angle)
- The opposite angles of any quadrilateral inscribed in a circle are supplementary.

28.1 Angle in a segment of a circle

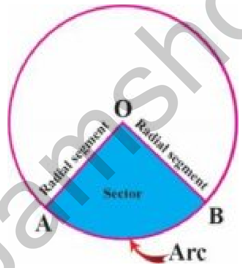
We have already studied most of the terms related to circle like chord, minor arc and major arc etc.

Now, we have to define some more terms in order to understand the theorems related to the angle in a segment of a circle.

Definitions:

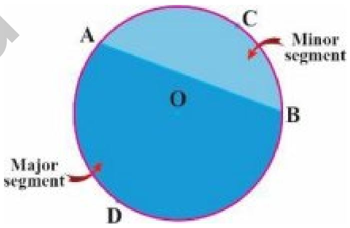
i. Sector of a circle:

A sector of a circle is the part of circular region bounded by two radial segments and the arc which they intercept.
In the given figure, AOB is a sector of the given circle, with centre O.



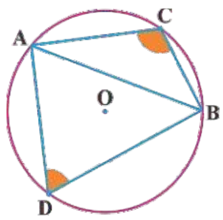
ii. Segment of a circle:

A segment of a circle is the part of circular region bounded by an arc and its chord.
A segment is called major segment if its arc is major arc. Similarly, a segment is called minor segment if its arc is minor arc. In the adjacent figure, ACB is a minor segment and ADB is a major segment.



iii. Angle in a segment of circle:

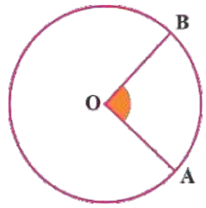
An angle in a segment of a circle is the angle which is subtended by the chord of the segment at a point other than the end points on the arc of the segment.
In the adjacent figure, $\angle ACB$ is an angle in the minor segment ACB.
Whereas $\angle ADB$ is an angle in the major segment ADB.



Note:
Angle in a segment of a circle is also known as inscribed angle of the arc of the segment or simply angle subtended by the arc

iv. Central angle of an arc:

A central angle of an arc of a circle is the angle subtended by the arc at the centre of the circle.
In the figure, $\angle AOB$ is the central angle of the arc AB.



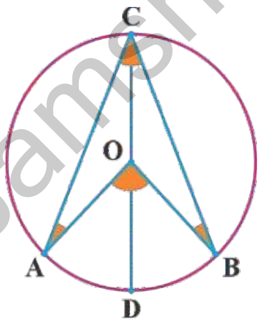
Understand the following theorems along with their corollaries and apply them to solve allied problems.

Theorem 28.1

The measure of a central angle of a minor arc of a circle is double that of the angle subtended by the corresponding major arc.

OR

The central angle of a minor arc is double in measure of the inscribed angle of the corresponding major arc.



Given:

In a circle with centre O, $\angle AOB$ is the central angle of minor arc AB.
$\angle ACB$ is the angle subtended by corresponding major arc ACB.

To prove:

$m\angle AOB = 2m\angle ACB$

Construction:

Draw $\overline{CO}$ and produce it to point D of the circle.

Proof:

Statements	Reasons
In ΔAOC, $\overline{OA} \cong \overline{OC}$	Radial segments of same circle
$m\angle OAC = m\angle OCA$ ...(i)	Angles opposite to congruent sides
Now $m\angle AOD = m\angle OAC + m\angle OCA$	Exterior angle is sum of opposite interior angles of a triangle.
$= 2m\angle OCA$ ...(ii)	Using eq:(i)
Similarly $m\angle BOD = 2m\angle OCB$ ...(iii)	By same process
Now $m\angle AOB = m\angle AOD + m\angle BOD$	Angle addition postulate
$= 2m\angle OCA + 2\,m\angle OCB$	Using (ii) and (iii)
$= 2(m\angle OCA + m\angle OCB)$	Taking 2 common
$m\angle AOB = 2\,m\angle ACB$	Angle addition postulate

Q.E.D

Corollary 1:

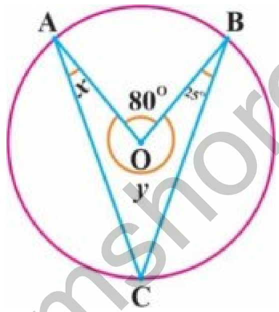
Measure of a central angle of a major arc is double that of the inscribed angle of the corresponding minor arc.

Corollary 2:

Measure of the central angle of a semi-circle is double that of the inscribed angle of the corresponding semi-circle.

Example 1:

In the adjacent figure, point O is the centre of the circle. Find the value of x if $m\angle AOB = 80°$ and $m\angle OBC = 25°$



Solution: In the figure, $\angle AOB$ is the central angle of $\widehat{AB}$ and $\angle ACB$ is inscribed angle of corresponding $\widehat{ACB}$

Central angle of $\widehat{AB}$ is double of the inscribed angle of the corresponding ACB.

$\therefore \quad m\angle AOB = 2m\angle ACB$

i.e. $\quad 80° = 2m\angle ACB$

$\Rightarrow \quad m\angle ACB = 40°$

As we know that the sum of all angles around a point is $360°$.

So,

$80° + y = 360°$

$\Rightarrow \quad y = 280°$

In quadrilateral AOBC

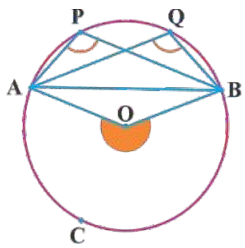
$x + y + 25° + m\angle ACB = 360°$

$\Rightarrow \quad x + 280° + 25° + 40° = 360°$

or $\quad x + 345° = 360°$

$\Rightarrow \quad x = 15°$

Theorem 28.2



Any two angles in the same segment of a circle are equal.

Given: $\angle APB$ and $\angle AQB$ are the two angles in the same segment APQB of circle with centre O.

To prove: $m\angle APB = m\angle AQB$

Construction: Join O with A and B so that $\angle AOB$ is the central angle of $\widehat{ACB}$.

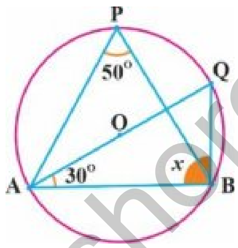
Proof:

Statements	Reasons
$\angle AOB$ is the central angle of $\widehat{ACB}$.	Construction
$\angle APB$ and $\angle AQB$ are the angles in the same segment	Given
Now $m\angle AOB = 2m\angle APB$	Central angle of major arc is double of inscribed angle of corresponding minor arc.
Also $m\angle AOB = 2m\angle AQB$	By same process
$2m\angle APB = 2m\angle AQB$	By transitive property
$m\angle APB = m\angle AQB$	Dividing both sides by 2

Corollary 1: Any two angles in a major segment are equal.

Example:

In the adjacent figure, $\angle P$ and $\angle Q$ are the two angles in the same segment APQB of circle with centre O. Find the value of x or $m\angle ABQ$ where the measures of angles are indicated in the figure.



Solution:

$\angle P$ and $\angle Q$ are the angles on the same segment

$\therefore \quad m\angle Q = m\angle P$

i.e. $\quad m\angle Q = 50^\circ \qquad (\because m\angle P = 50^\circ)$

In ΔABQ,

$m\angle A + m\angle Q + x = 180^\circ$

i.e. $\quad 30^\circ + 50^\circ + x = 180^\circ$

$\Rightarrow \quad x = 100^\circ$

EXERCISE 28.1

1. In figure 1, point O is the centre of the circle. Find x where $m\angle C = 30^\circ$.

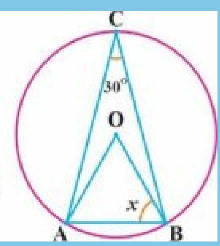


Figure 1

2. In figure 2, point O is the centre of the circle. Find x where $m\angle Q = 50^\circ$.

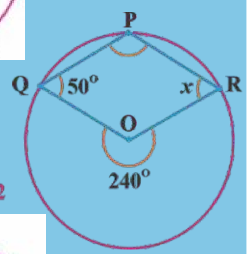


Figure 2

3. In figure 3, point O is the centre of the circle. Find $x + y$ when $m\angle D = 30^\circ$.

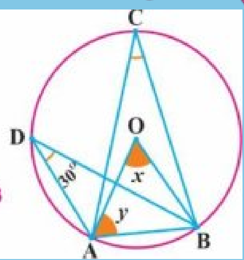


Figure 3

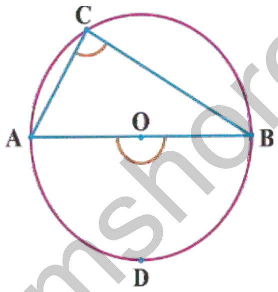
4. The inscribed angles of two congruent major arcs of two congruent circles are congruent. Prove it.
5. Prove that the inscribed angles of major arc and its corresponding minor arc in a circle are supplementary.

Theorem 28.3 (a)

The angle in a semi-circle is a right angle.

OR

The angle inscribed in a semi-circle is right angle.



Given:

In a circle with centre O, $\angle AOB$ is the central angle of semi-circle ADB.

$\angle ACB$ is the inscribed angle of corresponding semi-circle.

To prove: $\angle ACB$ is a right angle.

Proof:

Statements	Reasons
$m\angle AOB = 180°$ ...(i)	Central angle of semi-circle
Now $m\angle AOB = 2m\angle ACB$	Central angle of semi circle is double of inscribed angle of corresponding semi-circle
$\Rightarrow$ $2m\angle ACB = 180°$	Using eq: (i)
or $m\angle ACB = 90°$	Dividing both sides by 2
i.e. $\angle ACB$ is right angle	By definition of right angle.

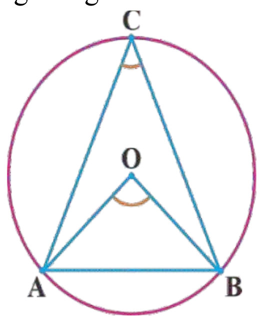
Q.E.D

Theorem 28.3 (b)

The angle in a segment greater than the semi-circle is less than a right angle.

OR

The angle inscribed in a major arc is acute.



Given:

In a circle with centre O, $\angle ACB$ is the angle in the segment ACB greater than semi-circle.

$\angle AOB$ is the central angle of corresponding minor segment AB.

To prove: $\angle ACB$ is an acute angle.

Proof:

Statements	Reasons
$m\angle AOB < 180°$ ...(i)	Central angle of minor arc
Now $m\angle AOB = 2m\angle ACB$	Central angle of minor arc is double of inscribed angle of corresponding major arc.
$\Rightarrow$ $2m\angle ACB < 180°$	Using eq: (i)
or $m\angle ACB < 90°$	Dividing both sides by 2
i.e. $\angle ACB$ is an acute angle	By definition of acute angle.

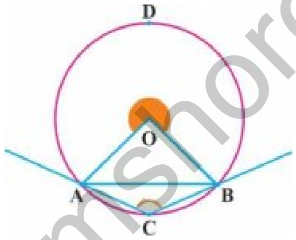
Q.E.D

Theorem 28.3 (c)

The angle in a segment less than a semi-circle is greater than a right angle.

OR

The angle inscribed in a minor arc is obtuse.



Given:

In a circle with centre O, $\angle ACB$ is the inscribed angle of minor arc AB.
$\angle AOB$ is the central angle of corresponding major arc ADB.

To prove: $\angle ACB$ is obtuse.

Proof:

Statements	Reasons
$m\angle AOB > 180^\circ$ ...(i)	Central angle of major arc
Now $m\angle AOB = 2m\angle ACB$	Central angle of major arc is double of inscribed angle of corresponding minor arc.
$\Rightarrow$ $2m\angle ACB > 180^\circ$	Using eq: (i)
or $m\angle ACB > 90^\circ$	Dividing both sides by 2
i.e. $\angle ACB$ is obtuse	By the definition of obtuse angle.

Q.E.D

Example:

In figure 1, Point O is the centre of the circle. Find x and y where $m\angle BCO = 30^\circ$. $\overline{AC}$ and $\overline{BD}$ are diameters.

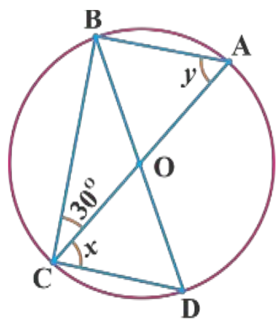


Figure 1

Solution:

$\overline{BD}$ is diameter

$\therefore$ $\angle BCD$ is inscribed angle of semi-circle

Hence $m\angle BCD = 90^\circ$

$\Rightarrow$ $x + 30^\circ = 90^\circ$

$\Rightarrow$ $x = 60^\circ$

$\overline{AC}$ is diameter

$m\angle CBA$ is inscribed angle of semi-circle

So $m\angle CBA = 90^\circ$

In ΔABC,

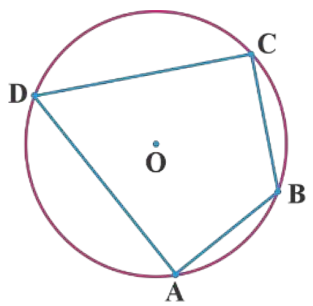
$30^\circ + m\angle CBA + y = 180^\circ$

$30^\circ + 90^\circ + y = 180^\circ$

$\Rightarrow$ $y = 60^\circ$

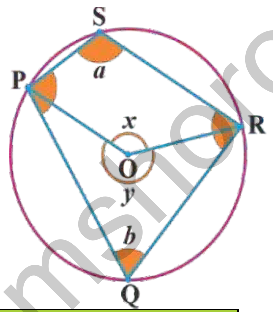
Inscribed Quadrilateral or Cyclic Quadrilateral:

A quadrilateral is called inscribed quadrilateral or cyclic quadrilateral if all of its vertices lie on the same circle. In the figure, ABCD is a cyclic quadrilateral.



The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Given: PQRS is a quadrilateral inscribed in a circle with center O.



To prove:

$m\angle Q + m\angle S = 180°$

and $m\angle P + m\angle R = 180°$

Construction:

Draw $\overline{OP}$ and $\overline{OR}$ so that $\angle POR$ or $\angle x$ is central angle of $\overparen{PR}$ and $\angle y$ is central angle of corresponding $\overparen{PQR}$.

Proof:

Statements	Reasons
$\angle S$ or $\angle a$ is inscribed angle of $\overparen{PR}$	By definition
$\angle y$ is central angle of corresponding $\overparen{PQR}$.	Construction
$m\angle y = 2m\angle a$	Central angle of major arc is double of inscribed angle of corresponding minor arc.
$\Rightarrow \quad m\angle a = \frac{1}{2} m\angle y$...(i)	Dividing both sides by 2
Now	
$\angle Q$ or $\angle b$ is inscribed angle of $\overparen{PQR}$.	By definition
$\angle x$ is central angle of corresponding $\overparen{PR}$.	Construction
So, $m\angle x = 2m\angle b$	Central angle of minor arc is double of inscribed angle of corresponding major arc.
i.e. $\quad m\angle b = \frac{1}{2} m\angle x$...(ii)	Dividing both sides by 2
Now	
$m\angle a + m\angle b = \frac{1}{2} m\angle x + \frac{1}{2} m\angle y$	Adding eq: (i) and eq: (ii)
$= \frac{1}{2}(m\angle x + m\angle y)$	Taking $\frac{1}{2}$ common
$= \frac{1}{2}(360°)$	Sum of all angles around a point is 360°.
i.e. $\quad m\angle a + m\angle b = 180°$	
or $\quad m\angle Q + m\angle S = 180°$	$\angle a \cong \angle S$ and $\angle b \cong \angle Q$
Similarly	
$m\angle P + m\angle R = 180°$	By same process

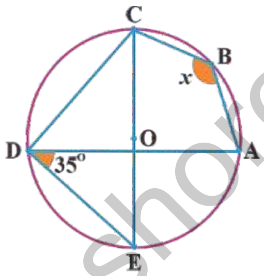
Q.E.D

Corollary:

A parallelogram inscribed in a circle is rectangle.

Example:

In given figure, point O is the centre of the circle. $\overline{CE}$ is its diameter and ABCD is cyclic quadrilateral. Find x if $m\angle ODE = 35^\circ$.



Solution:

$\angle CDE$ is inscribed angle of semi-circle

$\therefore \quad m\angle CDE = 90^\circ$

i.e. $\quad 35^\circ + m\angle ADC = 90^\circ$

$\Rightarrow \quad m\angle ADC = 55^\circ$

Now

In cyclic quadrilateral ABCD

$x + m\angle ADC = 180^\circ$

$\Rightarrow \quad x + 55^\circ = 180^\circ$

$\Rightarrow \quad x = 180^\circ - 55^\circ$

$\Rightarrow \quad x = 125^\circ$

EXERCISE 28.2

1. In figure 1, O is the centre of the circle, whereas $\overline{AC}$ is its diameter. Find x.

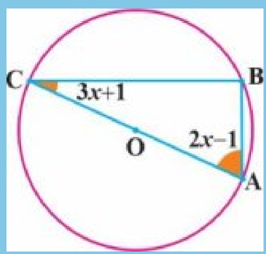


Figure 1

2. In figure 2, $\overset{\frown}{ACB}$ is a major arc of circle with centre O and $\angle C$ is its inscribed angle. Find x where $x \in \mathbb{R}$.

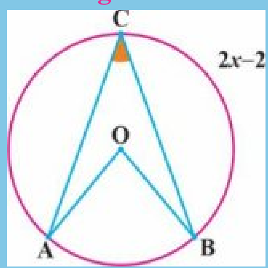


Figure 2

3. In a circle, $7x-1$ is the measure of inscribed angle of a minor arc of a circle. Find x where $x \in \mathbb{R}$.
4. Prove that a rhombus inscribed in a circle is a square.
5. Prove that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Review Exercise 28

1. Tick the correct option.

i. Inscribed angle of --------- is obtuse.
(a) minor arc (b) major arc (c) semi-circle (d) all of these

ii. -----------is the circular region bounded by arc and its chord
(a) Sector (b) Segment
(c) Major arc (d) Circumference

iii. If $2x$ is the measure of inscribed angle of minor arc then the central angle of corresponding major arc is ---------.
(a) x (b) $2x$ (c) $3x$ (d) $4x$

iv. If $2x$ and 60° are the measures of inscribed angles of same segment then $x =$-------.
(a) 120° (b) 60° (c) 20° (d) 30°

v. The inscribed angle of minor arc of circle is ------------ angle.
(a) acute (b) obtuse
(c) right (d) reflex

vi. The inscribed angle of major arc of circle is ------------ angle.
(a) acute (b) obtuse
(c) right (d) reflex

vii. Sum of opposite angles of cyclic quadrilateral is -------------.
(a) 90° (b) 180° (c) 270° (d) 360°

viii. A parallelogram inscribed in a circle is --------------.
(a) kite (b) trapezium
(c) rectangle (d) rhombus

ix. The central angle of an arc is ----- than inscribed angle of corresponding arc.
(a) less (b) greater
(c) less or equal (d) greater or equal

x. The sum of central angles of all the arcs of a circle is -----------.
(a) 90° (b) 180° (c) 360° (d) 1000°

SUMMARY

- The circular region bounded by an arc and two radial segments is called sector.
- The circular region bounded by an arc and its chord is called a segment.
- Angle in a segment of circle is also known as inscribed angle of the segment.
- Angle subtended by an arc at the centre of circle is called the central angle.
- The central angle of minor (or major) arc is double than inscribed angle of the corresponding major (or minor) arc respectively.
- Any two angles in the same segment of a circle are equal.
- The angle inscribed in semi-circle is right angle.
- The angle inscribed in major arc is acute.
- The angle inscribed in minor arc is obtuse.
- All the vertices of a cyclic quadrilateral are on the same circle.
- The opposite angles of cyclic quadrilateral are supplementary.

PRACTICAL GEOMETRY - CIRCLES

Unit 29

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Locate the centre of a given circle
- Draw a circle passing through three given non-collinear points.
- Complete the circle:
  - By finding the centre,
  - Without finding the centre, when a part of its circumference is given.
- Circumscribe a circle about a given triangle.
- Inscribe a circle in a given triangle.
- Escribe a circle to a given triangle.
- Circumscribe an equilateral triangle about agiven circle.
- Inscribe an equilateral triangle in a given triangle.
- Inscribe an equilateral triangle in a given circle
- Circumscribe a square about a given circle.
- Circumscribe a regular hexagon about a givencircle.
- Inscribe a regular hexagon in a given circle.
- Draw a tangent to a given arc, without using thecentre, through a given point P, when P is :
  - The middle point of the arc,
  - At the end of the arc,
  - Outside the arc.
- Draw a tangent to a given circle from a point *P*, when *P* lies
  - On the circumference,
  - Outside the circle.
- Draw two tangents to a circle meeting each otherat a given angle.
- Draw
  - Direct common tangent or external tangent,
  - Transverse common tangent or internaltangent to two equal circles,
- Draw
  - Direct common tangent or external tangent,
  - Transverse common tangent or internaltangent to two unequal circles,
- Draw a tangent to
  - To unequal touching circles,
  - Two un equal intersecting circles,
- Draw a circle which touches
  - Both the arms of a given angle,
  - Two converging lines and passes through agiven point between them,
  - Three converging lines.

Introduction:

We know that Practical Geometry is an important branch of Geometry which is used in architecture, computer graphics, art etc. We have already learnt to construct angles, triangles, rectangle, pentagon etc in previous classes. Let us learn the construction of circles and related figures.

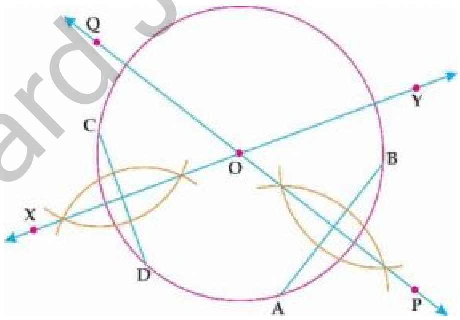
29.1 Construction of circles:

We know that a circle can easily be drawn by using compass when its centre and radius are given. In this section we will also learn to draw circles when centre and radius are not given

29.1(i) Locate the centre of a given circle:

In order to locate the centre of a given circle, we will use the fact that the right bisectors of two non-parallel chords of a circle always intersect each other at the centre of the circle. Method of locating the centre of a given circle is explained with the help of the following example.

Example: Locate the centre of given circle as shown in the figure.



Given: A circle

Required: To locate the centre of the given circle.

Steps of Construction:

(i) Draw two non-parallel chords $\overline{AB}$ and $\overline{CD}$ of the given circle.

(ii) Draw the right bisector $\overleftrightarrow{PQ}$ of the chord $\overline{AB}$.

(iii) Draw the right bisector $\overleftrightarrow{XY}$ of the chord $\overline{CD}$

(iv) Both the right bisectors $\overleftrightarrow{PQ}$ and $\overleftrightarrow{XY}$ intersect each other at point O. The point O is the centre of the given circle.

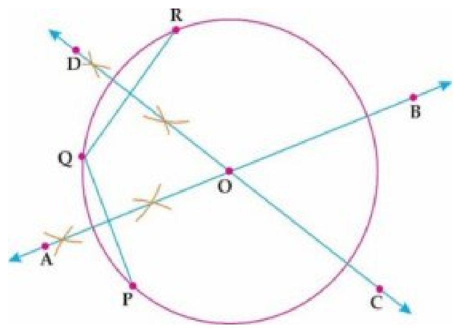
29.1(ii) Draw a circle passing through three given non-collinear points.

In order to draw a circle passing through three given non-collinear points, we will first locate the centre and then draw the circle as explained in the following example.

Example: Draw a circle passing through three non-collinear point P, Q and R

Given: Three non-collinear points P, Q and R.

Required: To draw a circle which passes through P, Q and R.



Steps of Construction:

(i) Take any three non-collinear points P, Q and R

(ii) Draw $\overline{PQ}$ and $\overline{QR}$

(iii) Draw the right bisector $\overleftrightarrow{AB}$ of $\overline{PQ}$

(iv) Draw the right bisector $\overleftrightarrow{CD}$ of $\overline{QR}$

(v) Both right bisectors $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect each other at point O which is the centre of the required circle.

(vi) With O as centre and radius equal to $m\overline{OP}$ or $m\overline{OQ}$ or $m\overline{OR}$, draw a circle.
This circle is the required circle.

29.1(iii) Complete the Circle:

- **By finding the centre.**
- **Without finding the centre.**

When a part of its circumference is given.

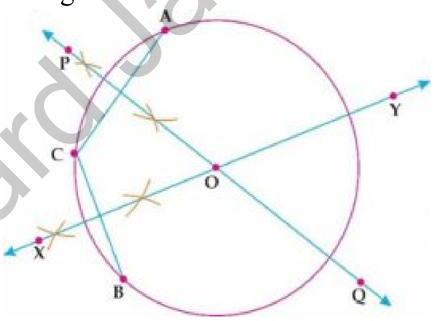
(a) Completion of circle by finding the centre when a part of its circumference is given.

We can complete a circle when a part of its circumference or an arc is given by finding its centre with the help of its any three non-collinear points. The method is explained in the following example.

Example: Complete a circle whose arc AB is given by finding the centre.

Given: A part of circumference or an arc AB is given.

Required: To complete a circle whose arc is given.



Steps of Construction:

(i) Draw the arc AB

(ii) Take a point C on AB other than A and B.

(iii) Draw $\overline{AC}$ and $\overline{BC}$.

(iv) Draw the right bisector $\overset{\frown}{PQ}$ of $\overline{AC}$

(v) Draw the right bisector $\overleftrightarrow{XY}$ of $\overline{BC}$

(vi) Both $\overleftrightarrow{PQ}$ and $\overleftrightarrow{XY}$ interest each other at point O which is the centre of the circle.

(vii) With point O as centre and radius equal to $m\overline{AO}$ or $m\overline{OB}$ or $m\overline{OC}$,
draw the remaining part of the circle.

Thus we have a complete circle of the given arc $\overset{\frown}{AB}$.

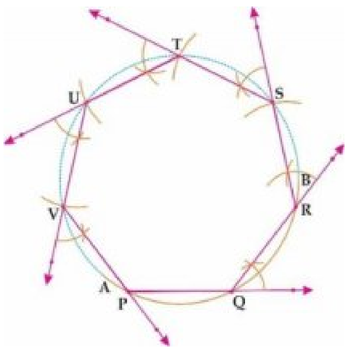
(b) Completion of a circle without finding the centre when a part of its circumference is given.

We can complete a circle without finding the centre when an arc is given with the help of regular polygon. We draw regular polygon with the help of either exterior angles or interior angles which are congruent. The method is explained with the help of the following example.

Example: Complete a circle whose arc AB is given without finding centre.

Given: An arc AB of a circle.

Required: To complete the circle of AB



Steps of Construction:

(i) Draw two congruent chords $\overline{PQ}$ and $\overline{QR}$ of given $\overset{\frown}{AB}$

(ii) Draw exterior $\angle R$ congruent to $\angle Q$ and draw $\overline{RS}$ congruent to $\overline{PQ}$

(iii) Draw exterior $\angle S$ congruent to $\angle R$ and draw $\overline{ST}$ congruent to $\overline{PQ}$

(iv) By the same way, we get point S, T, U and V

(v) So we get a regular heptagon PQRSTUV

(vi) Draw dotted arcs of $\overline{RS}$, $\overline{ST}$, $\overline{TU}$, $\overline{UV}$ and $\overline{VP}$

These dotted arcs complete the circle.

EXERCISE: 29.1

1. Draw a circle with the help of a circular object. Locate its centre.
2. Take three non-collinear points X, Y, Z and draw a circle which passes through these points.
3. Draw a minor arc AB by a circular object. Complete the circle by finding the centre.
4. Draw a major arc ABC by a circular object. Complete the circle by finding the centre.
5. Draw a minor arc AB by a circular object. Complete the circle without finding the centre.

29.2 Circles attached to polygons:

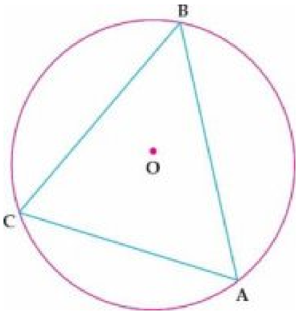
Recall that polygon is a three or more sided closed figure for example, triangle, quadrilateral, pentagon, hexagon etc.

The circles attached to polygons are those circles which either pass through all the vertices of the polygon or touch the sides of the polygon.

29.2 (i) Circumscribe a circle about a given triangle:

Circumcircle of a triangle:

A circle which passes through all the three vertices of a triangle is called circumcircle of the triangle. In the given figure, there is circumcircle of ΔABC with centre O which is called circumcentre.



Example: Draw the circumcircle of ΔABC in which $m\overline{AB} = 6cm$, $m\overline{BC} = 5cm$ and $m\overline{AC} = 7cm$.

Given: A triangle ABC in which $m\overline{AB} = 6cm$, $m\overline{BC} = 5cm$ and $m\overline{AC} = 7cm$.

Required: To draw a circumcircle of ΔABC.

Steps of Construction:

(i) Construct a ΔABC with the help of given data.

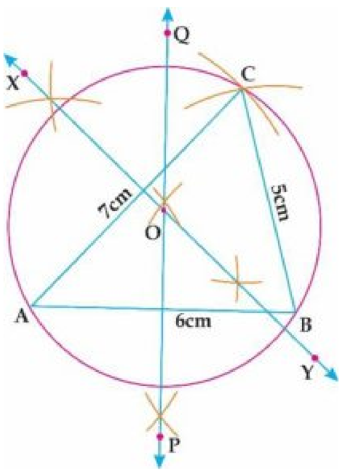
(ii) Draw the right bisector $\overleftrightarrow{PQ}$ of $\overline{AB}$

(iii) Draw the right bisector $\overleftrightarrow{XY}$ of $\overline{AC}$

(iv) Both the right bisectors $\overleftrightarrow{PQ}$ and $\overleftrightarrow{XY}$ intersect each other at point O.

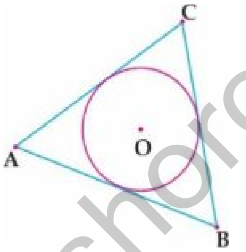
(v) With O as centre and radius equal to $m\overline{OA}$ or $m\overline{OB}$ or $m\overline{OC}$, draw a circle which passes through A, B and C.

This is the required circumcircle of ΔABC.



29.2(ii) Inscribe a circle in a given triangle:

Inscribed circle or Incircle of a triangle:



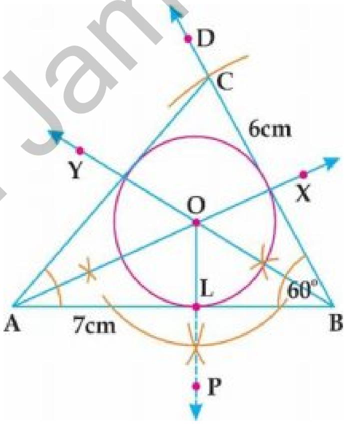
A circle which touches all the sides of a triangle is called incircle or inscribed circle of the triangle. In the adjacent figure, there is an incircle of ΔABC with centre O which is called the incentre

Example: Draw an inscribed circle of ΔABC in which $m\overline{AB} = 7cm$, $m\angle B = 60^\circ$ and $m\overline{BC} = 6$ cm

Given: A triangle ABC with $m\overline{AB} = 7cm$, $m\angle B = 60^\circ$ and $m\overline{BC} = 6cm$

Required: To draw an inscribed circle of ΔABC

Steps of Construction:



(i) Construct a ΔABC with the help of given data.

(ii) Draw the internal bisector $\overrightarrow{AX}$ of $\angle A$

(iii) Draw the internal bisector $\overrightarrow{BY}$ of $\angle B$

(iv) Both the internal bisectors $\overrightarrow{AX}$ and $\overrightarrow{BY}$ intersect each other at point O.

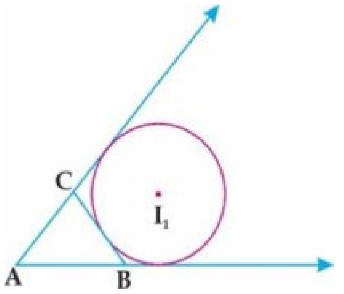
(v) Draw perpendicular OP on $\overline{AB}$ from point O. OP intersects $\overline{AB}$ at point L

(vi) With point O as centre and radius equal to $m\overline{OL}$, draw a circle which touches all the three sides of ΔABC.

This circle is the required inscribed circle.

29.2(iii) Escribe a circle to a given triangle

Escribed Circle or Excircle.



A circle which touches one side of a triangle externally and the two produced sides internally is called an escribed circle or excircle of the triangle.

In the given figure, there is an escribed circle of ΔABC opposite to vertex A with centre I_1, which is called excentre.

Similarly I_2 and I_3 are excenters of excircles of ΔABC opposite to vertices B and C respectively.

Example: Draw an escribed circle opposite to vertex A of ΔABC where $m\overline{AB} = 5cm$, $m\angle A = 30^\circ$ and $m\angle B = 60^\circ$

Given: A triangle ABC in which $m\overline{AB} = 5cm$, $m\angle A = 30^\circ$ and $m\angle B = 60^\circ$

Required: To draw an escribed circle opposite to vertex A of ΔABC.

Steps of Construction.

(i) Construct a ΔABC with the help of given data.

(ii) Produce $\overline{AC}$ beyond C and produce $\overline{AB}$ beyond B making two exterior angles BCE and CBF respectively.

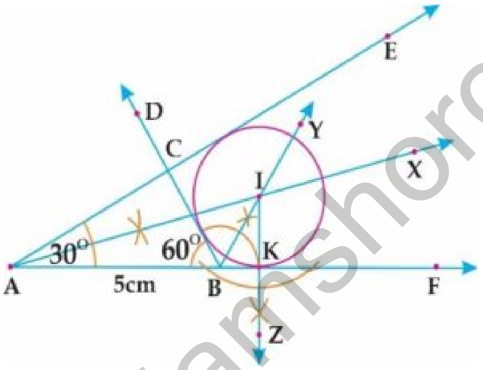
(iii) Draw the internal bisector $\overrightarrow{AX}$ of $\angle A$.

(iv) Draw the internal bisector $\overrightarrow{BY}$ of $\angle CBF$.

(v) Both bisectors $\overrightarrow{AX}$ and $\overrightarrow{BY}$ intersect each other at point I

(vi) Draw a perpendicular $\overrightarrow{IZ}$ on $\overrightarrow{AF}$ from I which cuts $\overrightarrow{AF}$ at point K.

(vii) With I as centre and radius equal to $m\overline{IK}$, draw a circle which touches $\overline{BC}$ externally and $\overrightarrow{AE}$ and $\overrightarrow{AF}$ internally.



This is the required escribed circle of ΔABC opposite to vertex A.

29.2(iv) Circumscribe an equilateral triangle about a given circle.

We will draw an equilateral triangle which circumscribes about a given circle by using the fact that if a circle is divided into three congruent arcs and tangents drawn at the points of division will be the sides of the equilateral triangle circumscribing about the circle. The method is explained in the following example.

Example: Draw a circle of radius 3cm with centre at point O and circumscribe an equilateral triangle about this circle.

Given: A circle of radius 3cm with centre at point O.

Required: To circumscribe an equilateral triangle about the given circle.

Steps of Construction:

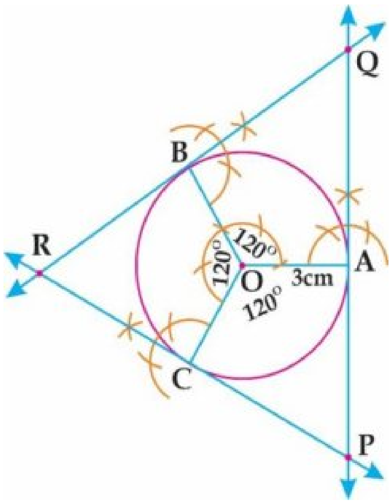
(i) Draw a circle of radius 3cm with centre at point O.

(ii) Divide the circle into three congruent arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{AC}$ by dividing central angle into three congruent angles.

(iii) Draw tangents at points A,B and C by making right angles at these points.

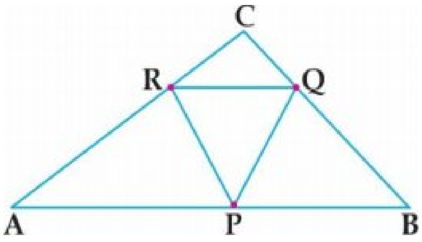
(iv) These tangents intersect each other at points P, Q and R.

Now PQR is the required equilateral triangle.



29.2(v) Inscribe an equilateral triangle in a given triangle

We know that an equilateral triangle is said to be inscribed in a given triangle if all the vertices of the equilateral triangle lie on different sides of the given triangle. In the adjacent figure, ΔPQR is the equilateral triangle inscribed in the given ΔABC.



Example: Draw an equilateral triangle inscribed in a ΔABC where $m\overline{AB} = 4cm$, $m\overline{BC} = 5cm$ and $m\overline{AC} = 7cm$.

Given: A triangle ABC in which $m\overline{AB}$=4cm, $m\overline{BC}$=5cm and $m\overline{AC}$=7cm.

Required: To draw an equilateral triangle inscribed in ΔABC.

Steps of Construction:

(i) Construct ΔABC with given data.

(ii) Draw the internal bisector BD of $\angle B$ which is the suitable angle (usually the largest angle)

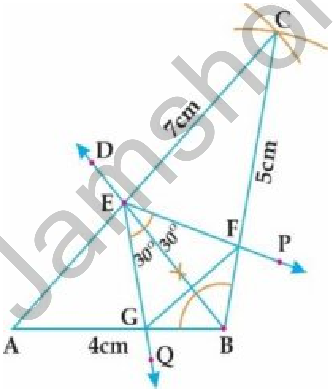
(iii) The internal bisector $\overrightarrow{BD}$ intersects $\overline{AC}$ at point E

(iv) At point E, draw two angles BEP and BEQ each of measure $30°$

(v) EP and EQ intersect $\overline{BC}$ and $\overline{AB}$ at points F and G respectively.

(vi) Draw $\overline{FG}$

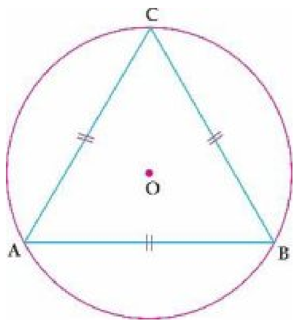
Now EFG is the required inscribed equilateral triangle of ΔABC



29.2(vi) Inscribe an equilateral triangle in a given circle

Inscribed equilateral triangle in a circle

An equilateral triangle is called inscribed equilateral triangle in a circle if the circle passes through its all the vertices. In the adjacent figure ABC is an inscribed equilateral triangle of circle with centre O.



Example: Draw an inscribed equilateral triangle of a circle with centre at O and radius of 3cm

Given: A circle of radius 3cm and with centre at point O.

Required: To draw an inscribed equilateral triangle of the given circle.

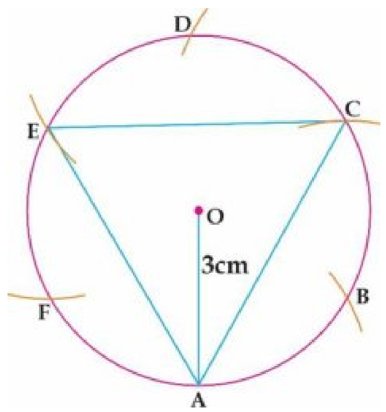
Steps of Construction:

(i) Draw a circle of radius 3cm with centre at point O.

(ii) Take a point A on the circle. Starting with A as centre and radius of 3cm divide the circle in six congruent parts at point B, C, D, E and F

(iii) Draw $\overline{AC}, \overline{AE}$ and $\overline{CE}$

Now AEC is the required inscribed equilateral triangle of the given circle.



EXERCISE: 29.2

1. Construct the ΔABC and draw its circumcircle in each case.
   (i) $m\overline{AB} = 5.5cm$, $m\overline{AC} = 6cm$ and $m\angle A = 50°$
   (ii) $m\overline{AB} = 6cm$, $m\overline{BC} = 4.5cm$ and $m\overline{AC} = 5cm$
2. Construct the ΔPQR and draw its incircle in each case
   (i) $m\overline{PQ} = 5cm$, $m\overline{QR} = 6.5cm$ and $m\overline{RP} = 5.5cm$
   (ii) $m\overline{PQ} = 6cm$, $m\angle P = 60$ and $m\angle Q = 50°$
3. Construct the ΔXYZ and draw its escribed circle opposite to $\angle Y$ in each case.
   (i) $m\overline{XY} = 4.5cm$, $m\overline{YZ} = 5cm$ and $m\angle Y = 30°$
   (ii) $m\overline{XY} = 5.5cm$, $m\overline{YZ} = 6cm$ and $m\overline{XZ} = 2.5cm$
4. Draw a circle of radius 3.5cm with centre at point O and circumscribe an equilateral triangle about this circle.
5. Draw an equilateral triangle inscribed in ΔPQR where
   $m\overline{PQ} = 4.5cm$, $m\overline{QR} = 5.5cm$ and $m\overline{PR} = 8cm$.
6. Draw an inscribed equilateral triangle of a circle with radius 3.8cm with centre O.

29.2(vii): Circumscribe a square about a given circle.

Before going to circumscribe or inscribe a square or hexagon or any other polygon about or in the given circle respectively. We must know the following theorem regarding regular polygons attached to the circles.

If the circumference of a circle be divided into n equal arcs then

(i) the points of division are the vertices of a regular n-gon inscribed in the circle.
(ii) the tangents drawn to the circle at these points will be the sides of a regular n-gon circumscribing about the circle.

It means if we have to circumscribe or inscribe a square, regular pentagon, regular hexagon etc with a given circle then we have to divide the given circle into four, five, six etc equal parts respectively.

Let us learn how to circumscribe a square about a given circle with the help of the following example.

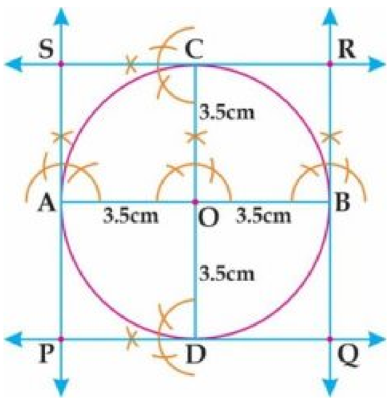
Example: Circumscribe a square about a circle whose radius is 3.5cm and centre at point O.

Given: A circle with radius 3.5cm and centre at point O.

Required: To construct a square which circumscribes the given circle.

Step of construction:

(i) With point O as centre, draw a circle of radius 3.5cm.
(ii) Draw a diameter $\overline{AB}$.
(iii) Draw another diameter $\overline{CD}$ which is perpendicular to $\overline{AB}$.



(iv) Draw tangents at points A,B,C and D.

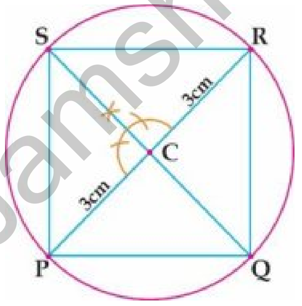
(v) These tangents intersect each other at point P,Q,R and S.
Now PQRS is the required square which circumscribes about the given circle.

Let us learn how to inscribe a square in a given circle with the help of the following example.

Example: Construct a square inscribed in the circle whose radius is 3cm and centre at point C.

Given: A circle with radius 3cm and centre at C.

Required: To construct a square inscribed in the given circle.



Step of construction:

(i) With C as centre , draw a circle of radius 3cm

(ii) Draw a diameter $\overline{PR}$

(iii) Draw a diameter $\overline{QS}$ which is perpendicular to $\overline{PR}$

(iv) Given circle has been divided into four equal parts at point P,Q,R and S

(v) Draw $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ and $\overline{SP}$. Now PQRS is the required inscribed square in the given circle.

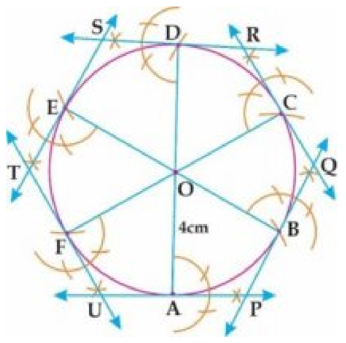
29.2(viii) circumscribe a regular hexagon about a given circle:

The method of circumscribing a regular hexagon about a given circle is explained with the help of the following example.

Example: Circumscribe a regular hexagon about a circle of radius 4cm with centre at point O.

Given: A circle of radius 4cm with centre at point O

Required: To construct a regular hexagon which circumscribes the given circle.



Steps of construction:

(i) With O as centre draw a circle of radius 4cm

(ii) Take any point A on the circle.

(iii) Starting with A and radius equal to $m\overline{OA}$ draw arcs to divide given circle in six equal parts with points of division are A,B,C,D,E and F

(iv) Draw tangents at these points of division

(v) These tangents intersect each other at points P,Q,R,S,T and U
Now PQRSTU is the required regular hexagon circumscribing the given circle.

29.2 (ix) Inscribe a regular hexagon in a given circle:

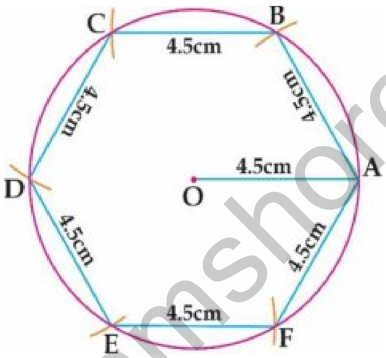
The method of inscribing a regular hexagon is explained with the help of the following example.

Example: Construct a regular hexagon inscribed in a circle of radius 4.5 cm and centre at point O.

Given: A circle of radius 4.5cm and with centre at point O.

Required: To construct a regular hexagon which inscribes the given circle.

Steps of construction:



(i) With centre at point O, draw a circle of radius 4.5cm

(ii) Take any point A on the circle.

(iii) Starting with point A and radius equal to $m\overline{AO}$ or 4.5cm draw arcs to divide the given circle in six equal parts at points A ,B, C, D, E and F

(iv) Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$ and $\overline{AF}$

Now ABCDEF is the required regular hexagon which is inscribed in the given circle.

EXERCISE 29.3

1. Circumscribe a square about a circle of radius 4.2 cm and with centre at point C.
2. Inscribe a square in a circle of radius 3.6 cm with centre at point O.
3. Circumscribe a regular hexagon about a circle of radius 4.8 cm with centre at point O.
4. Inscribe a regular hexagon in a circle of radius 4.4 cm with centre at point C
5. Inscribe a regular pentagon in a circle of radius 4.3 cm and centre O
   (Hint: Draw five congruent angles at centre)

29.3 Tangents to a circle

Recall that tangent to a circle is a line which touches the circle at a single point. Tangent and radial segment of a circle are always perpendicular to each other at the point of contact

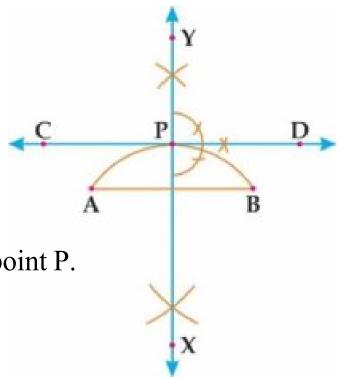
29.3 (i) Draw a tangent to a given arc without using the centre through a given point P when P is

- **The middle point of the arc**
- **At the end of the arc.**
- **Outside the arc**

Case1. When P is the middle point of the arc

The method of drawing tangent to a given arc without using the centre through a point P which is middle point of the arc , is explained with the help of the following example.

Example: Take an arc AB. Draw tangent to the arc AB through the middle point P of AB without using centre.



Given: Point P is the mid-point of $\overset{\frown}{AB}$

Required: To draw a tangent through P without using centre

Step of construction:

(i) Take an arc $\overset{\frown}{AB}$

(ii) Draw the chord AB

(iii) Draw the right bisector $\overleftrightarrow{XY}$ of $\overline{AB}$ which cuts the arc at point P.

(iv) At point P, draw a perpendicular $\overleftrightarrow{CD}$ on $\overleftrightarrow{XY}$

Thus $\overleftrightarrow{CD}$ is the required tangent

Case2. When P is at the end of the arc

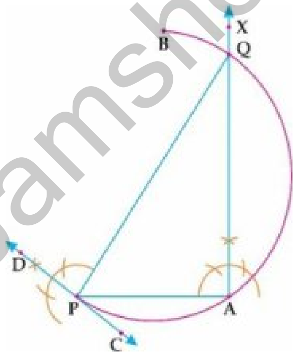
The method of drawing tangent to an arc without using centre through an end point P of the arc is explained with the help of the following example

Example: Take an arc $\overset{\frown}{PAB}$

Draw a tangent to the $\overset{\frown}{PAB}$ through its end-point P without using centre.

Given: Point P is the end point of $\overset{\frown}{PAB}$

Required: To draw a tangent to $\overset{\frown}{PAB}$ at the end point P without using centre.



Steps of Construction:

(i) Draw a $\overset{\frown}{PAB}$ with radius and centre of your choice

(ii) Draw chord $\overline{AP}$

(iii) Draw a perpendicular $\overrightarrow{AX}$ on $\overline{PA}$ at point A which cuts the given arc at point Q.

(iv) Draw $\overline{PQ}$ which is the diameter of the arc

(v) At point P draw a perpendicular $\overrightarrow{CD}$ on $\overline{PQ}$

Now $\overrightarrow{CD}$ is the required tangent.

Case 3. When P is outside the arc

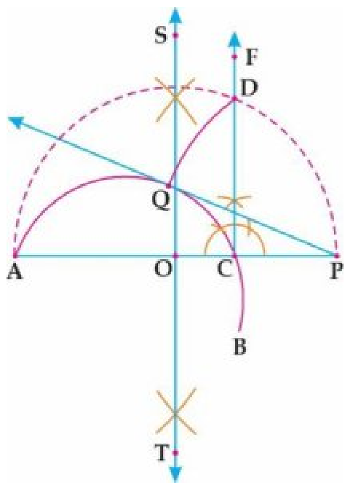
The method of drawing tangent to an arc through point P which is outside the arc without using centre is explained with the help of the following example.

Example: Take an arc $\overset{\frown}{AB}$

Draw a tangent to the arc $\overset{\frown}{AB}$ through point P which is outside the arc without using centre

Given: A point P outside an arc $\overleftrightarrow{AB}$

Required: To draw a tangent to the $\overset{\frown}{AB}$ from the point P without using centre



Steps of Construction:

(i) Draw an arc AB of your choice

(ii) Take a point P outside the AB

(iii) Draw $\overline{AP}$ which cuts the given arc at point C.

(iv) Draw the right bisector ST of $\overline{AP}$ which cuts $\overline{AP}$ at point O.

(v) With O as centre and radius equal to $m\overline{AO}$ or $m\overline{OP}$, draw a semi-circle

(vi) Draw a perpendicular $\overleftrightarrow{CF}$ on $\overline{AP}$ at point C which cuts the semi-circle at point D.

(vii) With P as centre and radius equal to $m\overline{PD}$, draw an arc which cuts $\overset{\frown}{AB}$ at point Q.

(viii) Draw $\overrightarrow{PQ}$

Now $\overrightarrow{PQ}$ is the required tangent.

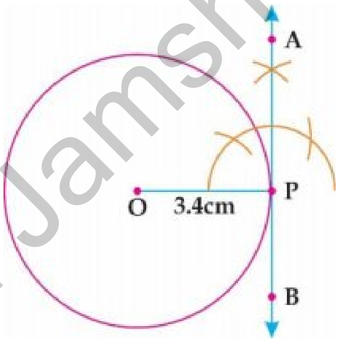
29.3 (ii) Draw a tangent to a given circle from a point P, when P lies

- On the circumference
- Outside the circle

Case1. When point P is on the circumference

The method of drawing a tangent to a given circle from a point P when P lies on the circumference is explained with the help of the following example.

Example: Draw a circle of radius 3.4cm with centre at O. Draw tangent to the circle at a point P of its circumference.



Given: A circle of radius 3.4cm with centre at point O and a point P on it circumference

Required: To draw a tangent to the circle at point P.

Steps of Construction:

(i) With point O as centre, draw a circle of radius 3.4cm

(ii) Take a point P on its circumference

(iii) Draw $\overline{OP}$

(iv) At point P, draw a perpendicular $\overleftrightarrow{AB}$ on $\overline{OP}$

Now $\overleftrightarrow{AB}$ is the required tangent.

Case2. When point P is outside the circle

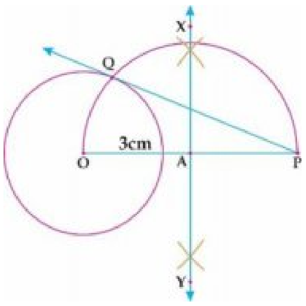
The method of drawing a tangent to a given circle from a point P which lies outside the circle is explained with the help of the following example.

Example: Draw a circle of radius 3cm with centre O. Draw a tangent to the circle from a point P outside the circle.

Given: A circle with centre O and radius of 3cm. Also a point P outside the circle

Required: To draw a tangent to the circle from point P outside it

Steps of Construction:



(i) With O as centre, draw a circle of radius 3cm

(ii) Take point P outside the circle

(iii) Draw $\overline{OP}$

(iv) Draw the right bisector $\overleftrightarrow{XY}$ of $\overline{OP}$ which cuts it at point A.

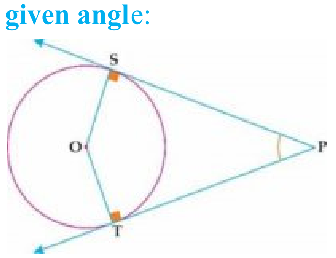
(v) With A as centre and radius equal to $m\overline{AO}$, draw a semi-circle which cuts the given circle at point Q

(vi) Draw $\overrightarrow{PQ}$

Now $\overrightarrow{PQ}$ is the required tangent

29.3 (iii) Draw two tangents to a circle meeting each other at a given angle:

Consider a circle with centre O. $\overrightarrow{PS}$ and $\overrightarrow{PT}$ are two tangents to the circle where $\angle P$ is the angle between the tangents.



Here $m\angle PSO = 90°$ ($\overline{OS} \perp \overrightarrow{PS}$)

and $m\angle PTO = 90°$ ($\overline{OT} \perp \overrightarrow{PT}$)

Now, in quadrilateral PSOT

$$m\angle P + m\angle PSO + m\angle O + m\angle PTO = 360^\circ$$

i.e., $m\angle P + 90^\circ + m\angle O + 90^\circ = 360^\circ$

$$\Rightarrow\ m\angle P + m\angle O = 180^\circ$$

$$\Rightarrow\ \boxed{m\angle O = 180^\circ - m\angle P}$$

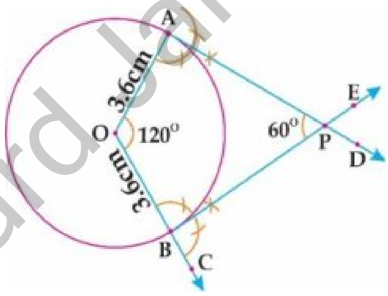
i.e., the central angle of circle joining the points of contact is supplement of the angle between the tangents

Using this fact, we will draw two tangents to a circle meeting each other at a given angle. The method is explained with the help of the following example.

Example: Draw two tangents to a circle of radius 3.6cm and centre at point O meeting each other at angle of measure 60°

Given: A circle with centre O and radius of 3.6cm whereas angle between the tangents is 60°

Required: To draw two tangents to the given circle meeting each other at angle of 60°



Steps of Construction:

(i) With centre O , draw a circle of radius 3.6cm

(ii) Take a point A on the circle and join it with O

(iii) Draw $\angle AOC$ of 120° (i.e $180^\circ - 60^\circ = 120^\circ$) such that $\overrightarrow{OC}$ cuts the given circle at point B

(iv) Draw perpendicular $\overrightarrow{AD}$ on $\overline{OA}$ at point A.

(v) Draw perpendicular $\overrightarrow{BE}$ on $\overline{OB}$ at point B.

(vi) Both $\overrightarrow{AD}$ and $\overrightarrow{BE}$ intersect each other at point P

Thus $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are the required tangents such that $m\angle P = 60^\circ$

EXERCISE 29.4

1. Take a minor arc $\overset{\frown}{PQ}$. Draw a tangent to $\overset{\frown}{PQ}$ through its midpoint A without using centre.
2. Take a major arc $\overset{\frown}{PQR}$. Draw a tangent to $\overset{\frown}{PQR}$ through its endpoint Q without using centre.
3. Take a minor arc $\overset{\frown}{XY}$. Draw a tangent to $\overset{\frown}{XY}$ through point Z which is outside arc without using centre
4. Draw a circle of radius 2.9cm with centre C. Draw a tangent to the circle from a point P of the circle.
5. Draw a circle of radius 3.2cm with centre P. Draw tangent to the circle from a point Q which is at a distance of 8cm from P.
6. Draw two tangents to a circle of radius 3.3cm with centre C meeting each other at an angle of measure (i) 50° (ii) 63°

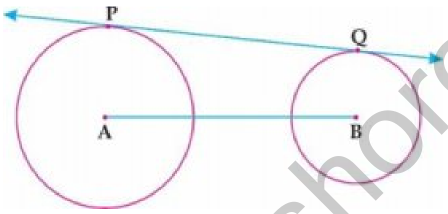
29.3(iv) Draw

- **Direct common tangents or external tangents.**
- **Transverse common tangents or internal tangents to two equal circles.**

(a) Drawing of direct common tangents or external tangents to two equal circles

Direct Common Tangent:

A common tangent is called direct common tangent or external tangent to two circles if it does not intersect the line segment joining the centres of the two given circles.



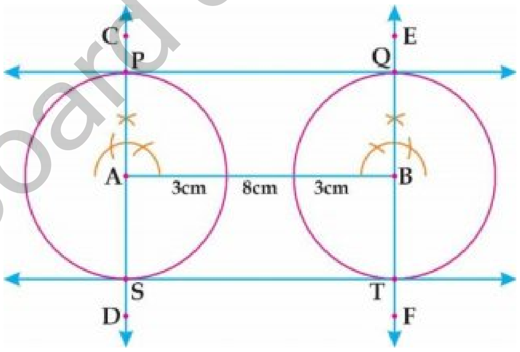
In the figure, $\overleftrightarrow{PQ}$ is a direct common tangent to two circles with centres at A and B.

Note that $\overleftrightarrow{PQ}$ does not intersect $\overline{AB}$. The method of drawing direct common tangents to two equal circles is explained by the following example.

Example: Draw two direct common tangents to two equal circles with centres at points A and B and having radius 3cm each such that $m\overline{AB} = 8cm$

Given: Two equal circles of radius 3cm each and centres at points A and B such that $m\overline{AB} = 8cm$

Required: To draw two direct common tangents to the given circles



Steps of Construction:

(i) Draw $\overline{AB}$ of 8cm

(ii) Draw two equal circles, each of radius 3cm at points A and B.

(iii) Draw perpendicular $\overleftrightarrow{CD}$ on $\overline{AB}$ at point A which cuts the circle at points P and S

(iv) Draw perpendicular EF on $\overline{AB}$ at point B which cuts the other circle at points Q and T

(v) Draw $\overleftrightarrow{PQ}$ and $\overleftrightarrow{ST}$

Now $\overleftrightarrow{PQ}$ and $\overleftrightarrow{ST}$ are the required direct common tangents to the given circles.

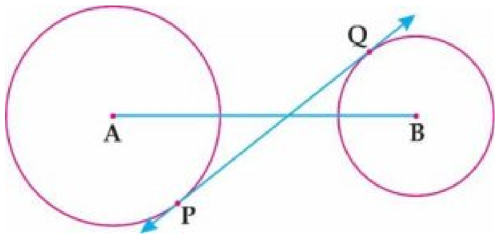
(b) Drawing of transverse common tangents or internal tangents to two equal circles

Transverse Common Tangent:

A common tangent is called transverse common tangent or internal tangent to two circles if it intersects the line segment joining the centres of the two given circles.

In the figure, $\overleftrightarrow{PQ}$ is a transverse common tangent to two given circles with centres A and B.

Note that $\overleftrightarrow{PQ}$ intersects $\overline{AB}$. The method of drawing transverse common tangents to two equal circles is explained with the help of the following example.



Example: Draw two transverse common tangents to two equal circles each of radius 2.8cm with centres at points A and B whereas $m\overline{AB} = 7.8$cm

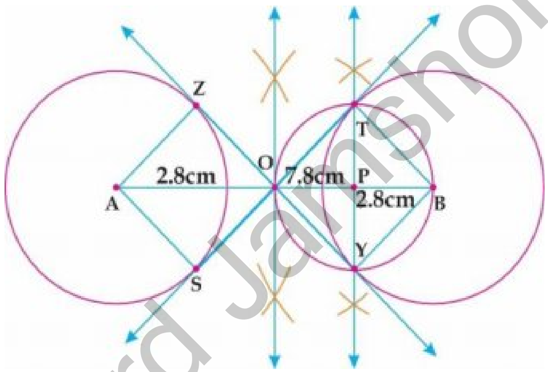
Method 1

Given:

Two circles each of radius 2.8cm with centres at A and B where $m\overline{AB}$=7.8cm

Required:

To draw two transverse common tangents to the given circles



Steps of Construction:

(i) Draw $\overline{AB}$ of 7.8cm

(ii) At points A and B , draw two circles each of radius 2.8cm

(iii) Draw the right bisector of $\overline{AB}$ which meets it at point O.

(iv) Draw a right bisector of $\overline{OB}$ meeting it at point P

(v) With P as centre and radius equal to $m\overline{OP}$, draw a circle which cuts the circle of centre B at points T and Y.

(vi) Draw $\overline{BT}$ and $\overline{BY}$

(vii) Draw $\overline{AZ}$ $\overline{BY}$ and $\overline{AS}$ $\overline{BT}$ using set squares.

(viii) Draw $\overleftrightarrow{ST}$ and $\overleftrightarrow{YZ}$.

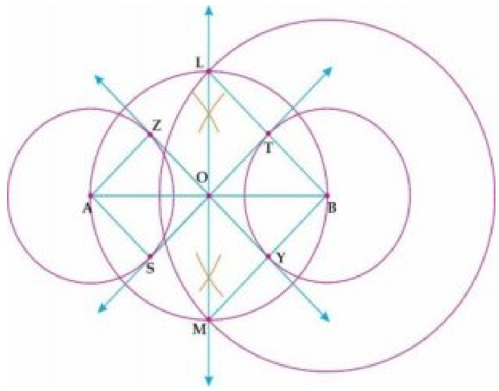
Now $\overleftrightarrow{ST}$ and $\overleftrightarrow{YZ}$ are the required transverse common tangents.

Method-2

Given: Two circles each of radius 2.8cm with centres at A and B such that $m\overline{AB} = 7.8$cm

Required: To draw two transverse common tangents to the given circles

Steps of Construction:



(i) Draw $\overline{AB}$ of 7.8cm.

(ii) At points A and B, draw two circles each of radius 2.8cm.

(iii) Bisect $\overline{AB}$ at O and draw a circle of radius equal to $m\overline{AO}$ with centre O.

(iv) With B as centre and radius equal to 5.6cm (double of 2.8cm), draw a circle which cuts the previous circle of centre O at points L and M.

(v) Draw $\overline{BL}$ and $\overline{BM}$ which cut the smaller circle with centre B at points T and Y respectively.

(vi) Draw $\overline{AS}$ $\overline{BT}$ and $\overline{AZ}$ $\overline{BY}$ by using set squares.

(vii) Draw $\overleftrightarrow{ST}$ and $\overleftrightarrow{YZ}$.

Now $\overleftrightarrow{ST}$ and $\overleftrightarrow{YZ}$ are the required transverse common tangents.

29.3(v) Draw

- **Direct common tangents or external tangents.**
- **Transverse common tangents or internal tangents to two unequal circles.**

(a) Drawing of direct common tangents or external tangents to two unequal circles.

Following example is suitable for learning the method of drawing direct common tangents to two unequal circles

Example: Draw direct common tangents to the given circles of radius 3.4cm and 2.1cm, where distance between their centres is 7.5 cm.

Method 1

Given: Two circles with centres A and B having radii 3.4 cm and 2.1 cm respectively such that $m\overline{AB} = 7.5cm$

Required: To draw direct common tangents to the given circles.

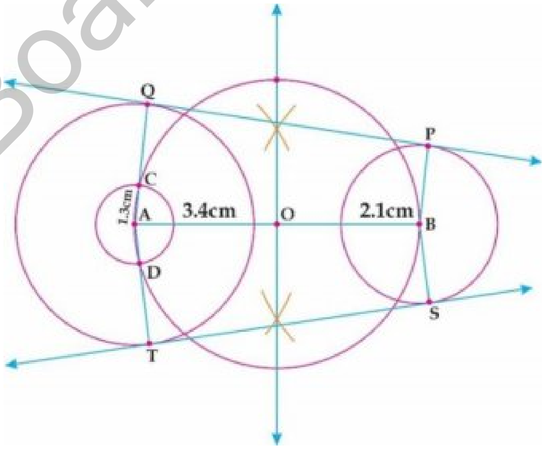
Steps of Construction:

(i) Draw $\overline{AB}$ of 7.5cm

(ii) With centres A and B, draw two circles of radii 3.4 cm and 2.1cm respectively.

(iii) Bisect $\overline{AB}$ at point O and draw a circle of radius equal to $m\overline{AO}$ with centre O.

(iv) With centre A of larger circle, draw a circle of radius 1.3cm $(\text{i.e. } 3.4 - 2.1 = 1.3)$ which cuts the previous circle at points C and D.



(v) Draw $\overline{AC}$ and produce it to intersect the given concentric circle at point Q

(vi) Draw $\overline{AD}$ and produce it to intersect the given concentric circle at point T

(vii) Draw $\overline{BP} \parallel \overline{AQ}$ and $\overline{BS} \parallel \overline{AT}$ by using set squares

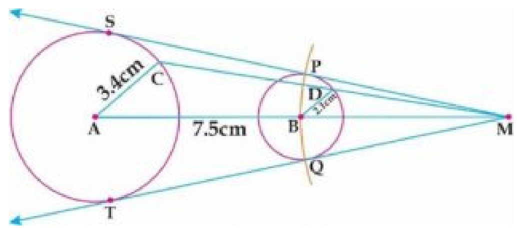
(viii) Draw $\overleftrightarrow{PQ}$ and $\overleftrightarrow{ST}$

Now $\overleftrightarrow{PQ}$ and $\overleftrightarrow{ST}$ are the required direct common tangents.

Method 2

Given: Two circles with centres, A and B and having radii 3.4cm and 2.1 cm respectively such that $m\overline{AB} = 7.5cm$

Required: To draw direct common tangents to the given circles.



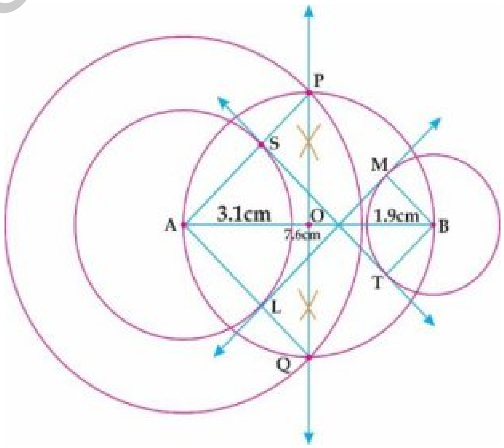
Steps of Construction:

(i) Draw $\overline{AB}$ of 7.5cm.
(ii) Draw circles of radii 3.4cm and 2.1cm at points A and B respectively.
(iii) Take a point C on circle with centre A and draw $\overline{AC}$.
(iv) Draw $\overline{BD} \parallel \overline{AC}$ using set squares.
(v) Draw $\overline{CD}$ and produce it beyond D.
(vi) Produce $\overline{AB}$ to intersect CD at point M.
(vii) With M as centre and radius equal to $m\overline{MB}$, draw an arc which cuts the circle of centre B at points P and Q.
(viii) Draw $\overline{MP}$ and produce it to meet other circle at S.
(ix) Draw $\overline{MQ}$ and produce it to meet the other circle at point T.
Now $\overleftrightarrow{SP}$ and $\overleftrightarrow{TQ}$ are the required direct common tangents to the given circles.

(b) Drawing of transverse common tangents or internal tangents to two unequal circles.

We explain the method of drawing transverse common tangents to two unequal circles by the following example.

Example: Draw the transverse common tangents to two circles with centres A and B, also having radii 3.1cm and 1.9cm such that the distance between their centres is 7.6cm



Given: Two circles with centres A and B having radii 3.1cm and 1.9cm respectively and $m\overline{AB}$ =7.6cm

Required: To draw transverse common tangents to two given circles.

Steps of Construction:

(i) Draw $\overline{AB}$ of 7.6cm.
(ii) Draw circles of radii 3.1cm and 1.9cm at points A and B respectively.
(iii) Bisect $\overline{AB}$ at point O and draw a circle of radius equal to $m\overline{OA}$ with centre O.
(iv) With centre A (centre of bigger circle) and radius of 5 cm (i.e,3.1cm+1.9cm=5cm), draw a circle which cuts the previous circle at points P and Q.
(v) Draw $\overline{AP}$ which cuts the given larger circle at point S.
(vi) Draw $\overline{AQ}$ which cuts the given larger circle at point L.
(vii) Draw $\overline{BM}$ $\overline{AQ}$ and $\overline{BT}$ $\overline{AP}$.
(viii) Draw $\overleftrightarrow{ST}$ and $\overleftrightarrow{LM}$.

Now $\overleftrightarrow{ST}$ and $\overleftrightarrow{LM}$ are the required transverse common tangents.

29.3(vi) Draw a tangent to

- Two unequal touching circles
- Two unequal intersecting circles

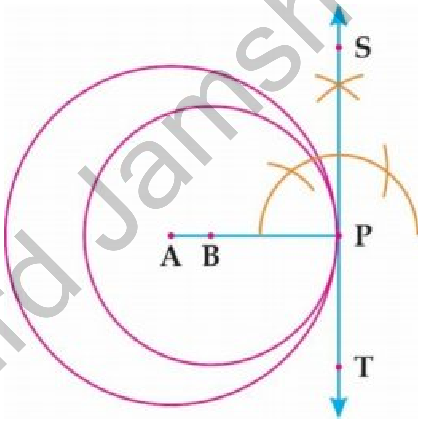
(a) Drawing of a tangent to two unequal touching circles

There are two cases

Case 1: when circles touch internally.

The method is explained by the following example

Example: Draw a tangent to two unequal circles of radii 3cm and 2cm with centres A and B respectively whereas circles touch internally.



Given: Two circles with centres A and B having radii 3cm and 2cm respectively which touch internally.

Required: To draw a tangent to these two circles

Steps of Construction:

(i) With centre at point A, draw larger circle of radius 3cm

(ii) Take any point P on the circle and draw $\overline{AP}$

(iii) Take a point B on $\overline{AP}$ at a distance of 2cm from point P

(iv) With B as centre and radius of 2cm draw a circle which touches the larger circle at point P

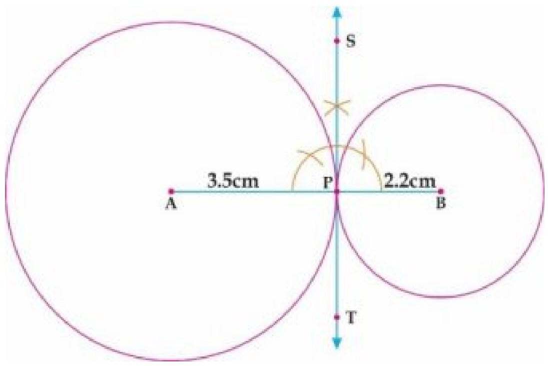
(v) Draw a perpendicular $\overleftrightarrow{ST}$ on $\overline{AP}$ at point of contact P

Now ST is the required tangent.

Case 2: When circles touch externally.

The method is explained in the following example

Example: Draw a tangent to two unequal circles of radii 3.5cm and 2.2cm with centres A and B respectively whereas both circles touch each other externally.



Given: Two circles of radii 3.5cm and 2.2cm with centres at A and B whereas circles touch each other externally.

Steps of Construction:

(i) With centre A, draw a circle of radius 3.5cm.

(ii) Take a point P on the circle.

(iii) Draw $\overline{AP}$ and produce it to point B such that $m\overline{PB}$=2.2cm

(iv) With P as centre and radius of 2.2cm, draw a circle which touches first circle at point P

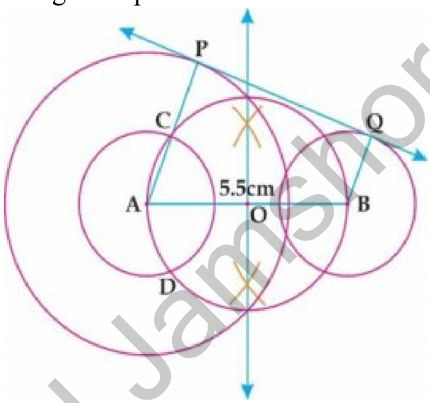
(v) Draw a perpendicular $\overleftrightarrow{ST}$ on $\overline{AP}$ at point of contact P

Now $\overleftrightarrow{ST}$ is the required tangent.

(b) Drawing of a tangent to two unequal intersecting circles.

The method is explained with the help of the following example

Example: Draw a tangent to two unequal interesting circles of radii 4cm and 2cm with centres A and B respectively such that $m\overline{AB}=5.5\text{cm}$



Given: Two unequal intersecting circles of radii 4cm and 2cm with centre A and B respectively such that $m\overline{AB}=5.5\text{cm}$

Required: To draw a tangent to both the circles

Steps of Construction:

(i) Draw $\overline{AB}$ of 5.5cm.

(ii) Draw circles of radii 4cm and 2cm with centres A and B respectively.

(iii) With centre A (centre of larger circle), draw a circle of radius $2\text{cm}(4\text{cm}-2\text{cm}=2\text{cm})$

(iv) Bisect $\overline{AB}$ at point O. With O as centre and radius equal to $m\overline{AO}$, draw a circle which cuts the smaller circle of centre A at point C and D.

(v) Draw $\overline{AC}$ and produce it to meet the larger concentric circle of centre A at point P.

(vi) Draw $\overline{BQ} \parallel \overline{AP}$ using set squares.

(vii) Draw $\overleftrightarrow{PQ}$

Now $\overleftrightarrow{PQ}$ is the required tangent.

29.3(vii) Draw a circle which touches

- **Both the arms of a given angle**
- **Two converging lines and passing through a given point between them**
- **Three converging lines**

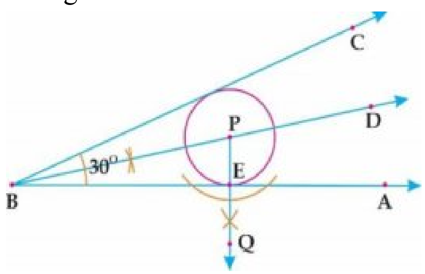
(a) Drawing a circle which touches both the arms of a given angle.

We explain the method of drawing a circle which touches both the arms of given angle in the following example

Example: Draw a circle which touches both the arms of an angle ABC of measure 30°

Given: An angle ABC of measure 30°

Required: To draw a circle which touches both the arms of the angle.



Steps of Construction:

(i) Draw an $\angle ABC$ of 30°

(ii) Draw the internal bisector $\overrightarrow{BD}$ of $\angle ABC$

(iii) Take any point P on $\overrightarrow{BD}$

(iv) From point P, draw a prependiculer $\overrightarrow{PQ}$ on $\overrightarrow{BA}$ which cuts it at point E

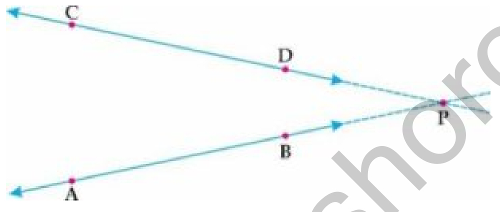
(v) With P as centre and radius equal to $m\overline{PE}$, draw a circle which touches $\overrightarrow{BA}$ and $\overrightarrow{BC}$

This circle is the required circle.

(b) Drawing a circle which touches two converging lines and passing through a given point between them.

Converging Lines:

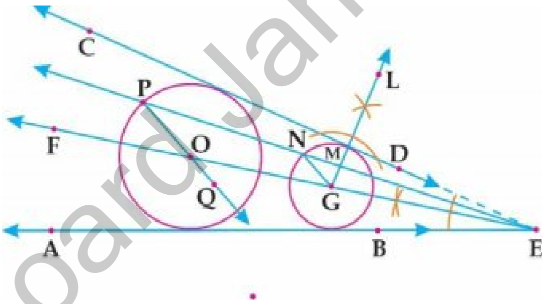
Two or more lines which get closer and closer to each other and finally meet at a point, are said to be converging lines, In the adjacent figure, two lines AB and CD are converging lines and they finally meet at a point P.



Example: Draw a circle which passes through a point P between the converging lines AB and CD and touch both the lines.

Given: Two converging lines AB and CD. And a point P between them

Required: To draw a circle which passes through point P and touches both the given converging lines.



Steps of Construction:

(i) Draw two converging lines AB and CD which meet at point E.

(ii) Take any point P between these lines.

(iii) Draw the internal bisector $\overrightarrow{EF}$ of $\angle$E.

(iv) Take any point G on $\overrightarrow{EF}$.

(v) From point G, draw a perpendicular $\overrightarrow{GL}$ on $\overrightarrow{EC}$ meeting it at point M

(vi) With G as centre and radius equal to $m\overline{GM}$, draw a circle which touches both the converging lines.

(vii) Draw $\overrightarrow{EP}$ which cuts this circle at point N.

(viii) Draw $\overline{NG}$. Also draw $\overline{PQ} \parallel \overline{NG}$.

(ix) $\overrightarrow{PQ}$ cuts $\overrightarrow{EF}$ at point O.

(x) With O a centre and radius equal to $m\overline{OP}$, draw a circle which passes through P and touches the two converging lines.

This is the required circle.

(c)Drawing a circle which touches three converging lines

Drawing a circle which touches three converging lines in a plane is not possible. However, it is possible in space which is beyond the scope of this book.

EXERCISE 29.5

1. Draw two equal circles each of radius 3.3cm with centres at points A and B such that $m\overline{AB} = 7.8$cm.

(a) Draw direct common tangents to these circles.

(b) Draw transverse common tangents to these circles.

2. Draw two unequal circles of radii 3.3cm and 2.1cm with centres, A and B respectively such that $m\overline{AB}=8cm$.
(a) Draw direct common tangents to these circles.
(b) Draw transverse common tangents to these circles.

3. Draw a tangent to two unequal circles of radii 3.8cm and 2.2cm with centres A and B respectively whereas
(i) circle touch internally
(ii) circles touch externally
(iii) circles intersect each other and $m\overline{AB}=5.6cm$.

4. Draw a circle which touches both the arms of an angle of measure
(i) $35°$ (ii) $40°$

5. Draw a circle which passes through a point M which lies between two converging lines PQ and ST such that the circle also touches both the lines.

REVIEW EXERCISE 29

Tick the correct option

(i) Right bisectors of non-parallel___________ intersect at the centre of the circle.
(a) Radial segments (b) Chords (c) Tangents (d) none of these

(ii) ______________circles can pass through a single point.
(a) One (b) Two (c) Three (d) Infinite

(iii)_____________circles can pass through three non-collinear points.
(a) One (b) Two (c) Three (d) Infinite

(iv)_____________angles of regular polygon are equal in measure.
(a) Interior (b) Exterior (c) both a and b (d) none of these

(v) Each interior angle of regular hexagon is equal to______.
(a) $90°$ (b) $108°$ (c) $120°$ (d) $135°$

(vi)A circle which touches all the sides of triangle is called___________.
(a) circumcircle (b) incircle (c) excircle (d) tricircle

(vii) A circle which touches one side of triangle externally and two produced sides internally is called_____________.
(a) excircle (b) circumcircle (c) incircle (d) tricircle

(viii) The centre of inscribed circle is called________________
(a) excentre (b) incentre (c) centroid (d) orthocentre

(ix)Right bisectors of sides of a triangle intersect each other at___________
(a) incentre (b) excentre (c) centroid (d) circumcentre

(x) The point of intersection of internal bisectors of angles of a triangle is called
(a) excentre (b) circum centre (c) centroid (d) incentre

(xi)If a regular hexagon is inscribed in a circle then length of each side of hexagon __________ radius of the circle
(a) $<$ (b) $>$ (c) $=$ (d) $\leq$

(xii) Angle between tangent and radial segment at the point of contact is ___________ angle
(a) right (b) obtuse (c) acute (d) reflex

(xiii) The central angle of circle joining the points of contact of tangents is ___________ of the angle between the tangents

(a) complement (b) supplement (c) square (d) cube

(xiv) ___________ common tangents do not intersect the line segment joining the centres of the circle

(a) internal (b) transverse (c) direct (d) vertical

(xv) Direct common tangents of two equal circles are ____________

(a) intersecting (b) coincident (c) equal (d) parallel

(xvi) _____________ common tangents of two equal circles intersect at the midpoint of the line segment joining the centres of the circles

(a) direct (b) transverse (c) external (d) parallel

(xvii) If two circles of radii 5cm and 2cm touch each other externally then the distance between their centers is _____________

(a) 5cm (b) 10cm (c) 3cm (d) 7cm

(xviii) If two circles of radii 5cm and 2cm touch each other internally then the distance between their centres is______________

(a) 5cm (b) 10cm (c) 3cm (d) 7cm

(xix) Two or more converging lines always intersect each other at ________

(a) single point (b) two points

(c) more than one point (d) none of these

(xx) Three or more sided closed figure is called ___________

(a) pentagon (b) hexagon (c) heptagon (d) polygon

SUMMARY

- Centre of a circle can be located by drawing the right bisectors of two non-parallel chords which meet each other at the centre.
- From three non-collinear points, a unique circle can be drawn.
- A circle can be completed without finding the centre when an arc is given with the help of regular polygon.
- Circumcircle always passes through all the vertices of a polygon.
- Incircle of a triangle always touches all the sides of the triangle.
- Excircle of a triangle touches one side of a triangle externally and two produced sides internally.
- Three excircles of a triangle can be constructed.
- Tangent is a line which touches circle at a single point
- Secant always cuts the circle at two points.
- Tangent to a circle is always perpendicular to the radial segment at the point of contact.
- Only one tangent can be drawn to a circle from a point of its circumference.
- Only two tangents to a circle can be drawn from a point outside the circle.
- The central angle of a circle joining the points of contact is supplement of the angle between the tangents.
- If a line is tangent to more than one circle, it is called common tangent.

- Direct common tangents do not intersect the line segment joining the centres of the two given circles.
- Transverse common tangents always intersect the line segment joining the centres of the two given circles.
- If two circles touch each other externally then the distance between their centers is the sum of their radii.
- If two circles touch each other internally then the distance between their centers is equal to the difference of their radii.
- Converging lines get closer and closer to each other and finally meet at a point.

INTRODUCTION TO TRIGONOMETRY

Unit 30

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Measure an angle in sexagesimal system (degrees, minutes and seconds).
- Convert an angle given in $D^{\circ}\ M'\ S''$ form into decimal form (upto two decimal places) and vice- versa.
- Define a radian (measure of an angle in circular system) and prove the relationship between radian and degree measures
- Establish the rule $l = r\Theta$, where r is the radius of the circle, l the length of the circular arc and Θ is the central angle measured in radians.
- Prove that the area of the sector of a circle is $\frac{1}{2}r^2\theta$ or $\frac{1}{2}l\theta$
- Define and identify:
  - General angle (coterminal angles)
  - Angle in standard position.
- Recognize quadrants and quadrantal angles.
- Define trigonometric ratios and their reciprocalswith the help of a unit circle.
- Recall the values of the trigonometric ratios for45º, 30º, 60º.
- Recognize signs of trigonometric ratios in different quadrants.
- Find the values of remaining trigonometric ratios ifone trigonometric ratio is given.
- Calculate the values of trigonometric ratios for0º, 90º, 180º, 270º, 360º.
- Prove the trigonometric identities and apply them to showdifferent trigonometric relations.
- Find angles of elevation and depression.
- Solve real life problems involving angles ofelevation and depression.

Introduction:

The word trigonometry has been derived from Greek word, “tri”(Three), gono (angles) and “metron” (measurement). Literally, trigonometry means measurement of the triangle. This branch of mathematics was developed by Greeks in 330 B.C. It deals with solution of triangle. It has vast application in Physics, navigation astronomy surveying etc.

30.1 Measurement of an Angle:

An angle is defined as union of two rays which have a common point (called vertex), one of the rays is called “initial side” and other is called “terminal side”.

Measuring of angle depends upon the direction of the rotation from the initial side to the terminal side. Angle measured in the anti clock-wise direction is taken to be “positive angle” as shown in figure (a) and angle measured in the clock-wise direction is to be negative angle, as shown in figure (b).

Note: The rays are also called arms of the angle and their common end point is also known as vertex of the angle.

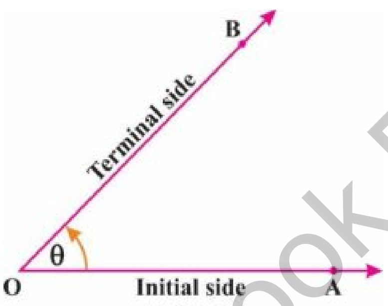


Figure (a)

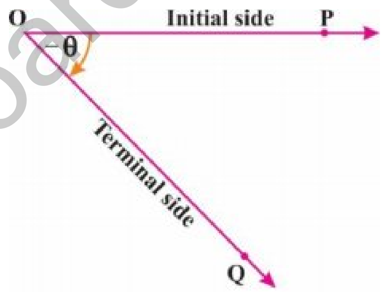


Figure (b)

An angle is said to be in standard position if its initial side is on the positive of x-axis and its vertex is at origin.

30.1.(i) Measure an angle in Sexagesimal system (Degrees, minutes and seconds)

If the initial ray (arm) $\overrightarrow{OA}$ completes one rotation in anticlock-wise direction, then the angle formed is said to be 360 degree or 360°. Hence, the circumference of a circle is divided into 360 equal arcs. The angle subtended at the centre of the circle by such an arc is called one degree and is denoted by 1°. Each degree is divided into 60 equal parts, each part is called 1 minute is denoted as $1'$. Furthermore, each minute is divided into further 60 parts and each part is 1 second and is denoted by $1''$. Hence 1 minute is the 60^{th} part of 1 degree and 1 second is $360^{\circ\,th}$ part of 1 degree. The system in which angle measures in degree, minutes and seconds is called sexagesimal system.

Thus one degree is defined as the measure $\frac{1}{360}$th of a complete rotation and it is denoted by 1°, it is further sub-divided as under

$$1^\circ = 60 \text{ minutes} = 60'$$

$$1' = 60 \text{ seconds} = 60''$$

e.g, 30 degrees, 20 minutes and 10 seconds are written symbolically as $30^\circ 20' 10''$

30.1.(ii) Convert an angle given in $D^\circ M' S''$ into decimal form (up to two decimal places) and vice versa

In this section we convert minutes and seconds into degrees by dividing minute with 60 and second with 3600. After simplifying we get required decimal form.

Example 1

Convert $30^\circ 30' 10''$ into degrees and write in decimal form.

Solution:

$$\text{Since } 1' = \left(\frac{1}{60}\right)^\circ$$

$$\therefore \quad 30' = \left(\frac{30}{60}\right)^\circ = \left(\frac{1}{2}\right)^\circ$$

$$\text{and } 1'' = \left(\frac{1}{3600}\right)^\circ$$

$$10'' = \left(\frac{10}{3600}\right)^\circ = \left(\frac{1}{360}\right)^\circ$$

$$\therefore \quad 30^\circ 30' 10'' = \left(30 + \frac{1}{2} + \frac{1}{360}\right)^\circ$$

$$= [30 + 0.5 + 0.003]^\circ$$

$$= 30.503^\circ$$

$$= 30.50^\circ \quad (\text{round off upto two decimal places})$$

Example 2

Convert 12.51° into $D^\circ M' S''$ form.

Solution:

$$12.51^\circ = 12^\circ + (0.51)^\circ$$

$$= 12^\circ + (0.51 \times 60)' \qquad (\because 1^\circ = 60')$$

$$= 12^\circ + (30.6)'$$

$$= 12^\circ + 30' + (0.6)'$$

$$= 12^\circ 30' + (0.6 \times 60)'' \qquad (\because 1' = 60'')$$

$$= 12^\circ 30' 36''$$

30.1.(iii) Define a radian (measure of an angle in circular system) and prove the relationship between radian and degree measures.

Radian is another unit to measure an angle which is equal to ratio of arc length to the radius of the circle. i.e., $\theta = \frac{l}{r}$, where θ is central angle in radians, l is the arc length and r is the radius of the circle. The system where angle is measured in radian is called circular system. Thus, the angle measure in circular system is said to be of 1 radian if it is subtended by an arc equal to radius of circle. (as shown in figure 30.1).

we have $\theta = \frac{l}{r}$,

when $l = r$

then $\theta = \frac{l}{l}$

$\boxed{\theta = 1 \quad \text{radian}}$

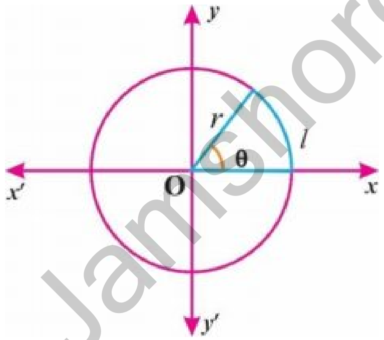


Fig: 30.1

For a complete rotation the arc length is the circumference of the circle i.e., $l = 2\pi r$.

Now,

$$\theta = \frac{l}{r}$$

$$\theta = \frac{2\pi r}{r}$$

$$\boxed{\theta = 2\pi \text{ radian}}$$

Hence for a complete rotation, the angle measured in radians is 2π.

Relationship between degree and radian

We know that, for a complete rotation the angle measured in degree is 360°, and angle measured in radians is 2π.

Therefore

$360^\circ = 2\pi$ radians

$\Rightarrow \quad 180^\circ = \pi$ radians

$\Rightarrow \quad 1^\circ = \frac{\pi}{180}$ radians ≈ 0.01745 radians

or, $\quad 1 \text{ radians} = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ$

Some standard angles in degrees and radians are as under

degree	30°	45°	60°	90°	180°	270°	360°
radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Example 1

Convert the following into radian measure.

(a) 120° (b) $10^\circ 30'$ (c) $24^\circ 32' 30''$

Solution (a):

we know that

$\because \quad 1° = \dfrac{\pi}{180}$ radians

$\therefore \quad 120° = 120 \times \dfrac{\pi}{180}$ radians

$\Rightarrow \quad 120° = \dfrac{2\pi}{3}$ radians

Solution (b):

we know that

$\because \quad 1' = \left(\dfrac{1}{60}\right)^{\circ}$

$\therefore \quad 30' = \left(\dfrac{30}{60}\right)^{\circ} = \left(\dfrac{1}{2}\right)^{\circ}$

Now $\quad 10°30' = \left(10 + \dfrac{30}{60}\right)^{\circ} = \left(10 + \dfrac{1}{2}\right)^{\circ} = \left(\dfrac{21}{2}\right)^{\circ}$

$\therefore \quad = \dfrac{21}{2} \times \dfrac{\pi}{180}$ radians $\qquad (\because 1^{\circ} = \dfrac{\pi}{180}$ radians$)$

$\Rightarrow \quad = \dfrac{7\pi}{120}$ radians

Solution (c): $24°32'30''$

$= 24° + \left(\dfrac{32}{60}\right)^{\circ} + \left(\dfrac{30}{3600}\right)^{\circ} \qquad 1' = \left(\dfrac{1}{60}\right)^{\circ}$ and $1'' = \left(\dfrac{1}{3600}\right)^{\circ}$

$= \left(24 + \dfrac{32}{60} + \dfrac{30}{3600}\right)^{\circ} = \left(24 + \dfrac{8}{15} + \dfrac{1}{120}\right)^{\circ}$

$= \left(\dfrac{24 \times 120 + 64 + 1}{120}\right)^{\circ}$

$= \left(\dfrac{2945}{120}\right)^{\circ} \qquad \left(\because 1° \cong \dfrac{\pi}{180} \text{ radian}\right)$

Now $\quad \left(\dfrac{2945}{120}\right)^{\circ} = \dfrac{2945}{120} \times \dfrac{\pi}{180}$ radians

$\therefore \quad = \dfrac{589}{4320}\pi$ radians

So, $24°32'30'' = \dfrac{589}{4320}\pi$ radians

EXERCISE 30.1

1. Convert the following into degree, and write in decimal form.

(i)	$32° 15'$	(ii)	$10° 30'$	(iii)	$8° 15' 30''$
(iv)	$45° 21' 36''$	(v)	$25° 30'$	(vi)	$18° 6' 21''$

2. Convert the following angles into $D° M' S''$ form.

(i)	$32.25°$	(ii)	$47.36°$	(iii)	$57.325°$
(iv)	$-67.58°$	(v)	$22.5°$	(vi)	$225.60°$

3. Express the following angles into degree measures.

(i)	$\frac{\pi}{4}$ radians	(ii)	$\frac{\pi}{3}$ radians	(iii)	$-\frac{3\pi}{4}$ radians
(iv)	3 radian	(v)	$\frac{3}{\pi}$ radians	(vi)	4.5 radians

4. Convert the following angles to radian measures.

(i)	$30°$	(ii)	$45°$	(iii)	$60°$
(iv)	$\left(22\frac{1}{2}\right)°$	(v)	$-225°$	(vi)	$60° 35' 48''$

30.2 Sector of a circle:

Definitions:

(i) A part of the circumference of a circle is called an arc.

(ii) A part of the circle bounded by the two radii and an arc is called sector of the circle.

The following figures help to understand the above mentioned definitions.

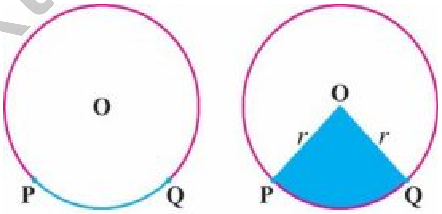


Fig: 30.2 (i) **Fig: 30.2 (ii)**

In fig: 30.2 (i) PQ ia an arc and in fig: 30.2 (ii) POQ is a sector.

30.2.(i) Establish the rule $l = r\theta$, where r is the radius of the circle, l the length of the circular arc and θ is the central angle measured in radian.

In a circle of radius r, the arc length l is directly proportional to the central angle θ measured in radians. (as shown in figure 30.3).

i.e., $l \propto \theta$

$\Rightarrow \quad l = c\theta \qquad \ldots(i)$

For complete rotation

$$l = 2\pi r$$

and $\theta = 2\pi$

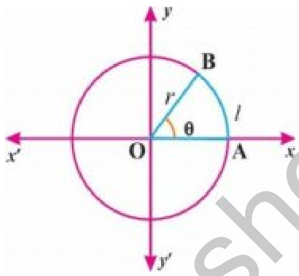


Fig: 30.3

By using equation (i), we have,

$$2\pi r = c(2\pi)$$

$$\Rightarrow \quad c = r$$

so equation (i) becomes $l = r\theta$.

Note: (i) l and r are measured in same units

(ii) θ is measured in radian

Example 1: Find the length of an arc of a circle of radius 5cm which subtends an angle of $\frac{2\pi}{5}$ radian.

Solution: Given that

$$r = 5\text{cm}, \ \theta = \frac{2\pi}{5} \text{ radian and } l = ?$$

Since, $l = r\theta$

$$\therefore \quad l = 5 \times \frac{2\pi}{5}$$

$$\Rightarrow \quad l = 2\pi \cong 2\left(\frac{22}{7}\right) \approx 6.28\text{cm}.$$

Example 2:

How far does a boy on a bicycle travel in 10 revolutions if the diameters of the wheels of his bicycle each equal to 56cm?

Solution:

We know that

one revolution $= 2\pi$ radian

$\therefore$ 10 revolutions $= 20\pi$ radians

i.e. $\theta = 20\pi$ radian

also, diameter of the wheel, d = 56cm

$$\therefore \quad r = \frac{d}{2} = \frac{56}{2}\text{cm} = \frac{56}{2 \times 100}\text{m} \qquad (\because \ 1\text{m} = 100\text{cm})$$

$$l = r\theta$$

$$\therefore \quad l = \frac{56}{2 \times 100} \times 20\pi$$

$$\Rightarrow \quad l \approx \frac{56}{2 \times 100} \times 20 \times \frac{22}{7} \qquad (\because \ \pi \cong \frac{22}{7})$$

$$\Rightarrow \quad l \approx 17.6\text{m}.$$

Thus, the boy travels 17.6m in 10 revolutions.

Example 3:

Find r, when $l = 4$cm and $\theta = \frac{1}{4}$ radians

Solution: Given that:

$$l = 4\text{cm}$$

$$\theta = \frac{1}{4} \text{ radians}$$

$$\text{and } r = ?$$

$$l = r\theta$$

$$\therefore \quad r = \frac{l}{\theta}$$

$$\Rightarrow \quad r = \left(4 \div \frac{1}{4}\right) = 4 \times \frac{4}{1} = 16\text{cm.}$$

30.2.(ii) Prove that the area of the sector of a circle is $\frac{1}{2}r^2\theta$ or $\frac{1}{2}lr$.

Proof:

Consider a circle of radius r with centre O, AB is an arc which subtends an angle θ radians at the centre as shown in the figure 30.4.

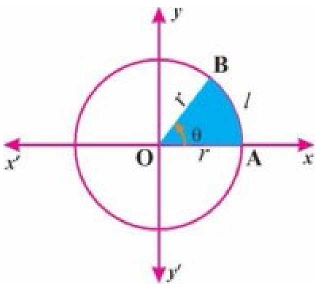


Fig: 30.4

We know that

$$\text{Area of circle } = \pi r^2$$

$$\text{Angle of one revolution } = 2\pi \text{ radian}$$

$$\text{Angle of the sector } = \theta \text{ radian}$$

Then by elementary geometry, using law of proportion, we have

$$\frac{\text{area of sector AOB}}{\text{area of a circle}} = \frac{\text{angle of sector}}{\text{angle of a circle}}$$

$$\therefore \quad \frac{\text{Area of sector AOB}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\Rightarrow \quad \text{Area of sector AOB} = \frac{\pi r^2 \times \theta}{2\pi}$$

$$\Rightarrow \quad \text{Area of sector AOB} = \frac{1}{2} r^2\theta$$

$$or \quad \text{Area of sector AOB} = \frac{1}{2} r\theta \times r$$

$$\Rightarrow \quad \text{Area of sector AOB} = \frac{1}{2} rl \qquad (\because r\theta = l)$$

Hence proved.

Example 1:

Find the area of sector of a circle of radius 5cm with central angle of 60°.

Solution: Given that:

$r = 5\text{cm}$

$\theta = 60^\circ = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3}\text{radians}$

and area of sector = ?

$\because$ Area of sector $= \frac{1}{2}r^2\theta$

$\therefore$ Area of sector $= \frac{1}{2}(5)^2 \times \frac{\pi}{3}$

$\Rightarrow$ Area of sector $\approx \frac{1}{2} \times 25 \times \frac{22}{7 \times 3}$ $\quad \therefore \pi \approx \frac{22}{7}$

$\Rightarrow$ Area of sector $= 13.09\text{cm}^2$

Example 2:

Find the area of sector whose radius is 4cm, with central angle of 12 radians.

Solution: Given that:

$r = 4\text{cm}$

$\theta = 12$ radian

and Area of sector = ?

$\because$ area of sector $= \frac{1}{2}r^2\theta$

$\therefore$ area of sector $= \frac{1}{2}(4)^2 \times 12$

$\Rightarrow$ area of sector $= 16 \times 6 = 96\text{cm}^2$

EXERCISE 30.2

1. Find θ, if
   (i) $l = 20\text{cm}$ and $r = 5\text{cm}$ (ii) $l = 30.2\text{cm}$ and $r = 2\text{cm}$
   (iii) $l = 6\text{cm}$ and $r = 2.87\text{cm}$ (iv) $l = 4.5\text{cm}$ and $r = 2.5\text{cm}$
2. Find l, if
   (i) $r = 1.01\text{cm}$ and $\theta = 2.1$ radian
   (ii) $r = 5.1\text{cm}$ and $\theta = 2$ radian
   (iii) $r = 6\text{cm}$ and $\theta = 30^\circ$
   (iv) $r = 15\text{cm}$ and $\theta = 60^\circ\ 30'$
3. Find r, if
   (i) $\theta = 60^\circ$ and $l = 2\text{m}$ (ii) $\theta = 3.5$ radians and $l = \frac{7}{4}\text{m}$
   (iii) $\theta = \frac{1}{4}$ radians and $l = 4\text{cm}$ (iv) $\theta = 180^\circ$ and $l = 15.4\text{cm}$

4. Find the arc length of a unit circle, corresponding to the central angle measuring.
   (i) 30° (ii) 45° (iii) 60° (iv) 90°
5. The arc of a circle subtends an angle of $\frac{\pi}{6}$ radians at the centre. The radius of a circle is 5cm, find;
   (i) length of an arc (ii) area of a circular sector
6. A point is moving on the circle of radius 10cm. If it makes 3.5 revolutions, find the distance travelled by the point.
7. Find the area of the sector with central angle of $\frac{\pi}{4}$ radian in circle of radius 4cm.
8. If a point on the rim of a 21cm diameter fly wheel travels 5040 meters per minute through, how many radian does the wheel turn in a second?
9. A car is running on a circular path of radius equal to double the arc of the circle travelled by the car. Find the angle subtended by the arc at the centre of the circular path.
10. In a circle of radius 12cm, an arc subtends a central angle of $84°$. Find its arc length and also calculate area of the sector.

30.3 Trigonometric Ratios

30.3.(i) Define and identify:

(a) General angle (Coterminal angles)

(b) Angle in standard position

30.3.1(a) Angles having the same initial and terminal sides are called coterminal angles and they differ by multiple of 2π radians. They are also called general angles.

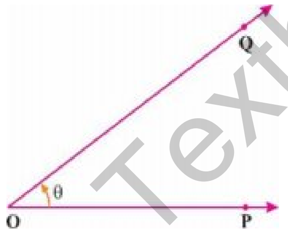


Fig: 30.5 (i)

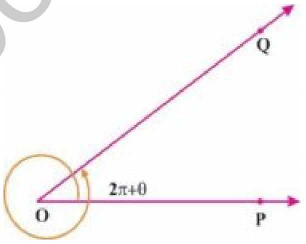


Fig: 30.5 (ii)

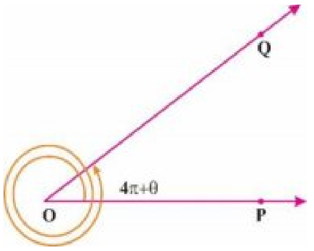


Fig: 30.5 (iii)

$m\angle POQ = \theta$ radian, where $0 \le \theta < 2\pi$.

(i) θ radian , (Zero revolution)

(ii) $(2\pi+\theta)$ radians , $(\text{After one revolution})$

(iii) $(4\pi+\theta)$ radians , $(\text{After two revolution})$

Hence $\theta, 2\pi+\theta$ and $4\pi+\theta$ and coterminal angles as shown in figure 30.5.

However, if the rotation are made in the clock-wise directions as mentioned in the below figures

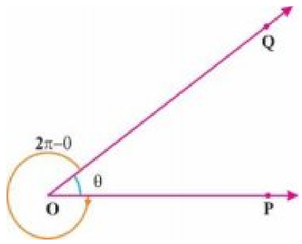


Fig: 30.6 (i)

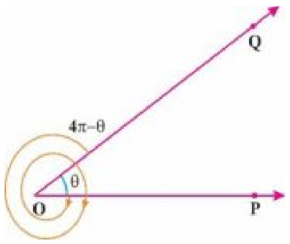


Fig: 30.6 (ii)

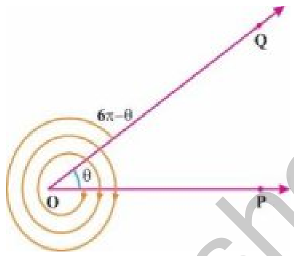


Fig: 30.6 (iii)

(i) $2\pi-\theta$ radians (no revolution)

(ii) $4\pi-\theta$ radians (after one clockwise revolution)

(iii) $6\pi-\theta$ radians (after two clock wise revolution)

$m\angle POQ=\theta$ radian, where $0\leq\theta<2\pi$.

Hence $2\pi-\theta$, $4\pi-\theta$ and $6\pi-\theta$ and coterminal angles as shown in figure 30.6.

Examples:

Which of the following angles are coterminal with $120°$?

$-240°, 480°, \frac{14\pi}{3}$ and $-\frac{14\pi}{3}$.

Solution:

$120°-(-240°)=120°+240°=360°$, yes $-240°$ is coterminal angle of $120°$

$480°-120°=360°$, yes $480°$ is coterminal angle of $120°$

$\frac{14\pi}{3}-120°=\frac{14\pi}{3}-\frac{2\pi}{3}=\frac{12\pi}{3}=4\pi$, yes $\frac{14\pi}{3}$ is coterminal angle of $120°$

$120°-\left(\frac{14\pi}{3}\right)=\frac{2\pi}{3}+\frac{14\pi}{3}=\frac{16\pi}{3}$, it is not multiple of 2π,

so, $-\frac{14\pi}{3}$ is not coterminal angle of $120°$.

30.3. (i)(b) An angle is said to in standard position if its vertex is at origin and its initial side on positive x-axis as shown in figure 30.7.

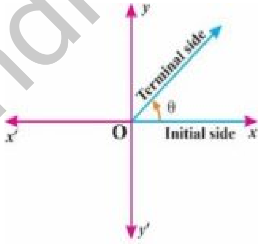


Fig: 30.7 (i)

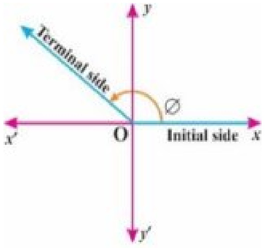


Fig: 30.7 (ii)

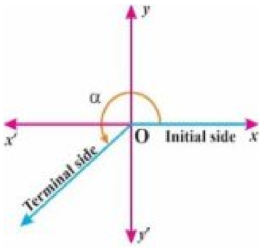


Fig: 30.7 (iii)

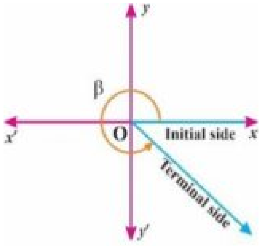


Fig: 30.7 (iv)

30.3.(ii) Recognize Quadrants and quadrantal angles:

30.3.(ii)(a) In rectangular coordinate system two axes divide the plane into equal parts, each part is called quadrant. Hence, four quadrants are formed as shown in figure 30.8.

Angles between 0° and 90° are in Quadrant I

i.e., $0 < \theta < \frac{\pi}{2}$

Angles between 90° and 180° are in Quadrant II

i.e., $90° < \theta < 180°$

Angles between 180° and 270° are in Quadrant III

i.e., $180° < \theta < 270°$

Angles between 270° and 360° are in Quadrant IV

i.e., $270° < \theta < 360°$

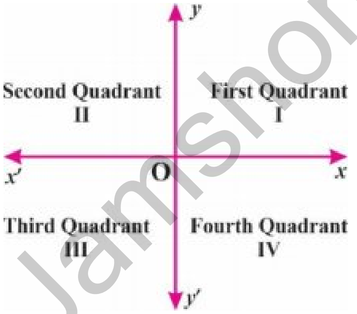


Fig: 30.8

An angle in standard position is said to be in a quadrant, if its terminal side lies in that quadrant.

30.3.(ii) (b) Quadrantal angles.

If the terminal side of the standard angle coincides with x-axis or y-axis, then it is called a quadrantal angle, for example $90°, 180°, 270°$ and $360°$ which are shown in the following figure.

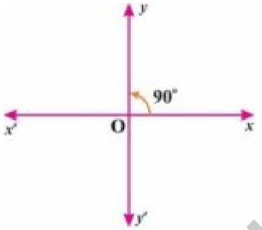


Fig: 30.9 (i)

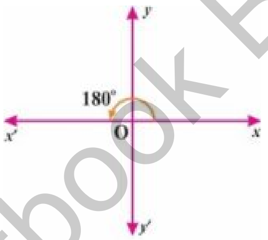


Fig: 30.9 (ii)

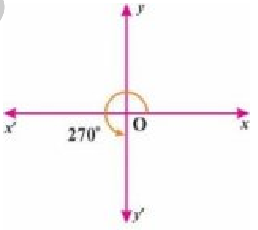


Fig: 30.9 (iii)

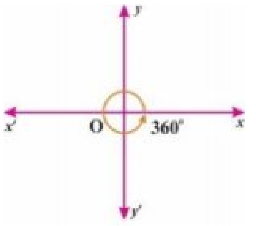


Fig: 30.9 (iv)

30.3.(iii) Define trigonometric ratios and their reciprocals with the help of a unit circle.

We have already learnt about trigonometric ratios for any acute angle θ of a right-angled triangle ABC in previous classes, which are as under.

1. $\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}} = \frac{a}{c}$

2. $\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}} = \frac{b}{c}$

3. $\tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}} = \frac{a}{b}$

and their reciprocals are respectively

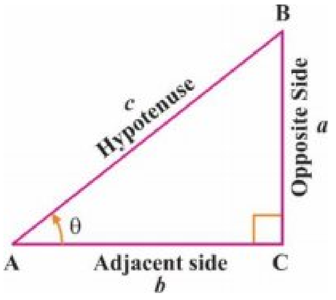


Fig: 30.10

4. $\operatorname{cosec}\theta = \dfrac{\text{length of hypotenuse}}{\text{length of opposite side}} = \dfrac{1}{\sin\theta} = \dfrac{c}{a}$

5. $\sec\theta = \dfrac{\text{length of hypotenuse}}{\text{length of adjacent side}} = \dfrac{1}{\cos\theta} = \dfrac{c}{b}$

6. $\cot\theta = \dfrac{\text{length of adjacent side}}{\text{length of opposite side}} = \dfrac{1}{\tan\theta} = \dfrac{b}{a}$

For each point of the circle, there always exist an angle corresponds to it.

Suppose that $P(x, y)$ be any point on the unit circle lying in the 1st quadrant and θ is the corresponding angle a shown in figure 30.11.

We define the trigonometric ratios as under

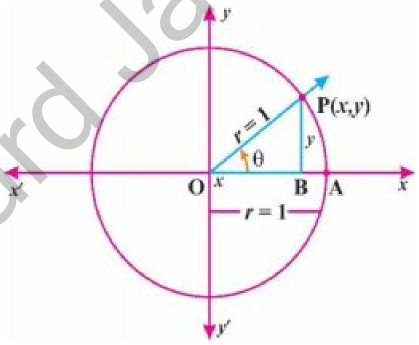


Fig: 30.11

$$\sin\theta = \frac{|\overline{BP}|}{|\overline{OP}|} = \frac{y}{1} = y,$$

$$\text{and } \cos\theta = \frac{|\overline{OB}|}{|\overline{OP}|} = \frac{x}{1} = x,$$

$$\text{Now, } \tan\theta = \frac{\overline{BP}}{\overline{OB}} = \frac{y}{x}, \text{ provided } x \neq 0$$

similarly,

$$\cot\theta = \frac{x}{y} \qquad \text{provided } y \neq 0$$

$$\sec\theta = \frac{1}{x} \qquad \text{provided } x \neq 0$$

$$\operatorname{cosec}\theta = \frac{1}{y} \qquad \text{provided } y \neq 0$$

Reciprocal ratios, listed below:

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} \qquad \text{or} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \text{or} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \text{or} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Note:

- All trigonometric ratios in terms of x and y are called trigonometric function.
- The function $\cos\theta$ and $\sin\theta$ are also called circular functions.

Example:

Find the value of trigonometric ratios at angle θ , if point P (6, 8) is on the terminal side of θ

Solution:

Here $P(x, y) = (6,8)$

$\Rightarrow$ $x = 6$ and $y = 8$ as shown in figure 30.12.

By using Pythogras theorem,

$$r^2 = x^2 + y^2$$

$$\therefore \quad r^2 = (6)^2 + (8)^2$$

$$\Rightarrow \quad r^2 = 36 + 64 = 100$$

$$\Rightarrow \quad r = \sqrt{100} = 10, \text{ where } r = |\overline{OP}|$$

Thus, values of all six trigonometric ratios are

$$\sin\theta = \frac{y}{r} = \frac{8}{10} = \frac{4}{5}, \qquad \csc\theta = \frac{5}{4}$$

$$\cos\theta = \frac{x}{r} = \frac{6}{10} = \frac{3}{5}, \qquad \sec\theta = \frac{5}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{8}{6} = \frac{4}{3} \quad \text{and} \quad \cot\theta = \frac{3}{4}$$

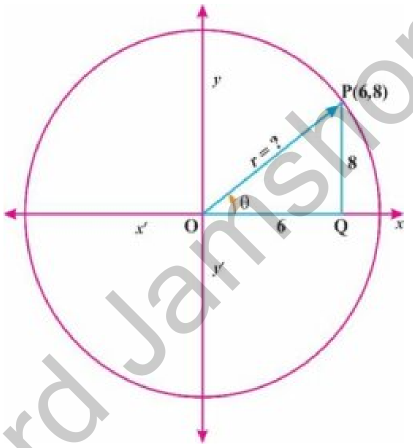


Fig: 30.12

30.3.(iv) Recall the values of trigonometric ratios for 45°,30° and 60°

We have already learnt that values of trigonometric ratios for $30°, 45°$ and $60°$ which are given in the table below

θ	$30°$	$45°$	$60°$
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Now we can find the values of reciprocals trigonometric ratios with the help of above table.

$$\csc 30° = \frac{1}{\sin 30°} = \frac{1}{\frac{1}{2}} = 2$$

$$\sec 30° = \frac{1}{\cos 30°} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$$\cot 30° = \frac{1}{\tan 30°} = \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3} \qquad \operatorname{cosec} 45° = \frac{1}{\sin 45°} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

$$\sec 45° = \frac{1}{\cos 45°} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2} \qquad \cot 45° = \frac{1}{\tan 45°} = \frac{1}{1} = 1$$

$$\operatorname{cosec} 60° = \frac{1}{\sin 60°} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \qquad \sec 60° = \frac{1}{\cos 60°} = \frac{1}{\frac{1}{2}} = 2$$

and $$\cot 60° = \frac{1}{\tan 60°} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

30.3.(v) Recognize signs of trigonometric rations in different quadrants.

If θ is not a quadrantal angles, then signs of trigonometric ratios can be determined according to point $P(x, y)$ on its terminal side.

1. If θ lies in first quadrant, then point (x, y) on its terminal side has x and y coordinates positive.
   i.e. $x > 0$ and $y > 0$.

$\therefore$ All trigonometric functions/ratios are positive in the first quadrant.

2. In 2nd quadrant $x < 0$ and $y > 0$. Therefore $\sin\theta$ and $\operatorname{cosec}\theta$ are positive in second quadrant and remaining ratios are negative.
3. In 3rd quadrant $x < 0$ and $y < 0$. Therefore $\tan\theta$ and $\cot\theta$ are positive in third quadrant and remaining ratios are negative.
4. In 4th quadrant $x > 0$ and $y < 0$. Therefore $\cos\theta$ and $\sec\theta$ are positive in fourth quadrant and remaining ratios are negative.

Example 1: Find the signs of the following trigonometric functions/ratios

(i) $\cos 120°$ (ii) $\sin 1030°$ (iii) $\tan 229°$ (iv) $\tan\left(-\frac{\pi}{4}\right)$ (v) $\operatorname{cosec}\left(-\frac{\pi}{3}\right)$

Solution:

(i) $\cos 120°$

$\because$ $120°$ lies in 2nd quadrant

$\therefore$ $\cos 120°$ is negative

(i) $\cos 120°$

$\because$ $120°$ lies in Quadrant II, and $\cos\theta$ is negative in II Quadrant

$\therefore$ $\cos 120°$ is negative.

(ii) $\sin 1030° = \sin(720^0 + 310^0)$

$\because$ $130°$ lies in Quadrant IV and Sin θ is negative in IV Quadrant,

$\therefore$ Sine $1030°$ is negative.

(iii) $\tan 229^\circ$

$\because$ 229° lies in Quadrant III and $\tan\theta$ is positive in Quadrant III,

$\therefore$ $\tan 229^\circ$ is positive.

(iv) $\tan\left(-\frac{\pi}{4}\right)$

$\because$ $\frac{-\pi}{4}$ lies in Quadrant IV and $\tan\theta$ is negative in Quadrant IV,

$\therefore$ $\tan\frac{-\pi}{4}$ is negative.

(v) $\operatorname{cosec}\left(-\frac{\pi}{3}\right)$

$\because$ $\frac{-\pi}{3}$ lies in IV Quadrant and $\operatorname{cosec}\theta$ is negative in IV Quadrant,

$\therefore$ $\operatorname{cosec}\frac{-\pi}{3}$ is of negative "–" sign.

30.3.(vi) Find the values of remaining trigonometric ratios if one trigonometric ratio is given.

The method is illustrated by the following examples:

Example 1: If $\sin\theta = \frac{3}{5}$ then find values of remaining trigonometric function/ratios.

Solution: Given that:

Since $\sin\theta = \frac{y}{r}$

By using figure 30.13

$$\frac{3}{5} = \frac{y}{r}$$

$\Rightarrow$ $y = 3$ and $r = 5$

Since $x^2 + y^2 = r^2$

$$x^2 + (3)^2 = (5)^2$$

$$x^2 + 9 = 25$$

$$x^2 = 25 - 9$$

$\Rightarrow$ $x^2 = 16$

$\Rightarrow$ $x = 4$

Now

$$\cos\theta = \frac{x}{r}$$

$$\cos\theta = \frac{4}{5}$$

$$\sec\theta = \frac{r}{x} = \frac{5}{4}$$

$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{5}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{3}{4}$$

$$\cot\theta = \frac{x}{y} = \frac{4}{3}$$

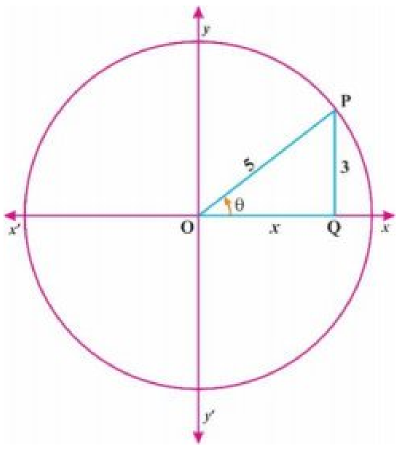


Fig: 30.13

Example 2: If $\tan\theta = 1$ and θ lies in the 3[rd] quadrant. Find values of remaining trigonometric functions.

Solution: Given that:

By using figure 30.14

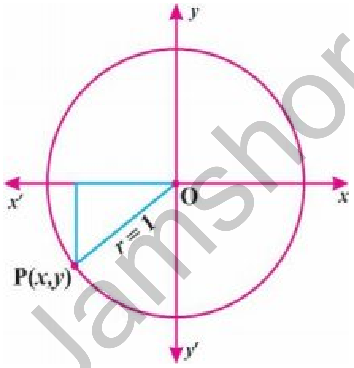


Fig: 30.14

$$\tan\theta = 1 \qquad \therefore\ \cot\theta = \frac{1}{\tan\theta} = \frac{1}{1} = 1,$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} = 1,$$

$$\Rightarrow \quad y = x$$

$$\because \quad x^2 + y^2 = 1$$

$$\therefore \quad x^2 + x^2 = 1 \qquad \because\ y = x$$

$$\Rightarrow \quad 2x^2 = 1$$

$$\Rightarrow \quad x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

Here we take negative value, because

θ lies in 3rd quadrant

$$x = \cos\theta = -\frac{1}{\sqrt{2}} \text{ and } y = \sin\theta = -\frac{1}{\sqrt{2}}$$

Therefore remaining trigonometric functions are

$$\cot\theta = 1,$$

$$\sin\theta = -\frac{1}{\sqrt{2}},$$

$$\text{cosec}\theta = \frac{1}{\sin\theta} = -\sqrt{2},$$

$$\cos\theta = -\frac{1}{\sqrt{2}}$$

$$\text{and} \quad \sec\theta = \frac{1}{\cos\theta} = -\sqrt{2}$$

Example 3: Find the remaining trigonometric functions/ratios, using the unit circle. If

(i) $\cos\theta = \dfrac{2}{3}$ and θ is in 1st quadrant.

(ii) $\sin\theta = 0.6$ and $\tan\theta$ is negative.

Solution (i): Given that:

$$\cos\theta = x = \frac{2}{3}, \qquad \therefore\ \sec\theta = \frac{1}{\cos\theta} = \frac{3}{2},$$

θ lies in 1st quadrant $\therefore\ p(x, y) \in$ Quadrant I

and $x^2 + y^2 = 1$ (Given)
Here $x = \cos\theta$ and $y = \sin\theta$

$$\therefore \quad \left(\frac{2}{3}\right)^2 + y^2 = 1$$

$$\Rightarrow \quad y^2 = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\Rightarrow \quad y = \pm\frac{\sqrt{5}}{3}$$

Here, we take +ve value, because θ lies in 1st quadrant

$$\therefore \quad y = \frac{\sqrt{5}}{3} = \sin\theta \ ; \qquad \therefore \ \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{3}{\sqrt{5}},$$

and $$\tan\theta = \frac{y}{x} = \frac{\frac{\sqrt{5}}{3}}{\frac{2}{3}} = \frac{\sqrt{5}}{2}; \qquad \therefore \ \cot\theta = \frac{1}{\tan\theta} = \frac{2}{\sqrt{5}},$$

Therefore required remaining trigonometric ratios are

1. $\sec\theta = \frac{3}{2}$ 2. $\sin\theta = \frac{\sqrt{5}}{3}$ 3. $\operatorname{cosec}\theta = \frac{3}{\sqrt{5}}$

4. $\tan\theta = \frac{\sqrt{5}}{2}$ and 5. $\cot\theta = \frac{2}{\sqrt{5}}$

Solution (ii): Given that:

$$\sin y = 0.6 = \frac{6}{10} = \frac{3}{5}, \qquad \therefore \ \operatorname{cosec}\theta = \frac{5}{3} = 1.6,$$

since $\sin\theta > 0$ and $\tan\theta < 0$

$$x^2 + y^2 = 1$$

$$\therefore \quad x^2 + \left(\frac{3}{5}\right)^2 = 1$$

$$\Rightarrow \quad x^2 = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\Rightarrow \quad x^2 = \pm\frac{4}{5}$$

Here, we take negative value because $\rho(\theta)$ is in 2nd quadrant

$$\therefore \quad x = \cos\theta = -\frac{4}{5} = -0.8 \ \text{ and } \ \sec\theta = \frac{1}{\cos\theta} = -\frac{5}{4} = -1.25$$

Now, $$\tan\theta = \frac{y}{x} = \frac{\frac{3}{5}}{\frac{-4}{5}} = -\frac{3}{4} = -0.75 \ \text{ and } \ \cot\theta = \frac{1}{\tan\theta} = -\frac{4}{3} = 1.3$$

Therefore required remaining trigonometric ratios are

1. $\text{cosec}\theta = 1.6$ 2. $\cos\theta = -0.8$ 3. $\sec\theta = -1.25$

4. $\tan\theta = -0.75$ 5. $\cot\theta = -1.3$

30.3.(vii) Calculate the values of trigonometric ratios of 0°, 90°, 180°, 270° and 360°.

We have already discussed quadrantal angles in section 30.3.(ii)(b). Here we calculate trigonometric ratios of the quadrantal angle

When $\theta = 0°$

In a unit circle, the point $P(1,0)$ lies on terminal side of an angle $\theta°$.

$\therefore$ Here $x = 1$, $y = 0$ and $r = 1$. Where r is the radius of unit circle.

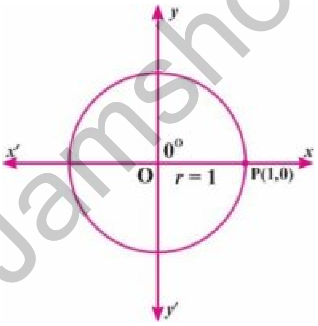


Fig: 30.15

$\therefore \quad \sin 0° = \frac{y}{r} = \frac{0}{1} = 0$; $\quad \therefore \quad \text{cosec}0° = \frac{r}{y} = \frac{1}{0}$(undefined)

$\therefore \quad \cos 0° = \frac{x}{r} = \frac{1}{1} = 1$; $\quad \therefore \quad \sec 0° = \frac{r}{x} = \frac{1}{1} = 1$

$\therefore \quad \tan 0° = \frac{y}{x} = \frac{0}{1} = 0$; $\quad \therefore \quad \cot 0° = \frac{x}{y} = \frac{1}{0}$(undefined)

When $\theta = 90°$

In a unit circle, the point $P(0,1)$ lies on terminal sides of an angle $90°$ and coincide with the (+ve) y-axis.

$\therefore$ Here $x = 0, y = 1$ and $r = 1$.

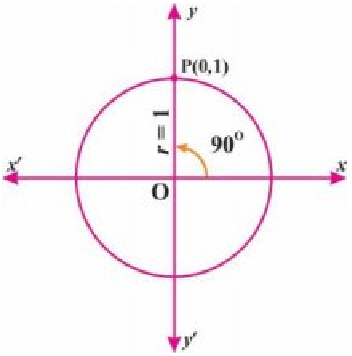


Fig: 30.16

$\therefore \quad \sin 90° = \frac{y}{r} = \frac{1}{1} = 1;$

$\text{cosec}90° = \frac{r}{y} = \frac{1}{1} = 1$

$\cos 90° = \frac{x}{r} = \frac{0}{1} = 0;$

$\sec 90° = \frac{r}{x} = \frac{1}{0}$(undefined)

$\tan 90° = \frac{y}{x} = \frac{1}{0}$(undefined)

and $\quad \cot 90° = \frac{x}{y} = 0$

When $\theta = 180°$

In a unit circle, the point $P(-1,0)$ lies on terminal sides of an angle 180° and coincide with the –ve x-axis. In this case

$P(-1,0) \Rightarrow x=-1, y=0$ and $r=1$.

$\sin 180^\circ = \frac{y}{r} = \frac{0}{1} = 0;$

$\therefore \quad \text{cosec}180^\circ = \frac{r}{y}$ (undefined)

$\cos 180^\circ = \frac{x}{r} = \frac{-1}{1} = -1;$

$\sec 180^\circ = \frac{r}{x} = \frac{1}{-1} = -1$

$\tan 180^\circ = \frac{y}{x} = \frac{0}{-1} = 0;$

and $\cot 180^\circ = \frac{x}{y} = \frac{1}{0}$ (undefined)

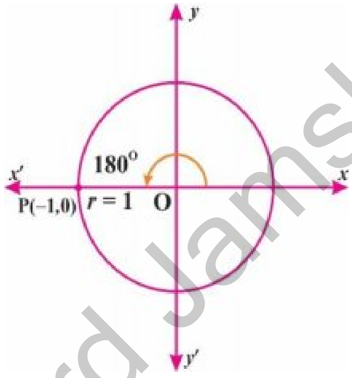


Fig: 30.17

When $\theta = 270^\circ$

In a unit circle, the point $P(0,-1)$ lies on terminal sides of an angle 180° and coincide with the –ve y-axis. In this case

$\therefore \quad P(0,-1) \Rightarrow x=0, y=-1$ and $r=1$.

$\therefore \quad \sin 270^\circ = \frac{y}{r} = \frac{-1}{1} = -1;$

$\text{cosec}270^\circ = \frac{r}{y} = \frac{1}{-1} = -1$

$\cos 270^\circ = \frac{y}{r} = \frac{0}{1} = 0;$

$\sec 270^\circ = \frac{r}{x} = \frac{1}{0}$ (undefined)

$\tan 270^\circ = \frac{y}{x} = \frac{-1}{0} =$ undifined

and $\cot 270^\circ = \frac{0}{-1} = 0$

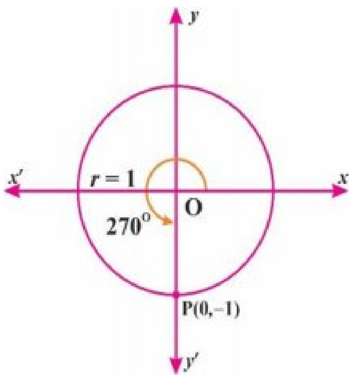


Fig: 30.18

When $\theta = 360^\circ$

In a unit circle, the point $P(1,0)$ lies on terminal sides of an angle 360° and coincide with the +ve x-axis. In this case,

$P(1,0) \Rightarrow x=1, y=0$ and $r=1$.

$\therefore \quad \sin 360^\circ = \frac{y}{r} = \frac{0}{1} = 0;$

$\text{cosec}360^\circ = \frac{1}{\sin 360^\circ} = \frac{1}{0}\text{(undefined)}$

$\cos 360^\circ = \frac{x}{r} = \frac{1}{1} = 1;$

$\sec 360^\circ = \frac{1}{\cos 360^\circ} = \frac{1}{1} = 1$

$\tan 360^\circ = \frac{y}{x} = \frac{0}{1} = 0;$

and $\quad \cot 360^\circ = \frac{1}{\tan 360^\circ} = \frac{1}{0}\text{(undefined)}$

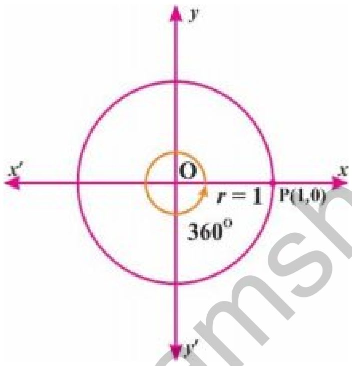


Fig: 30.19

Summarizing all the results, we have a table

θ	0°	30°	45°	60°	90°	180°	270°	360°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	−∞	0
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	−1	∞
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	−1	∞	1
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞

EXERCISE 30.3

1. Find coterminal angles of the following angles.

(i) 55° (ii) $\frac{\pi}{6}$ (iii) -45° (iv) $\frac{-3\pi}{4}$

2. Identify the quadrants of the following angles.

(i) $\frac{8\pi}{5}$ (ii) 75° (iii) -818° (iv) 1090° (v) $\frac{7\pi}{6}$ (vi) $-\frac{5\pi}{4}$

3. Find the signs of the following.

(i) $\cos 120°$ (ii) $\sin 340°$ (iii) $\sec 200°$

(iv) $\text{cosec} 198°$ (v) $\tan\left(\frac{-\pi}{3}\right)$ (vi) $\cot\left(-\frac{2}{3}\pi\right)$

4. In which quadrant does θ lie if:

(i) $\cos\theta < 0$ and $\sin\theta > 0$ (ii) $\tan\theta > 0$ and $\cos\theta < 0$

(iii) $\sin\theta < 0$ and $\sec\theta < 0$ (iv) $\cot\theta > 0$ and $\sec\theta < 0$

(v) $\cos\theta < 0$ and $\tan\theta < 0$ (vi) $0 < \cot\theta < 1$

5. If $\cos\theta = \frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, then find the remaining trigonometric ratios.

6. Find remaining trigonometric functions/ratios, if:

(i) $\sin\theta = \frac{\sqrt{3}}{2}$ and θ lies in second quadrant

(ii) $\cos\theta = \frac{2}{3}$ and θ lies in fourth quadrant

(iii) $\tan\theta = -\frac{1}{2}$ and θ lies in second quadrant

(iv) $\sec\theta =$ $\text{cosec}\theta = \sqrt{2}$ and θ lies in first quadrant

(v) $\cos\theta = \frac{1}{2}$ and $\tan\theta$ is positive.

7. Find the values of:

(i) $\tan 30° \tan 60° + \tan 45° \cot 45°$ (ii) $\frac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°}$

(iii) $\frac{\sin 45°}{\sin 45° + \cot 45°}$ (iv) $\cos\frac{\pi}{6}\cdot\cos\frac{\pi}{3} + \sin\frac{\pi}{6}\cdot\cos\frac{\pi}{3}$

(v) $\frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}}$ (vi) $\frac{\tan 45° + \cot 45°}{\sin 60° \cos 30° - \cos 60° \sin 30°}$

30.4 Trigonometric Identities:

An identity is an equation which is true for all values of the variable (except where value is not defined).

30.4.1 Prove the trigonometric identities and apply them to show different trigonometric relations.

For any real number θ related to the right triangle in a unit circle, we have the following fundamental trigonometric identities

(i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $\sec^2\theta = 1 + \tan^2\theta$

(iii) $\text{cosec}^2\theta = 1 + \cot^2\theta$

(i) **Proof:**

In a unit circle, consider a right ΔOAP, in which $\angle AOP = \theta$ radians is in standard position. Let $P(x, y)$ be on the terminal side of the angle.

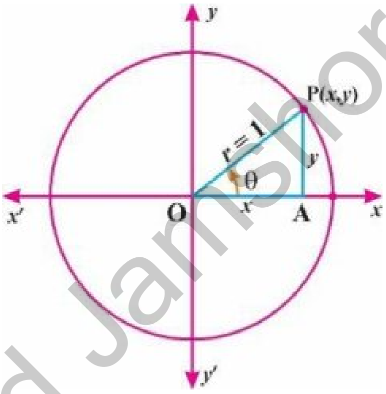


Fig: 30.20

By Pythagoras theorem, we have

$x^2 + y^2 = 1$

$\Rightarrow \quad \cos^2\theta + \sin^2\theta = 1 \qquad [\because \; x = \cos\theta \text{ and } y = \sin\theta]$

Hence proved.

(ii) **Proof:**

By Pythagoras theorem

$1 = x^2 + y^2$

dividing both sides by x^2, we have,

$\Rightarrow \quad \dfrac{1}{x^2} = 1 + \dfrac{y^2}{x^2}$ or $\dfrac{1}{x^2} = 1 + \left(\dfrac{y}{x}\right)^2$

$\Rightarrow \quad \dfrac{1}{\cos^2\theta} = 1 + \tan^2\theta, \qquad$ (Provided $x = \cos\theta \neq 0$)

$\Rightarrow \quad \boxed{\sec^2\theta = 1 + \tan^2\theta}$

Hence proved.

(iii) **Proof:**

By Pythagoras theorem

$1 = x^2 + y^2$

Dividing both sides by y^2, we have,

$\dfrac{1}{y^2} = \dfrac{x^2}{y^2} + 1, \quad$ or $\left(\dfrac{1}{y}\right)^2 = \left(\dfrac{x}{y}\right)^2 + 1$

$\Rightarrow \quad \left(\dfrac{1}{\sin\theta}\right)^2 = (\cot\theta)^2 + 1, \qquad$ (Provided $\sin\theta \neq 0$)

$\Rightarrow \quad \boxed{\operatorname{cosec}^2\theta = 1 + \cot^2\theta} \qquad \left(\because \; \dfrac{\cos\theta}{\sin\theta} = \cot\theta\right)$

Hence proved.

Example 1:

Prove that: $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Proof:-

$$\begin{aligned} \text{L.H.S} &= (\sin\theta + \cos\theta)^2 \\ &= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta \\ &= 1 + 2\sin\theta\cos\theta \qquad [\because \; \sin^2\theta + \cos^2\theta = 1] \\ &= \text{R.H.S} \end{aligned}$$

$$\text{L.H.S} = \text{R.H.S}$$

$\therefore \quad (\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Hence proved.

Example 2:

Prove that: $\sin\theta = \sqrt{1-\cos^2\theta}$

Proof:

$$\text{L.H.S} = \sin\theta$$

$$= \sqrt{(\sin\theta)^2}$$

$$= \sqrt{\sin^2\theta},$$

$$= \sqrt{1-\cos^2\theta}, \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \text{R.H.S}$$

$\because \quad \text{L.H.S} = \text{R.H.S}$

$\therefore \quad \sin\theta = \sqrt{1-\cos^2\theta}$

Hence proved.

Example 3:

Prove that $\dfrac{1+\sec\theta}{1-\sec\theta} = \dfrac{\sin\theta + \tan\theta}{\sin\theta - \tan\theta}$, (Provided $\cos\theta \neq 0$)

Proof:

$$\text{L.H.S} = \frac{1+\sec\theta}{1-\sec\theta},$$

$$= \frac{1+\dfrac{1}{\cos\theta}}{1-\dfrac{1}{\cos\theta}},$$

$$= \frac{\cos\theta + 1}{\cos\theta - 1}$$

$$= \frac{\sin\theta(\cos\theta+1)}{\sin\theta(\cos\theta-1)} \qquad \text{(Provided } \sin\theta \neq 0)$$

$$= \frac{\sin\theta\cos\theta + \sin\theta}{\sin\theta\cos\theta - \sin\theta}$$

$$= \frac{\dfrac{\sin\theta\cos\theta}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}}{\dfrac{\sin\theta\cos\theta}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta}}, \text{ Dividing numerator and denominator by } \cos\theta$$

$$= \frac{\sin\theta + \tan\theta}{\sin\theta - \tan\theta}$$

$$= \text{R.H.S}$$

$\because$ L.H.S = R.H.S

$\therefore$ $$\frac{1+\sec\theta}{1-\sec\theta} = \frac{\sin\theta + \tan\theta}{\sin\theta - \tan\theta}$$

Hence proved.

Example4: Prove that $\sin\theta = \tan\theta \cdot \cos\theta$

Proof:

L.H.S = $\sin\theta$,

$$= \frac{\sin\theta}{\cos\theta} \cdot \cos\theta, \qquad (\text{Provided } \cos\theta \neq 0)$$

$$= \tan\theta \cdot \cos\theta,$$

$$= \text{R.H.S}$$

L.H.S = R.H.S

$\therefore$ $\sin\theta = \tan\theta \cdot \cos\theta$

Hence proved.

EXERCISE 30.4

1. Prove the following trigonometric identities:

(i) $\sin^2\theta = (\sec^2\theta - 1)\cos^2\theta$, $\quad (\cos\theta \neq 0)$

(ii) $\frac{1+\sin\theta}{\cos\theta} = \tan\theta + \sec\theta$, $\quad (\cos\theta \neq 0)$

(iii) $\tan\theta = \sin\theta\sqrt{1+\tan^2\theta}$, $\quad (\cos\theta \neq 0)$

(iv) $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

(v) $\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$

(vi) $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{\sin\theta}{1-\cos\theta}$, $\quad (\cos\theta \neq 1)$

(vii) $\sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = \frac{\sec\theta+1}{\tan\theta}$, $\quad (\tan\theta \neq 0)$

(viii) $\frac{\sin^2\theta}{\cos\theta} + \cos\theta = \sec\theta$, $\quad (\tan\theta \neq 0)$

(ix) $\sec\theta - \cos\theta = (\cot\theta + \cos\theta)(\tan\theta - \sin\theta)$, $(\sin\theta \neq 0 \text{ and } \cos\theta \neq 0)$

(x) $\dfrac{\cot\theta + \text{cosec}\theta}{\sin\theta + \tan\theta} = \text{cosec}\theta\cot\theta$, $(\sin\theta \neq 0)$

(xi) $\sin^2\theta = \dfrac{\tan^2\theta}{1+\tan^2\theta}$, $(\cos\theta \neq 0)$

(xii) $\cos^2\theta = \dfrac{\cot^2\theta}{1+\cot^2\theta}$, $(\sin\theta \neq 0)$

(xiii) $\sin\theta\cos\theta\tan\theta + \sin\theta\cos\theta\cot\theta = 1$, $(\sin\theta \neq 0 \text{ and } \cos\theta \neq 0)$

2. Transform the first expression into second one:

(i) $(\sec\theta+1)(\sec\theta-1)$ into $\tan^2\theta$

(ii) $\dfrac{\cos\theta}{\sin\theta} - \dfrac{\text{cosec}\,\theta}{\cos\theta}$ into $-\tan\theta$

(iii) $\text{cosec}\,\theta + \cot\theta$ into $\dfrac{1+\cos\theta}{\sin\theta}$

30.5 Angles of Elevation and Depression:

30.5.(i) Find angles of elevation and depression.

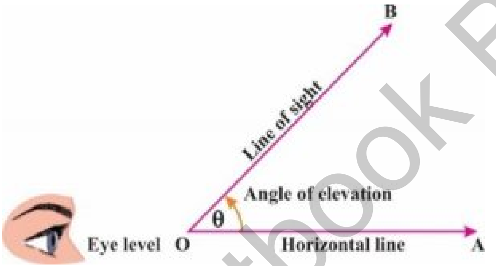


Fig: 30.21 (i)

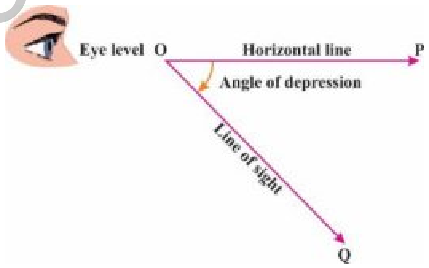


Fig: 30.21 (ii)

When an object is above the eye level of an observer, then the counter clockwise angle from the horizontal line to the line of sight is the angle of elevation as shown in Fig 30.21 (i). When the object is below the eye level, then the clockwise angle from the horizontal line to the line of sight is the angle of depression as shown in Fig 30.21 (ii)

In our daily life, it is used to find distance and angle relationships for object sighted a long distance from observer either above or below the observer's position. This involve uses of angles of elevation or angle of depression.

30.5. (ii) Solve real life problems involving angles of elevation and depression.

Example 1: The angle of elevation of the top of the electric pole from a point on the ground 18 meter away from its base is $30°$. Find the height of the electric pole.

Solution:

Angle of elevation from a point P (above the eye level) to the top of the tower B is $30°$ and let "h" be the height of the pole. Pole is 18m away from the the top of the tower point P. In ΔPAB, we have $\theta = 30°$ and $|\overline{PA}| = 18$ m

$$\therefore \quad \tan 30° = \frac{h}{|PA|} = \frac{h}{18}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{18}$$

$$\Rightarrow \quad h = \frac{18}{\sqrt{3}}$$

$$\Rightarrow \quad h = \frac{18\sqrt{3}}{3} = 6\sqrt{3} \text{ meter.}$$

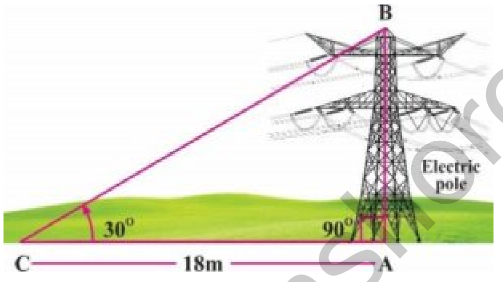


Fig: 30.22

Thus, height of the electric pole is $6\sqrt{3}$ meter.

Example 2: Find the height of an object if the angle of elevation of the sun is $19°$ and the length of the shadow of the object is 1.7 meters.

Solution: Given that angle of elevation of sun i.e. $\theta = 19°$ and the length of shadow $a = 1.7$m as shown in the figure.

Then,

$$\tan 19° = \frac{h}{a} = \frac{h}{1.7}$$

$$\Rightarrow \quad h = (1.7) \cdot \tan 19°$$

$$\Rightarrow \quad h = (1.7) \cdot (0.3443)$$

$\Rightarrow \quad h = 0.585$m approx is the required height of the object.

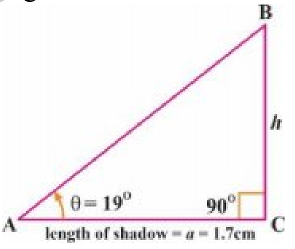


Fig: 30.23

Example 3: From the top of a tower, the angle of depression to the nearest port of a ship at its waterline is $40°$. If the height of tower is 35m, find the distance between the ship and the foot of the tower.

Solution:

Let 'x' be the distance of the ship from point B to the foot of the tower at point A as shown in the figure and "h" be the height of the tower.

Here,

$h = 35$m, height of the tower,

$\theta = 40°$, angle of depression,

and $x = ?$

In ΔABC $\quad \tan 40° = \frac{h}{x} = \frac{35}{x}$

$$\Rightarrow \quad x = \frac{35}{\tan 40°} = \frac{35}{0.8391} = 41.71\text{m (approx),}$$

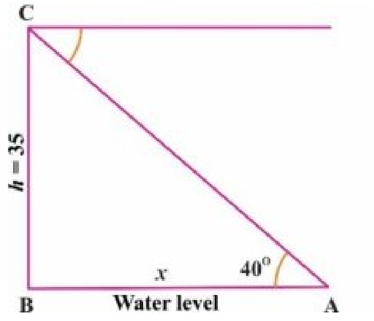


Fig: 30.24

Thus, the distance of ship to the foot of tower is 41.71 meters, (approx).

EXERCISE 30.5

1. From the top of a light house 102 meters high, measure of the angle of depression of a ship is $18°30'$. How far is the ship from the light house.
2. A ladder makes angle $60°$ with the ground and reaches a height of 6m on the wall, find the length of the ladder.
3. Find the angle of elevation when a 6m high bamboo makes a shadow of length $2\sqrt{3}$m.
4. An angle of elevation of the top of cliff is $30°$. Walking 210 meter from the point towards the cliff, the angle of elevation is $45°$. Find the height of cliff.
5. An observation balloon is 4280m above the ground and 9613m away from a farm house. Find angle of depression of the farm house as observed from the balloon.

REVIEW EXERCISE 30

1. Multiple Choice Question M.C.Qs.
   Spot the correct option.

i. The system of measurement in which angle is measured in radian is called:
   (a) CGS System (b) Sexagesimal
   (c) MKS system (d) circular system

ii. The union of two non-collinear rays at common vertex is called
   (a) an angle (b) a degree
   (c) a minute (d) radian

iii. $10° =$ _________
   (a) $600''$ (b) $3600''$ (c) $600'$ (d) 600

iv. $5\frac{\pi}{4}$ radians = _________
   (a) $115°$ (b) $225°$ (c) $135°$ (d) $45°$

v. $\frac{1}{2}\sec 45° =$ _________
   (a) $\sqrt{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) $\frac{2}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}}$

vi. $\operatorname{cosec}\theta \cdot \sin\theta =$ _________
   (a) 1 (b) 0 (c) -1 (d) 0.5

vii. $\sec^2\theta - \tan^2\theta =$ _________
   (a) 0 (b) 1 (c) -1 (d) $\cos^2\theta$

viii. In unit circle for point (x, y), $\sin\theta =$ _________

(a) x (b) $-x$ (c) y (d) $-y$

ix. If $\sin\theta = -\frac{1}{\sqrt{2}}$ and θ lies in the 3rd quadrant, then $\tan\theta = ?$

(a) -1 (b) 1 (c) ± 1 (d) $\frac{1}{\sqrt{2}}$

x. If an object is above the level of an observer then angle formed between the horizontal line and observer's line of sight is called:

(a) an angle of depression (b) an angle of elevation
(c) an obtuse angle (d) None of these

2. Define an angle and angle in standard position.
3. Convert the $70°30'90''$ into radians.
4. How many degrees are there in $(7200)'$.
5. Covert $\frac{2\pi}{3}$ radians to degree measure.
6. Covert $\frac{3}{\pi}$ radians to $D°M'S''$ form.
7. Find r, when $l = 7\text{cm}$ and $\theta = \frac{\pi}{4}$ radian
8. Find θ, when arc length equal to radius of a circle.
9. Prove that $(1-\cos^2\theta)(1+\cot^2\theta) = 1$
10. Calculate area of the circular sector when $r = 2\text{cm}$ and $\theta = 3$ radians.

SUMMARY

- Division of circumference of a circle into 360equal parts and angle subtended at centre by each part is called one degree and it denoted by $1°$.
- The angle subtended at the centre of the circle such that arc length and radius of the circle are equal then angle measured is of one radiation.
- Sub-division of radian/degree measures,
  - $1° = \frac{\pi}{180}$ radian ≈ 0.01745 radians and
  - $1° = \left(\frac{180°}{\pi}\right) \approx 57.3°$
- Formula for finding arc length of circle i.e. $l = r\theta$.
- Formula for area of sector $= \frac{1}{2}r^2\theta$ or $= \frac{1}{2}lr$.

- Angles having the same initial and terminal sides are called coterminal angles and they differ by multiple of 2π radians. They are also called general angles.
- An angle is said to in standard position if its vertex is at origin and its initial side on positive x-axis.
- In rectangular coordinate system two axes divide the plane into equal parts, each part is called quadrant. Hence, four quadrants are formed.

    Angles between 0° and 90° are in Quadrant I
    Angles between 90° and 180° are in Quadrant II
    Angles between 180° and 270° are in Quadrant III
    Angles between 270° and 360° are in Quadrant IV

- If the terminal side of the standard angle coincides with x-axis or y-axis, then it is called a quadrantal angle.
- There are six basic trigonometric ratios for angle θ. These are $\sin\theta, \cos\theta, \tan\theta, \text{cosec}\theta$ $\sec\theta$ and $\cot\theta$
- Reciprocal ratios, listed below:

$$\sin\theta = \frac{1}{\text{cosec}\theta} \quad \text{or} \quad \text{cosec}\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \quad \text{or} \quad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \quad \text{or} \quad \cot\theta = \frac{1}{\tan\theta}$$

- An identity is an equation which is true for all values of the variable (except where value is not defined).
- Three fundamental trigonometric identities are:

    (i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $\sec^2\theta = 1 + \tan^2\theta$
    (iii) $\text{cosec}^2\theta = 1 + \cot^2\theta$

- When an object is above the eye level of an observer, then the counter clockwise angle from the horizontal line to the line of sight is the angle of elevation.
- When the object is below the eye level, then the clockwise angle from the horizontal line to the line of sight is the angle of depression.