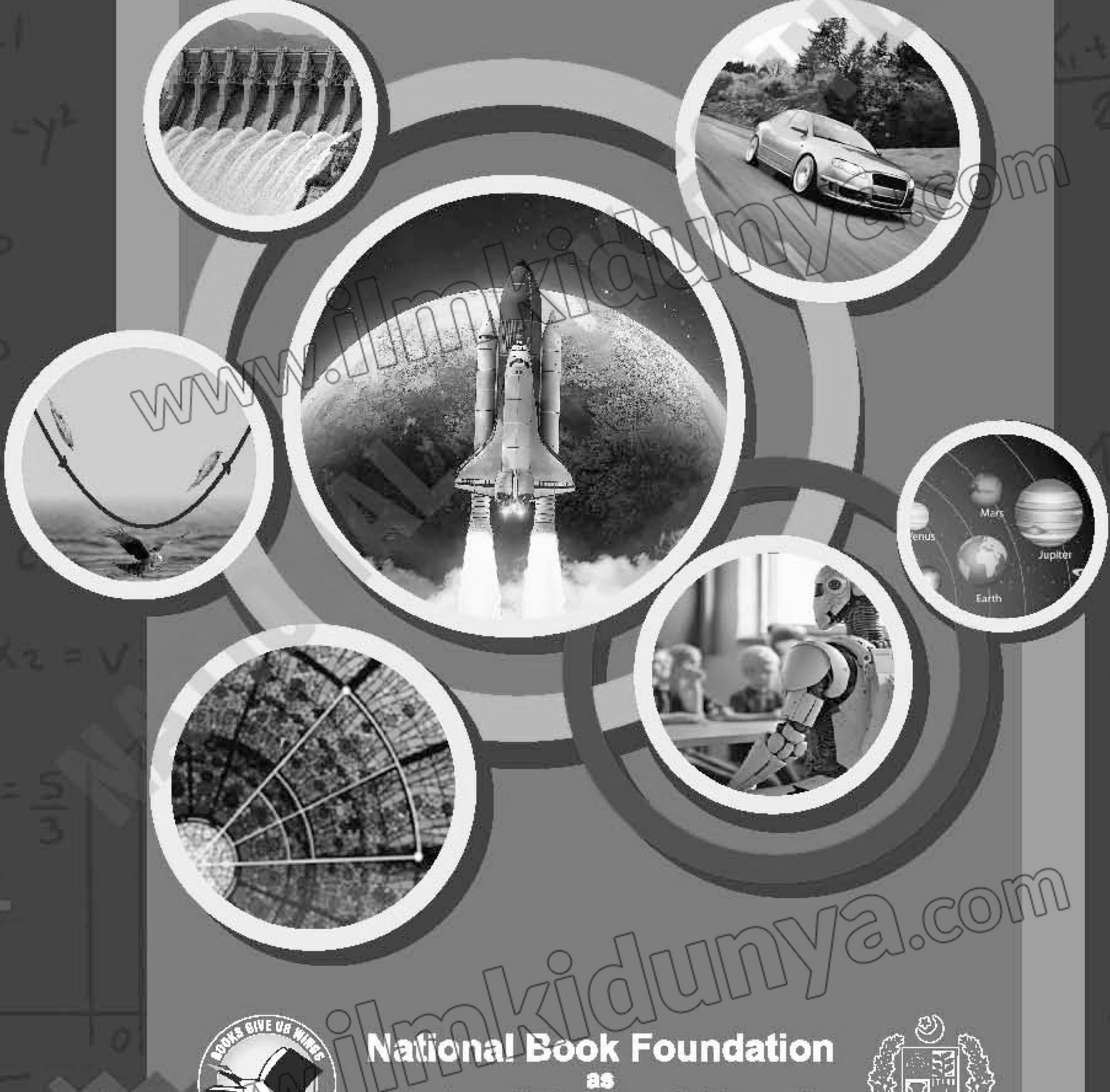
12

Based on National Curriculum of Pakistan 2022-23

Textbook of
MATHEMATICS



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Based on National Curriculum of Pakistan 2022-23

Textbook of
Mathematics
Science Group

12

National Curriculum Council
Ministry of Federal Education and Professional Training

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Preface

This Textbook for Mathematics Grade 12 has been developed by NBF according to the National Curriculum of Pakistan 2022-2023. The aim of this textbook is to enhance learning abilities through inculcation of logical thinking in learners, and to develop higher order thinking processes by systematically building the foundation of learning from the previous grades. A key emphasis of the present textbook is creating real life linkage of the concepts and methods introduced. This approach was devised with the intent of enabling students to solve daily life problems as they grow up in the learning curve and also to fully grasp the conceptual basis that will be built in subsequent grades.

After amalgamation of the efforts of experts and experienced authors, this book was reviewed and finalized after extensive reviews by professional educationists. Efforts were made to make the contents student friendly and to develop the concepts in interesting ways.

The National Book Foundation is always striving for improvement in the quality of its textbooks. The present textbook features an improved design, better illustration and interesting activities relating to real life to make it attractive for young learners. However, there is always room for improvement, the suggestions and feedback of students, teachers and the community are most welcome for further enriching the subsequent editions of this textbook.

May Allah guide and help us (Ameen).

Dr. Kamran Jahangir
Managing Director

Application of Mathematics

Functions and Graphs: Functions represent relationships between variables and their graphs provide a visual representation of these relationships. They are used to study trends like profit versus cost in businesses, population growth over time or changes in speed in physics. Functions help in predicting outcomes and analyzing real-world data effectively.

Limit, Continuity and Derivative: Limits describe how functions behave near specific points, continuity ensures smooth graphs without breaks, and derivatives measure rates of change like speed or growth. These concepts are essential in physics to calculate instantaneous velocity, in economics for cost optimization and in biology for population growth modeling.

Integration: Integration helps calculate areas under curves, volumes, or accumulated quantities. It is widely used to determine total distance from velocity, analyze energy consumption and compute areas in construction projects. Integration also has applications in physics, such as finding work done by a variable force.

Differential Equations: Differential equations describe changes in dynamic systems, modeling real-world processes like population growth, chemical reactions and motion. They are used in engineering to design systems, in physics to describe heat flow or wave motion and in biology to model disease spread.

Kinematics of Motion in a Straight Line: This studies the motion of objects along a straight path using concepts like displacement, velocity and acceleration. Applications include calculating stopping distances of vehicles, analyzing free-fall under gravity and predicting the motion of objects in linear transport systems.

Analytical Geometry: Analytical geometry combines algebra and geometry to study the properties of shapes on the coordinate plane. It is used in designing structures, solving distance and slope problems and planning urban layouts. It plays a significant role in architecture and engineering.

Conic Section: Conic sections include circles, ellipses, parabolas, and hyperbolas, which have various applications. Ellipses describe satellite orbits, parabolas are used in designing headlights and bridges and hyperbolas are applied in communication systems and radar design.

Inverse Trigonometric Functions and Their Graphs: Inverse trigonometric functions calculate angles from trigonometric values, essential in navigation, surveying, and architecture. Their graphs help solve problems involving slopes, elevation angles and real-world measurements requiring precision.

Solution of Trigonometric Equations: Trigonometric equations model periodic phenomena such as sound and light waves. Solving these equations is crucial in designing musical instruments, analyzing alternating electrical currents and studying wave patterns in physics and engineering.

Numerical Methods: Numerical methods approximate solutions for complex problems using algorithms. They are widely used in weather forecasting, structural analysis of buildings and financial risk modeling. These methods make it possible to solve equations that are difficult to handle analytically.

بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ

اللہ کے نام سے شروع جو بڑا مہربان، نہایت رحم والا ہے

CONTENT

UNIT
01

FUNCTIONS AND GRAPHS

After studying this unit, students will be able to:

- Recall definition of function, find its domain, codomain, range and its types.
- Find inverse of a function and demonstrate its domain and range with examples.
- Know linear, quadratic and square root functions.
- Sketch graphs of linear and non-linear functions.
- Plot graph of function of the type $y = x^n$ when (i) n is a + ve or – ve integer and $x \neq 0$, (ii) n is a rational number for $x > 0$.
- Plot graph of quadratic function of the form $y = ax^2 + bx + c$, where a, b, c are integers and $a \neq 0$.
- Draw graph using factors and predict functions from their graphs.
- Find the intersecting points graphically when intersection occurs between (i) a linear function and coordinate axes (ii) two linear functions (iii) a linear function and a quadratic function.
- Draw the graph of modulus functions.
- Solve graphically appropriate problems from daily life.
- Classify algebraic and transcendental functions and describe trigonometric, inverse trigonometric, logarithmic and exponential functions.
- Define logarithm, and derive and apply laws of logarithm.
- Graph and analyse exponential and logarithmic functions.
- Apply the concept of exponential functions to find compound interest.
- Solve problems involving exponential and logarithmic equations.
- Identify the domain and range of transcendental functions through graphs.
- Interpret the relation between a one-one function and its inverse through a graph.
- Demonstrate the transformation of a graph through horizontal shift, vertical shift and scaling.

Functions have many applications in real life. One is the use of function in signal processing applications in engineering, including noise reduction, modulation, and filtering. For example, functions in audio processing are used to analyze and alter sound waves, which makes it possible to design devices like equalizers and noise-canceling headphones. Similar to this, functions are essential to the encoding, transmitting, and decoding of signals for wireless communication systems in the telecommunications industry.

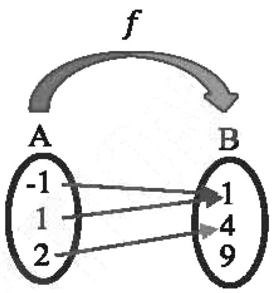
1.1 Function

A function f from a set A to a set B assigns to each element of A exactly one element of B. The set A is called the domain of the function and the set B is called the co-domain of the function.

Mathematically, it is written as f : A → B and is read as f is a function from A to B.

For example, if we are given two sets A = $\{-1, 1, 2\}$ and B = $\{1, 4, 9\}$,

then $f = \{(-1, 1), (1, 1), (2, 4)\}$ is a function because each element of the set A is assigned to exactly one element of the set B. i.e. there is no repetition in the first element of ordered pairs in f. The first element of each ordered pair in f is called pre-image while its corresponding second element is called image of the first element. For example, in (2, 4), 2 is the pre-image of 4 and 4 is the image of 2.



If x is independent variable and y is dependent variable, then in general, a function f from A to B is written as $f(x) = y$.

For example, in the above example,

$f(-1) = 1, f(1) = 1$ and $f(2) = 4$

Explanation: As discussed above, a function relates an input to an output.

For example, if a tree grows 15 cm every year and the height h of the tree is related to its age as follows:

$$h(\text{age}) = \text{age} \times 15$$

then the height of the tree after 10 year is $h(10) = 10 \times 15 = 150$ cm

∴ '$h(10) = 150$' is like saying 10 is related to 150 or 10 → 150

Here, 10 is the input and 150 is the output of the function.

1.1.1 Domain of a Function

The set of all possible values of independent variable which qualify as inputs to a function is known as the domain of the function. In the above example,

Domain of function f = Dom $f = \{-1, 1, 2\}$

Key Facts

A function in which real numbers are used is called a real valued function.

How to Find the Domain of a Function

To find the domain, we ensure that there is no zero in the denominator of a fraction and no negative sign inside a square root. In general, the set of all real numbers is considered as the domain of a function subject to some restrictions. For example:

(i) When the given function is of the form $f(x) = 3x + 8$ or $f(x) = x^3 + 2x - 5$, the domain will be "the set of all real numbers".

(ii) When the given function is of the form $f(x) = \frac{1}{x-2}$ the domain will be the set of all real numbers except 2.

(iii) In some cases, the interval be specified along with the function such as, $f(x) = x + 1$, $0 < x < 10$. Here, x can take the values between 0 and 10 in the domain.

(iv) Domain restrictions refer to the values for which the given function cannot be defined.

1.1.2 Range of a Function

The set of all the outputs of a function is known as the range of the function or after substituting the domain, the entire set of all possible values as outcomes of the dependent variable.

In a function $y = f(x)$, the spread of all the values y from minimum to maximum is the range of the function. In the above example,

Range of function f = Rang $f = \{1, 4\}$

How to Find the Range of a Function

- Substitute all the values of x in the function to check whether it is positive, negative or equal to other values. Eliminate the values of x for which the function is not defined.
- Find the minimum and maximum values for y.

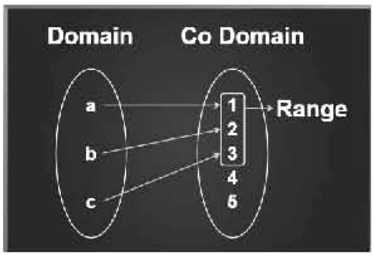
1.1.3 Codomain of a Function

The codomain is the set of all possible outcomes of the given function.

In general, the range is the subset of the codomain. But sometimes the codomain is also equal to the range of the function. In above example,

Codomain of function $f = \{1, 4, 9\}$

In short:



- ➢ What can go into a function is called the domain.
- ➢ What may possibly come out of a function is called the codomain.
- ➢ What actually comes out of a function is called the range.

In the adjoining figure,

- The set {a, b, c} is the domain.
- The set $\{1, 2, 3, 4, 5\}$ is the codomain.
- The set $\{1, 2, 3\}$ is the range.

Example 1: Find the domain and range of a function $f(x) = 2x^2 - 4$.

Solution: $f(x) = 2x^2 - 4$

The given function has no undefined values of x.

Thus, the domain is the set of all real numbers.

Domain = $(-\infty, \infty) = R$

If we put $x = 0$ in the given function, we get $f(x) = -4$.

For all real values of x, other than 0, we get an output greater than -4.

Hence, the range of $f(x)$ is $[-4, \infty)$.

Key Facts

A function is like a machine that takes an input and produces a corresponding output. For example, the distance a car has traveled (the output) is dependent on how long that car has been driving (the input).

Example 2:

Find the domain and range of function $g(x) = \sqrt{x-3}$.

Solution: $g(x) = \sqrt{x-3}$

The given function is defined for all real numbers x greater than or equal to 3.

Thus, the domain of $g(x)$ is $[3, \infty)$.

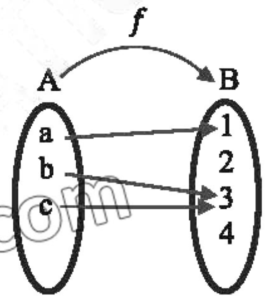
If we put $x = 3$ in the given function, we get $g(3) = 0$.

For all real values of x, greater than 3, we get an output greater than 0.

Hence, the range of $g(x)$ is $[0, \infty)$.

1.2 Types of Functions

1.2.1 Into Function



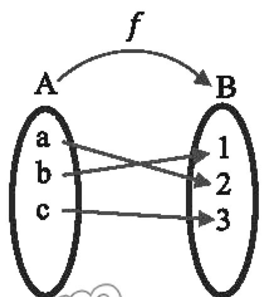
A function f: A → B is said to be into function if there exists at least one element or more than one element in B, which does not have any pre-images in A, which simply means that every element of the codomain is not mapped with elements of the domain. i.e., range$(f) \neq$ B.

In the adjoining diagram, f = {(a, 1), (b, 3), (c, 3)} is an into function.

Some examples of into functions are:

- $f(x) = \sin x$ where f: R → R is into function because it doesn't cover all values in the interval R.
- $g(x) = x^2$ where g: R → R is into function because it doesn't map to any negative real numbers.
- $h(x) = e^x$ where h: R → $[0, \infty)$ is into function because it doesn't map to zero.

1.2.2 Onto (Surjective) Function



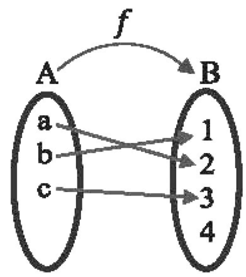
For any two non-empty sets A and B, a function f : A → B will be onto if every element of set B is an image of some element of set A. i.e., for every $y \in$ B there exists an element x in A such that $f(x) = y$ which implies rang(f) = B.

In the adjoining diagram, f = {(a, 2), (b, 1), (c, 3)} is an onto function.

Some examples of onto functions are:

- $f(x) = x$ (Identity function)
- $g(x) = e^x$ when g: R → R^+ (Exponential function)
- $h(x) = x^2$ (Square function)
- $m(x) = x^3$ (Cubic function)
- $p(x) = c$ (Constant function)

1.2.3 One to One (Injective) Function



For any two non-empty sets A and B, a function $f: A \rightarrow B$ will be one-to-one if distinct elements of set A have distinct images in set B. In the adjoining diagram, $f = \{(a, 2), (b, 1), (c, 3)\}$ is a one-to-one function.

A function $f: A \rightarrow B$ is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

i.e., an image of a distinct element of A under f mapping (function) is distinct.

Some examples of one-one functions are:

- $f(x) = x$ (Identity function)
- $g(x) = 2x + 3$ (Linear function)
- $h(x) = e^x$ (Exponential function)
- $m(x) = \sqrt{x}$ (Square root function, defined for $x \geq 0$)

1.2.4 Injective Function

A function which is both into and one-one is called an injective function.

1.2.5 Bijective Function

A function which is both onto and one-one is called a bijective function.

Bijective function shows one-one correspondence between the elements of two sets.

Key Facts

Various types of functions are mentioned in the below table:

Based on elements	One-one function, Many-one function, Onto function, Bijective Function, Into function, constant Function
Based on the equation	Identity function, Linear function, Quadratic function, Cubic function, Polynomial functions
Based on the range	Modulus function, Rational function, Even and odd functions, Periodic functions, Greatest and smallest integer function, Inverse function, Composite functions
Based on the domain	Algebraic functions, Trigonometric functions, Logarithmic functions, Exponential functions

Example 3: Check whether the function $f(x) = 2x + 3$, is one-to-one or not if domain = $\{0.5, 1, 2\}$ and codomain = $\{4, 5, 7\}$

Solution: Putting 1, 2 and 0.5 in $f(x) = 2x + 3$, we get $f(0.5) = 4$, $f(1) = 5$ and $f(2) = 7$

As, for every value of x, we get a unique $f(x)$ thus, the function $f(x)$ is one to one.

Example 4: Check whether the function is one-to-one or not: $f(x) = 2x^2 + 1$.

Solution: To check whether the function is one to one or not, let:

$$f(x_1) = f(x_2)$$

$$2(x_1)^2 + 1 = 2(x_2)^2 + 1$$

$$(x_1)^2 = (x_2)^2$$

Since $(x_1)^2 = (x_2)^2$ is not always true, therefore the function is not one to one function.

Example 5: Check the type of function $f(x) = x^2 - 1$ if Dom $f(x) = \{1, -1, 2, -2\}$ and Codom $f(x) = \{0, 3, -3\}$

Solution: Given $f(x) = x^2 - 1$ with Dom $f(x) = \{1, -1, 2, -2\}$ and Codom $f(x) = \{0, 3, -3\}$

Substituting the elements of the domain in the function, we get:

$$f(1) = 1^2 - 1 = 0$$
$$f(-1) = (-1)^2 - 1 = 0$$
$$f(2) = 2^2 - 1 = 3$$
$$f(-2) = (-2)^2 - 1 = 3$$

Therefore, Rang $f(x) = \{0, 3\}$. As, Rang $f(x) = \{0, 3\} \neq \{0, 3, -3\} =$ Codom $f(x)$.
So, the given function is an into function.

Example 6: Find the type of the function $f(x) = 3x + 2$ defined on $f: R \to R$.

Solution: Let, $f(x) = y \quad \Rightarrow \quad y = 3x + 2 \quad \Rightarrow \quad y - 2 = 3x \quad \Rightarrow \quad x = \frac{y-2}{3}$

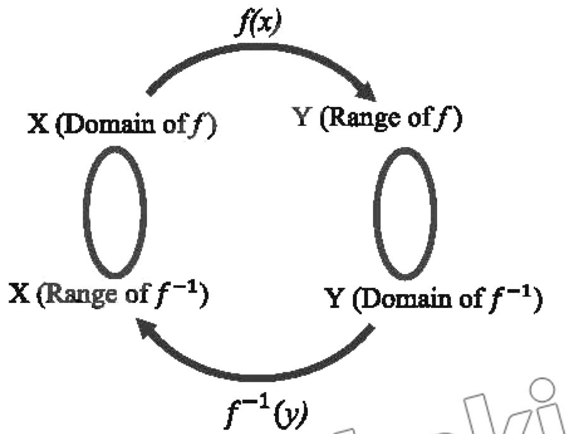
Substituting the value of x in the given function $f(x)$, we get:

$$f(x) = f\left(\frac{y-2}{3}\right) = 3\left(\frac{y-2}{3}\right) + 2 = y - 2 + 2 = y$$

Since, we get back y after putting the value of x in the function. Hence the given function is an onto function.

1.3 Inverse Function

The inverse of any function $f(x)$ is a function denoted by $f^{-1}(x)$ which reverses the effect of $f(x)$ and it undoes what $f(x)$ does. In mathematics, the inverse function is also denoted by f^{-1}. If f: X → Y, then f^{-1}: Y → X. i.e., If the application of a function f to x as input gives an output of y, then the application of inverse function f^{-1} to y should give back the value of x.
It can be illustrated in the following diagram as:



Key Facts

- If $y = f(x)$ is bijective function then $x = f^{-1}(y)$.
- If $fog(x) = gof(x) = x$, then $g = f^{-1}$ and $f = g^{-1}$
- $(f^{-1})^{-1} = f$

From the above diagram:

$$\text{dom } f = \text{rang } f^{-1} \quad \text{and} \quad \text{rang } f = \text{dom } f^{-1}$$

i.e., The domain of the given function becomes the range of the inverse function, and the range of the given function becomes the domain of the inverse function.

Note that f^{-1} is not the reciprocal of f and not every function has an inverse. If a function $f(x)$ has an inverse, then $f(x)$ never takes the same value twice. In simple words, the inverse function exists only when f is both one-one and onto function. Can we say that the inverse function is also a bijective function?

Moreover, the composition of the function f and the inverse function f^{-1} gives the domain value of x.

$$fof^{-1}(x) = f^{-1}of(x) = x$$

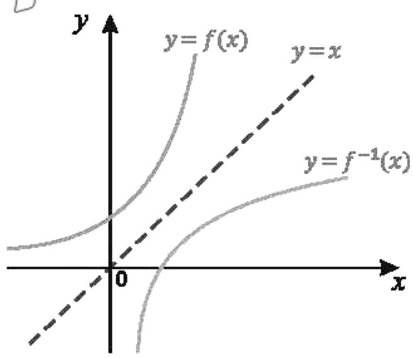
1.3.1 Steps to Find an Inverse Function

Consider a function $f(x) = ax + b$.

- Replace $f(x)$ with y, to obtain $y = ax + b$.
- Solve the expression for x to obtain $x = \frac{y-b}{a}$.
- Replace x with $f^{-1}(y)$ to get $f^{-1}(y) = \frac{y-b}{a}$.
- Interchange y with x in the function $f^{-1}(y) = \frac{y-b}{a}$ and get inverse function $f^{-1}(x) = \frac{x-b}{a}$.

1.3.2 Graph of an Inverse Function

If the graphs of both functions are symmetric with respect to the line $y = x$ then we say that the two functions are inverses of each other. This is because of the fact that if (x, y) lies on the function, then (y, x) lies on its inverse function.



Example 7:

Find the inverse function of $f(x) = \frac{x}{x-2}$ defined on $f: R \to R$.

(i) Find domain and range of function and its inverse.

(ii) Prove that $fof^{-1}(x) = f^{-1}of(x) = x$

Solution:

(i) Given function is $f(x) = \frac{x}{x-2}$

Dom $f(x) = R - \{2\}$

To find inverse function, let:

$$y = \frac{x}{x-2} \quad \Rightarrow \quad y(x-2) = x \quad \Rightarrow \quad xy - x = 2y$$

$$\Rightarrow \quad x(y-1) = 2y \quad \Rightarrow \quad x = \frac{2y}{y-1}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{2y}{y-1} \quad \ldots\ldots \quad (x = f^{-1}(y))$$

$$\Rightarrow \quad f^{-1}(x) = \frac{2x}{x-1} \quad \ldots\ldots \text{ Replacing } y \text{ with } x.$$

From the inverse function, we see that:
Dom $f^{-1}(x) = R - \{1\}$
Hence,
Dom $f = R - \{2\} =$ Rang f^{-1} and Dom $f^{-1} = R - \{1\} =$ Rang f

(ii) $fof^{-1}(x) = f(f^{-1}(x)) = f\left(\frac{2x}{x-1}\right) = \frac{\frac{2x}{x-1}}{\frac{2x}{x-1}-2} = \frac{2x}{2x-2x+2} = \frac{2x}{2} = x$ (i)

$f^{-1}of(x) = f^{-1}(f(x)) = f^{-1}\left(\frac{x}{x-2}\right) = \frac{2\left(\frac{x}{x-2}\right)}{\frac{x}{x-2}-1} = \frac{2x}{x-x+2} = \frac{2x}{2} = x$ (ii)

From (i) and (ii), we get:

$$fof^{-1}(x) = f^{-1}of(x) = x$$

Exercise 1.1

1. Find the domain of following functions.
   (i) $f(x) = x^2 - 6$ (ii) $g(x) = \frac{x}{x+3}$ (iii) $h(x) = \frac{x+4}{x^2-9}$
   (iv) $i(x) = \frac{x}{5x+2}$ (v) $j(x) = \frac{x}{x^2+4}$ (vi) $k(x) = \sqrt{x+1}$
2. Find the domain and range of the functions.
   (i) $f(x) = x + 7$ (ii) $f(x) = 2x^2 + 1$ (iii) $f(x) = 2\sqrt{x-5}$
   (iv) $f(x) = |x-2| - 3$ (v) $f(x) = 1 + \sin x$ (vi) $f(x) = 3 + \sqrt{x-2}$
   (vii) $f(x) = \frac{3e^x}{7}$ (viii) $f(x) = \frac{x^2-16}{x+4}$ (ix) $f(x) = (x-1)^2 + 1$
   (x) $f(x) = \frac{1}{x-1}$ (xi) $f(x) = \frac{x-2}{x+3}$ (xii) $f(x) = \frac{x^2-x-6}{x-3}$
3. Given that A = {0, 1, 2, 3}, B = {p, q, r, s} and f = {(0, p), (1, q), (2, r), (3, s)}. Check whether the function is one to one, onto and/or into.
4. A = {2, 3, 4, 5}, B = {b, c, d, e}. The function is defined as f = {(2, b), (3, c), (4, e), (5, e)}. Check whether the function is one to one, into or onto.
5. Check whether the functions are one-to-one or not.
   (i) $f(x) = 4x - 7$ (ii) $f(x) = 6x^2 + 2$ (iii) $f(x) = \frac{x^3-1}{2}$
6. Check the type of function $g(x) = 2x^2 + 3x + 1$ if Dom $g(x) = \{0, 1, 2, 3\}$ and Rang $g(x) = \{1, 6, 15, 28, 35\}$
7. Find the type of the function $h(x) = 2x + 1$ defined on $h: R \to R$.
8. If f: A → B is defined by $f(x) = \frac{x-2}{x-3}$ for all $x \in$ A where A = R – {3} and B = R – {1}. Then show that the function f is bijective.

9. Find the domain and range of inverse functions when:

(i) $f(x) = 4x - 3$ (ii) $f(x) = \frac{x}{x-5}$ (iii) $f(x) = \frac{x+2}{x-1}$

(iv) $f(x) = \sqrt{x+2}$ (v) $f(x) = x^2 + 6$ (vi) $f(x) = \frac{2x-1}{x+4}$

Also prove that $f(f^{-1}(x)) = f^{-1}(f(x))$.

1.4 Linear, Quadratic and Square Root Functions

1.4.1 Linear Function

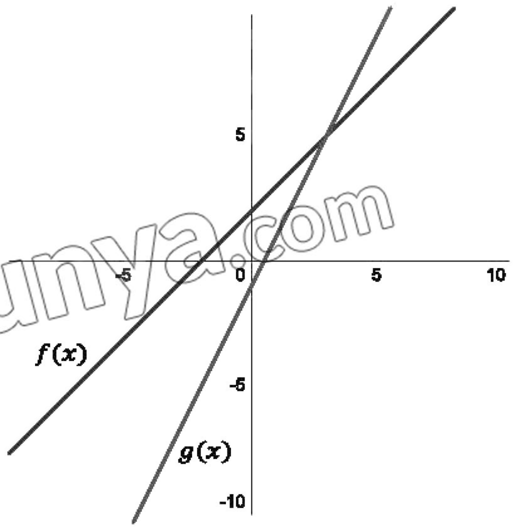
The function of the form,

$$y = ax + b;\ a, b \in R \wedge a \neq 0$$

is called linear. It is a polynomial function of degree one.

For example, $f(x) = x + 2,\ g(x) = 2x - 1$ are linear functions.

The graph of a linear function is a straight line and the slope of any two points on the line is the same. The domain and range of the linear function is R.



1.4.2 Non-linear Function

A function that is not linear is called a non-linear function. A nonlinear function is a function whose plotted graph form a curved line. For example, quadratic function, cubic function, square root function and exponential function etc. The slope of every two points on the graph of non-linear is not the same. Let us recall the shapes of the graphs of quadratic and square root functions here.

(a) Quadratic Function

The function is of the form,

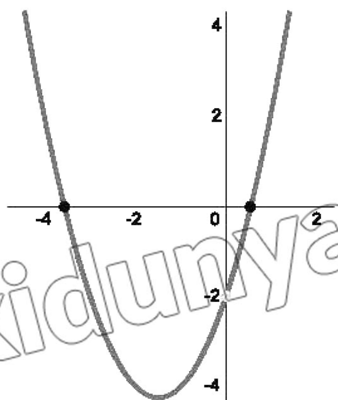
$$y = ax^2 + bx + c;\ a, b, c \in R \wedge a \neq 0$$

is called quadratic.

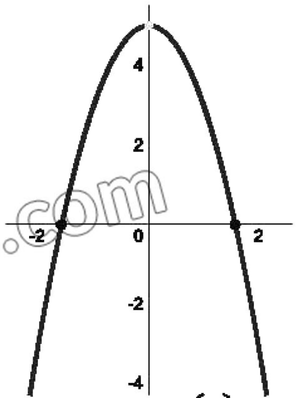
It is a polynomial function of degree two.

For example, $f(x) = x^2 + 3x - 2$ and $g(x) = 5 - 2x^2$ are quadratic functions.

The domain and range of the quadratic function is R. The graph of a quadratic equation is U-shaped and is parabolic in nature.



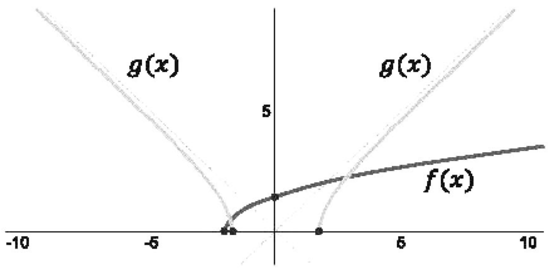
$y = f(x)$



$y = g(x)$

(b) Square Root Function

The function of the form $y = \sqrt{x}$, where $x \geq 0$ is called a square root function.
For example,
$f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{x^2 - 3}$ are square root functions.



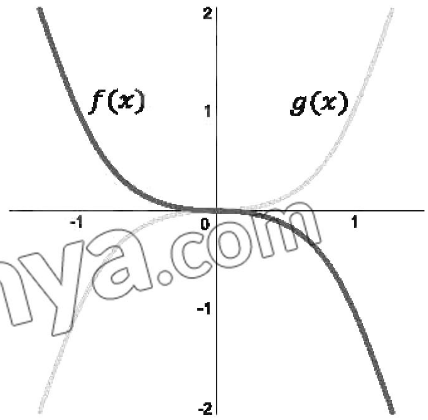
The domain of square root function depends upon its formation.

1.5 Plotting Graph of Function of the Type $y = x^n$

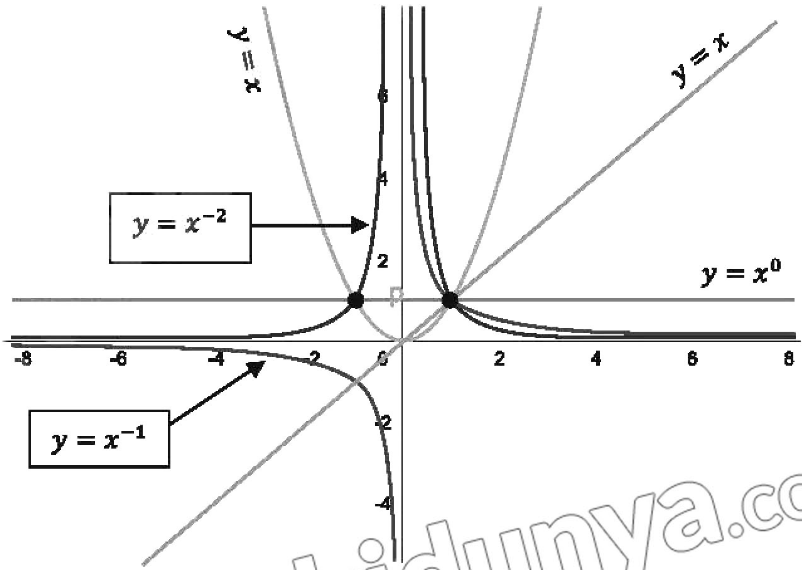
1.5.1 Graph of the Function $y = x^n$; $n \in Z \wedge x \neq 0$

For plotting the graph of the function $y = x^n$; $n \in Z \wedge x \neq 0$, we take different integral values of n. For example, the table for the function $y = \pm x^3$ is:

x	0	0.5	1	2
$f(x) = x^3$	0	0.125	1	8
$g(x) = -x^3$	0	−0.125	−1	−8



We observe that graphs of $y = x^3$ and $y = -x^3$ have the same shape but opposite behavior.
The graph of $y = x^n$ for $n = -2, -1, 0, 1, 2$ is shown below.



From the above graphs, we observe that:

- the graph of $y = x^0$ is a horizontal line passing through $y = 1$.
- the graph of $y = x^1$ is a straight line bisecting first and third quadrant.
- the graph of $y = x^{-1}$ is a hyperbola passing through first and third quadrant.

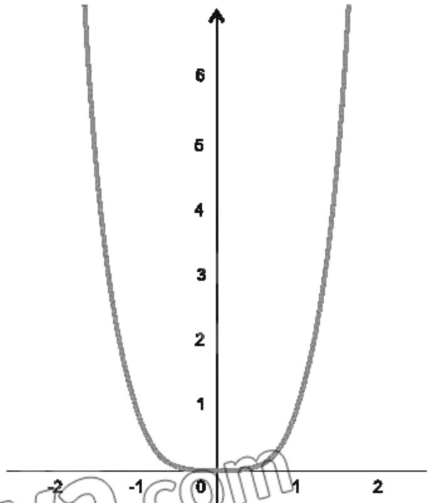
- the graph of $y = x^2$ is a parabola starting from origin and opening upwards.
- the graph of $y = x^{-2}$ has exponential behavior with two branches.
- The graphs of $y = x^n$ pass through (1, 1).

Example 8: Draw the graph of $y = x^4$.

Solution:

Table for some values of x and y for the function is:

x	-1.5	-1	0	1	1.5
y	5.06	1	0	1	5.06

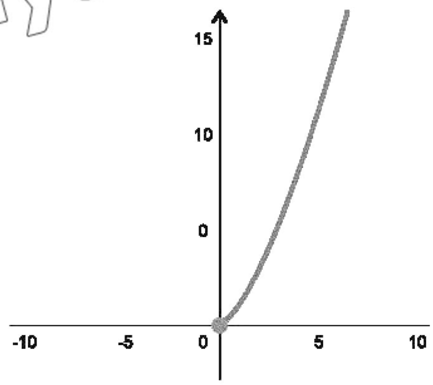


From the figure, we can see that the graph of $y = x^4$ is U-shaped which opens upward starting from origin. The value of y increases slowly for real numbers $-0.5 < x < 0.5$.
After that it increases abruptly.

1.5.2 Graph of the Function $y = x^n$; $n \in Q \wedge x > 0$

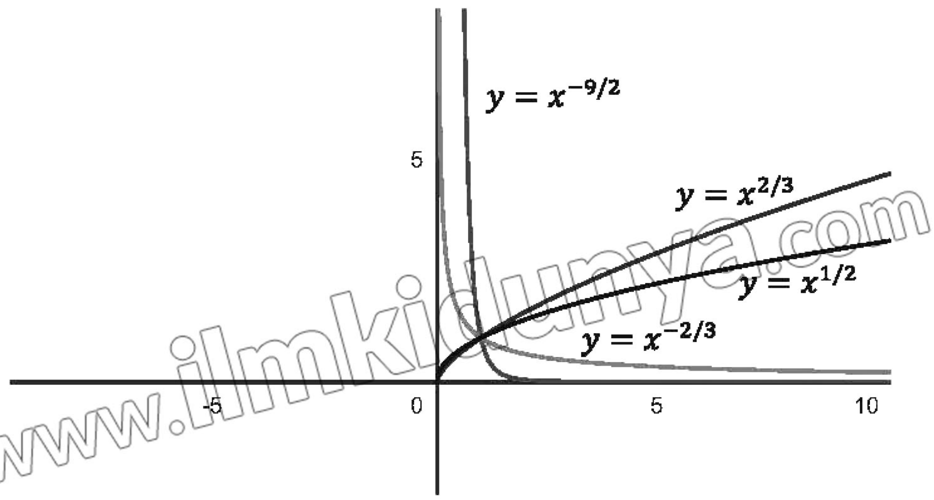
For plotting the graph of the function $y = x^n$; $n \in Q \wedge x > 0$, we take different integral values of n. For example, the table for the function $y = x^{3/2}$ is:

x	0.5	1	4	6
$f(x) = x^{3/2}$	0.35	1	8	14.7



We observe that graphs of $y = x^{3/2}$ has exponential behavior.

The graph of $y = x^n$ for $n = -\frac{9}{2}, -\frac{2}{3}, 0, \frac{2}{3}, \frac{1}{2}$ is shown below.



From the above graphs, we observe that:

- the graph of $y = x^{-2/3}$ is closer to y-axis.
- the graph of $y = x^{-9/2}$ moves away from y-axis as compared with the graph of $y = x^{-2/3}$.
- the graph of $y = x^{1/2}$ is closer to x-axis.
- the graph of $y = x^{2/3}$ moves away from x-axis as compared with the graph of $y = x^{1/2}$.
- The graphs of $y = x^n$ pass through (1, 1).

1.5.3 Graph of Quadratic Function

We know that the polynomial function of degree two is called a quadratic function. This function is of the form:

$$f(x) = ax^2 + bx + c \; ; a, b, c \in R \text{ and } a \neq 0$$

The graph of the quadratic equation is a parabola.

For example, $y = x^2 + 2x + 1$ and $y = 2 - 3x^2$ are quadratic functions.

Understanding the Graph

- a: The coefficient a, affects the direction and width of the parabola. If $a > 0$, the parabola opens upwards. If $a < 0$, the parabola opens downwards. The larger the absolute value of a, the narrower the parabola.
- b: This coefficient affects the position of the vertex horizontally (left or right) and slope of the parabola at the vertex.
- c: This is the y-intercept. So, the point $(0, c)$ is on the parabola.
- $x = -\frac{b}{2a}$ is equation of axis of symmetry and is also the x-coordinate of the vertex.

Example 9: Draw the graph of $y = 2x^2 + 3x - 2$.

Solution:

$y = 2x^2 + 3x - 2$ (i)

Comparing (i) with $y = ax^2 + bx + c$, we have $a = 2, b = 3$ and $c = -2$

Step 1: Determine whether parabola opens upwards or downwards.

As $a > 0$, therefore parabola opens upwards.

Step 2: Find and draw the axis of symmetry.

Equation of axis of symmetry is:

$$x = -\frac{b}{2a} = -\frac{3}{2(2)} = -\frac{3}{4}$$

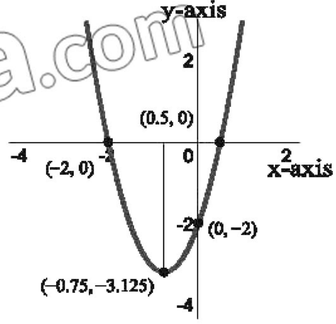
Step 3: Find and plot the vertex.

The x-coordinate of vertex is $x = -\frac{3}{4}$

To find the y-coordinate, substitute value of x in equation (i).

$$y = 2\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right) - 2 = \frac{9}{8} - \frac{9}{4} - 2 = -\frac{25}{8}$$

So, the vertex is $\left(-\frac{3}{4}, -\frac{25}{8}\right)$.



Step 4: Find some more points if needed and plot the graph.

Table for some values of x and y for the function is:

x	-2	-1	0	1
y	0	-3	-2	3

Form the figure, it is clear that the graph of function $y = 2x^2 + 3x - 2$ is parabola.

Example 10: Draw the graph of $y = 4 - 2x^2$

Solution: $y = -2x^2 + 0x + 4$ (i)

Comparing (i) with $y = ax^2 + bx + c$, we have $a = -2, b = 0$ and $c = 4$

Step 1: Determine whether parabola opens upwards or downwards.

As $a < 0$, therefore parabola opens downwards.

Step 2: Find and draw the axis of symmetry. Equation of axis of symmetry is:

$x = -\frac{b}{2a} = -\frac{0}{2(-2)} = 0$

Step 3: Find and plot the vertex.

The x-coordinate of vertex is $x = 0$. To find the y-coordinate, substitute value of x in equation (i). We get $y = -2(0)^2 + 0(0) + 4 = 4$

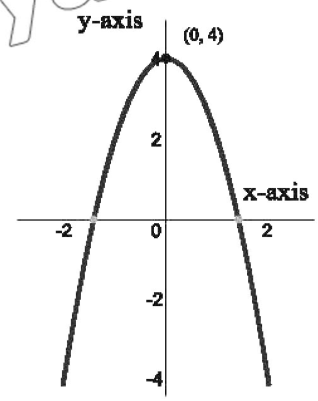
So, the vertex is $(0, 4)$.

Step 4: Find some more points if needed and plot the graph.

Table for some values of x and y for the function is:

x	-2	-1	0	1	2
y	-4	2	4	2	-4

The sketch of the graph is shown in the figure.



1.6 Drawing Graph Using Factors

Let us draw the graph of quadratic function using factors.

We know that $y = ax^2 + bx + c$ is a quadratic function where $a \neq 0$ and the graph of such a function is a parabola. These graphs can be tricky to sketch manually, but factoring the quadratic gives us all of the information we need to do so successfully.

Procedure:

The points where any parabola intersects the x-axis will be the solutions to the equation:

$$ax^2 + bx + c = 0 \quad \text{(i)}$$

Now if we can factor equation (i) in the format:

$$(x - x_1)(x - x_2) = 0$$

then by the zero product property, we get:

$$x = x_1 \quad \text{and} \quad x = x_2$$

This means that x_1 and x_2 are the x-intercepts. In other words, graph will intersect x-axis at $(x_1, 0)$ and $(x_2, 0)$. The constant c tells us what the y-intercept will be. More specifically, the constant term c places the y-intercept at $(0, c)$, giving us a third specific point on y-axis. Likewise, the value of a tells us whether our parabola will open up or down. If a is positive, the parabola opens upward. If it is negative, the parabola opens downward.

We can also figure out the vertex of the parabola by factoring the quadratic equation.

The x-coordinate of the vertex of a parabola is the arithmetic mean of two x-intercepts = $\frac{x_1+x_2}{2}$.

Once we have the x-coordinate, we can determine the y-coordinate by plugging the x value into the given function and solving for y. With these four specific points including both x-intercepts, y-intercept and the vertex of the parabola, we can create an accurate sketch of the graph quickly and easily.

Example 10: Draw the graph of $y = x^2 - 8x + 12$ using factors.

Solution:

Given function is: $y = x^2 - 8x + 12$

To get x-intercepts, put $x^2 - 8x + 12 = 0$

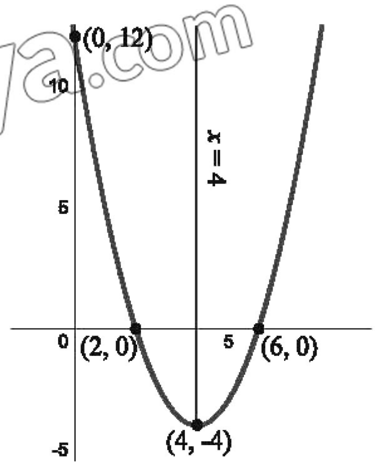
After factorization, we get:

$$x_1 = 2, \ \ x_2 = 6$$

⇒ The graph intersects x-axis at (2, 0) and (6, 0).

To find y-intercept, put $x = 0$ in the given function which gives $y = 12$.

Therefore y-intercept is (0, 12).



Vertex: x-coordinate of vertex = $\frac{x_1+x_2}{2} = \frac{2+6}{2} = 4$

Substituting, $x = 4$ in given function, we get:

y-coordinate of vertex = $4^2 - 8(4) + 12 = -4$

∴ Vertex = $(4, -4)$

As $a = 1 > 0$, therefore parabola opens upward.

The graph is symmetric about $x = 4$. The sketch of the graph is shown in the figure.

Example 11: Draw the graph of $y = -x^2 + 4x - 4$ using factors.

Solution:

Given function is: $y = -x^2 + 4x - 4$

To get x-intercepts, put:

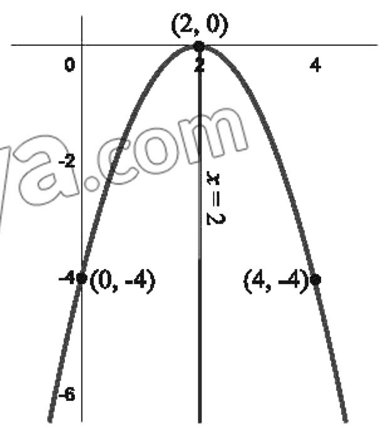
$-x^2 + 4x - 4 = 0 \Rightarrow -(x^2 - 4x + 4) = 0 \Rightarrow x^2 - 4x + 4 = 0$

After factorization, we get repeated roots:

$$x_1 = 2, \ \ x_2 = 2$$

This shows that the graph intersects x-axis at (2, 0), which is the vertex of the parabola.

To find y-intercept, put $x = 0$ in the given function which gives $y = -4$.



Therefore y-intercept is $(0, -4)$.
$a = -1 < 0$, shows thar the parabola opens downwards. As the graph is symmetric about $x = 4$, therefore parabola also passes through $(4, -4)$. The sketch of the graph is shown in the figure.

Check Point

Draw the graph of $y = 2x^2 + 6x + 4$ using factors.

1.7 Predicting Functions from Their Graphs

The method is explained with the help of following examples.

Example 12: Predict function from the graph.

Solution:

The graph shows a line passing through points $(2, 0)$ and $(0, -2)$.

Slope of the line is:

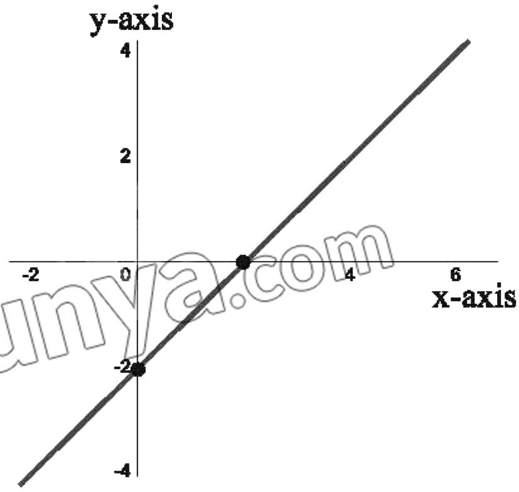
$m = \frac{0+2}{2-0} = 1$

Equation of the line is:

$y - 0 = 1(x - 2)$ (Point slope form of the line)

$y = x - 2$

Which is the required function.



Example 13: Predict function from the graph.

Solution:

The graph shows a parabola passing through points $(2, 0)$ and $(-1, 0)$ and $(0, -4)$. We know that the equation of the parabola is quadratic.

Now, the equation of the parabola passing through $(p, 0)$ and $(q, 0)$ is of the form:

$y = a(x - p)(x - q)$ where $p = 2, q = -1$ and $a > 0$ as the parabola opens upwards.

Substituting the values of p and q in the above equation, we get:

$y = a(x - 2)(x + 1)$ (i)

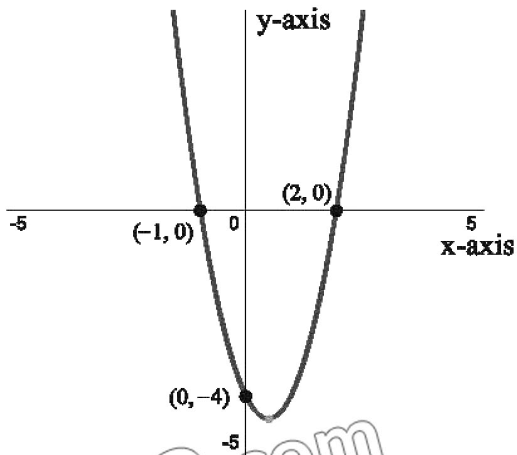
Since the parabola passes through $(0, -4)$, therefore from (i), we have:

$-4 = a(0 - 2)(0 + 1) \Rightarrow -4 = a(-2) \Rightarrow a = 2$

Therefore from (i):

$y = 2(x - 2)(x + 1) \Rightarrow y = 2(x^2 - x - 2)$

Which is the required function.



Key Facts

Equation of parabolic function passing through (p, r) and (q, r) is:
$y - r = a(x - p)(x - q)$ where p, q and r are positive.

1.8 Graph of Modulus Functions

A modulus function (absolute valued function) determines a number's magnitude regardless of its sign. If x is a real number, then the modulus function is denoted by:

$$y = |x| \quad \text{or} \quad f(x) = |x| \text{ where } x \in \mathbb{R}.$$

The modulus function takes the actual value of x if it is more than or equal to 0 and the function takes the minus of the actual value x if it is less than 0.

1.8.1 Domain and Range of Modulus Function

The domain of modulus function is R while its range is the set of non-negative real numbers, denoted as $[0, \infty)$. Any real number can be modulated using the modulus function.

Example: Consider the modulus function $f(x) = |x|$. Then:

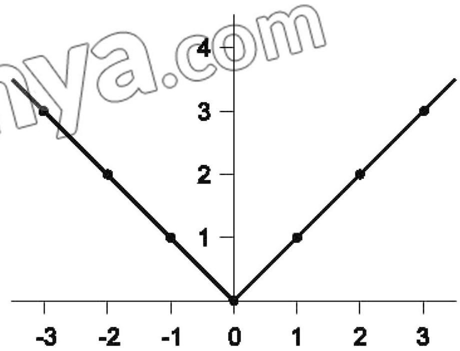
- If $x = -5$, then $y = f(x) = -(-5) = 5$, since x is less than zero.
- If $x = 6$, then $y = f(x) = 6$, since x is greater than zero.
- If $x = 0$, then $y = f(x) = 0$, since x is equal to zero.

1.8.2 Graph of Modulus Function

Consider the modulus function $f(x) = |x|$

Table shows the values of $f(x)$ below:

x	-3	-2	-1	0	1	2	3
$f(x)$	3	2	1	0	1	2	3



It can be inferred that for all possible values of x, the function $f(x)$ remains positive.

1.9 Finding the Intersecting Points Graphically

1.9.1 Intersection Point between a Linear Function and Coordinate Axes

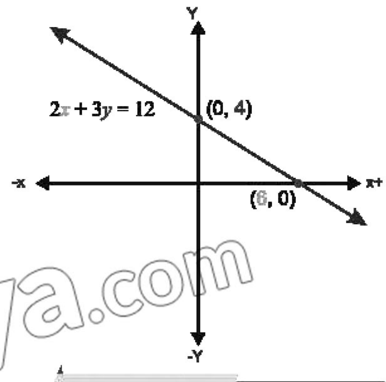
As we know that the graph of a linear equation is a straight line and the points of intersection of the line with axes are called intercepts.

Example 14:

Find x-intercept and y-intercept of the function: $f(x) = \frac{12-2x}{3}$

Solution:

x	6	0	3
$f(x)$	0	4	2



The graph of line $f(x) = \frac{12-2x}{3}$ is shown in the adjoining figure. From the graph it is clear that:

- The line crosses the x-axis at (6, 0).
  So, its x-intercept is 6.
- The line crosses the y-axis at (0, 4).
  So, its y-intercept is 4.

Check Point

x and y-intercepts of a line are -3 and -5 respectively. Find points of intersections of line with axes.

1.9.2 Intersection Point between two Linear Functions

While solving simultaneous linear functions graphically, keep in mind the following points.

1. Draw each linear function on the same set of axes.
2. Find the coordinates where the lines intersect.

Example 15: Find the graphical solution of $f(x) = \frac{6-x}{2}$ and $g(x) = x - 3$.

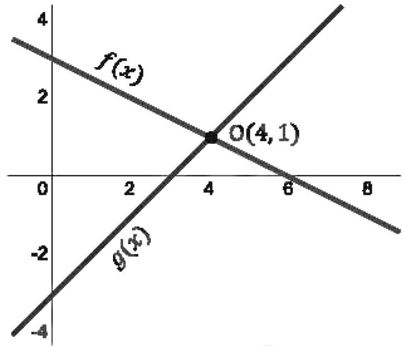
Solution:

Table of values of function $f(x) = \frac{6-x}{2}$ is:

x	6	0	4
$f(x)$	0	3	1

Table of values of function $g(x) = x - 3$ is:

x	3	0	5
$g(x)$	0	−3	2



The graph of functions $f(x) = \frac{6-x}{2}$ and $g(x) = x - 3$ is shown in the adjoining figure.

From the graph it is clear that the both linear functions intersect each other at point O(4, 1).

Therefore, point O(4, 1) is the graphical solution of the given linear functions.

1.9.3 Intersection Point between a Linear Function and a Quadratic Function

As we know that the graph of a quadratic equation is a curve. The point of intersection of a linear function and a quadratic function is a point where both the graphs intersect each other.

Example 16: Solve $f(x) = 3x + 4$ and $g(x) = 5 + 3x - 2x^2$ graphically.

Solution:

Table of values for $f(x) = 3x + 4$ is:

x	0	−1	1
$f(x)$	4	1	7

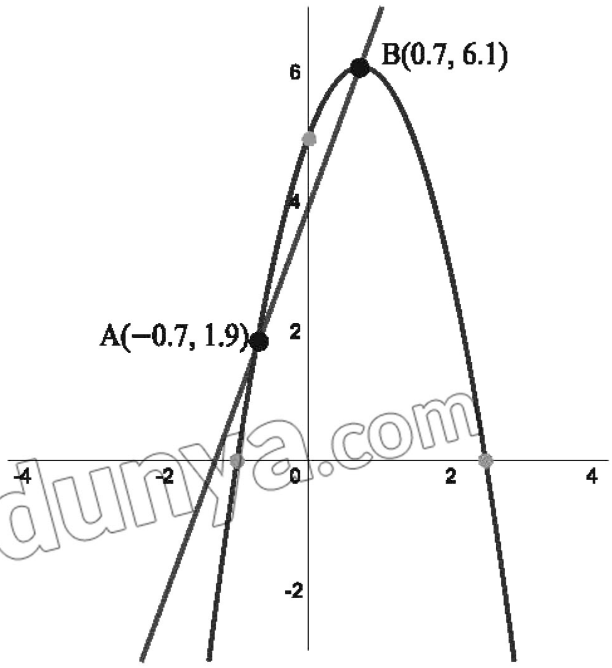
Comparing the graph of $g(x) = 5 + 3x - 2x^2$, with $y = ax^2 + bx + c$, we have:

$a = -2, b = 3$ and $c = 5$

Here, $a = -2 < 0$, so the curve will open downward.

Table of values for $g(x) = 5 + 3x - 2x^2$

x	−2	−1	0	1	2	3
$g(x)$	−9	0	5	6	3	−4



Both the graphs intersect each other at:

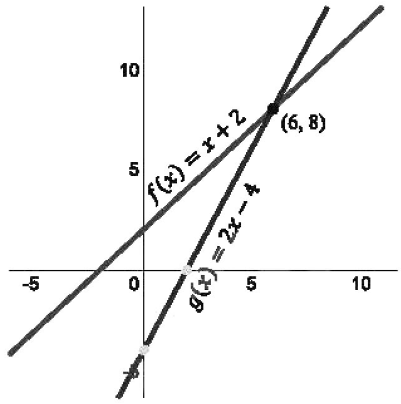
A(−0.7, 1.9) and B(0.7, 6.1).

Hence, solution set is: {(−0.7, 1.9), (0.7, 6.1)}

1.9.4 Solving Problems Graphically from Daily Life

See the following example to understand the method.

Example 17: Two airplanes are moving along the paths representing $f(x) = x + 2$ and $g(x) = 2x - 4$ respectively. Draw the graph of paths of both planes and find the point, where both planes pass.



Solution:

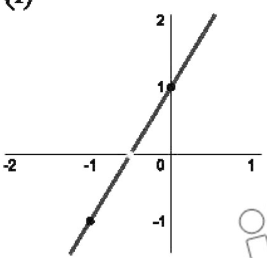
In the figure, red line represents the path of first plane while blue line represents the path of second plane.

The graph shows that both planes pass through the point (6, 8).

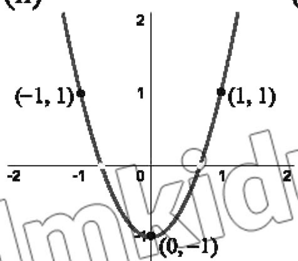
Exercise 1.2

1. Plot the graph of the functions:
   (i) $f(x) = 3x - 2$ (ii) $f(x) = 3x$ (iii) $f(x) = 1 - 2x$
   (iv) $g(x) = x^2 + 4$ (v) $g(x) = x^2 - x - 6$ (vi) $g(x) = \sqrt{2x+1}$
2. Plot the graph of following functions.
   (i) $f(x) = -x^2 + 1$ (ii) $f(x) = 2x^3$ (iii) $f(x) = 1 + x^{-2}$
   (iv) $f(x) = 3x^{\frac{1}{2}}$ (v) $f(x) = 2 - x^{-\frac{1}{2}}$ (vi) $f(x) = x^{\frac{5}{2}}$
3. Find possible x-intercept, y-intercept and vertex of the following functions and then plot.
   (i) $f(x) = x^2 + 2x + 1$ (ii) $f(x) = -2x^2 + 2x - 1$
   (iii) $f(x) = x^2 + 2x$ (iv) $f(x) = 9 - x^2$
4. Draw the graph of following function using factors.
   (i) $f(x) = x^2 - 2x + 1$ (ii) $f(x) = x^2 - 7x + 12$
   (iii) $f(x) = x^2 - 2x$ (iv) $f(x) = -2x^2 + x + 3$
   (v) $f(x) = 4x^2 - 4x$ (iv) $f(x) = 6 - x^2 - x$
5. Predict algebraic functions from the following graphs.

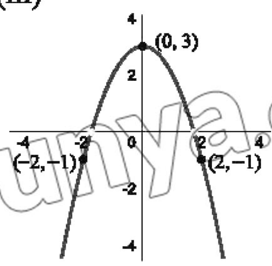
(i)



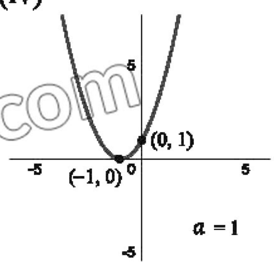
(ii)



(iii)



(iv)



6. Plot the graph of following and find point of intersection of function with axes.
   (i) $y = x + 3$ (ii) $y = 6 - 3x$ (iii) $y = x^2 - 5x$

7. Find graphical solution of:
   (i) $f(x) = 4 - 3x,\ g(x) = -x + 1$ (ii) $f(x) = 2(2 + x),\ g(x) = x^2 + 1$
   (iii) $f(x) = 5 + 3x,\ g(x) = -x^2 + 5$ (iv) $f(x) = 1,\ g(x) = -2x^2 + 2x + 5$
   (v) $f(x) = 2 + 3x + x^2,\ g(x) = 5 + 3x - 2x^2$
8. Draw the graph of following modulus functions.
   (i) $f(x) = -1.5|x|$ (ii) $f(x) = 1 + 2|x|$ (iii) $f(x) = 3|x| + x$
9. The equations for supply and demand are given by two linear equations:
   Supply equation: $S(x) = 2x + 10$; where x is quantity and $S(x)$ is the price.
   Demand equation: $D(x) = -3x + 40$; where x is quantity and $D(x)$ is the price.
   Find the equilibrium point where the price of supply equals the price of demand by drawing the graphs of both equations.
10. Suppose a ball is thrown into the air and its height h(t) after t seconds is given by the parabolic trajectory: $h(t) = -6t^2 + 10t + 5$. If this ball hits a wall 10 m high representing the equation: $h(t) = 9t$. By drawing the graphs, find out when and where the ball reaches the wall.
11. Two asteroids are following the parabolic paths represented by $f(x) = x^2 - 7x + 12$ and $g(x) = x(x - 3)$. By drawing the graphs of both trajectories, find out the place from where, both asteroids will pass.

1.10 Algebraic and Transcendental Functions

1.10.1 Algebraic Function

An algebraic function is a function that involves only algebraic operations. These operations include addition, subtraction, multiplication, division, and exponentiation.

Types of Algebraic Functions

Main types of algebraic functions are:

(i) Polynomial Functions

A function of the from $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where $a_0, a_1, \ldots, a_n$ are constants and n is integer, is called a polynomial function. Some examples are:

- $f(x) = 3x + 7$ (linear function)
- $f(x) = x^2 - 2x + 5$ (quadratic function)
- $f(x) = x^3 - 7x + 7$ (cubic function)
- $f(x) = x^4 - 5x^2 + 2x - 8$ (biquadratic function)
- $f(x) = x^5 - 7x + 3$ (quintic function)

(ii) Rational Functions

A function that is composed of two functions and expressed in the form of a fraction is a rational function. If $f(x)$ is a rational function, then $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function. Some examples are:

$$f(x) = \frac{x - 4}{2x + 3}, \quad g(x) = \frac{7}{x^2 + 5x + 1}$$

(iii) Power Functions

The power functions are of the form $f(x) = k\,x^a$ where 'k' and 'a' are any real numbers. Since 'a' is a real number, the exponent can be either an integer or a rational number.

Some examples are:

$f(x) = x^2$, $f(x) = x^{-1}$ (reciprocal function) and $f(x) = \sqrt{x-2} = (x-2)^{\frac{1}{2}}$

Properties

- Algebraic functions are closed under addition, subtraction, multiplication, division and composition.
- Algebraic functions are easy to solve, differentiate and integrate.

Application

- Physics and Engineering: Simple mechanical system, the motion of objects under constant acceleration.
- Geometry: Many curves such as circles and ellipses.

1.10.2 Transcendental Functions

The functions which are not algebraic are called transcendental. These functions can only be expressed in terms of infinite series. Some examples are:

- Exponential functions: $f(x) = e^x$, $g(x) = a^{3x}$
- Logarithmic functions: $f(x) = log_a x$, $g(x) = ln x$; where base a is a positive constant.
- Trigonometric functions: $f(x) = \sin x$, $g(x) = \cos x$, $h(x) = \tan x$
- Inverse trigonometric functions: $f(x) = \sin^{-1}x$, $g(x) = \cos^{-1}x$
- Hyperbolic functions: $f(x) = \sinh x$, $g(x) = \cosh x$, $h(x) = \tanh x$
- Inverse hyperbolic functions: $f(x) = \sinh^{-1}x$, $g(x) = \cosh^{-1}x$
- Special functions: Bessel functions, Gamma functions, error functions etc.

Properties

- These functions are not expressible in terms of a finite combination of algebraic operation of addition, subtraction, division, multiplication, raising to a power and extracting a root.
- These functions often exhibit more complex behavior like periodicity (in the case of trigonometric functions) and rapid growth (in the case of exponential function).

Application

- Science and Engineering: Exponential and logarithmic functions are critical in modelling growth, decay and oscillation in natural systems.
- Signal processing: Trigonometric functions are fundamental in analysing waves, sounds and signals.
- Mathematical analysis: Many problems in calculus, differential equations and complex analysis involve transcendental functions.

1.10.3 Logarithmic Functions

Logarithmic functions form a fundamental class of transcendental functions. These functions are inverse of exponential functions. They play a crucial role in mathematics, science, engineering and many applied fields.

Definition: If you have an exponential function of the form $y = a^x$ where $a > 0$ and $a \neq 1$, then the logarithmic function is defined as:

$$x = \log_a(y)$$

Replacing y with x, we have:

$$y = \log_a(x)$$

Here, $\log_a(x)$ is read as logarithmic of x to the base a.

Base of the Logarithms

The base of logarithm determines its specific type. Some types are:

- Natural logarithm: It is written as $\log_e(x) = \ln x$ where $e = 2.71828\ldots$ is called Euler's number.
- Common logarithm: It is written as $\log_{10}(x)$ where $a = 10$.
- Binary logarithm: It is written as $\log_2(x)$ where $a = 2$.

Properties

- The logarithm is the inverse of exponential. If $y = a^x$, then $x = \log_a(y)$. This means $\log_a(a^x) = x$ and $a^{\log_a(y)} = y$.
- The domain of $\log_a(x)$ is $x > 0$ because we cannot take the logarithm of zero or a negative real number.

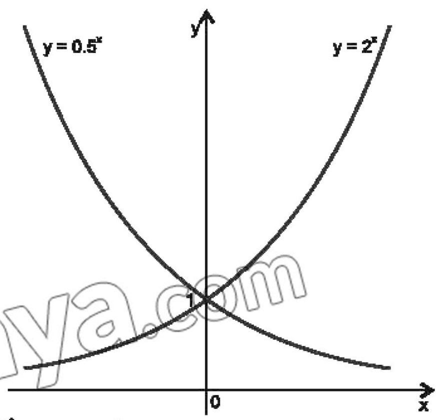
Laws of Logarithms

- Product Rule: $\log_a(xy) = \log_a(x) + \log_a(y)$
- Quotient Rule: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- Power Rule: $\log_a(x^n) = n\log_a(x)$
- Change of Base Rule: For any positive bases $a \neq 1$ and $b \neq 1$:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

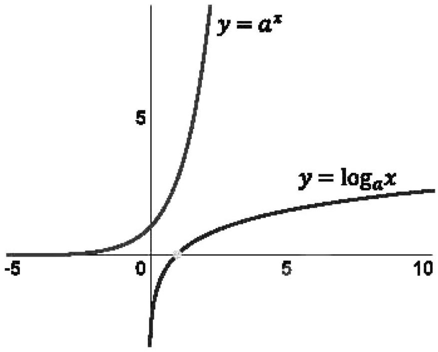
Note: As $a^x = 1$, therefore $\log_a(1) = 0$.

Graph of Exponential Function



- If the base, a is greater than 1, then the function increases exponentially at a growth rate of a. This is known as exponential growth.
- If the base, a is less than 1 (but greater than 0) the function decreases exponentially at a rate of a. This is known as exponential decay.
- If the base, a is equal to 1, then the function trivially becomes $y = 1$. This means exponential function always passes through (0, 1).
- The points (0, 1) and (1, a) are always on the graph of the function $y = a^x$.
- Exponential function takes only positive values and its graph never touches x-axis.
- The domain of the exponential function is the set of all real numbers, whereas the range of this function is the set of positive real numbers.

Graph of Logarithmic Function



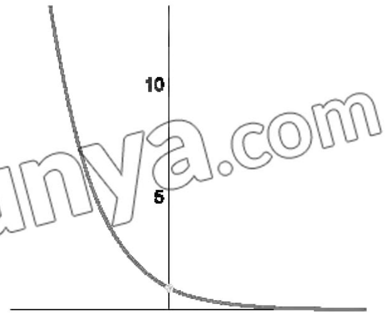
- When graphed, the logarithmic function is similar in shape to the square root function.
- The logarithmic function always passes through the point $(1, 0)$ because $\log_a(1) = 0$.
- The curve approaches to y-axis but never touches it.
- The domain of the logarithmic function is the set of all positive real numbers, whereas the range of this function is the set of all real numbers.
- For $a > 1$, the value of function increases as x increases.
- For $0 < a < 1$, the value of function decreases as x increases.

Example 18: Draw the graph of $f(x) = e^{-0.5x}$.

Solution:

Table of values for $f(x) = e^{-0.5x}$

x	-5	-2	-1	0	1	2	3
$g(x)$	12	2.7	1.6	1	0.6	0.4	0.2



Graph is shown in the adjoining figure.

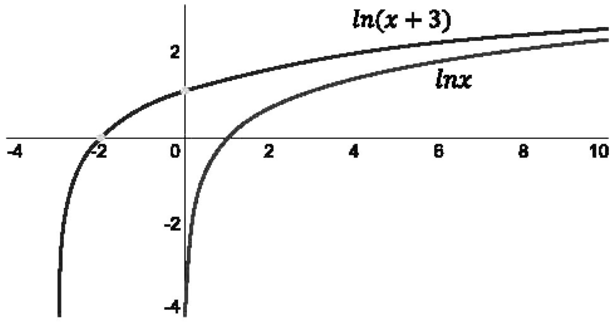
Example 19: Draw the graph of:

(i) $f(x) = lnx$ (ii) $g(x) = ln(x + 3)$

Solution:

Table of values for $f(x)$ and $g(x)$ is:

x	0	0.1	0.5	1	4	10
$f(x)$	-	-2.3	-0.7	0	1.4	2.3
$g(x)$	1.09	1.13	1.3	1.4	1.9	2.6



Graph of both functions is shown in the adjoining figure.

Applications

- Growth and Decay Model: Logarithm functions are used to model phenomenon that grow rapidly at first and then slow down such as population growth or the spread of deceases.
- pH Measurement in Chemistry: The pH of a solution is the logarithmic measure of the hydrogen in cm concentration.

$$\text{pH} = -\log_{10}[\text{H}^+]$$

- Sound Intensity (Decibels): The decibel scale which measures intensity, is a logarithmic scale:

$$\text{Decibels} = 10 \times \log_{10}\left(\frac{\text{I}}{\text{I}_o}\right)$$

here I is the intensity of the sound and I_o is the reference intensity.

- Information Theory: Logarithms are used in information theory to measure information content and entropy.

- Financial Models: Logarithmic functions are used in finance particularly in modelling the time value of money and compound interest.
- Computer Science: Logarithmic functions appear in algorithms and data structures.

Conclusions

Logarithmic functions are powerful tools for dealing with exponential growth and decay as well as for measuring and comparing quantities on vastly different scales. Their unique properties and applications make them essential in both theoretical and applied fields. Most of the applications, we find, are in the fields of engineering and computer technology.

Example 20: Suppose that Rs. 30,000 is invested at 8% interest compounded annually. In t years, it will grow to the amount $A(t)$ given by the function: $A(t) = 30{,}000\ (1.08)^t$

(i) How long will it take until then is Rs. 150,000 in the account?

(ii) Let T be the amount of time it takes for the Rs.30,000 to double itself. Find T.

Solution:

(i) We set $A(t) = 150{,}000$ and solve for t.

$$150{,}000 = 30{,}000\ (1.08)^t \quad \Rightarrow \quad (1.08)^t = \frac{150{,}000}{30{,}000} = 5$$

Taking natural log on both sides, we get:

$$ln(1.08)^t = ln5 \quad \Rightarrow \quad t\, ln(1.08) = \ln 5$$

$$\Rightarrow \quad t = \frac{\ln 5}{\ln(1.08)} = \frac{1.6094}{0.07696} \approx 20.9$$

Therefore, it will take almost 20.9 years for Rs. 30,000 to grow to Rs. 150,000.

(ii) To find the doubling time T, we set A(t) = Rs. 60,000, t = T and solve for T.

$$60{,}000 = 30{,}000\ (1.08)^{\mathrm{T}} \quad \Rightarrow \quad (1.08)^{\mathrm{T}} = \frac{60{,}000}{30{,}000} = 2$$

Taking natural log on both sides, we get:

$$ln(1.08)^{\mathrm{T}} = ln2 \quad \Rightarrow \mathrm{T}\ ln(1.08) = ln2$$

$$\Rightarrow \quad \mathrm{T} = \frac{\ln 2}{\ln(1.08)} = \frac{0.6931}{0.07696} \approx 9$$

Therefore, doubling time is about 9 years.

Example 21: In 2020, the population of the country was 249 million and the exponential growth rate was 0.9% per year. If $P(t) = P_0 e^{rt}$ is exponential growth function, then:

(i) Find the exponential growth function for the given data.

(ii) What would you expect the population to be in the year 2028?

Solution:

(i) Here $P_0 = 249,\ r = 9\% = 0.009$

The population growth function, gives:

$P(t) = 249 \times e^{0.009t}$ (a)

(ii) In 2028, we have $t = 8$,

To find the population in 2028, we substitute 8 for t in (a).

$$P(8) = 249 \times e^{0.009 \times 8} = 249 \times e^{0.072}$$

$$\approx 249 \times 1.0747 = 267.6$$

Therefore, population of the city in 2028, will be about 267.6 million.

Key Facts

The function $P(t) = P_0 e^{rt}$ models the growth in the quantity while the function $P(t) = P_0 e^{-rt}$ models the decay or decline in the quantity where $r > 0$.

Example 22: The radioactive element Carbon-14 has a half-life of 5750 years. The percentage of Carbon-14 present in the bones of dead animals can be used to determine the time of death of that animal. How old is the animal bone that has lost 40% of its Carbon-14?

Solution:

First of all, we find constant r using the concept of half-life. When $t = 5750$ (half-life), $P(t)$ will be half of P_0. That is $P(t) = 0.5P_0$

Therefore, the decay function $P(t) = P_0e^{-rt}$ implies:

$$0.5P_0 = P_0e^{-r\times 5750} \quad \Rightarrow \quad 0.5 = e^{-r\times 5750}$$

Taking natural log on both sides, we get:

$$\ln(0.5) = \ln e^{-r\times 5750} \quad \Rightarrow \quad \ln(0.5) = -r \times 5750$$

$$\Rightarrow \quad r = -\frac{\ln(0.5)}{5750} \approx 0.00012$$

We have the formula: $P(t) = P_0e^{-0.00012t}$

If an animal bone has lost 40% of its Carbon-14 from an initial amount P_0, then 60% of P_0 is the amount present. To find the age t of the bone, we solve the decay function for t.

$$0.6P_0 = P_0e^{-0.00012t} \qquad [P(t) = 60\% \text{ of } P_0 = P(t) = 0.6P_0]$$

$$0.6 = e^{-0.00012t} \quad \Rightarrow \quad \ln(0.6) = \ln e^{-0.00012t} \quad \Rightarrow \quad -0.5108 = -0.00012t$$

$$\Rightarrow \quad t = \frac{0.5108}{0.00012} = 4257$$

Therefore, the animal bone is about 4257 years old.

Exercise 1.3

1. Draw the graphs of functions.

   (i) $f(x) = e^{2x}$ (ii) $g(x) = e^{0.5x}$ (iii) $h(x) = 2 - e^x$

   (iv) $h(x) = 1 + e^{-2x}$ (v) $f(x) = \ln(2x)$ (vi) $g(x) = \log(x + 1)$

   (vii) $h(x) = 3 + \log(x)$ (viii) $f(x) = e^{0.6x}$ and $g(x) = \ln(0.6x)$

2. The number of compact discs N (in million) purchased each year increasing exponentially is given by:

   $$N(t) = 7.5(6)^{0.5t}$$

   Where $t = 0$ corresponds to 2024, $t = 1$ corresponds to 2025 and so on, t being the number of years after 2024.

   a. After what amount of time will one billion compact discs be sold in a year?

   b. What is the doubling time on the sale of compact discs?

3. Suppose that Rs. 50,000 is invested at 6% interest compounded annually. After t years, it grows to the amount A given by the function:

   $$A(t) = 50{,}000(1.06)^t$$

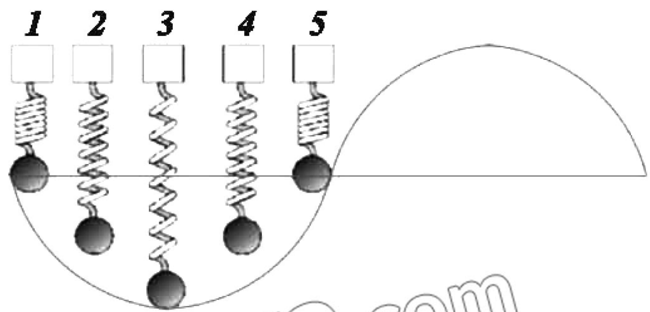
   a. After what amount of time will Rs. 50,000 grows to Rs. 450,000?

   b. Find the doubling time.

4. The exponential growth rate of the population of the city is 1% per year. After how many years, the population will be doubled?
5. The population of the world was 5.2 billion in 1990. The exponential growth rate was 1.6% per year at that time.
   a. Find the exponential growth function.
   b. Find the population of the world in 2000.
   c. In which year the world population was 8 billion?
6. Students in a mathematics class took a final exam in monthly intervals thereafter. The average score $S(t)$, after t months was given by:
$$S(t) = 68 - 20\log(t+1);\ \ t \geq 0$$
   a. What was the average score when they initially took the test $(t = 0)$?
   b. What was the average score (i) after 4 months (ii) after 24 months?
   c. Graph the function.
   d. After what time was the average score 50?
7. If $P(t) = P_0e^{kt}$ denotes the growth function of oil and the exponential growth rate of the demand for oil is 10% per year, when will the demand be doubled?
8. Approximately two third of all Aluminum cans distributed are recycled each year. A beverage company distributes 250,000 cans. The number still in use after t years is given by the function:
$$N(t) = 250{,}000\left(\frac{2}{3}\right)^t$$
   a. After how many years will 60,000 cans be in use?
   b. After what amount of time will only 1,000 cans be in use?

1.11 Domain and Range of Transcendental Functions through Graphs

If a weight is attached to a spring and the weight is pushed up or pulled down and released, it tends to rise and fall alternately. The weight is said to be oscillating in harmonic motion. If the position of the weight y is graphed over time the result is the graph of a sine or cosine curve.

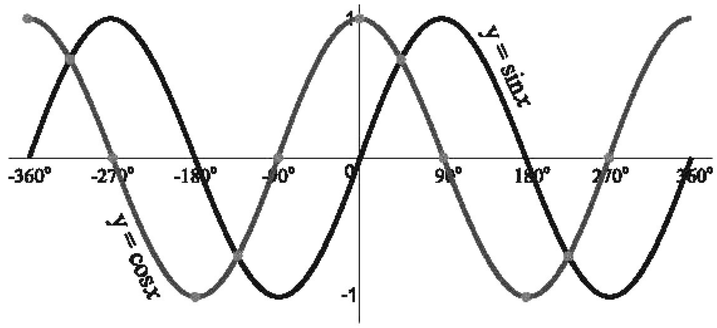


1.11.1 Graph of y = sinθ and y = cosθ

To graph the sine or cosine function, we use the horizontal axis for the values of θ expressed in either degrees or radians and vertical axis for the values of sinθ or cosθ. Ordered pairs for these points are of the form (θ, sinθ) or (θ, cosθ).

θ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
sinθ	0	0.3	0.5	0.7	0.87	0.97	1	0.97	0.87	0.7	0.5	0.3	0
cosθ	1	0.97	0.87	0.7	0.5	0.3	0	-0.3	-0.5	-0.7	-0.87	-0.97	-1

θ	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°
$\sin\theta$	- 0.3	-0.5	-0.7	-0.87	-0.97	-1	-0.97	-0.87	-0.7	-0.5	-0.3	0
$\cos\theta$	-0.97	-0.87	-0.7	-0.5	-0.3	0	0.3	0.5	0.7	0.87	0.97	1



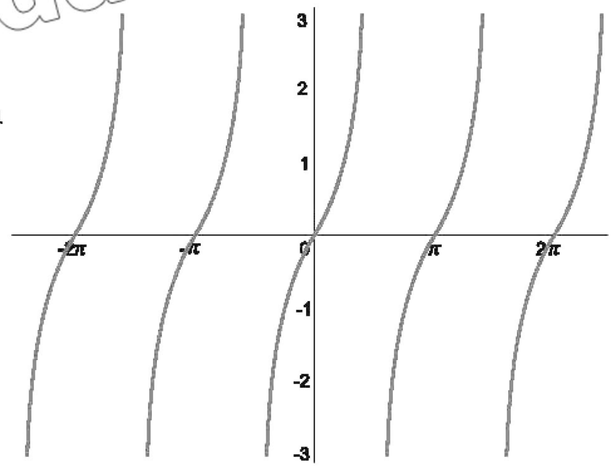
From the behavior of graphs of sine and cosine functions, we can easily predict the domain and range of both functions which are:

Function	Domain	Range
y = sinθ	$R = \theta \in (-\infty, \infty) = -\infty < \theta < \infty$	$y \in [-1, 1] = -1 \leq y \leq 1$
y = cosθ	$R = \theta \in (-\infty, \infty) = -\infty < \theta < \infty$	$y \in [-1, 1] = -1 \leq y \leq 1$

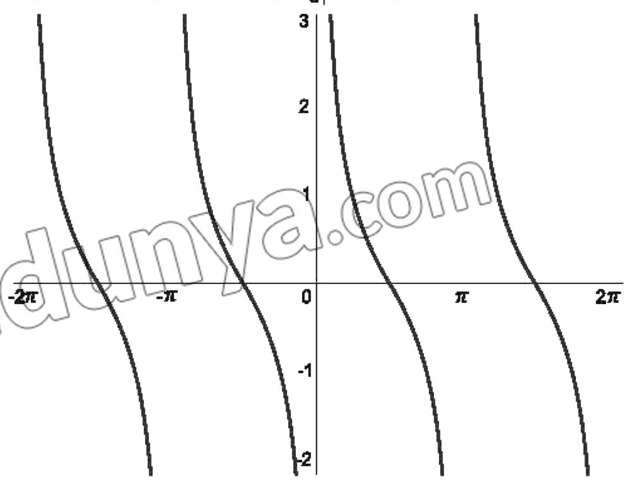
1.11.2 Graph of y = tanθ and y = cotθ

Similarly, by drawing the graph of y = tanθ and y = cotθ , we can easily predict the domain and range of both functions as follows.

Function	$y = \tan\theta$
Domain	$\theta \neq (2n+1)\frac{\pi}{2}; n \in Z$
Range	R

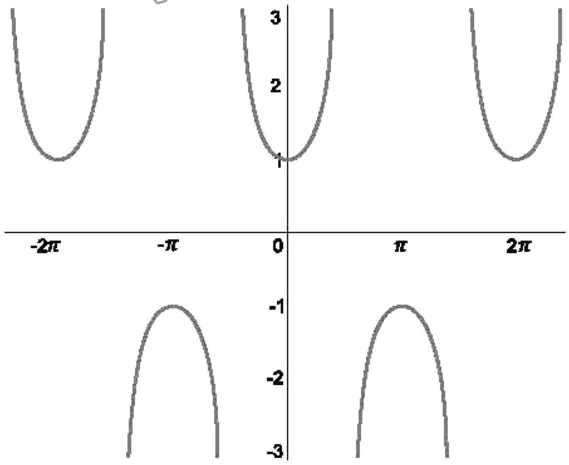


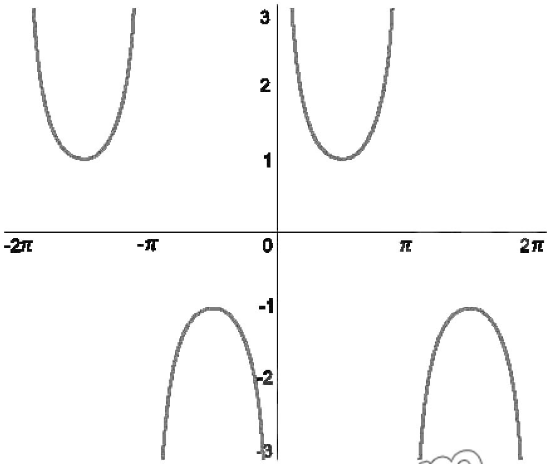
Function	$y = \cot\theta$
Domain	$\theta \neq n\pi; n \in Z$
Range	R



1.11.3 Graph of y = secθ and y = cosecθ

Domain and range of y = secθ and y = cosecθ is obvious form the graphs of both functions shown below.





Function	$y = \sec\theta$
Domain	$\theta \neq (2n+1)\frac{\pi}{2}; n \in Z$
Range	$y \leq -1, y \geq 1$ or $y \in (-\infty, -1] \cup [1, \infty)$

Function	$y = \csc\theta$
Domain	$\theta \neq n\pi; n \in Z$
Range	$y \leq -1, y \geq 1$ or $y \in (-\infty, -1] \cup [1, \infty)$

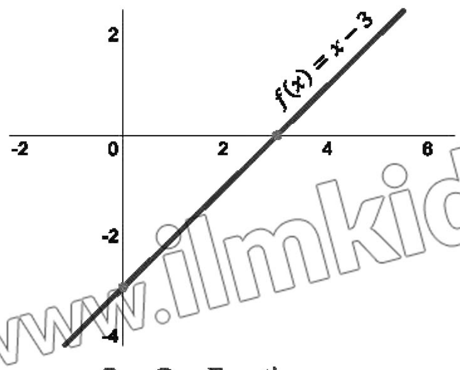
1.12 Relation Between a 1-1 Function and its Inverse through Graphs

1.12.1 One-One Function and its Graph

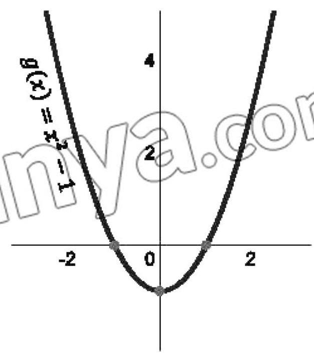
One to one function is a special function that maps every element of the range to exactly one element of its domain i.e., the outputs never repeat.

Examples: (i) The function $f(x) = x - 3$ is a one-to-one function since it produces a different answer for every input.

(ii) The function $g(x) = x^2 - 1$ is not a one-to-one function since it produces one output 0 for the two inputs 1 and -1.



One-One Function



Not a One-One Function

1.12.2 Horizontal Line Test

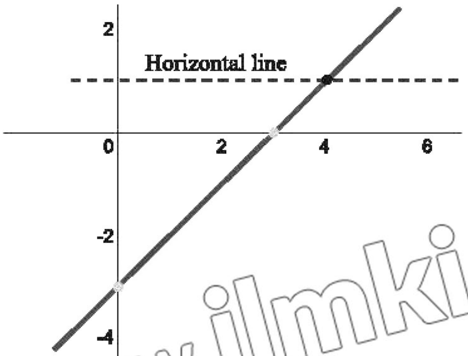
The horizontal line test is used to determine whether a function is one-one when its graph is given. To test whether the function is one-one from its graph just take a horizontal line (consider a horizontal stick) and make it pass through the graph.

- If the horizontal line does not pass through more than one point of the graph, then the function is one-one.
- If the horizontal line passes through more than one point of the graph, then the function is not one-one.

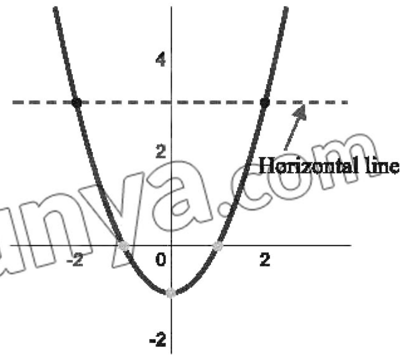
Examples: If we draw horizontal lines on the above graphs, we observe that:

(i) The graph of $f(x) = x - 3$ passes horizontal line test, so it is ono-one function.

(ii) The graph of $g(x) = x^2 - 1$ fails horizontal line test, so it is not ono-one function.



$f(x)$ is one-one function.



$g(x)$ is not a one-one function.

Check Point

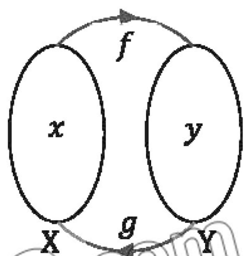
By using horizontal line test, check whether the function $y = x^3$ is 1-1 function or not.

1.12.3 Inverse of One-One Function

Suppose $f: X \to Y$ is a one-one function. Since every element y of Y corresponds with precisely one element x of X, the function f must determines a "reverse function" $g: Y \to X$ whose domain is Y and range is X. Then f and g imply that:

$f(x) = y$ and $g(y) = x$

or $f(g(y)) = y$ and $g(f(x)) = x$



The function g is given the formal name as "inverse of f".

From the above discussion it is clear that:

Dom f = Rang g and Rang f = Dom g

Definition:

Let f be a one-to-one function with domain X and range Y. The inverse of f is a function g with domain Y and range X for which:

$f(g(y)) = y$ for every y in Y and $g(f(x)) = x$ for every x in X.

Symbolically the inverse of a function f is denoted by f^{-1}. Thus, $g(x) = f^{-1}(x)$. It is to be noted that $f^{-1}(x)$ is not the same as $[f(x)]^{-1}$. In terms of this new notation, we have:

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

1.12.4 Properties of the Inverse of One to One Function

Here are the properties of the inverse of one to one function:

- The function f has an inverse function if and only if f is a one to one function.
- If the functions f and g are inverses of each other then, both these functions are one to one.
- f and g are inverses of each other if and only if $f(g(x)) = x, x$ in the domain of g and $g(f(x)) = x,\ x$ in the domain of f.
- If f and g are inverses of each other then the domain of f is equal to the range of g and the range of g is equal to the domain of f.
- If f and g are inverses of each other then their graphs will make reflections of each other on the line $y = x$.
- If the point (a, b) is on the graph of f then point (b, a) is on the graph of f^{-1}.

Example 23: Find the inverse of $f(x) = \frac{1}{2x-3}; x \neq \frac{3}{2}$, then represent f and f^{-1} graphically.

Solution: Given that $f(x) = \frac{1}{2x-3}; x \neq \frac{3}{2}$

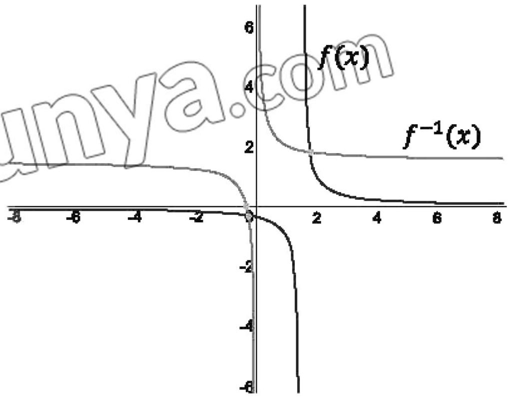
Since f is a one to one function, therefore:

$f(f^{-1}(x)) = \frac{1}{2f^{-1}(x)-3}$ [Replacing x with $f^{-1}(x)$]

Solving for $f^{-1}(x)$, we get:

$\Rightarrow\ x = \frac{1}{2f^{-1}(x)-3} \quad \Rightarrow\ 2f^{-1}(x) - 3 = \frac{1}{x}$

$\Rightarrow\ 2f^{-1}(x) = \frac{1}{x} + 3 \quad \Rightarrow\ f^{-1}(x) = \frac{1+3x}{2x}$



Graph of function $f(x)$ and $f^{-1}(x)$ are shown in the adjoining figure. From the graph it is clear that if any point (a, b) is on the graph of $f(x)$ then point (b, a) is on the graph of $f^{-1}(x)$.

Challenge: Can you find inverse of $f(x)$ given in example 23, by any other method?

Example 24: Given that $f(x) = 3 - 4x$ is one to one. Find its inverse and represent f and f^{-1} graphically.

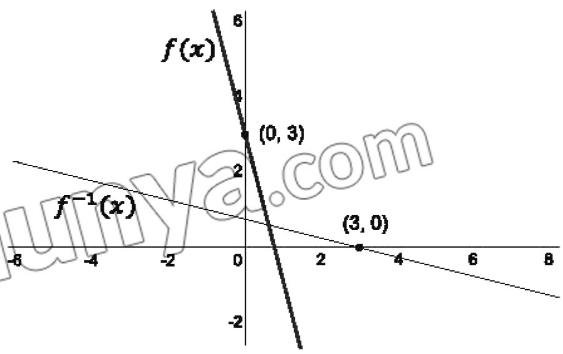
Solution: Given that $f(x) = 3 - 4x$ or $y = 3 - 4x$

Solving for x, we get:

$\Rightarrow\ 4x = 3 - y \quad \Rightarrow\ x = \frac{3-y}{4}$

$\Rightarrow\ f^{-1}(y) = \frac{3-y}{4}$

$\Rightarrow\ f^{-1}(x) = \frac{3-x}{4}$ [Replacing y with x.]



Graph of function $f(x)$ and $f^{-1}(x)$ are shown in the adjoining figure. From the graph it is clear that the point $(3, 0)$ is on the graph of $f(x)$ and the point $(0, 3)$ is on the graph of $f^{-1}(x)$. Therefore, both the graphs are reflections of each other.

Exercise 1.4

1. Find the domain and range of the functions graphically.
   (i) $f(x) = \sin\left(\frac{x}{2}\right)$ (ii) $g(x) = 3\cos\left(\frac{x}{3}\right)$ (iii) $h(x) = 2\tan x$
   (iv) $y = \cot\left(\frac{x}{4}\right)$ (v) $y = 2\sec(2x)$ (vi) $y = \sin(2x)$
2. Determine whether the given function is one to one by examining its graph. If the function is one to one, find its inverse. Also draw the graphs of inverse function.
   (i) $f(x) = \frac{1}{3}x + 3$ (ii) $g(x) = x(x-5)$ (iii) $h(x) = (x+1)^2$
   (iv) $f(x) = x^3 - 8$ (v) $g(x) = 4 \div x$ (vi) $h(x) = \frac{1}{3x+5}$
   (vii) $f(x) = x^4 + 2$ (viii) $g(x) = 5$ (ix) $h(x) = |x|$

1.13 Transformation of a Graph through Vertical Shift, Horizontal Shift and Scaling

1.13.1 Vertical and Horizontal Shift

A shift is a rigid translation as it does not change the shape or size of the graph of the function. A shift only changes the location of the graph.

Vertical Shift: A vertical shift adds/subtracts a positive constant to/from every y-coordinate while leaving the x-coordinate unchanged.

Horizontal Shift: A horizontal shift adds/subtracts a positive constant to/from every x-coordinate while leaving the y-coordinate unchanged.

Key Facts

Vertical and horizontal shifts can be combined into one expression.
Shifts are added/subtracted to the x or $f(x)$ components. If the positive constant is grouped with the x, then it is a horizontal shift, otherwise it is a vertical shift.

In this section, we will discuss the geometric effects on the graph of $y = f(x)$ by adding or subtracting a positive constant c to f or to its independent variable x.

The summary of vertical and horizontal shift is elaborated in the table 1.1 below.

Original function $y = f(x)$	Add a positive constant c to $f(x)$.	Subtract a positive constant c from $f(x)$.	Add a positive constant c to x.	Subtract a positive constant c from x.
$y = f(x)$	$y = f(x) + c$	$y = f(x) - c$	$y = f(x+c)$	$y = f(x-c)$
Geometric effects	Shifts the graph c units up.	Shifts the graph c units down.	Shifts the graph c units left.	Shifts the graph c units right.

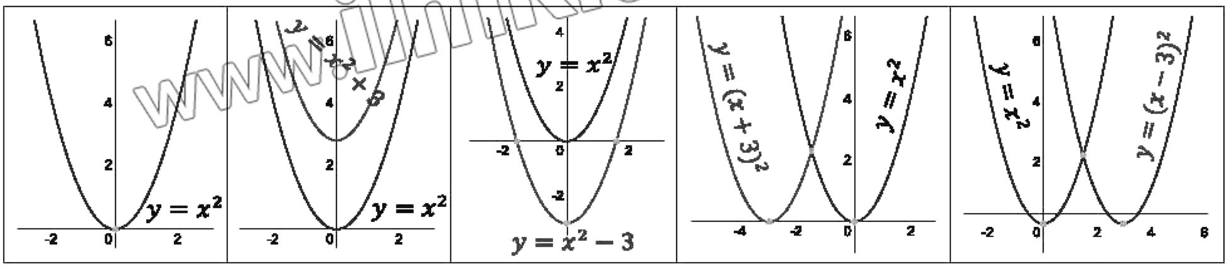
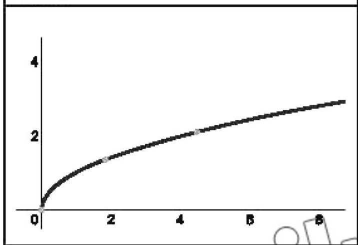
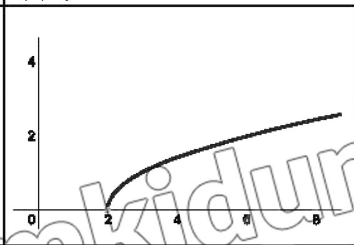
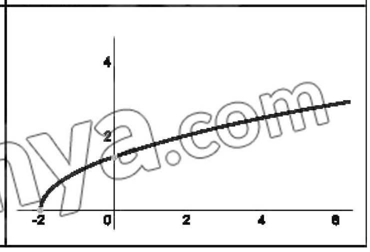


Table 1.1

Example 25: Sketch the graph of (a) $y = \sqrt{x}$ (b) $y = \sqrt{x-2}$ (c) $y = \sqrt{x+2}$
Which kind of shift did you observe after sketching the graphs.

Solution:

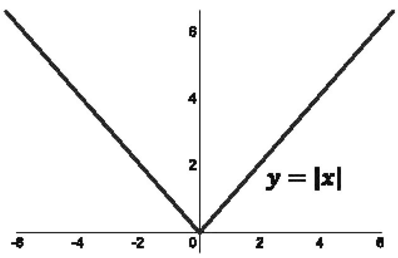
(a) $y = \sqrt{x}$	(b) $y = \sqrt{x-2}$	(c) $y = \sqrt{x+2}$
		

Above graphs show a horizontal shift. The graph of the function $y = \sqrt{x-2}$ can be obtained by transforming the graph of given function 2 units right to the origin while the graph of $y = \sqrt{x+2}$ can be obtained by transforming the graph of given function 2 units left to the origin.

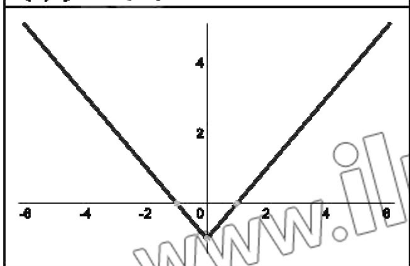
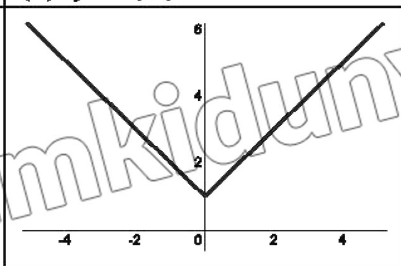
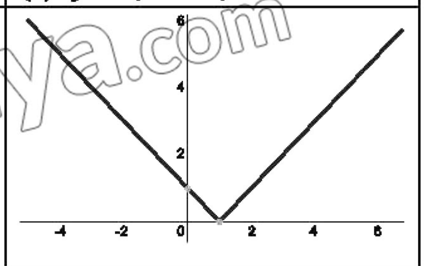
Example 26: Draw the graph of $y = |x|$ and then sketch the graphs of:
(a) $y = |x| - 1$ (b) $y = |x| + 1$ (c) $y = |x-1|$ (d) $y = |x+1|$
(e) $y = |x-1| - 1$ (f) $y = |x+1| - 1$
Which kind of shift did you observe after sketching the graphs.

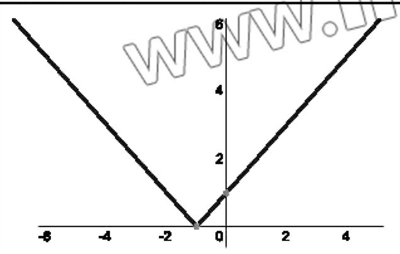
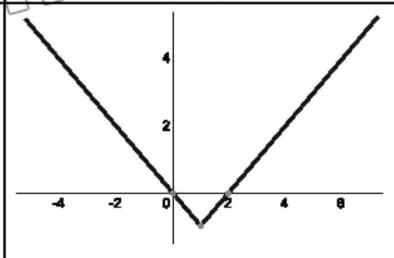
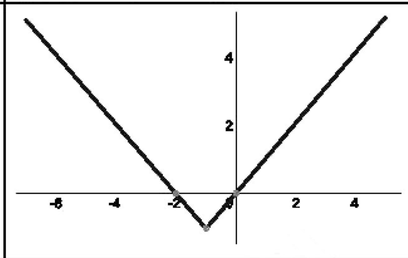
Solution: Table of the values of the function $y = |x|$ is given as:

x	0	± 1	± 2	± 3	± 4	± 5	± 6
$f(x)$	0	1	2	3	4	5	6



The graph is shown in the adjoining figure.
Sketch of other graphs is shown in the table below.

(a) $y = \|x\| - 1$	(b) $y = \|x\| + 1$	(c) $y = \|x - 1\|$
		
Vertical shift 1 unit down	Vertical shift 1 unit up	Horizontal shift 1 unit right

(d) $y = \lvert x+1 \rvert$	(e) $y = \lvert x-1 \rvert - 1$	(f) $y = \lvert x+1 \rvert - 1$
		
Horizontal shift 1 unit left	Horizontal shift 1 unit right Vertical shift 1 unit down	Horizontal shift 1 unit left Vertical shift 1 unit down

1.13.2 Scaling (Stretching/Compressing)

Scaling is a non-rigid translation in which the shape and size of the graph of the function is altered. A scale will multiply/divide coordinates and this will change the appearance as well as the location.

Vertical Scaling: A vertical scaling multiplies/divides every y-coordinate by a constant while leaving the x-coordinate unchanged.

Horizontal Scaling: A horizontal scaling multiplies/divides every x-coordinate by a constant while leaving the y-coordinate unchanged.

Note: The vertical and horizontal scaling can be combined into one expression.

In this section, we will discuss the geometric effects on the graph of $y = f(x)$ by multiplying or dividing with a positive constant c to f or to its independent variable x.

The summary of vertical and horizontal scaling is elaborated in the tables 1.2 and 1.3 below.

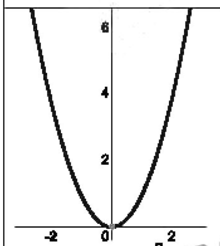
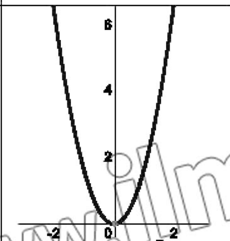
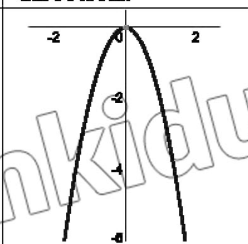
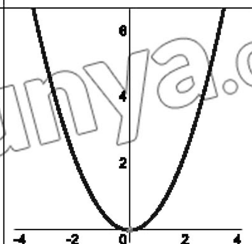
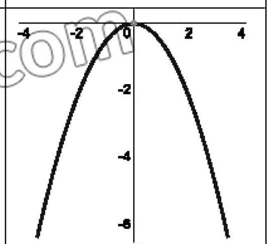
Original function $y = f(x)$	Multiply $f(x)$ by a positive constant c.	Multiply $f(x)$ by a negative constant c.	Divide $f(x)$ by a positive constant c.	Divide $f(x)$ by a negative constant c.
$y = f(x)$	$y = cf(x)$; $c > 0$	$y = cf(x)$; $c < 0$	$y = \dfrac{f(x)}{c}; c > 0$	$y = \dfrac{f(x)}{c}; c < 0$
Geometric effects	Figure is compressed by changing y-values by 2 in the same direction.	Figure is compressed by changing y-values by 2 in the opposite direction.	Figure is stretched by changing y-values by 2 in the same direction.	Figure is stretched by changing y-values by 2 in the opposite direction.
 $y = x^2$	 $y = 2x^2$	 $y = -2x^2$	 $y = x^2/2$	 $y = x^2/-2$

Table 1.2

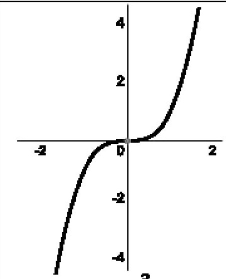
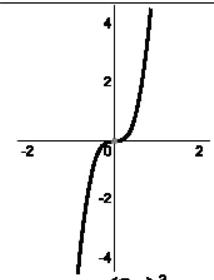
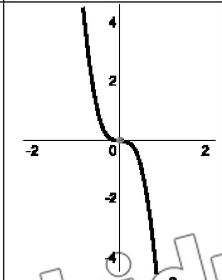
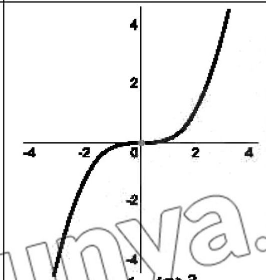
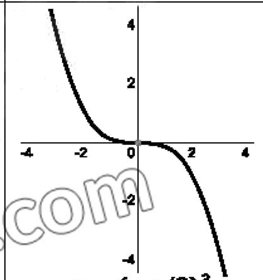
Original function $y = f(x)$	Multiply x by a positive constant c.	Multiply x by a negative constant c.	Divide x by a positive constant c.	Divide x by a negative constant c.
$y = f(x)$	$y = f(cx)$; $c > 0$	$y = f(cx)$; $c < 0$	$y = f(x/c)$; $c > 0$	$y = f(x/c)$; $c < 0$
Geometric effects	Figure is compressed by changing x-values by 2 in the same direction.	Figure is compressed by changing x-values by 2 in the opposite direction.	Figure is stretched by changing x-values by 2 in the same direction.	Figure is stretched by changing x-values by 2 in the opposite direction.
 $y = x^3$	 $y = (2x)^3$	 $y = (-2x)^3$	 $y = (x/2)^3$	 $y = (-x/2)^3$

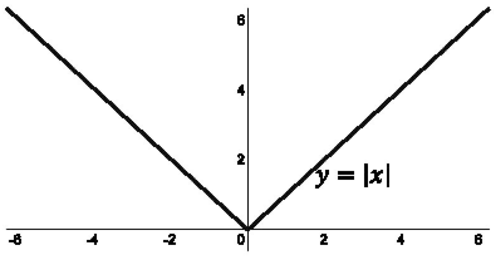
Table 1.3

Example 27: Draw the graph of $y = |x|$ and then sketch the graphs of:
(a) $y = |1.5x|$ and $y = |-1.5x|$ (b) $y = |x| \div 1.5$ (c) $y = |x| \div (-1.5)$
Which kind of scaling did you observe after sketching the graphs.

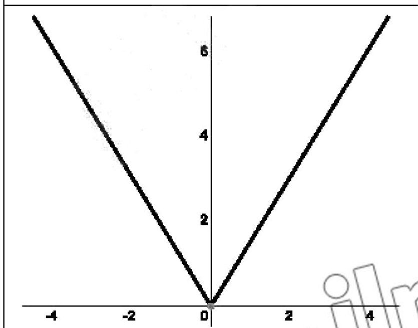
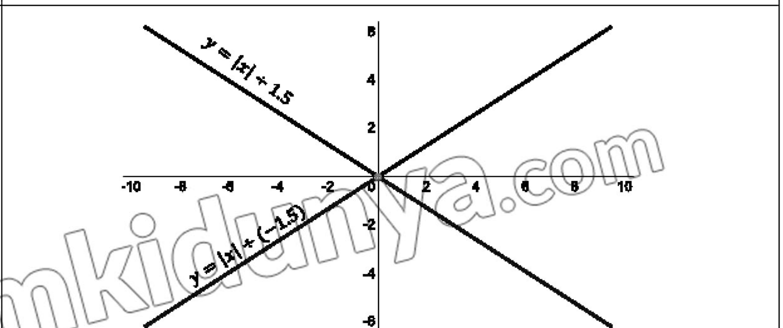
Solution:

Table of the values of the function $y = |x|$ is given as:

x	0	±1	±2	±3	±4	±5	±6
$f(x)$	0	1	2	3	4	5	6



The graph is shown in the adjoining figure.
Sketch of other graphs is shown in the table below.

(a) $y = \|1.5x\| = \|-1.5x\|$	(b) $y = \|x\| \div 1.5$ and (c) $y = \|x\| \div (-1.5)$
	 y = |x| ÷ 1.5, y = |x| ÷ (−1.5)
Figure is compressed by changing x-values by 1.5 in both cases.	Figure is stretched by changing y-values by 1.5 in both cases but with opposite behavior.

Exercise 1.5

Draw the graphs of the given functions and then sketch the graphs of other functions using translation. Verify the results using graphical calculator.

1. $y = |x|$ (a) $y = |x+2|$ (b) $y = |x-2|$ (c) $y = |x|+2$ (d) $y = |x|-2$
2. $y = x^2$ (a) $y = x^2+4$ (b) $y = x^2-4$ (c) $y = (x-4)^2$ (d) $y = (x+4)^2$
3. $y = \sqrt{x}$ (a) $y = \sqrt{x+3}$ (b) $y = \sqrt{x-3}$ (c) $y = \sqrt{x}+3$ (d) $y = \sqrt{x}-3$
4. $y = x$ (a) $y = x+5$ (b) $y = x-5$ (c) $y = 5x$ (d) $y = -5x$
5. $y = x^3$ (a) $y = x^3+1$ (b) $y = x^3-1$ (c) $y = (x-1)^3$ (d) $y = (x+1)^3$
6. $y = x^2+4$
   (a) $y = (x^2+4)-3$ (b) $y = (x^2+4)+3$
   (c) $y = (x-3)^2+4$ (d) $y = (x+3)^2+4$
7. $y = x^2$ (a) $y = 3x^2$ (b) $y = -3x^2$ (c) $y = \frac{x^2}{3}$ (d) $y = -\frac{x^2}{3}$
8. $y = x^2$ (a) $y = (3x)^2$ (b) $y = (-3x)^2$ (c) $y = \left(\frac{x}{3}\right)^2$ (d) $y = \left(-\frac{x}{3}\right)^2$
9. $y = \sqrt{x}$ (a) $y = \sqrt{2x}$ (b) $y = 2\sqrt{x}$ (c) $y = 2\sqrt{x}+3$ (d) $y = \sqrt{2x+5}$

Review Exercise

1. Tick the correct option in each of the following.

   (i) Which of the following is an example of exponential growth function?
   (a) $f(x) = 3x+4$ (b) $f(x) = 3^x \times 5$ (c) $f(x) = x^3$ (d) $f(x) = x^2$

   (ii) The exponential decay function is expressed by:
   (a) $f(x) = a.b^x; 0 < b < 1$ (b) $f(x) = a.b^x; b > 1$
   (c) $f(x) = a.b^x; 0 < a < 1$ (d) $f(x) = a.b^x; a > 1$

   (iii) The logarithmic function $f(x) = \log_b x$ is defined for:
   (a) all real numbers (b) $x < 0$ (c) $x > 0$ (d) $x \geq 0$

   (iv) What is the value of $\log_5 125$?
   (a) 25 (b) 5 (c) 4 (d) 3

   (v) A function $f: A \longrightarrow B$ is said to be onto if:
   (a) Every element of the set A has a unique image in the set B.
   (b) Every element in the set B has a preimage in the set A.
   (c) Some elements of the set B have no preimage in the set A.
   (d) f is both one to one and onto.

   (vi) The function $f(x) = x+1$, where $f: \{1, 2, 3\} \longrightarrow \{2, 3, 4\}$, is:
   (a) one to one but not onto (b) onto but not one to one
   (c) both one to one and onto (d) neither one to one nor onto

(vii) The function $f: R \rightarrow [0, \infty)$ defined by $f(x) = x^2 + 1$, is:
(a) onto but not on to one (b) one to one but not onto
(c) neither one to one nor onto (d) both one to one and onto

(viii) A function $f: A \rightarrow B$ has an inverse if and only if:
(a) f is one to one (b) f is onto
(c) f is both one to one and onto (d) f is neither one to one and onto

(ix) The inverse function of $f(x) = x^3$, is:
(a) $f^{-1}(x) = x^{-3}$ (b) $f^{-1}(x) = \sqrt{x^{-3}}$ (c) $f^{-1}(x) = \sqrt[3]{x^3}$ (d) $f^{-1}(x) = \sqrt[3]{x}$

(x) The function $f(x) = \sin x$, where $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$, is:
(a) one to one but not onto (b) onto but not one to one
(c) both one to one and onto (d) neither one to one nor onto

(xi) The inverse function of $f(x) = \frac{1}{x}; x \neq 0$, is:
(a) $f^{-1}(x) = 1$ (b) $f^{-1}(x) = -x$ (c) $f^{-1}(x) = x$ (d) $f^{-1}(x) = \frac{1}{x}$

(xii) Scaling refers to:
(a) increasing the size of an object.
(b) decreasing the size of an object.
(c) maintaining the properties while resizing an object.
(d) changing the shape of an object.

(xiii) Which of the following statements is true for the uniform scaling?
(a) both width and height change proportionally.
(b) only the width changes.
(c) only the height changes.
(d) width and height remain unchanged.

(xiv) What is the effect on the graph of $f(x)$ when it is replaced by $f(x+2)$?
(a) It shifts 2 units to the right. (b) It shifts 2 units to the left.
(c) It shifts 2 units up (d) It shifts 2 unit down.

(xv) The domain of $y = \sin^{-1}(x)$, is:
(a) $[0, \infty)$ (b) $(-\infty, \infty)$ (c) $[-1, 1]$ (d) $[0, 1]$

2. Find the domain of the given functions.
(a) $f(x) = 4 + \sqrt{x+2}$ (b) $f(x) = x\sqrt{2x-3}$
(c) $f(x) = \frac{x}{x-2}$ (d) $f(x) = \sqrt{x^2 - 5x + 4}$

3. Find the domain and range of the given functions.
(a) $f(x) = 1 + x^2$ (b) $f(x) = (2x+1)^2$
(c) $f(x) = 9 + \sqrt{x}$ (d) $f(x) = 3 + \sqrt{4 - x^2}$

4. Draw the graph of $f(x) = \sqrt{x}$, then sketch the graphs of the following functions.
   (a) $f(x) = \sqrt{x-2}$ (b) $f(x) = \sqrt{x} + 4$ (c) $f(x) = -\sqrt{x}$
   (d) $f(x) = 1 + \sqrt{x-2}$ (e) $f(x) = 4\sqrt{x}$ (f) $f(x) = -\frac{1}{3}\sqrt{x}$
5. Graph the given functions.
   (a) $y = 2 + 2\sin x$ (b) $y = -\frac{1}{2}\tan x$
   (c) $y = 3 - \operatorname{cosec} x$ (d) $y = \cos(x + \pi)$
6. Find the domain and range of the inverse function of $f(x) = \log(x^2 + 1)$.
7. Show that $f(g(x)) = g(f(x)) = x$, when:
   $f(x) = e^x$ and $g(x) = \ln x$.
8. The population of a town grows exponentially according to the formula $P(t) = 1000\, e^{0.05t}$ where t is the time in years. After how many years, will the population reach 5000?
9. A company has the following cost and revenue functions:
   $C(x) = 5x + 10$; where x is the number of units produced.
   $D(x) = 15x$; where x is the number of units sold.
   Find the equilibrium point where cost equals revenue.