12

Based on National Curriculum of Pakistan 2022-23

Textbook of

MATHEMATICS

National Book Foundation
as
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Islamabad

Based on National Curriculum of Pakistan 2022-23

Textbook of
Mathematics
Science Group

12

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Preface

This Textbook for Mathematics Grade 12 has been developed by NBF according to the National Curriculum of Pakistan 2022-2023. The aim of this textbook is to enhance learning abilities through inculcation of logical thinking in learners, and to develop higher order thinking processes by systematically building the foundation of learning from the previous grades. A key emphasis of the present textbook is creating real life linkage of the concepts and methods introduced. This approach was devised with the intent of enabling students to solve daily life problems as they grow up in the learning curve and also to fully grasp the conceptual basis that will be built in subsequent grades.

After amalgamation of the efforts of experts and experienced authors, this book was reviewed and finalized after extensive reviews by professional educationists. Efforts were made to make the contents student friendly and to develop the concepts in interesting ways.

The National Book Foundation is always striving for improvement in the quality of its textbooks. The present textbook features an improved design, better illustration and interesting activities relating to real life to make it attractive for young learners. However, there is always room for improvement, the suggestions and feedback of students, teachers and the community are most welcome for further enriching the subsequent editions of this textbook.

May Allah guide and help us (Ameen).

Dr. Kamran Jahangir
Managing Director

Application of Mathematics

Functions and Graphs: Functions represent relationships between variables and their graphs provide a visual representation of these relationships. They are used to study trends like profit versus cost in businesses, population growth over time or changes in speed in physics. Functions help in predicting outcomes and analyzing real-world data effectively.

Limit, Continuity and Derivative: Limits describe how functions behave near specific points, continuity ensures smooth graphs without breaks, and derivatives measure rates of change like speed or growth. These concepts are essential in physics to calculate instantaneous velocity, in economics for cost optimization and in biology for population growth modeling.

Integration: Integration helps calculate areas under curves, volumes, or accumulated quantities. It is widely used to determine total distance from velocity, analyze energy consumption and compute areas in construction projects. Integration also has applications in physics, such as finding work done by a variable force.

Differential Equations: Differential equations describe changes in dynamic systems, modeling real-world processes like population growth, chemical reactions and motion. They are used in engineering to design systems, in physics to describe heat flow or wave motion and in biology to model disease spread.

Kinematics of Motion in a Straight Line: This studies the motion of objects along a straight path using concepts like displacement, velocity and acceleration. Applications include calculating stopping distances of vehicles, analyzing free-fall under gravity and predicting the motion of objects in linear transport systems.

Analytical Geometry: Analytical geometry combines algebra and geometry to study the properties of shapes on the coordinate plane. It is used in designing structures, solving distance and slope problems and planning urban layouts. It plays a significant role in architecture and engineering.

Conic Section: Conic sections include circles, ellipses, parabolas, and hyperbolas, which have various applications. Ellipses describe satellite orbits, parabolas are used in designing headlights and bridges and hyperbolas are applied in communication systems and radar design.

Inverse Trigonometric Functions and Their Graphs: Inverse trigonometric functions calculate angles from trigonometric values, essential in navigation, surveying, and architecture. Their graphs help solve problems involving slopes, elevation angles and real-world measurements requiring precision.

Solution of Trigonometric Equations: Trigonometric equations model periodic phenomena such as sound and light waves. Solving these equations is crucial in designing musical instruments, analyzing alternating electrical currents and studying wave patterns in physics and engineering.

Numerical Methods: Numerical methods approximate solutions for complex problems using algorithms. They are widely used in weather forecasting, structural analysis of buildings and financial risk modeling. These methods make it possible to solve equations that are difficult to handle analytically.

بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ

اللہ کے نام سے شروع جو بڑا مہربان، نہایت رحم والا ہے

CONTENT

UNIT

01

FUNCTIONS AND GRAPHS

After studying this unit, students will be able to:

- Recall definition of function, find its domain, codomain, range and its types.
- Find inverse of a function and demonstrate its domain and range with examples.
- Know linear, quadratic and square root functions.
- Sketch graphs of linear and non-linear functions.
- Plot graph of function of the type $y = x^n$ when (i) n is a + ve or – ve integer and $x \neq 0$, (ii) n is a rational number for $x > 0$.
- Plot graph of quadratic function of the form $y = ax^2 + bx + c$, where a, b, c are integers and $a \neq 0$.
- Draw graph using factors and predict functions from their graphs.
- Find the intersecting points graphically when intersection occurs between (i) a linear function and coordinate axes (ii) two linear functions (iii) a linear function and a quadratic function.
- Draw the graph of modulus functions.
- Solve graphically appropriate problems from daily life.
- Classify algebraic and transcendental functions and describe trigonometric, inverse trigonometric, logarithmic and exponential functions.
- Define logarithm, and derive and apply laws of logarithm.
- Graph and analyse exponential and logarithmic functions.
- Apply the concept of exponential functions to find compound interest.
- Solve problems involving exponential and logarithmic equations.
- Identify the domain and range of transcendental functions through graphs.
- Interpret the relation between a one-one function and its inverse through a graph.
- Demonstrate the transformation of a graph through horizontal shift, vertical shift and scaling.

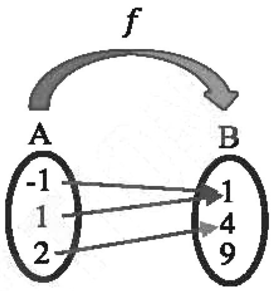
Functions have many applications in real life. One is the use of function in signal processing applications in engineering, including noise reduction, modulation, and filtering. For example, functions in audio processing are used to analyze and alter sound waves, which makes it possible to design devices like equalizers and noise-canceling headphones. Similar to this, functions are essential to the encoding, transmitting, and decoding of signals for wireless communication systems in the telecommunications industry.

1.1 Function

A function f from a set A to a set B assigns to each element of A exactly one element of B. The set A is called the domain of the function and the set B is called the co-domain of the function.

Mathematically, it is written as $f : A \rightarrow B$ and is read as f is a function from A to B.

For example, if we are given two sets $A = \{-1, 1, 2\}$ and $B = \{1, 4, 9\}$,



then $f = \{(-1, 1), (1, 1), (2, 4)\}$ is a function because each element of the set A is assigned to exactly one element of the set B. i.e. there is no repetition in the first element of ordered pairs in f. The first element of each ordered pair in f is called pre-image while its corresponding second element is called image of the first element. For example, in (2, 4), 2 is the pre-image of 4 and 4 is the image of 2.

If x is independent variable and y is dependent variable, then in general, a function f from A to B is written as $f(x) = y$.

For example, in the above example,

$f(-1) = 1, f(1) = 1$ and $f(2) = 4$

Explanation: As discussed above, a function relates an input to an output.

For example, if a tree grows 15 cm every year and the height h of the tree is related to its age as follows:

$$h(\text{age}) = \text{age} \times 15$$

then the height of the tree after 10 year is $h(10) = 10 \times 15 = 150$ cm

$\therefore$ '$h(10) = 150$' is like saying 10 is related to 150 or $10 \rightarrow 150$

Here, 10 is the input and 150 is the output of the function.

1.1.1 Domain of a Function

The set of all possible values of independent variable which qualify as inputs to a function is known as the domain of the function. In the above example,

Domain of function f = Dom $f = \{-1, 1, 2\}$

> **Key Facts**
> A function in which real numbers are used is called a real valued function.

How to Find the Domain of a Function

To find the domain, we ensure that there is no zero in the denominator of a fraction and no negative sign inside a square root. In general, the set of all real numbers is considered as the domain of a function subject to some restrictions. For example:

(i) When the given function is of the form $f(x) = 3x + 8$ or $f(x) = x^3 + 2x - 5$, the domain will be "the set of all real numbers".

(ii) When the given function is of the form $f(x) = \frac{1}{x-2}$ the domain will be the set of all real numbers except 2.

(iii) In some cases, the interval be specified along with the function such as, $f(x) = x + 1$, $0 < x < 10$. Here, x can take the values between 0 and 10 in the domain.

(iv) Domain restrictions refer to the values for which the given function cannot be defined.

1.1.2 Range of a Function

The set of all the outputs of a function is known as the range of the function or after substituting the domain, the entire set of all possible values as outcomes of the dependent variable.

In a function $y = f(x)$, the spread of all the values y from minimum to maximum is the range of the function. In the above example,

Range of function f = Rang $f = \{1, 4\}$

How to Find the Range of a Function

- Substitute all the values of x in the function to check whether it is positive, negative or equal to other values. Eliminate the values of x for which the function is not defined.
- Find the minimum and maximum values for y.

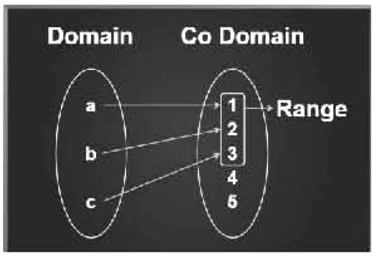
1.1.3 Codomain of a Function

The codomain is the set of all possible outcomes of the given function.

In general, the range is the subset of the codomain. But sometimes the codomain is also equal to the range of the function. In above example,

Codomain of function $f = \{1, 4, 9\}$

In short:



- ➢ What can go into a function is called the domain.
- ➢ What may possibly come out of a function is called the codomain.
- ➢ What actually comes out of a function is called the range.

In the adjoining figure,

- The set {a, b, c} is the domain.
- The set $\{1, 2, 3, 4, 5\}$ is the codomain.
- The set $\{1, 2, 3\}$ is the range.

Example 1: Find the domain and range of a function $f(x) = 2x^2 - 4$.

Solution: $f(x) = 2x^2 - 4$

The given function has no undefined values of x.
Thus, the domain is the set of all real numbers.
Domain = $(-\infty, \infty) = R$
If we put $x = 0$ in the given function, we get $f(x) = -4$.
For all real values of x, other than 0, we get an output greater than -4.
Hence, the range of $f(x)$ is $[-4, \infty)$.

Key Facts

A function is like a machine that takes an input and produces a corresponding output. For example, the distance a car has traveled (the output) is dependent on how long that car has been driving (the input).

Example 2:

Find the domain and range of function $g(x) = \sqrt{x-3}$.

Solution: $g(x) = \sqrt{x-3}$

The given function is defined for all real numbers x greater than or equal to 3.

Thus, the domain of $g(x)$ is $[3, \infty)$.

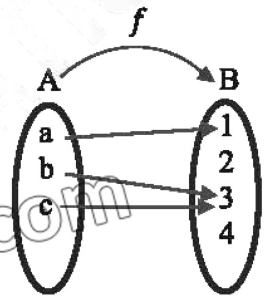
If we put $x = 3$ in the given function, we get $g(3) = 0$.

For all real values of x, greater than 3, we get an output greater than 0.

Hence, the range of $g(x)$ is $[0, \infty)$.

1.2 Types of Functions

1.2.1 Into Function



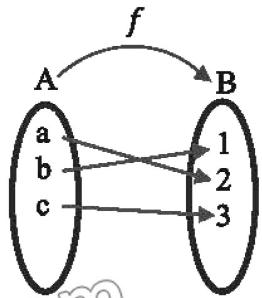
A function f: A → B is said to be into function if there exists at least one element or more than one element in B, which does not have any pre-images in A, which simply means that every element of the codomain is not mapped with elements of the domain. i.e., range$(f) \neq$ B.

In the adjoining diagram, f = {(a, 1), (b, 3), (c, 3)} is an into function.

Some examples of into functions are:

- $f(x) = \sin x$ where f: R → R is into function because it doesn't cover all values in the interval R.
- $g(x) = x^2$ where g: R → R is into function because it doesn't map to any negative real numbers.
- $h(x) = e^x$ where h: R → $[0, \infty)$ is into function because it doesn't map to zero.

1.2.2 Onto (Surjective) Function



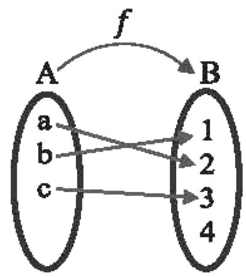
For any two non-empty sets A and B, a function f : A → B will be onto if every element of set B is an image of some element of set A. i.e., for every $y \in$ B there exists an element x in A such that $f(x) = y$ which implies rang(f) = B.

In the adjoining diagram, f = {(a, 2), (b, 1), (c, 3)} is an onto function.

Some examples of onto functions are:

- $f(x) = x$ (Identity function)
- $g(x) = e^x$ when g: R → R^+ (Exponential function)
- $h(x) = x^2$ (Square function)
- $m(x) = x^3$ (Cubic function)
- $p(x) = c$ (Constant function)

1.2.3 One to One (Injective) Function



For any two non-empty sets A and B, a function $f: A \rightarrow B$ will be one-to-one if distinct elements of set A have distinct images in set B. In the adjoining diagram, $f = \{(a, 2), (b, 1), (c, 3)\}$ is a one-to-one function.

A function $f: A \rightarrow B$ is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

i.e., an image of a distinct element of A under f mapping (function) is distinct.

Some examples of one-one functions are:

- $f(x) = x$ (Identity function)
- $g(x) = 2x + 3$ (Linear function)
- $h(x) = e^x$ (Exponential function)
- $m(x) = \sqrt{x}$ (Square root function, defined for $x \geq 0$)

1.2.4 Injective Function

A function which is both into and one-one is called an injective function.

1.2.5 Bijective Function

A function which is both onto and one-one is called a bijective function.

Bijective function shows one-one correspondence between the elements of two sets.

Key Facts

Various types of functions are mentioned in the below table:

Based on elements	One-one function, Many-one function, Onto function, Bijective Function, Into function, constant Function
Based on the equation	Identity function, Linear function, Quadratic function, Cubic function, Polynomial functions
Based on the range	Modulus function, Rational function, Even and odd functions, Periodic functions, Greatest and smallest integer function, Inverse function, Composite functions
Based on the domain	Algebraic functions, Trigonometric functions, Logarithmic functions, Exponential functions

Example 3: Check whether the function $f(x) = 2x + 3$, is one-to-one or not if domain = $\{0.5, 1, 2\}$ and codomain = $\{4, 5, 7\}$

Solution: Putting 1, 2 and 0.5 in $f(x) = 2x + 3$, we get $f(0.5) = 4$, $f(1) = 5$ and $f(2) = 7$

As, for every value of x, we get a unique $f(x)$ thus, the function $f(x)$ is one to one.

Example 4: Check whether the function is one-to-one or not: $f(x) = 2x^2 + 1$.

Solution: To check whether the function is one to one or not, let:

$$f(x_1) = f(x_2)$$
$$2(x_1)^2 + 1 = 2(x_2)^2 + 1$$
$$(x_1)^2 = (x_2)^2$$

Since $(x_1)^2 = (x_2)^2$ is not always true, therefore the function is not one to one function.

Example 5: Check the type of function $f(x) = x^2 - 1$ if Dom $f(x) = \{1, -1, 2, -2\}$ and Codom $f(x) = \{0, 3, -3\}$

Solution: Given $f(x) = x^2 - 1$ with Dom $f(x) = \{1, -1, 2, -2\}$ and Codom $f(x) = \{0, 3, -3\}$

Substituting the elements of the domain in the function, we get:

$$f(1) = 1^2 - 1 = 0$$
$$f(-1) = (-1)^2 - 1 = 0$$
$$f(2) = 2^2 - 1 = 3$$
$$f(-2) = (-2)^2 - 1 = 3$$

Therefore, Rang $f(x) = \{0, 3\}$. As, Rang $f(x) = \{0, 3\} \neq \{0, 3, -3\} =$ Codom $f(x)$.

So, the given function is an into function.

Example 6: Find the type of the function $f(x) = 3x + 2$ defined on $f: R \rightarrow R$.

Solution: Let, $f(x) = y \quad \Rightarrow \quad y = 3x + 2 \quad \Rightarrow \quad y - 2 = 3x \quad \Rightarrow \quad x = \frac{y-2}{3}$

Substituting the value of x in the given function $f(x)$, we get:

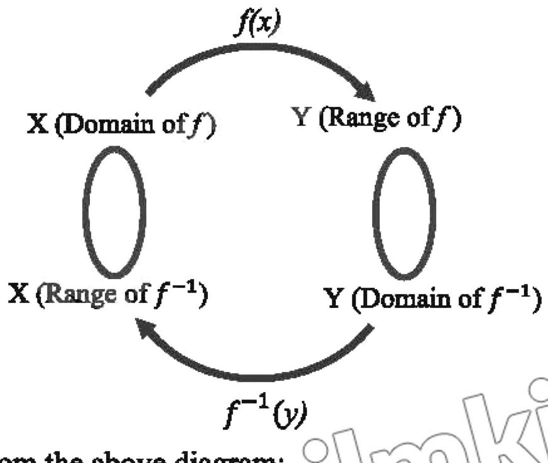
$$f(x) = f\left(\frac{y-2}{3}\right) = 3\left(\frac{y-2}{3}\right) + 2 = y - 2 + 2 = y$$

Since, we get back y after putting the value of x in the function. Hence the given function is an onto function.

1.3 Inverse Function

The inverse of any function $f(x)$ is a function denoted by $f^{-1}(x)$ which reverses the effect of $f(x)$ and it undoes what $f(x)$ does. In mathematics, the inverse function is also denoted by f^{-1}. If f: X → Y, then f^{-1}: Y → X. i.e., If the application of a function f to x as input gives an output of y, then the application of inverse function f^{-1} to y should give back the value of x.

It can be illustrated in the following diagram as:



Key Facts

- If $y = f(x)$ is bijective function then $x = f^{-1}(y)$.
- If $fog(x) = gof(x) = x$, then $g = f^{-1}$ and $f = g^{-1}$
- $(f^{-1})^{-1} = f$

From the above diagram:

$$\text{dom } f = \text{rang } f^{-1} \quad \text{and} \quad \text{rang } f = \text{dom } f^{-1}$$

i.e., The domain of the given function becomes the range of the inverse function, and the range of the given function becomes the domain of the inverse function.

Note that f^{-1} is not the reciprocal of f and not every function has an inverse. If a function $f(x)$ has an inverse, then $f(x)$ never takes the same value twice. In simple words, the inverse function exists only when f is both one-one and onto function. Can we say that the inverse function is also a bijective function?

Moreover, the composition of the function f and the inverse function f^{-1} gives the domain value of x.

$$fof^{-1}(x) = f^{-1}of(x) = x$$

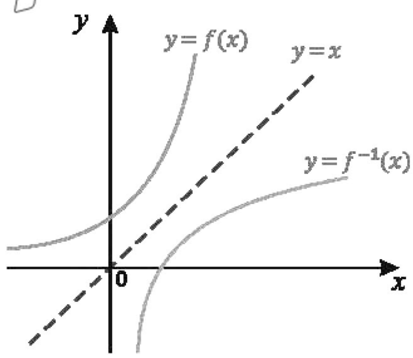
1.3.1 Steps to Find an Inverse Function

Consider a function $f(x) = ax + b$.

- Replace $f(x)$ with y, to obtain $y = ax + b$.
- Solve the expression for x to obtain $x = \frac{y-b}{a}$.
- Replace x with $f^{-1}(y)$ to get $f^{-1}(y) = \frac{y-b}{a}$.
- Interchange y with x in the function $f^{-1}(y) = \frac{y-b}{a}$ and get inverse function $f^{-1}(x) = \frac{x-b}{a}$.

1.3.2 Graph of an Inverse Function

If the graphs of both functions are symmetric with respect to the line $y = x$ then we say that the two functions are inverses of each other. This is because of the fact that if (x, y) lies on the function, then (y, x) lies on its inverse function.



Example 7:

Find the inverse function of $f(x) = \frac{x}{x-2}$ defined on $f: R \to R$.

(i) Find domain and range of function and its inverse.

(ii) Prove that $fof^{-1}(x) = f^{-1}of(x) = x$

Solution:

(i) Given function is $f(x) = \frac{x}{x-2}$

Dom $f(x) = R - \{2\}$

To find inverse function, let:

$$y = \frac{x}{x-2} \quad \Rightarrow \quad y(x-2) = x \quad \Rightarrow \quad xy - x = 2y$$

$$\Rightarrow \quad x(y-1) = 2y \quad \Rightarrow \quad x = \frac{2y}{y-1}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{2y}{y-1} \quad \text{......} \quad (x = f^{-1}(y))$$

$$\Rightarrow \quad f^{-1}(x) = \frac{2x}{x-1} \quad \text{...... Replacing } y \text{ with } x.$$

From the inverse function, we see that:
Dom $f^{-1}(x) = R - \{1\}$
Hence,
Dom $f = R - \{2\} =$ Rang f^{-1} and Dom $f^{-1} = R - \{1\} =$ Rang f

(ii) $fof^{-1}(x) = f(f^{-1}(x)) = f\left(\frac{2x}{x-1}\right) = \frac{\frac{2x}{x-1}}{\frac{2x}{x-1}-2} = \frac{2x}{2x-2x+2} = \frac{2x}{2} = x$ (i)

$f^{-1}of(x) = f^{-1}(f(x)) = f^{-1}\left(\frac{x}{x-2}\right) = \frac{2\left(\frac{x}{x-2}\right)}{\frac{x}{x-2}-1} = \frac{2x}{x-x+2} = \frac{2x}{2} = x$ (ii)

From (i) and (ii), we get:

$$fof^{-1}(x) = f^{-1}of(x) = x$$

Exercise 1.1

1. Find the domain of following functions.
   (i) $f(x) = x^2 - 6$ (ii) $g(x) = \frac{x}{x+3}$ (iii) $h(x) = \frac{x+4}{x^2-9}$
   (iv) $i(x) = \frac{x}{5x+2}$ (v) $j(x) = \frac{x}{x^2+4}$ (vi) $k(x) = \sqrt{x+1}$
2. Find the domain and range of the functions.
   (i) $f(x) = x + 7$ (ii) $f(x) = 2x^2 + 1$ (iii) $f(x) = 2\sqrt{x-5}$
   (iv) $f(x) = |x-2| - 3$ (v) $f(x) = 1 + \sin x$ (vi) $f(x) = 3 + \sqrt{x-2}$
   (vii) $f(x) = \frac{3e^x}{7}$ (viii) $f(x) = \frac{x^2-16}{x+4}$ (ix) $f(x) = (x-1)^2 + 1$
   (x) $f(x) = \frac{1}{x-1}$ (xi) $f(x) = \frac{x-2}{x+3}$ (xii) $f(x) = \frac{x^2-x-6}{x-3}$
3. Given that A = {0, 1, 2, 3}, B = {p, q, r, s} and f = {(0, p), (1, q), (2, r), (3, s)}. Check whether the function is one to one, onto and/or into.
4. A = {2, 3, 4, 5}, B = {b, c, d, e}. The function is defined as f = {(2, b), (3, c), (4, e), (5, e)}. Check whether the function is one to one, into or onto.
5. Check whether the functions are one-to-one or not.
   (i) $f(x) = 4x - 7$ (ii) $f(x) = 6x^2 + 2$ (iii) $f(x) = \frac{x^3-1}{2}$
6. Check the type of function $g(x) = 2x^2 + 3x + 1$ if Dom $g(x) = \{0, 1, 2, 3\}$ and Rang $g(x) = \{1, 6, 15, 28, 35\}$
7. Find the type of the function $h(x) = 2x + 1$ defined on $h: R \to R$.
8. If f: A → B is defined by $f(x) = \frac{x-2}{x-3}$ for all $x \in$ A where A = R – {3} and B = R – {1}. Then show that the function f is bijective.

9. Find the domain and range of inverse functions when:

(i) $f(x) = 4x - 3$ (ii) $f(x) = \frac{x}{x-5}$ (iii) $f(x) = \frac{x+2}{x-1}$

(iv) $f(x) = \sqrt{x+2}$ (v) $f(x) = x^2 + 6$ (vi) $f(x) = \frac{2x-1}{x+4}$

Also prove that $f(f^{-1}(x)) = f^{-1}(f(x))$.

1.4 Linear, Quadratic and Square Root Functions

1.4.1 Linear Function

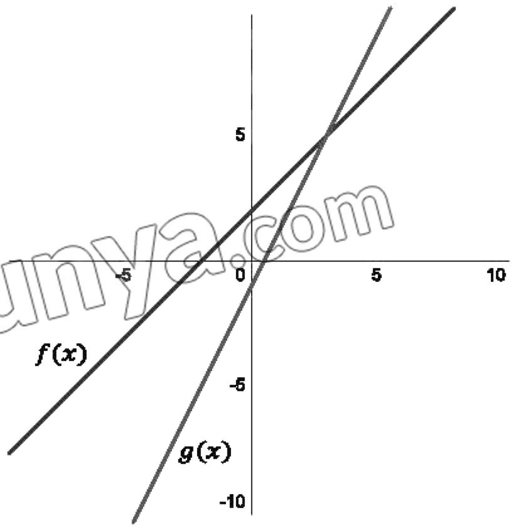
The function of the form,

$y = ax + b; a, b \in R \wedge a \neq 0$

is called linear. It is a polynomial function of degree one.

For example, $f(x) = x + 2,\ g(x) = 2x - 1$ are linear functions.

The graph of a linear function is a straight line and the slope of any two points on the line is the same. The domain and range of the linear function is R.



1.4.2 Non-linear Function

A function that is not linear is called a non-linear function. A nonlinear function is a function whose plotted graph form a curved line. For example, quadratic function, cubic function, square root function and exponential function etc. The slope of every two points on the graph of non-linear is not the same. Let us recall the shapes of the graphs of quadratic and square root functions here.

(a) Quadratic Function

The function is of the form,

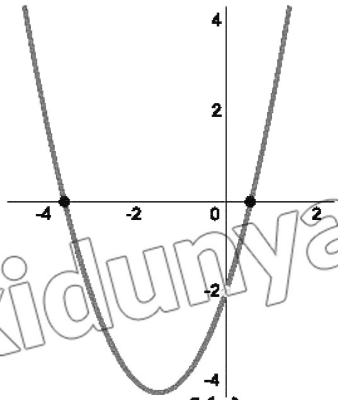
$y = ax^2 + bx + c; a, b, c \in R \wedge a \neq 0$

is called quadratic.

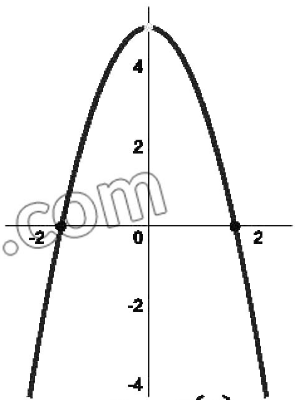
It is a polynomial function of degree two.

For example, $f(x) = x^2 + 3x - 2$ and $g(x) = 5 - 2x^2$ are quadratic functions.

The domain and range of the quadratic function is R. The graph of a quadratic equation is U-shaped and is parabolic in nature.



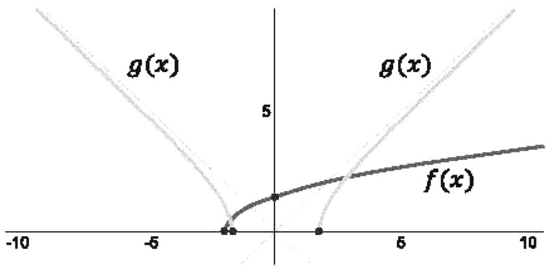
$y = f(x)$



$y = g(x)$

(b) Square Root Function

The function of the form $y = \sqrt{x}$, where $x \geq 0$ is called a square root function.
For example,
$f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{x^2 - 3}$ are square root functions.



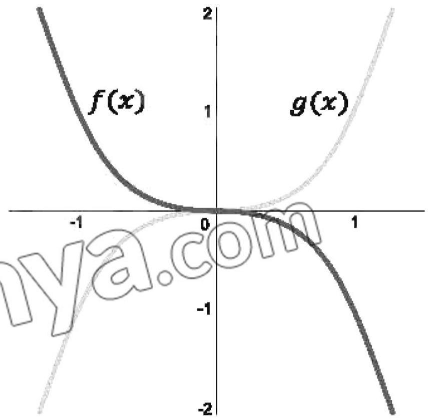
The domain of square root function depends upon its formation.

1.5 Plotting Graph of Function of the Type $y = x^n$

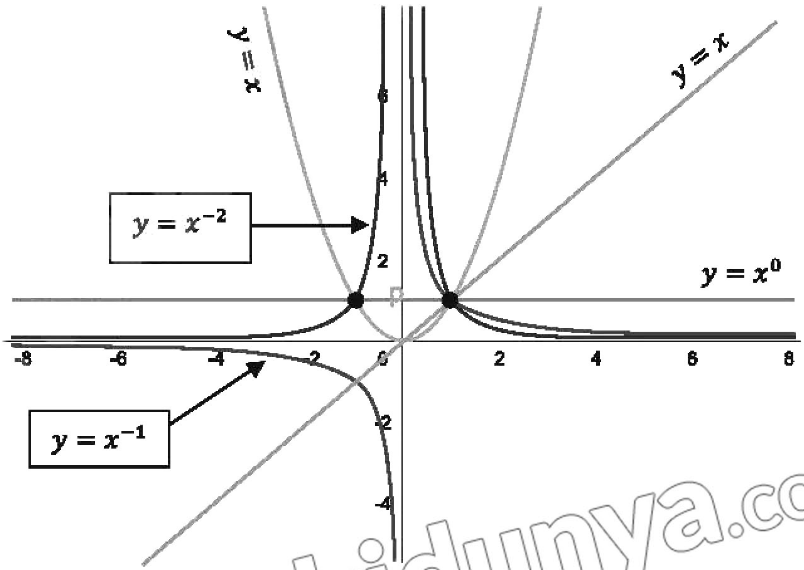
1.5.1 Graph of the Function $y = x^n$; $n \in Z \land x \neq 0$

For plotting the graph of the function $y = x^n$; $n \in Z \land x \neq 0$, we take different integral values of n. For example, the table for the function $y = \pm x^3$ is:

x	0	0.5	1	2
$f(x) = x^3$	0	0.125	1	8
$g(x) = -x^3$	0	−0.125	−1	−8



We observe that graphs of $y = x^3$ and $y = -x^3$ have the same shape but opposite behavior.
The graph of $y = x^n$ for $n = -2, -1, 0, 1, 2$ is shown below.



From the above graphs, we observe that:

- ➢ the graph of $y = x^0$ is a horizontal line passing through $y = 1$.
- ➢ the graph of $y = x^1$ is a straight line bisecting first and third quadrant.
- ➢ the graph of $y = x^{-1}$ is a hyperbola passing through first and third quadrant.

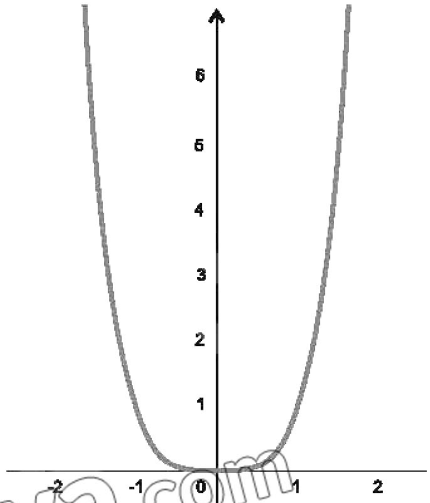
- the graph of $y = x^2$ is a parabola starting from origin and opening upwards.
- the graph of $y = x^{-2}$ has exponential behavior with two branches.
- The graphs of $y = x^n$ pass through (1, 1).

Example 8: Draw the graph of $y = x^4$.

Solution:

Table for some values of *x* and *y* for the function is:

x	-1.5	-1	0	1	1.5
y	5.06	1	0	1	5.06

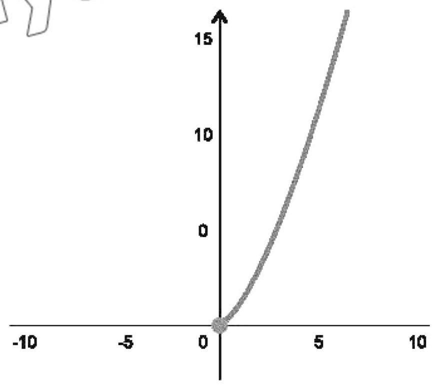


From the figure, we can see that the graph of $y = x^4$ is U-shaped which opens upward starting from origin. The value of y increases slowly for real numbers $-0.5 < x < 0.5$.
After that it increases abruptly.

1.5.2 Graph of the Function $y = x^n$; $n \in Q \wedge x > 0$

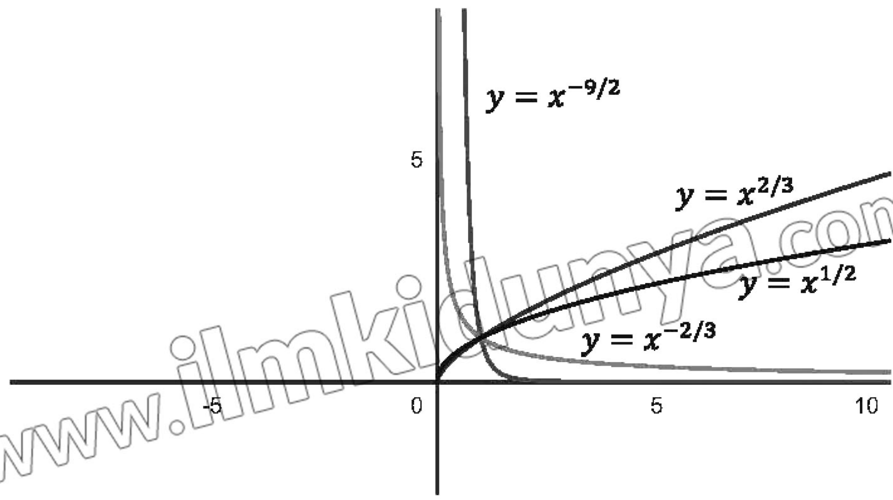
For plotting the graph of the function $y = x^n$; $n \in Q \wedge x > 0$, we take different integral values of *n*. For example, the table for the function $y = x^{3/2}$ is:

x	0.5	1	4	6
$f(x) = x^{3/2}$	0.35	1	8	14.7



We observe that graphs of $y = x^{3/2}$ has exponential behavior.

The graph of $y = x^n$ for $n = -\frac{9}{2}, -\frac{2}{3}, 0, \frac{2}{3}, \frac{1}{2}$ is shown below.



From the above graphs, we observe that:

- the graph of $y = x^{-2/3}$ is closer to y-axis.
- the graph of $y = x^{-9/2}$ moves away from y-axis as compared with the graph of $y = x^{-2/3}$.
- the graph of $y = x^{1/2}$ is closer to x-axis.
- the graph of $y = x^{2/3}$ moves away from x-axis as compared with the graph of $y = x^{1/2}$.
- The graphs of $y = x^n$ pass through (1, 1).

1.5.3 Graph of Quadratic Function

We know that the polynomial function of degree two is called a quadratic function. This function is of the form:

$$f(x) = ax^2 + bx + c\ ;\ a, b, c \in R \text{ and } a \neq 0$$

The graph of the quadratic equation is a parabola.

For example, $y = x^2 + 2x + 1$ and $y = 2 - 3x^2$ are quadratic functions.

Understanding the Graph

- a: The coefficient a, affects the direction and width of the parabola. If $a > 0$, the parabola opens upwards. If $a < 0$, the parabola opens downwards. The larger the absolute value of a, the narrower the parabola.
- b: This coefficient affects the position of the vertex horizontally (left or right) and slope of the parabola at the vertex.
- c: This is the y-intercept. So, the point (0, c) is on the parabola.
- $x = -\frac{b}{2a}$ is equation of axis of symmetry and is also the x-coordinate of the vertex.

Example 9: Draw the graph of $y = 2x^2 + 3x - 2$.

Solution:

$y = 2x^2 + 3x - 2$ (i)

Comparing (i) with $y = ax^2 + bx + c$, we have $a = 2, b = 3$ and $c = -2$

Step 1: Determine whether parabola opens upwards or downwards.

As $a > 0$, therefore parabola opens upwards.

Step 2: Find and draw the axis of symmetry.

Equation of axis of symmetry is:

$$x = -\frac{b}{2a} = -\frac{3}{2(2)} = -\frac{3}{4}$$

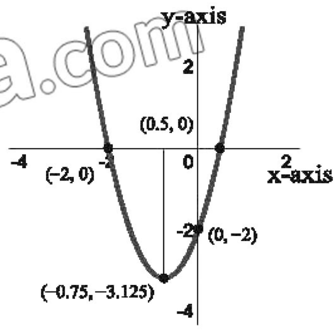
Step 3: Find and plot the vertex.

The x-coordinate of vertex is $x = -\frac{3}{4}$

To find the y-coordinate, substitute value of x in equation (i).

$$y = 2\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right) - 2 = \frac{9}{8} - \frac{9}{4} - 2 = -\frac{25}{8}$$

So, the vertex is $\left(-\frac{3}{4}, -\frac{25}{8}\right)$.



Step 4: Find some more points if needed and plot the graph.

Table for some values of x and y for the function is:

x	-2	-1	0	1
y	0	-3	-2	3

Form the figure, it is clear that the graph of function $y = 2x^2 + 3x - 2$ is parabola.

Example 10: Draw the graph of $y = 4 - 2x^2$

Solution: $y = -2x^2 + 0x + 4$ (i)

Comparing (i) with $y = ax^2 + bx + c$, we have $a = -2, b = 0$ and $c = 4$

Step 1: Determine whether parabola opens upwards or downwards.

As $a < 0$, therefore parabola opens downwards.

Step 2: Find and draw the axis of symmetry. Equation of axis of symmetry is:

$$x = -\frac{b}{2a} = -\frac{0}{2(-2)} = 0$$

Step 3: Find and plot the vertex.

The x-coordinate of vertex is $x = 0$. To find the y-coordinate, substitute value of x in equation (i). We get $y = -2(0)^2 + 0(0) + 4 = 4$

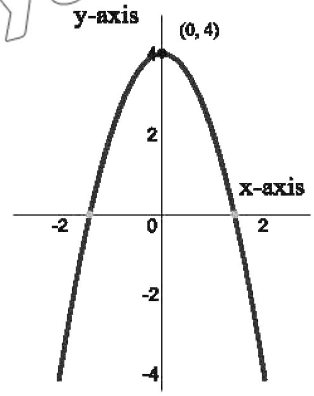
So, the vertex is $(0, 4)$.

Step 4: Find some more points if needed and plot the graph.

Table for some values of x and y for the function is:

x	-2	-1	0	1	2
y	-4	2	4	2	-4

The sketch of the graph is shown in the figure.



1.6 Drawing Graph Using Factors

Let us draw the graph of quadratic function using factors.

We know that $y = ax^2 + bx + c$ is a quadratic function where $a \neq 0$ and the graph of such a function is a parabola. These graphs can be tricky to sketch manually, but factoring the quadratic gives us all of the information we need to do so successfully.

Procedure:

The points where any parabola intersects the x-axis will be the solutions to the equation:

$$ax^2 + bx + c = 0 \quad \text{(i)}$$

Now if we can factor equation (i) in the format:

$$(x - x_1)(x - x_2) = 0$$

then by the zero product property, we get:

$$x = x_1 \quad \text{and} \quad x = x_2$$

This means that x_1 and x_2 are the x-intercepts. In other words, graph will intersect x-axis at $(x_1, 0)$ and $(x_2, 0)$. The constant c tells us what the y-intercept will be. More specifically, the constant term c places the y-intercept at $(0, c)$, giving us a third specific point on y-axis. Likewise, the value of a tells us whether our parabola will open up or down. If a is positive, the parabola opens upward. If it is negative, the parabola opens downward.

We can also figure out the vertex of the parabola by factoring the quadratic equation.

The x-coordinate of the vertex of a parabola is the arithmetic mean of two x-intercepts = $\frac{x_1+x_2}{2}$.

Once we have the x-coordinate, we can determine the y-coordinate by plugging the x value into the given function and solving for y. With these four specific points including both x-intercepts, y-intercept and the vertex of the parabola, we can create an accurate sketch of the graph quickly and easily.

Example 10: Draw the graph of $y = x^2 - 8x + 12$ using factors.

Solution:

Given function is: $y = x^2 - 8x + 12$

To get x-intercepts, put $x^2 - 8x + 12 = 0$

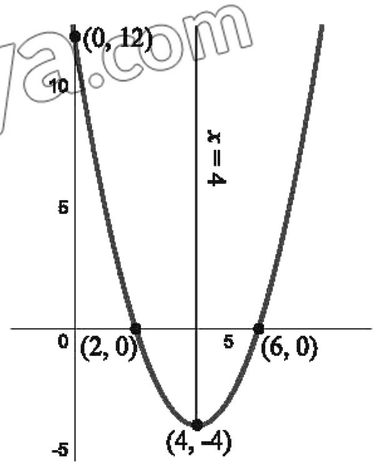
After factorization, we get:

$$x_1 = 2, \ x_2 = 6$$

⇒ The graph intersects x-axis at (2, 0) and (6, 0).

To find y-intercept, put $x = 0$ in the given function which gives $y = 12$.

Therefore y-intercept is (0, 12).



Vertex: x-coordinate of vertex = $\frac{x_1+x_2}{2} = \frac{2+6}{2} = 4$

Substituting, $x = 4$ in given function, we get:

y-coordinate of vertex = $4^2 - 8(4) + 12 = -4$

∴ Vertex = $(4, -4)$

As $a = 1 > 0$, therefore parabola opens upward.

The graph is symmetric about $x = 4$. The sketch of the graph is shown in the figure.

Example 11: Draw the graph of $y = -x^2 + 4x - 4$ using factors.

Solution:

Given function is: $y = -x^2 + 4x - 4$

To get x-intercepts, put:

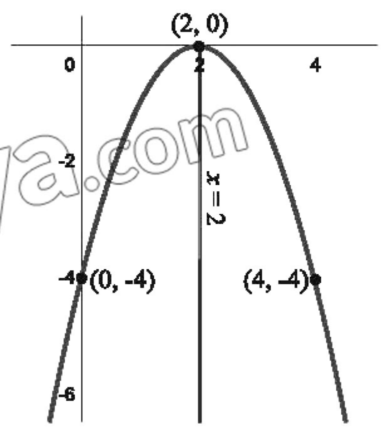
$-x^2 + 4x - 4 = 0 \Rightarrow -(x^2 - 4x + 4) = 0 \Rightarrow x^2 - 4x + 4 = 0$

After factorization, we get repeated roots:

$$x_1 = 2, \ x_2 = 2$$

This shows that the graph intersects x-axis at (2, 0), which is the vertex of the parabola.

To find y-intercept, put $x = 0$ in the given function which gives $y = -4$.



Therefore y-intercept is $(0, -4)$.
$a = -1 < 0$, shows that the parabola opens downwards. As the graph is symmetric about $x = 4$, therefore parabola also passes through $(4, -4)$. The sketch of the graph is shown in the figure.

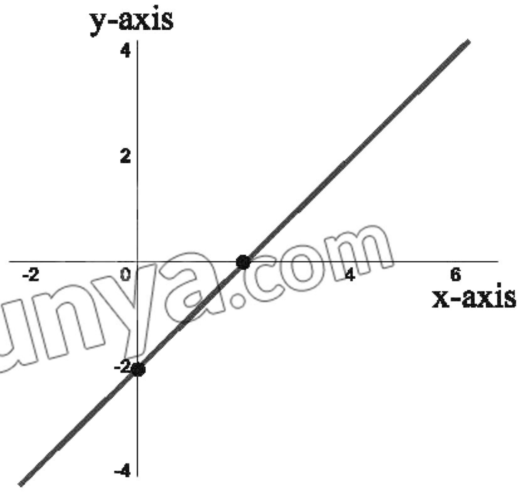
Check Point

Draw the graph of $y = 2x^2 + 6x + 4$ using factors.

1.7 Predicting Functions from Their Graphs

The method is explained with the help of following examples.

Example 12: Predict function from the graph.



Solution:

The graph shows a line passing through points $(2, 0)$ and $(0, -2)$.

Slope of the line is:

$m = \frac{0+2}{2-0} = 1$

Equation of the line is:

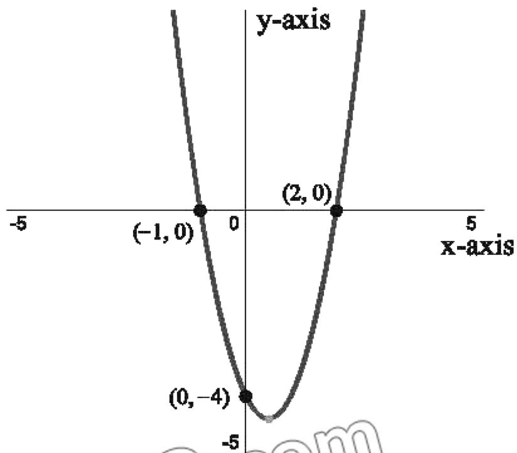
$y - 0 = 1(x - 2)$ (Point slope form of the line)

$y = x - 2$

Which is the required function.

Example 13: Predict function from the graph.

Solution:



The graph shows a parabola passing through points $(2, 0)$ and $(-1, 0)$ and $(0, -4)$. We know that the equation of the parabola is quadratic.

Now, the equation of the parabola passing through $(p, 0)$ and $(q, 0)$ is of the form:

$y = a(x - p)(x - q)$ where $p = 2, q = -1$ and $a > 0$ as the parabola opens upwards.

Substituting the values of p and q in the above equation, we get:

$y = a(x - 2)(x + 1)$ (i)

Since the parabola passes through $(0, -4)$, therefore from (i), we have:

$-4 = a(0 - 2)(0 + 1) \Rightarrow -4 = a(-2) \Rightarrow a = 2$

Therefore from (i):

$y = 2(x - 2)(x + 1) \Rightarrow y = 2(x^2 - x - 2)$

Which is the required function.

Key Facts

Equation of parabolic function passing through (p, r) and (q, r) is:
$y - r = a(x - p)(x - q)$ where p, q and r are positive.

1.8 Graph of Modulus Functions

A modulus function (absolute valued function) determines a number's magnitude regardless of its sign. If x is a real number, then the modulus function is denoted by:

$$y = |x| \quad \text{or} \quad f(x) = |x| \text{ where } x \in \mathrm{R}.$$

The modulus function takes the actual value of x if it is more than or equal to 0 and the function takes the minus of the actual value x if it is less than 0.

1.8.1 Domain and Range of Modulus Function

The domain of modulus function is R while its range is the set of non-negative real numbers, denoted as $[0, \infty)$. Any real number can be modulated using the modulus function.

Example: Consider the modulus function $f(x) = |x|$. Then:

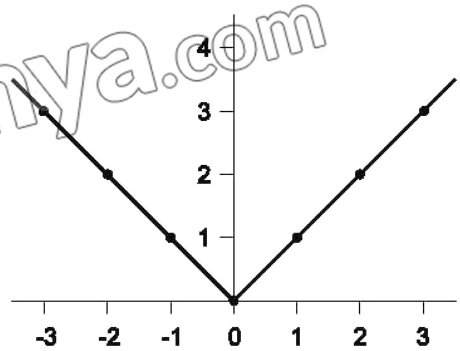
- If $x = -5$, then $y = f(x) = -(-5) = 5$, since x is less than zero.
- If $x = 6$, then $y = f(x) = 6$, since x is greater than zero.
- If $x = 0$, then $y = f(x) = 0$, since x is equal to zero.

1.8.2 Graph of Modulus Function

Consider the modulus function $f(x) = |x|$

Table shows the values of $f(x)$ below:

x	-3	-2	-1	0	1	2	3
$f(x)$	3	2	1	0	1	2	3



It can be inferred that for all possible values of x, the function $f(x)$ remains positive.

1.9 Finding the Intersecting Points Graphically

1.9.1 Intersection Point between a Linear Function and Coordinate Axes

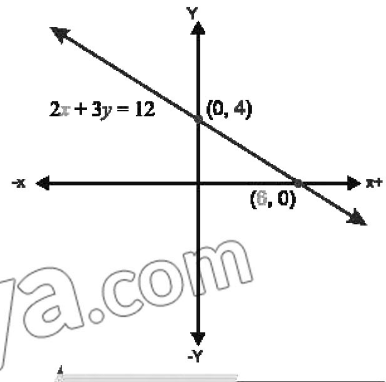
As we know that the graph of a linear equation is a straight line and the points of intersection of the line with axes are called intercepts.

Example 14:

Find x-intercept and y-intercept of the function: $f(x) = \frac{12-2x}{3}$

Solution:

x	6	0	3
$f(x)$	0	4	2



The graph of line $f(x) = \frac{12-2x}{3}$ is shown in the adjoining figure. From the graph it is clear that:

- The line crosses the x-axis at (6, 0). So, its x-intercept is 6.
- The line crosses the y-axis at (0, 4). So, its y-intercept is 4.

Check Point

x and y-intercepts of a line are -3 and -5 respectively. Find points of intersections of line with axes.

1.9.2 Intersection Point between two Linear Functions

While solving simultaneous linear functions graphically, keep in mind the following points.

1. Draw each linear function on the same set of axes.
2. Find the coordinates where the lines intersect.

Example 15: Find the graphical solution of $f(x) = \frac{6-x}{2}$ and $g(x) = x - 3$.

Solution:

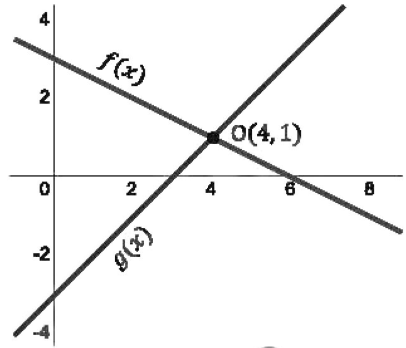


Table of values of function $f(x) = \frac{6-x}{2}$ is:

x	6	0	4
$f(x)$	0	3	1

Table of values of function $g(x) = x - 3$ is:

x	3	0	5
$g(x)$	0	-3	2

The graph of functions $f(x) = \frac{6-x}{2}$ and $g(x) = x - 3$ is shown in the adjoining figure.

From the graph it is clear that the both linear functions intersect each other at point O(4, 1).

Therefore, point O(4, 1) is the graphical solution of the given linear functions.

1.9.3 Intersection Point between a Linear Function and a Quadratic Function

As we know that the graph of a quadratic equation is a curve. The point of intersection of a linear function and a quadratic function is a point where both the graphs intersect each other.

Example 16: Solve $f(x) = 3x + 4$ and $g(x) = 5 + 3x - 2x^2$ graphically.

Solution:

Table of values for $f(x) = 3x + 4$ is:

x	0	-1	1
$f(x)$	4	1	7

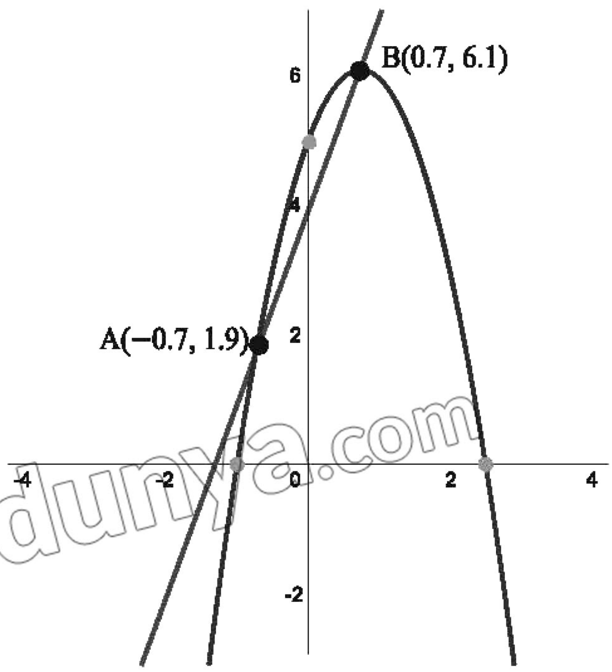
Comparing the graph of $g(x) = 5 + 3x - 2x^2$, with $y = ax^2 + bx + c$, we have:

$a = -2, b = 3$ and $c = 5$

Here, $a = -2 < 0$, so the curve will open downward.

Table of values for $g(x) = 5 + 3x - 2x^2$

x	-2	-1	0	1	2	3
$g(x)$	-9	0	5	6	3	-4



Both the graphs intersect each other at:

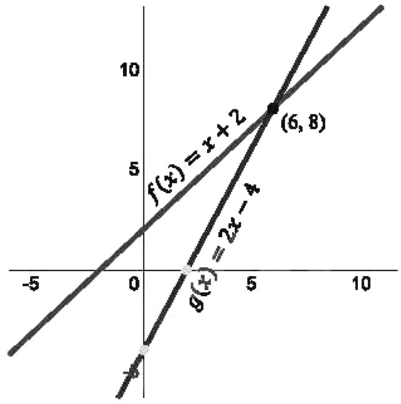
A(−0.7, 1.9) and B(0.7, 6.1).

Hence, solution set is: {(−0.7, 1.9), (0.7, 6.1)}

1.9.4 Solving Problems Graphically from Daily Life

See the following example to understand the method.

Example 17: Two airplanes are moving along the paths representing $f(x) = x + 2$ and $g(x) = 2x - 4$ respectively. Draw the graph of paths of both planes and find the point, where both planes pass.



Solution:

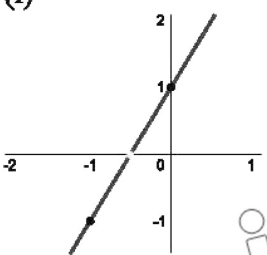
In the figure, red line represents the path of first plane while blue line represents the path of second plane.

The graph shows that both planes pass through the point (6, 8).

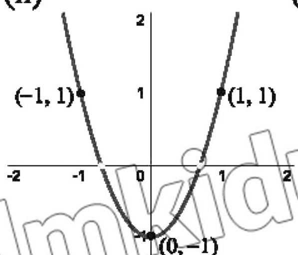
Exercise 1.2

1. Plot the graph of the functions:
   (i) $f(x) = 3x - 2$ (ii) $f(x) = 3x$ (iii) $f(x) = 1 - 2x$
   (iv) $g(x) = x^2 + 4$ (v) $g(x) = x^2 - x - 6$ (vi) $g(x) = \sqrt{2x + 1}$
2. Plot the graph of following functions.
   (i) $f(x) = -x^2 + 1$ (ii) $f(x) = 2x^3$ (iii) $f(x) = 1 + x^{-2}$
   (iv) $f(x) = 3x^{\frac{1}{2}}$ (v) $f(x) = 2 - x^{-\frac{1}{2}}$ (vi) $f(x) = x^{\frac{5}{2}}$
3. Find possible x-intercept, y-intercept and vertex of the following functions and then plot.
   (i) $f(x) = x^2 + 2x + 1$ (ii) $f(x) = -2x^2 + 2x - 1$
   (iii) $f(x) = x^2 + 2x$ (iv) $f(x) = 9 - x^2$
4. Draw the graph of following function using factors.
   (i) $f(x) = x^2 - 2x + 1$ (ii) $f(x) = x^2 - 7x + 12$
   (iii) $f(x) = x^2 - 2x$ (iv) $f(x) = -2x^2 + x + 3$
   (v) $f(x) = 4x^2 - 4x$ (iv) $f(x) = 6 - x^2 - x$
5. Predict algebraic functions from the following graphs.

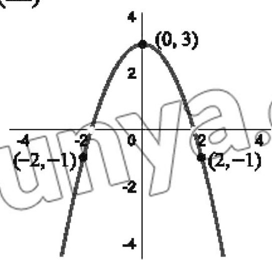
   (i)



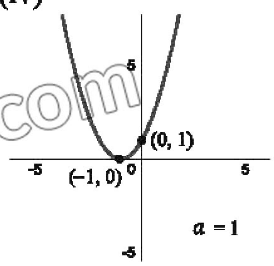
   (ii)



   (iii)



   (iv)



6. Plot the graph of following and find point of intersection of function with axes.
   (i) $y = x + 3$ (ii) $y = 6 - 3x$ (iii) $y = x^2 - 5x$

7. Find graphical solution of:
   (i) $f(x) = 4 - 3x,\ g(x) = -x + 1$ (ii) $f(x) = 2(2 + x),\ g(x) = x^2 + 1$
   (iii) $f(x) = 5 + 3x,\ g(x) = -x^2 + 5$ (iv) $f(x) = 1,\ g(x) = -2x^2 + 2x + 5$
   (v) $f(x) = 2 + 3x + x^2,\ g(x) = 5 + 3x - 2x^2$
8. Draw the graph of following modulus functions.
   (i) $f(x) = -1.5|x|$ (ii) $f(x) = 1 + 2|x|$ (iii) $f(x) = 3|x| + x$
9. The equations for supply and demand are given by two linear equations:
   Supply equation: $S(x) = 2x + 10$; where x is quantity and $S(x)$ is the price.
   Demand equation: $D(x) = -3x + 40$; where x is quantity and $D(x)$ is the price.
   Find the equilibrium point where the price of supply equals the price of demand by drawing the graphs of both equations.
10. Suppose a ball is thrown into the air and its height h(t) after t seconds is given by the parabolic trajectory: $h(t) = -6t^2 + 10t + 5$. If this ball hits a wall 10 m high representing the equation: $h(t) = 9t$. By drawing the graphs, find out when and where the ball reaches the wall.
11. Two asteroids are following the parabolic paths represented by $f(x) = x^2 - 7x + 12$ and $g(x) = x(x - 3)$. By drawing the graphs of both trajectories, find out the place from where, both asteroids will pass.

1.10 Algebraic and Transcendental Functions

1.10.1 Algebraic Function

An algebraic function is a function that involves only algebraic operations. These operations include addition, subtraction, multiplication, division, and exponentiation.

Types of Algebraic Functions

Main types of algebraic functions are:

(i) Polynomial Functions

A function of the from $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_0, a_1, \ldots, a_n$ are constants and n is integer, is called a polynomial function. Some examples are:

- $f(x) = 3x + 7$ (linear function)
- $f(x) = x^2 - 2x + 5$ (quadratic function)
- $f(x) = x^3 - 7x + 7$ (cubic function)
- $f(x) = x^4 - 5x^2 + 2x - 8$ (biquadratic function)
- $f(x) = x^5 - 7x + 3$ (quintic function)

(ii) Rational Functions

A function that is composed of two functions and expressed in the form of a fraction is a rational function. If $f(x)$ is a rational function, then $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function. Some examples are:

$$f(x) = \frac{x-4}{2x+3}, \quad g(x) = \frac{7}{x^2 + 5x + 1}$$

(iii) Power Functions

The power functions are of the form $f(x) = k\,x^a$ where 'k' and 'a' are any real numbers. Since 'a' is a real number, the exponent can be either an integer or a rational number.

Some examples are:

$f(x) = x^2,\ f(x) = x^{-1}$ (reciprocal function) and $f(x) = \sqrt{x-2} = (x-2)^{\frac{1}{2}}$

Properties

- Algebraic functions are closed under addition, subtraction, multiplication, division and composition.
- Algebraic functions are easy to solve, differentiate and integrate.

Application

- Physics and Engineering: Simple mechanical system, the motion of objects under constant acceleration.
- Geometry: Many curves such as circles and ellipses.

1.10.2 Transcendental Functions

The functions which are not algebraic are called transcendental. These functions can only be expressed in terms of infinite series. Some examples are:

- Exponential functions: $f(x) = e^x,\ g(x) = a^{3x}$
- Logarithmic functions: $f(x) = log_a x,\ g(x) = lnx$; where base a is a positive constant.
- Trigonometric functions: $f(x) = \sin x,\ g(x) = \cos x,\ h(x) = \tan x$
- Inverse trigonometric functions: $f(x) = \sin^{-1}x,\ g(x) = \cos^{-1}x$
- Hyperbolic functions: $f(x) = \sinh x,\ g(x) = \cosh x,\ h(x) = \tanh x$
- Inverse hyperbolic functions: $f(x) = \sinh^{-1}x,\ g(x) = \cosh^{-1}x$
- Special functions: Bessel functions, Gamma functions, error functions etc.

Properties

- These functions are not expressible in terms of a finite combination of algebraic operation of addition, subtraction, division, multiplication, raising to a power and extracting a root.
- These functions often exhibit more complex behavior like periodicity (in the case of trigonometric functions) and rapid growth (in the case of exponential function).

Application

- Science and Engineering: Exponential and logarithmic functions are critical in modelling growth, decay and oscillation in natural systems.
- Signal processing: Trigonometric functions are fundamental in analysing waves, sounds and signals.
- Mathematical analysis: Many problems in calculus, differential equations and complex analysis involve transcendental functions.

1.10.3 Logarithmic Functions

Logarithmic functions form a fundamental class of transcendental functions. These functions are inverse of exponential functions. They play a crucial role in mathematics, science, engineering and many applied fields.

Definition: If you have an exponential function of the form $y = a^x$ where $a > 0$ and $a \neq 1$, then the logarithmic function is defined as:

$$x = \log_a(y)$$

Replacing y with x, we have:

$$y = \log_a(x)$$

Here, $\log_a(x)$ is read as logarithmic of x to the base a.

Base of the Logarithms

The base of logarithm determines its specific type. Some types are:

- Natural logarithm: It is written as $\log_e(x) = \ln x$ where $e = 2.71828\ldots$ is called Euler's number.
- Common logarithm: It is written as $\log_{10}(x)$ where $a = 10$.
- Binary logarithm: It is written as $\log_2(x)$ where $a = 2$.

Properties

- The logarithm is the inverse of exponential. If $y = a^x$, then $x = \log_a(y)$. This means $\log_a(a^x) = x$ and $a^{\log_a(y)} = y$.
- The domain of $\log_a(x)$ is $x > 0$ because we cannot take the logarithm of zero or a negative real number.

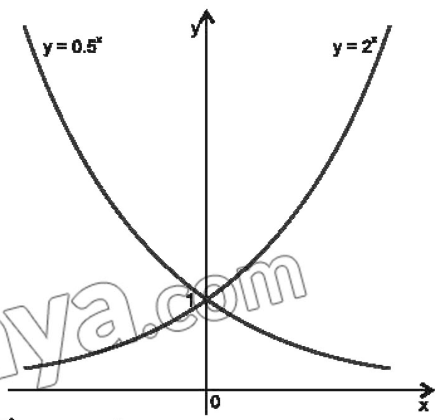
Laws of Logarithms

- Product Rule: $\log_a(xy) = \log_a(x) + \log_a(y)$
- Quotient Rule: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- Power Rule: $\log_a(x^n) = n\log_a(x)$
- Change of Base Rule: For any positive bases $a \neq 1$ and $b \neq 1$:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

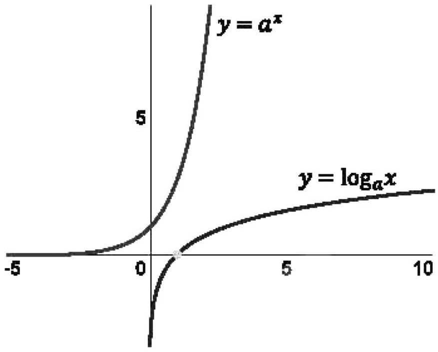
Note: As $a^x = 1$, therefore $\log_a(1) = 0$.

Graph of Exponential Function



- If the base, a is greater than 1, then the function increases exponentially at a growth rate of a. This is known as exponential growth.
- If the base, a is less than 1 (but greater than 0) the function decreases exponentially at a rate of a. This is known as exponential decay.
- If the base, a is equal to 1, then the function trivially becomes $y = 1$. This means exponential function always passes through (0, 1).
- The points (0, 1) and (1, a) are always on the graph of the function $y = a^x$.
- Exponential function takes only positive values and its graph never touches x-axis.
- The domain of the exponential function is the set of all real numbers, whereas the range of this function is the set of positive real numbers.

Graph of Logarithmic Function



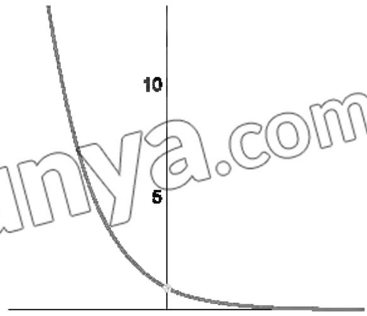
- When graphed, the logarithmic function is similar in shape to the square root function.
- The logarithmic function always passes through the point $(1, 0)$ because $\log_a(1) = 0$.
- The curve approaches to y-axis but never touches it.
- The domain of the logarithmic function is the set of all positive real numbers, whereas the range of this function is the set of all real numbers.
- For $a > 1$, the value of function increases as x increases.
- For $0 < a < 1$, the value of function decreases as x increases.

Example 18: Draw the graph of $f(x) = e^{-0.5x}$.

Solution:

Table of values for $f(x) = e^{-0.5x}$

x	-5	-2	-1	0	1	2	3
$g(x)$	12	2.7	1.6	1	0.6	0.4	0.2



Graph is shown in the adjoining figure.

Example 19: Draw the graph of:

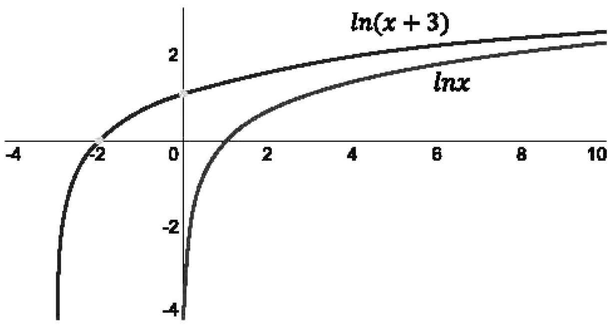
(i) $f(x) = lnx$ (ii) $g(x) = ln(x + 3)$

Solution:

Table of values for $f(x)$ and $g(x)$ is:

x	0	0.1	0.5	1	4	10
$f(x)$	-	-2.3	-0.7	0	1.4	2.3
$g(x)$	1.09	1.13	1.3	1.4	1.9	2.6

Graph of both functions is shown in the adjoining figure.



Applications

- Growth and Decay Model: Logarithm functions are used to model phenomenon that grow rapidly at first and then slow down such as population growth or the spread of deceases.
- pH Measurement in Chemistry: The pH of a solution is the logarithmic measure of the hydrogen in cm concentration.

$$\text{pH} = -\log_{10}[\text{H}^+]$$

- Sound Intensity (Decibels): The decibel scale which measures intensity, is a logarithmic scale:

$$\text{Decibels} = 10 \times \log_{10}\left(\frac{\text{I}}{\text{I}_0}\right)$$

here I is the intensity of the sound and I_0 is the reference intensity.

- Information Theory: Logarithms are used in information theory to measure information content and entropy.

- Financial Models: Logarithmic functions are used in finance particularly in modelling the time value of money and compound interest.
- Computer Science: Logarithmic functions appear in algorithms and data structures.

Conclusions

Logarithmic functions are powerful tools for dealing with exponential growth and decay as well as for measuring and comparing quantities on vastly different scales. Their unique properties and applications make them essential in both theoretical and applied fields. Most of the applications, we find, are in the fields of engineering and computer technology.

Example 20: Suppose that Rs. 30,000 is invested at 8% interest compounded annually. In t years, it will grow to the amount $A(t)$ given by the function: $A(t) = 30{,}000\ (1.08)^t$

(i) How long will it take until then is Rs. 150,000 in the account?

(ii) Let T be the amount of time it takes for the Rs.30,000 to double itself. Find T.

Solution:

(i) We set $A(t) = 150{,}000$ and solve for t.

$$150{,}000 = 30{,}000\ (1.08)^t \quad \Rightarrow \quad (1.08)^t = \frac{150{,}000}{30{,}000} = 5$$

Taking natural log on both sides, we get:

$$ln(1.08)^t = ln5 \quad \Rightarrow \quad t\, ln(1.08) = \ln 5$$

$$\Rightarrow \quad t = \frac{\ln 5}{\ln(1.08)} = \frac{1.6094}{0.07696} \approx 20.9$$

Therefore, it will take almost 20.9 years for Rs. 30,000 to grow to Rs. 150,000.

(ii) To find the doubling time T, we set A(t) = Rs. 60,000, t = T and solve for T.

$$60{,}000 = 30{,}000\ (1.08)^{\mathrm{T}} \quad \Rightarrow \quad (1.08)^{\mathrm{T}} = \frac{60{,}000}{30{,}000} = 2$$

Taking natural log on both sides, we get:

$$ln(1.08)^{\mathrm{T}} = ln2 \quad \Rightarrow \mathrm{T}\ ln(1.08) = ln2$$

$$\Rightarrow \quad \mathrm{T} = \frac{\ln 2}{\ln(1.08)} = \frac{0.6931}{0.07696} \approx 9$$

Therefore, doubling time is about 9 years.

Example 21: In 2020, the population of the country was 249 million and the exponential growth rate was 0.9% per year. If $P(t) = P_0e^{rt}$ is exponential growth function, then:

(i) Find the exponential growth function for the given data.

(ii) What would you expect the population to be in the year 2028?

Solution:

(i) Here $P_0 = 249,\ r = 9\% = 0.009$

The population growth function, gives:

$P(t) = 249 \times e^{0.009t}$ (a)

(ii) In 2028, we have $t = 8$.

To find the population in 2028, we substitute 8 for t in (a).

$$P(8) = 249 \times e^{0.009 \times 8} = 249 \times e^{0.072}$$

$$\approx 249 \times 1.0747 = 267.6$$

Therefore, population of the city in 2028, will be about 267.6 million.

Key Facts

The function $P(t) = P_0e^{rt}$ models the growth in the quantity while the function $P(t) = P_0e^{-rt}$ models the decay or decline in the quantity where $r > 0$.

Example 22: The radioactive element Carbon-14 has a half-life of 5750 years. The percentage of Carbon-14 present in the bones of dead animals can be used to determine the time of death of that animal. How old is the animal bone that has lost 40% of its Carbon-14?

Solution:

First of all, we find constant r using the concept of half-life. When $t = 5750$ (half-life), $P(t)$ will be half of P_0. That is $P(t) = 0.5P_0$

Therefore, the decay function $P(t) = P_0e^{-rt}$ implies:

$$0.5P_0 = P_0e^{-r \times 5750} \quad \Rightarrow \quad 0.5 = e^{-r \times 5750}$$

Taking natural log on both sides, we get:

$$\ln(0.5) = \ln e^{-r \times 5750} \quad \Rightarrow \quad \ln(0.5) = -r \times 5750$$

$$\Rightarrow \quad r = -\frac{\ln(0.5)}{5750} \approx 0.00012$$

We have the formula: $P(t) = P_0e^{-0.00012t}$

If an animal bone has lost 40% of its Carbon-14 from an initial amount P_0, then 60% of P_0 is the amount present. To find the age t of the bone, we solve the decay function for t.

$$0.6P_0 = P_0e^{-0.00012t} \qquad [P(t) = 60\% \text{ of } P_0 = P(t) = 0.6P_0]$$

$$0.6 = e^{-0.00012t} \quad \Rightarrow \quad \ln(0.6) = \ln e^{-0.00012t} \quad \Rightarrow \quad -0.5108 = -0.00012t$$

$$\Rightarrow \quad t = \frac{0.5108}{0.00012} = 4257$$

Therefore, the animal bone is about 4257 years old.

Exercise 1.3

1. Draw the graphs of functions.

   (i) $f(x) = e^{2x}$ (ii) $g(x) = e^{0.5x}$ (iii) $h(x) = 2 - e^x$

   (iv) $h(x) = 1 + e^{-2x}$ (v) $f(x) = \ln(2x)$ (vi) $g(x) = \log(x + 1)$

   (vii) $h(x) = 3 + \log(x)$ (viii) $f(x) = e^{0.6x}$ and $g(x) = \ln(0.6x)$

2. The number of compact discs N (in million) purchased each year increasing exponentially is given by:

   $$N(t) = 7.5(6)^{0.5t}$$

   Where $t = 0$ corresponds to 2024, $t = 1$ corresponds to 2025 and so on, t being the number of years after 2024.

   a. After what amount of time will one billion compact discs be sold in a year?

   b. What is the doubling time on the sale of compact discs?

3. Suppose that Rs. 50,000 is invested at 6% interest compounded annually. After t years, it grows to the amount A given by the function:

   $$A(t) = 50{,}000(1.06)^t$$

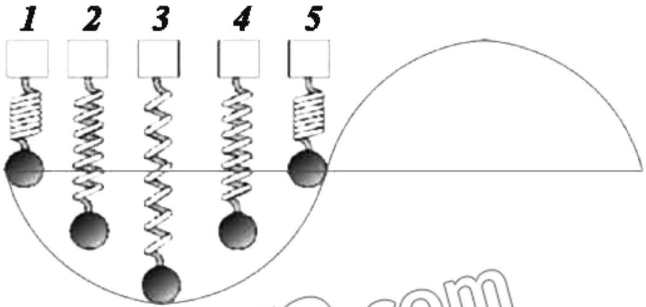
   a. After what amount of time will Rs. 50,000 grows to Rs. 450,000?

   b. Find the doubling time.

4. The exponential growth rate of the population of the city is 1% per year. After how many years, the population will be doubled?
5. The population of the world was 5.2 billion in 1990. The exponential growth rate was 1.6% per year at that time.
   a. Find the exponential growth function.
   b. Find the population of the world in 2000.
   c. In which year the world population was 8 billion?
6. Students in a mathematics class took a final exam in monthly intervals thereafter. The average score $S(t)$, after t months was given by:
$$S(t) = 68 - 20\log(t+1);\ \ t \geq 0$$
   a. What was the average score when they initially took the test ($t = 0$)?
   b. What was the average score (i) after 4 months (ii) after 24 months?
   c. Graph the function.
   d. After what time was the average score 50?
7. If $P(t) = P_0e^{kt}$ denotes the growth function of oil and the exponential growth rate of the demand for oil is 10% per year, when will the demand be doubled?
8. Approximately two third of all Aluminum cans distributed are recycled each year. A beverage company distributes 250,000 cans. The number still in use after t years is given by the function:
$$N(t) = 250{,}000\left(\frac{2}{3}\right)^t$$
   a. After how many years will 60,000 cans be in use?
   b. After what amount of time will only 1,000 cans be in use?

1.11 Domain and Range of Transcendental Functions through Graphs

If a weight is attached to a spring and the weight is pushed up or pulled down and released, it tends to rise and fall alternately. The weight is said to be oscillating in harmonic motion. If the position of the weight y is graphed over time the result is the graph of a sine or cosine curve.

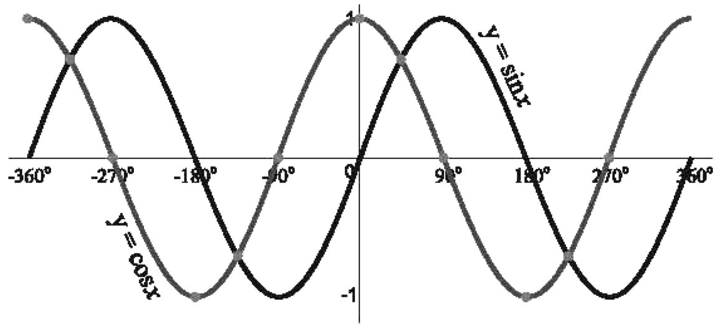


1.11.1 Graph of y = sinθ and y = cosθ

To graph the sine or cosine function, we use the horizontal axis for the values of θ expressed in either degrees or radians and vertical axis for the values of sinθ or cosθ. Ordered pairs for these points are of the form (θ, sinθ) or (θ, cosθ).

θ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
sinθ	0	0.3	0.5	0.7	0.87	0.97	1	0.97	0.87	0.7	0.5	0.3	0
cosθ	1	0.97	0.87	0.7	0.5	0.3	0	-0.3	-0.5	-0.7	-0.87	-0.97	-1

θ	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°
$\sin\theta$	- 0.3	-0.5	-0.7	-0.87	-0.97	-1	-0.97	-0.87	-0.7	-0.5	-0.3	0
$\cos\theta$	-0.97	-0.87	-0.7	-0.5	-0.3	0	0.3	0.5	0.7	0.87	0.97	1



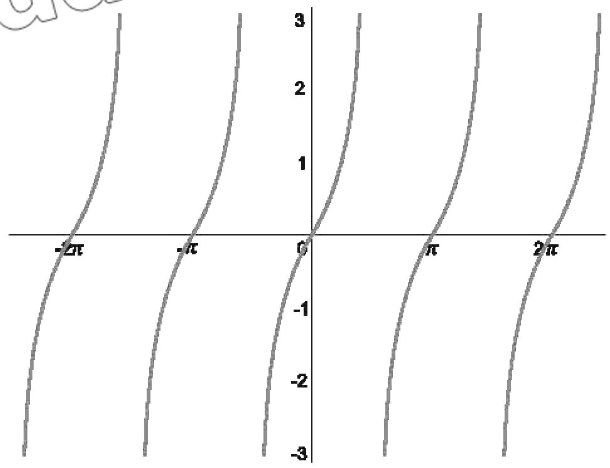
From the behavior of graphs of sine and cosine functions, we can easily predict the domain and range of both functions which are:

Function	Domain	Range
y = sinθ	$R = \theta \in (-\infty, \infty) = -\infty < \theta < \infty$	$y \in [-1, 1] = -1 \leq y \leq 1$
y = cosθ	$R = \theta \in (-\infty, \infty) = -\infty < \theta < \infty$	$y \in [-1, 1] = -1 \leq y \leq 1$

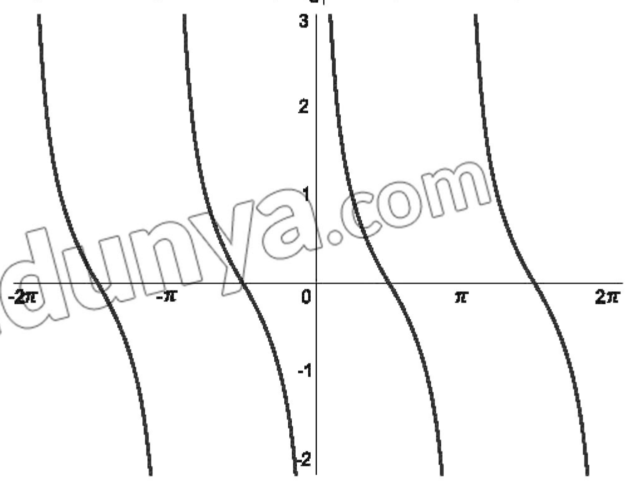
1.11.2 Graph of y = tanθ and y = cotθ

Similarly, by drawing the graph of y = tanθ and y = cotθ , we can easily predict the domain and range of both functions as follows.

Function	$y = \tan\theta$
Domain	$\theta \neq (2n+1)\dfrac{\pi}{2}; n \in Z$
Range	R

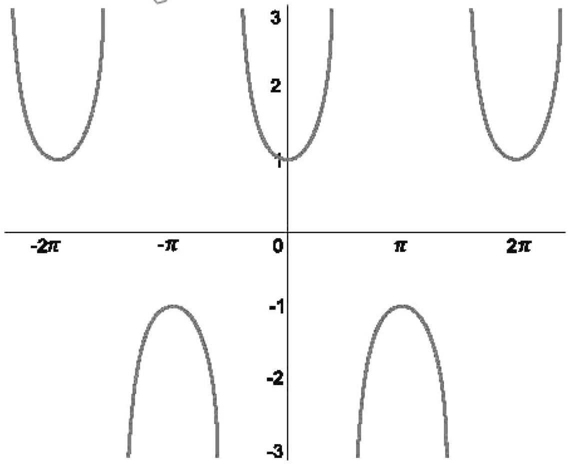


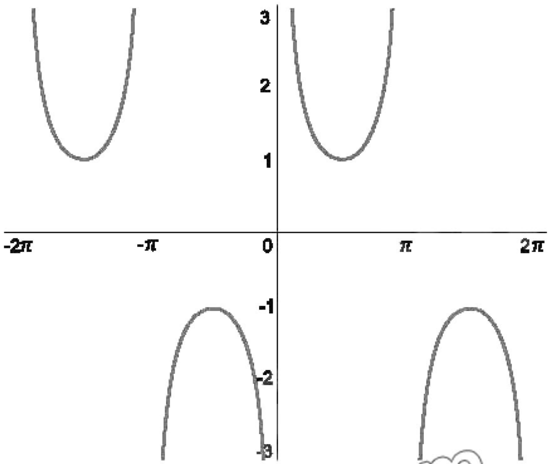
Function	$y = \cot\theta$
Domain	$\theta \neq n\pi; n \in Z$
Range	R



1.11.3 Graph of y = secθ and y = cosecθ

Domain and range of y = secθ and y = cosecθ is obvious form the graphs of both functions shown below.





Function	$y = \sec\theta$
Domain	$\theta \neq (2n+1)\frac{\pi}{2}; n \in Z$
Range	$y \leq -1, y \geq 1$ or $y \in (-\infty, -1] \cup [1, \infty)$

Function	$y = \csc\theta$
Domain	$\theta \neq n\pi; n \in Z$
Range	$y \leq -1, y \geq 1$ or $y \in (-\infty, -1] \cup [1, \infty)$

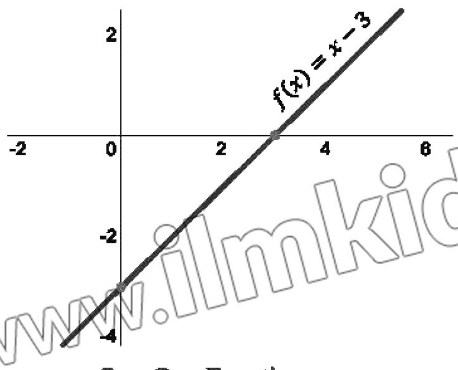
1.12 Relation Between a 1-1 Function and its Inverse through Graphs

1.12.1 One-One Function and its Graph

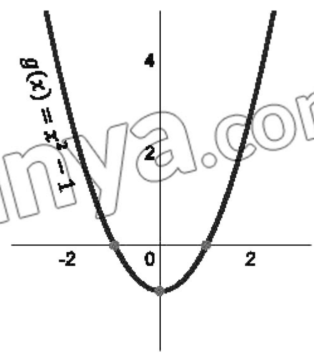
One to one function is a special function that maps every element of the range to exactly one element of its domain i.e., the outputs never repeat.

Examples: (i) The function $f(x) = x - 3$ is a one-to-one function since it produces a different answer for every input.

(ii) The function $g(x) = x^2 - 1$ is not a one-to-one function since it produces one output 0 for the two inputs 1 and -1.



One-One Function



Not a One-One Function

1.12.2 Horizontal Line Test

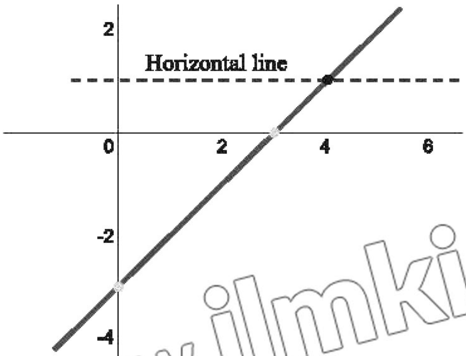
The horizontal line test is used to determine whether a function is one-one when its graph is given. To test whether the function is one-one from its graph just take a horizontal line (consider a horizontal stick) and make it pass through the graph.

- If the horizontal line does not pass through more than one point of the graph, then the function is one-one.
- If the horizontal line passes through more than one point of the graph, then the function is not one-one.

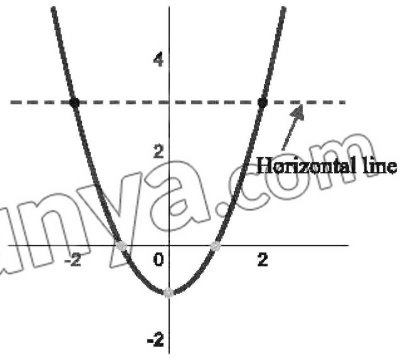
Examples: If we draw horizontal lines on the above graphs, we observe that:

(i) The graph of $f(x) = x - 3$ passes horizontal line test, so it is ono-one function.

(ii) The graph of $g(x) = x^2 - 1$ fails horizontal line test, so it is not ono-one function.



$f(x)$ is one-one function.



$g(x)$ is not a one-one function.

Check Point

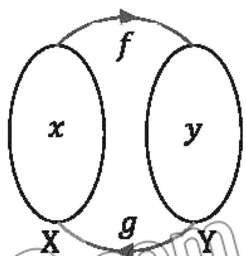
By using horizontal line test, check whether the function $y = x^3$ is 1-1 function or not.

1.12.3 Inverse of One-One Function

Suppose $f: X \to Y$ is a one-one function. Since every element y of Y corresponds with precisely one element x of X, the function f must determines a "reverse function" $g: Y \to X$ whose domain is Y and range is X. Then f and g imply that:

$f(x) = y$ and $g(y) = x$

or $f(g(y)) = y$ and $g(f(x)) = x$



The function g is given the formal name as "inverse of f".

From the above discussion it is clear that:

Dom f = Rang g and Rang f = Dom g

Definition:

Let f be a one-to-one function with domain X and range Y. The inverse of f is a function g with domain Y and range X for which:

$f(g(y)) = y$ for every y in Y and $g(f(x)) = x$ for every x in X.

Symbolically the inverse of a function f is denoted by f^{-1}. Thus, $g(x) = f^{-1}(x)$. It is to be noted that $f^{-1}(x)$ is not the same as $[f(x)]^{-1}$. In terms of this new notation, we have:

$$f\left(f^{-1}(x)\right) = f^{-1}(f(x)) = x$$

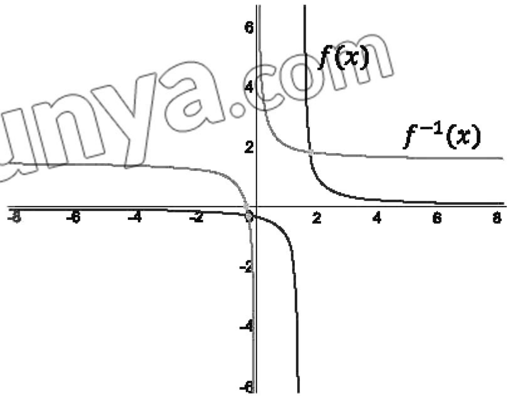
1.12.4 Properties of the Inverse of One to One Function

Here are the properties of the inverse of one to one function:

- The function f has an inverse function if and only if f is a one to one function.
- If the functions f and g are inverses of each other then, both these functions are one to one.
- f and g are inverses of each other if and only if $f(g(x)) = x, x$ in the domain of g and $g(f(x)) = x,\ x$ in the domain of f.
- If f and g are inverses of each other then the domain of f is equal to the range of g and the range of g is equal to the domain of f.
- If f and g are inverses of each other then their graphs will make reflections of each other on the line $y = x$.
- If the point (a, b) is on the graph of f then point (b, a) is on the graph of f^{-1}.

Example 23: Find the inverse of $f(x) = \frac{1}{2x-3}; x \neq \frac{3}{2}$, then represent f and f^{-1} graphically.

Solution: Given that $f(x) = \frac{1}{2x-3}; x \neq \frac{3}{2}$



Since f is a one to one function, therefore:

$f(f^{-1}(x)) = \frac{1}{2f^{-1}(x)-3}$ [Replacing x with $f^{-1}(x)$]

Solving for $f^{-1}(x)$, we get:

$\Rightarrow\ x = \frac{1}{2f^{-1}(x)-3} \quad \Rightarrow\ 2f^{-1}(x) - 3 = \frac{1}{x}$

$\Rightarrow\ 2f^{-1}(x) = \frac{1}{x} + 3 \quad \Rightarrow\ f^{-1}(x) = \frac{1+3x}{2x}$

Graph of function $f(x)$ and $f^{-1}(x)$ are shown in the adjoining figure. From the graph it is clear that if any point (a, b) is on the graph of $f(x)$ then point (b, a) is on the graph of $f^{-1}(x)$.

Challenge: Can you find inverse of $f(x)$ given in example 23, by any other method?

Example 24: Given that $f(x) = 3 - 4x$ is one to one. Find its inverse and represent f and f^{-1} graphically.

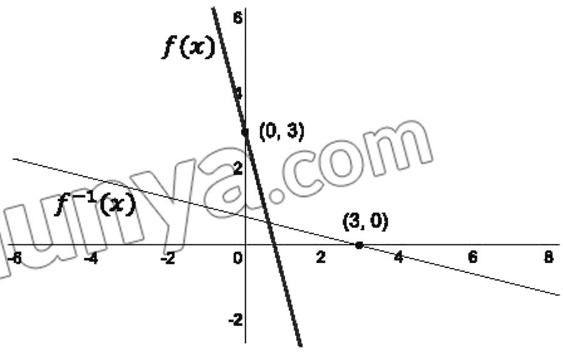
Solution: Given that $f(x) = 3 - 4x$ or $y = 3 - 4x$

Solving for x, we get:

$\Rightarrow\ 4x = 3 - y \quad \Rightarrow\ x = \frac{3-y}{4}$

$\Rightarrow\ f^{-1}(y) = \frac{3-y}{4}$

$\Rightarrow\ f^{-1}(x) = \frac{3-x}{4}$ [Replacing y with x.]



Graph of function $f(x)$ and $f^{-1}(x)$ are shown in the adjoining figure. From the graph it is clear that the point $(3, 0)$ is on the graph of $f(x)$ and the point $(0, 3)$ is on the graph of $f^{-1}(x)$. Therefore, both the graphs are reflections of each other.

Exercise 1.4

1. Find the domain and range of the functions graphically.
   (i) $f(x) = \sin\left(\frac{x}{2}\right)$ (ii) $g(x) = 3\cos\left(\frac{x}{3}\right)$ (iii) $h(x) = 2\tan x$
   (iv) $y = \cot\left(\frac{x}{4}\right)$ (v) $y = 2\sec(2x)$ (vi) $y = \sin(2x)$
2. Determine whether the given function is one to one by examining its graph. If the function is one to one, find its inverse. Also draw the graphs of inverse function.
   (i) $f(x) = \frac{1}{3}x + 3$ (ii) $g(x) = x(x-5)$ (iii) $h(x) = (x+1)^2$
   (iv) $f(x) = x^3 - 8$ (v) $g(x) = 4 \div x$ (vi) $h(x) = \frac{1}{3x+5}$
   (vii) $f(x) = x^4 + 2$ (viii) $g(x) = 5$ (ix) $h(x) = |x|$

1.13 Transformation of a Graph through Vertical Shift, Horizontal Shift and Scaling

1.13.1 Vertical and Horizontal Shift

A shift is a rigid translation as it does not change the shape or size of the graph of the function. A shift only changes the location of the graph.

Vertical Shift: A vertical shift adds/subtracts a positive constant to/from every y-coordinate while leaving the x-coordinate unchanged.

Horizontal Shift: A horizontal shift adds/subtracts a positive constant to/from every x-coordinate while leaving the y-coordinate unchanged.

Key Facts

Vertical and horizontal shifts can be combined into one expression.
Shifts are added/subtracted to the x or $f(x)$ components. If the positive constant is grouped with the x, then it is a horizontal shift, otherwise it is a vertical shift.

In this section, we will discuss the geometric effects on the graph of $y = f(x)$ by adding or subtracting a positive constant c to f or to its independent variable x.
The summary of vertical and horizontal shift is elaborated in the table 1.1 below.

Original function $y = f(x)$	Add a positive constant c to $f(x)$.	Subtract a positive constant c from $f(x)$.	Add a positive constant c to x.	Subtract a positive constant c from x.
$y = f(x)$	$y = f(x) + c$	$y = f(x) - c$	$y = f(x+c)$	$y = f(x-c)$
Geometric effects	Shifts the graph c units up.	Shifts the graph c units down.	Shifts the graph c units left.	Shifts the graph c units right.

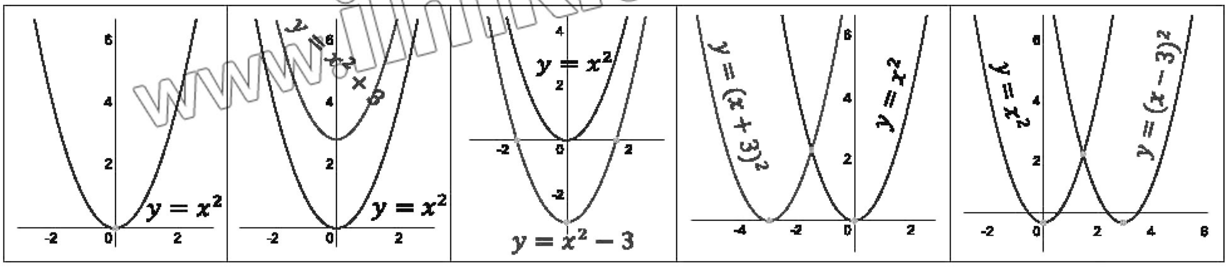
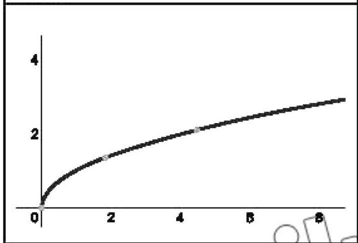
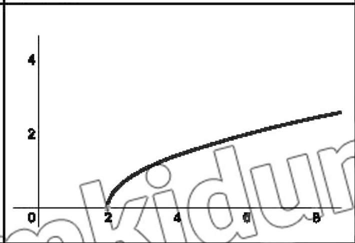
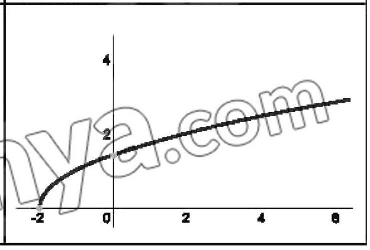


Table 1.1

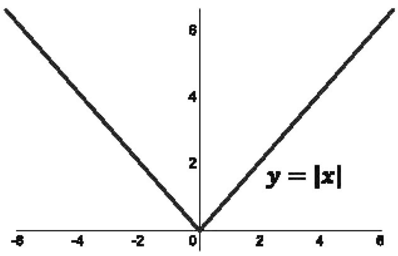
Example 25: Sketch the graph of (a) $y = \sqrt{x}$ (b) $y = \sqrt{x-2}$ (c) $y = \sqrt{x+2}$
Which kind of shift did you observe after sketching the graphs.
Solution:

(a) $y = \sqrt{x}$	(b) $y = \sqrt{x-2}$	(c) $y = \sqrt{x+2}$
		

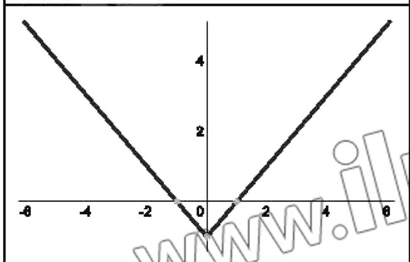
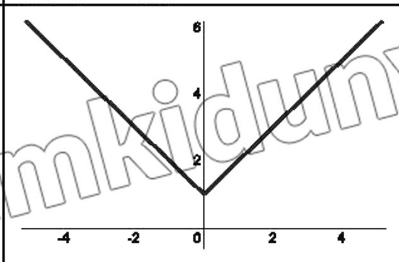
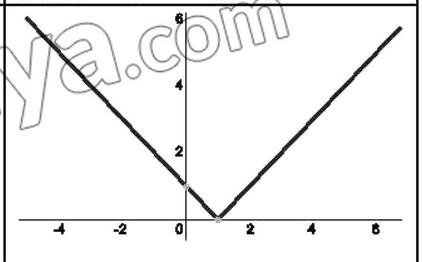
Above graphs show a horizontal shift. The graph of the function $y = \sqrt{x-2}$ can be obtained by transforming the graph of given function 2 units right to the origin while the graph of $y = \sqrt{x+2}$ can be obtained by transforming the graph of given function 2 units left to the origin.

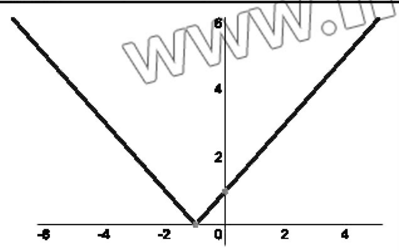
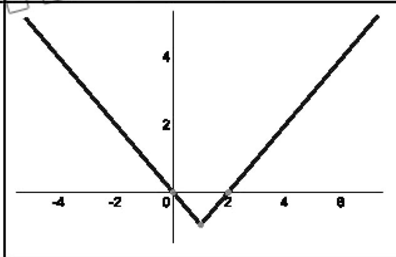
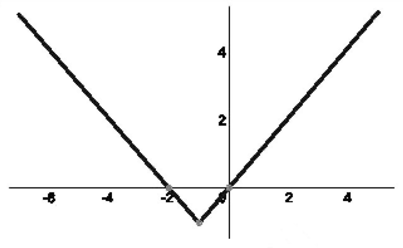
Example 26: Draw the graph of $y = |x|$ and then sketch the graphs of:
(a) $y = |x| - 1$ (b) $y = |x| + 1$ (c) $y = |x-1|$ (d) $y = |x+1|$
(e) $y = |x-1| - 1$ (f) $y = |x+1| - 1$
Which kind of shift did you observe after sketching the graphs.
Solution: Table of the values of the function $y = |x|$ is given as:

x	0	±1	±2	±3	±4	±5	±6
$f(x)$	0	1	2	3	4	5	6



The graph is shown in the adjoining figure.
Sketch of other graphs is shown in the table below.

(a) $y = |x| - 1$	(b) $y = |x| + 1$	(c) $y = |x-1|$
		
Vertical shift 1 unit down	Vertical shift 1 unit up	Horizontal shift 1 unit right

(d) $y = \|x+1\|$	(e) $y = \|x-1\| - 1$	(f) $y = \|x+1\| - 1$
		
Horizontal shift 1 unit left	Horizontal shift 1 unit right Vertical shift 1 unit down	Horizontal shift 1 unit left Vertical shift 1 unit down

1.13.2 Scaling (Stretching/Compressing)

Scaling is a non-rigid translation in which the shape and size of the graph of the function is altered. A scale will multiply/divide coordinates and this will change the appearance as well as the location.

Vertical Scaling: A vertical scaling multiplies/divides every y-coordinate by a constant while leaving the x-coordinate unchanged.

Horizontal Scaling: A horizontal scaling multiplies/divides every x-coordinate by a constant while leaving the y-coordinate unchanged.

Note: The vertical and horizontal scaling can be combined into one expression.

In this section, we will discuss the geometric effects on the graph of $y = f(x)$ by multiplying or dividing with a positive constant c to f or to its independent variable x.

The summary of vertical and horizontal scaling is elaborated in the tables 1.2 and 1.3 below.

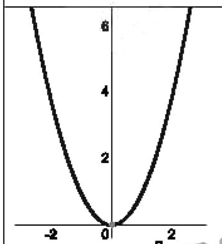
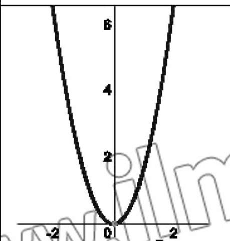
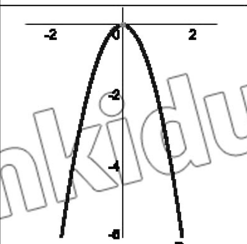
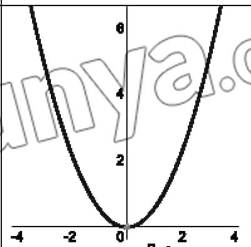
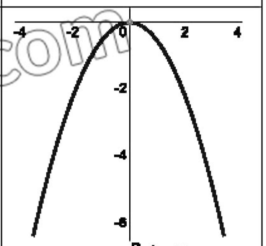
Original function $y = f(x)$	Multiply $f(x)$ by a positive constant c.	Multiply $f(x)$ by a negative constant c.	Divide $f(x)$ by a positive constant c.	Divide $f(x)$ by a negative constant c.
$y = f(x)$	$y = cf(x)$; $c > 0$	$y = cf(x)$; $c < 0$	$y = \dfrac{f(x)}{c}; c > 0$	$y = \dfrac{f(x)}{c}; c < 0$
Geometric effects	Figure is compressed by changing y-values by 2 in the same direction.	Figure is compressed by changing y-values by 2 in the opposite direction.	Figure is stretched by changing y-values by 2 in the same direction.	Figure is stretched by changing y-values by 2 in the opposite direction.
 $y = x^2$	 $y = 2x^2$	 $y = -2x^2$	 $y = x^2/2$	 $y = x^2/-2$

Table 1.2

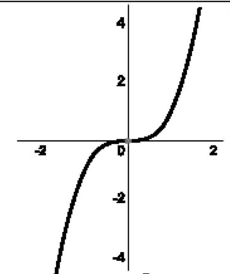
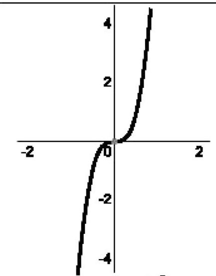
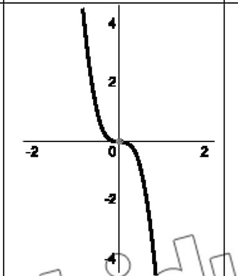
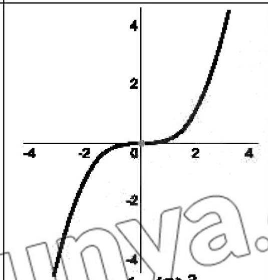
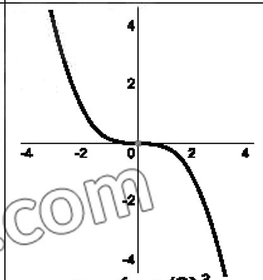
Original function $y = f(x)$	Multiply x by a positive constant c.	Multiply x by a negative constant c.	Divide x by a positive constant c.	Divide x by a negative constant c.
$y = f(x)$	$y = f(cx)$; $c > 0$	$y = f(cx)$; $c < 0$	$y = f(x/c)$; $c > 0$	$y = f(x/c)$; $c < 0$
Geometric effects	Figure is compressed by changing x-values by 2 in the same direction.	Figure is compressed by changing x-values by 2 in the opposite direction.	Figure is stretched by changing x-values by 2 in the same direction.	Figure is stretched by changing x-values by 2 in the opposite direction.
 $y = x^3$	 $y = (2x)^3$	 $y = (-2x)^3$	 $y = (x/2)^3$	 $y = (-x/2)^3$

Table 1.3

Example 27: Draw the graph of $y = |x|$ and then sketch the graphs of:

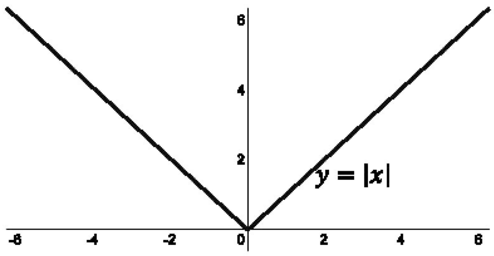
(a) $y = |1.5x|$ and $y = |-1.5x|$ (b) $y = |x| \div 1.5$ (c) $y = |x| \div (-1.5)$

Which kind of scaling did you observe after sketching the graphs.

Solution:

Table of the values of the function $y = |x|$ is given as:

x	0	± 1	± 2	± 3	± 4	± 5	± 6
$f(x)$	0	1	2	3	4	5	6



The graph is shown in the adjoining figure.

Sketch of other graphs is shown in the table below.

(a) $y = \|1.5x\| = \|-1.5x\|$	(b) $y = \|x\| \div 1.5$ and (c) $y = \|x\| \div (-1.5)$

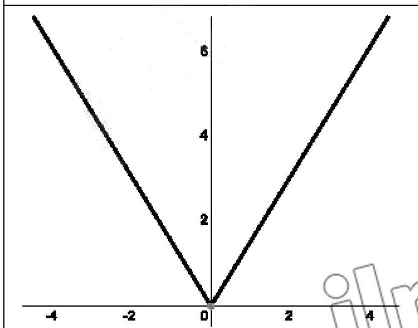
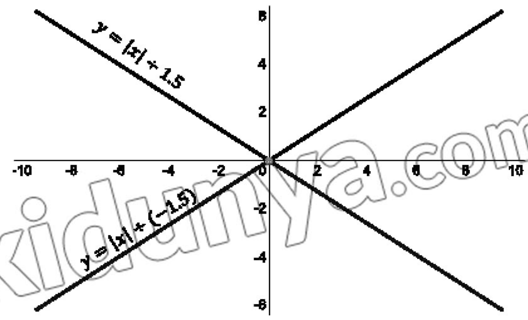

	

Figure is compressed by changing x-values by 1.5 in both cases.	Figure is stretched by changing y-values by 1.5 in both cases but with opposite behavior.

Exercise 1.5

Draw the graphs of the given functions and then sketch the graphs of other functions using translation. Verify the results using graphical calculator.

1. $y = |x|$ (a) $y = |x + 2|$ (b) $y = |x - 2|$ (c) $y = |x| + 2$ (d) $y = |x| - 2$
2. $y = x^2$ (a) $y = x^2 + 4$ (b) $y = x^2 - 4$ (c) $y = (x - 4)^2$ (d) $y = (x + 4)^2$
3. $y = \sqrt{x}$ (a) $y = \sqrt{x + 3}$ (b) $y = \sqrt{x - 3}$ (c) $y = \sqrt{x} + 3$ (d) $y = \sqrt{x} - 3$
4. $y = x$ (a) $y = x + 5$ (b) $y = x - 5$ (c) $y = 5x$ (d) $y = -5x$
5. $y = x^3$ (a) $y = x^3 + 1$ (b) $y = x^3 - 1$ (c) $y = (x - 1)^3$ (d) $y = (x + 1)^3$
6. $y = x^2 + 4$
   (a) $y = (x^2 + 4) - 3$ (b) $y = (x^2 + 4) + 3$
   (c) $y = (x - 3)^2 + 4$ (d) $y = (x + 3)^2 + 4$
7. $y = x^2$ (a) $y = 3x^2$ (b) $y = -3x^2$ (c) $y = \frac{x^2}{3}$ (d) $y = -\frac{x^2}{3}$
8. $y = x^2$ (a) $y = (3x)^2$ (b) $y = (-3x)^2$ (c) $y = \left(\frac{x}{3}\right)^2$ (d) $y = \left(-\frac{x}{3}\right)^2$
9. $y = \sqrt{x}$ (a) $y = \sqrt{2x}$ (b) $y = 2\sqrt{x}$ (c) $y = 2\sqrt{x} + 3$ (d) $y = \sqrt{2x + 5}$

Review Exercise

1. Tick the correct option in each of the following.
   (i) Which of the following is an example of exponential growth function?
   (a) $f(x) = 3x + 4$ (b) $f(x) = 3^x \times 5$ (c) $f(x) = x^3$ (d) $f(x) = x^2$
   (ii) The exponential decay function is expressed by:
   (a) $f(x) = a.b^x; 0 < b < 1$ (b) $f(x) = a.b^x; b > 1$
   (c) $f(x) = a.b^x; 0 < a < 1$ (d) $f(x) = a.b^x; a > 1$
   (iii) The logarithmic function $f(x) = \log_b x$ is defined for:
   (a) all real numbers (b) $x < 0$ (c) $x > 0$ (d) $x \geq 0$
   (iv) What is the value of $\log_5 125$?
   (a) 25 (b) 5 (c) 4 (d) 3
   (v) A function $f: A \longrightarrow B$ is said to be onto if:
   (a) Every element of the set A has a unique image in the set B.
   (b) Every element in the set B has a preimage in the set A.
   (c) Some elements of the set B have no preimage in the set A.
   (d) f is both one to one and onto.
   (vi) The function $f(x) = x + 1$, where $f: \{1, 2, 3\} \longrightarrow \{2, 3, 4\}$, is:
   (a) one to one but not onto (b) onto but not one to one
   (c) both one to one and onto (d) neither one to one nor onto

(vii) The function $f: R \longrightarrow [0, \infty)$ defined by $f(x) = x^2 + 1$, is:
(a) onto but not on to one (b) one to one but not onto
(c) neither one to one nor onto (d) both one to one and onto

(viii) A function $f: A \longrightarrow B$ has an inverse if and only if:
(a) f is one to one (b) f is onto
(c) f is both one to one and onto (d) f is neither one to one and onto

(ix) The inverse function of $f(x) = x^3$, is:
(a) $f^{-1}(x) = x^{-3}$ (b) $f^{-1}(x) = \sqrt{x^{-3}}$ (c) $f^{-1}(x) = \sqrt[3]{x^3}$ (d) $f^{-1}(x) = \sqrt[3]{x}$

(x) The function $f(x) = \sin x$, where $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$, is:
(a) one to one but not onto (b) onto but not one to one
(c) both one to one and onto (d) neither one to one nor onto

(xi) The inverse function of $f(x) = \frac{1}{x}; x \neq 0$, is:
(a) $f^{-1}(x) = 1$ (b) $f^{-1}(x) = -x$ (c) $f^{-1}(x) = x$ (d) $f^{-1}(x) = \frac{1}{x}$

(xii) Scaling refers to:
(a) increasing the size of an object.
(b) decreasing the size of an object.
(c) maintaining the properties while resizing an object.
(d) changing the shape of an object.

(xiii) Which of the following statements is true for the uniform scaling?
(a) both width and height change proportionally.
(b) only the width changes.
(c) only the height changes.
(d) width and height remain unchanged.

(xiv) What is the effect on the graph of $f(x)$ when it is replaced by $f(x + 2)$?
(a) It shifts 2 units to the right. (b) It shifts 2 units to the left.
(c) It shifts 2 units up (d) It shifts 2 unit down.

(xv) The domain of $y = \sin^{-1}(x)$, is:
(a) $[0, \infty)$ (b) $(-\infty, \infty)$ (c) $[-1, 1]$ (d) $[0, 1]$

2. Find the domain of the given functions.
(a) $f(x) = 4 + \sqrt{x + 2}$ (b) $f(x) = x\sqrt{2x - 3}$
(c) $f(x) = \frac{x}{x-2}$ (d) $f(x) = \sqrt{x^2 - 5x + 4}$

3. Find the domain and range of the given functions.
(a) $f(x) = 1 + x^2$ (b) $f(x) = (2x + 1)^2$
(c) $f(x) = 9 + \sqrt{x}$ (d) $f(x) = 3 + \sqrt{4 - x^2}$

4. Draw the graph of $f(x) = \sqrt{x}$, then sketch the graphs of the following functions.
   (a) $f(x) = \sqrt{x-2}$ (b) $f(x) = \sqrt{x} + 4$ (c) $f(x) = -\sqrt{x}$
   (d) $f(x) = 1 + \sqrt{x-2}$ (e) $f(x) = 4\sqrt{x}$ (f) $f(x) = -\frac{1}{3}\sqrt{x}$
5. Graph the given functions.
   (a) $y = 2 + 2\sin x$ (b) $y = -\frac{1}{2}\tan x$
   (c) $y = 3 - \operatorname{cosec} x$ (d) $y = \cos(x + \pi)$
6. Find the domain and range of the inverse function of $f(x) = \log(x^2 + 1)$.
7. Show that $f(g(x)) = g(f(x)) = x$, when:
   $f(x) = e^x$ and $g(x) = \ln x$.
8. The population of a town grows exponentially according to the formula $P(t) = 1000\, e^{0.05t}$ where t is the time in years. After how many years, will the population reach 5000?
9. A company has the following cost and revenue functions:
   $C(x) = 5x + 10$; where x is the number of units produced.
   $D(x) = 15x$; where x is the number of units sold.
   Find the equilibrium point where cost equals revenue.

UNIT
02

LIMIT, CONTINUITY AND DERIVATIVE

After studying this unit, students will be able to:

- Demonstrate and find the limit of a function.
- State and apply theorems on limit of sum, difference, product and quotient of functions to algebraic, exponential and trigonometric functions.
- Demonstrate and test continuity, discontinuity of a function at a point and in an interval.
- Apply concepts of transcendental functions, limit of a function and its continuity to real world problems.
- Calculate inflation over a period. Calculate depreciation with the help of straight-line method.
- Recognize the meaning of the tangent to a curve at a point.
- Calculate the gradient of a curve at a point. Identify the derivative as the limit of a difference quotient. Calculate the derivative of function. Estimate the derivative as rate of change of velocity, temperature and profit. Recognize the derivative function.
- State the connection between derivative and continuity.
- Find the derivative: function, square root, quadratic and logarithmic functions.
- Apply the differentiation rules to polynomials, rational and trigonometric functions.
- Apply the differentiation to state the increasing and decreasing function.
- Apply differentiation to real world problems.
- Find higher order derivatives of algebraic, implicit, parametric, trigonometric, inverse trigonometric functions. Describe the ability to approximate functions.
- Explain differentials to approximate the change in quantity. Calculate errors.
- Find extreme values by applying second derivative test. Explain and find critical point.
- Apply derivative and higher order derivative to real world problems.

The word calculus is a diminutive form of the Latin word calx, which means stone. In ancient civilization, small stones or pebbles were often used as a means for reckoning consequently, the word calculus can refer to any systematic method of computation. However, over the last. several hundred years, a definition of calculus means that the branch of mathematics concerned with the calculation and application of entities known as derivatives and integrals.

2.1 Limits of Functions

Two of the most fundamental concepts in the study of calculus are the notions of function and the limit of the function. In this first section, we shall be especially interested in determining whether the values $f(x)$ of a function f approach a fixed number L as x approaches a number 'a' using the symbol '$\rightarrow$' for the word **'approach'** we ask $f(x)\rightarrow L$ as $x\rightarrow a$.

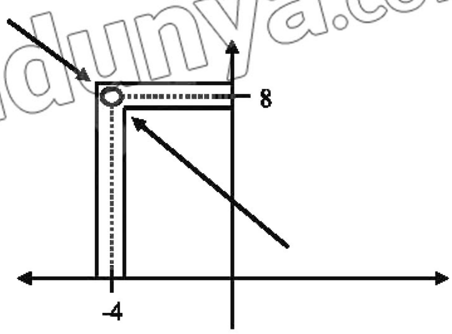
2.1.1 Limit of a Function as x Approaches to a Number

Consider a function: $f(x)=\dfrac{16-x^2}{4+x}$

Whose domain is set of all real numbers except -4. Although $f(-4)$ is not defined, nonetheless, $f(x)$ can be calculated for any value of x near -4. The table shows that, as x approaches to -4 from either the left or right, the functional values $f(x)$ approaches to 8. That is, when x is near -4, $f(x)$ is near 8. We say 8 is the limit of $f(x)$ as x approaches to -4. We can write as:

$$f(x)\rightarrow 8 \text{ as } x\rightarrow -4 \text{ or } \lim_{x\to -4}\frac{16-x^2}{4+x}=8$$

x	$f(x)$
-4.1	8.1
-4.01	8.01
-4.001	8.001
-3.9	7.9
-3.99	7.99
-3.999	7.999



For $x\neq -4$, f can be simplified by cancellation $f(x)=\dfrac{16-x^2}{4+x}=\dfrac{(4+x)(4-x)}{4+x}=4-x$.

The graph of f is essentially the graph of $y=4-x$ with the exception that the graph of f has a hole at the point that corresponds to $x=-4$. As x get closer and closer to -4, represented by the two arrowheads on the x-axis. The two arrowheads on the y-axis simultaneously get closer and closer to the number 8.

Intuitive Definition: If $f(x)$ can be made arbitrarily closer to a finite number by taking x sufficiently close to but different from a number a, from both the left and right side of a, then $\lim_{x\to a} f(x)=L$

$x\rightarrow a^-$ denote that x approaches a from the left and $x\rightarrow a^+$ denote that x approaches a from the right. Thus, if both sides have the common value L,

$$\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=L$$

We say that: $\lim_{x\to a} f(x)$ exist and write $\lim_{x\to a} f(x)=L$

Note: The existence of a limit of a function f at a does not depend on whether f is actually defined for a but only on whether f is defined for near a.

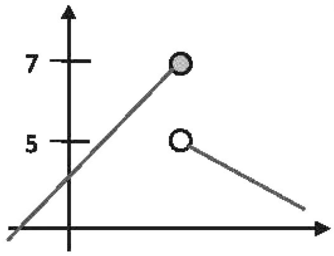
Example 1: Using the graph, check whether the limit of the function exists or not.

$$f(x) = \begin{cases} x+2 & x \le 5 \\ -x+10 & x > 5 \end{cases}$$

$\lim_{x\to 5^-} f(x) = x + 2 = 7$

$\lim_{x\to 5^+} f(x) = -x + 10 = -5 + 10 = 5$

$x \to 5^-$	$f(x)$
4.9	6.9
4.99	6.99
4.999	6.999



$x \to 5^+$	$f(x)$
5.1	4.9
5.01	4.99
5.001	4.999

Since $\lim_{x\to 5^-} f(x) \neq \lim_{x\to 5^+} f(x)$, we concluded that $\lim_{x\to 5} f(x)$ does not exist.

Example 2: Evaluate. a. $\lim_{x\to 0} \frac{\sin x}{x}$ b. $\lim_{x\to 0} \frac{1-\cos x}{x}$

Solution:

a.

$x \to 0^-$	$\frac{\sin x}{x}$
−0.1	0.998341
−0.01	0.9999833
−0.001	0.999998

a. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$x \to 0^+$	$\frac{\sin x}{x}$
0.1	0.998341
0.01	0.9999833
0.001	0.999998

b.

$x \to 0^-$	$\frac{1-\cos x}{x}$
−0.1	−0.0499583
−0.01	−0.0049999
−0.001	−0.0005001
−0.0001	−0.000510

$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

$x \to 0^+$	$\frac{1-\cos x}{x}$
0.1	0.0499583
0.01	0.0049999
0.001	0.0005001
0.0001	0.000510

Example 3: Evaluate: a. $\lim_{x\to 3} 15$ b. $\lim_{x\to 5} 10x$ c. $\lim_{x\to 5}(x^2 - 5x + 6)$

d. $\lim_{x\to -1} \frac{3x-1}{6x+2}$ e. $\lim_{x\to 1} \frac{x-1}{x^2+x-2}$ f. $\lim_{x\to 2}(3x-2)^6$

Solution:

a. $\lim_{x\to 3} 15 = 15$

b. $\lim_{x\to 5} 10x = 10(5) = 50$

c. $\lim_{x\to 5}(x^2-5x+6)=25-25+6=6$

d. $\lim_{x\to -1}\frac{3x-1}{6x+2}=\frac{3(-1)-1}{6(-1)+2}=\frac{-3-1}{-6+2}=1$

e. $\lim_{x\to 1}\frac{x-1}{x^2+x-2}$, $\lim_{x\to 1}x^2+x-2=0$

By simplifying first we can apply theorem v,

$$=\lim_{x\to 1}\frac{x-1}{x^2+x-2}=\lim_{x\to 1}\frac{(x-1)}{(x-1)(x+2)}$$

$$=\lim_{x\to 1}\frac{1}{(x+2)}=\frac{1}{3}$$

f. $\lim_{x\to 2}(3x-2)^6=(3(2)-2)^6=(4)^6=4096$

Theorems on Limits:

i. If c is constant, then $\lim_{x\to a}c=c$.

ii. If c is constant, then

$$\lim_{x\to a}cf(x)=c\lim_{x\to a}f(x)$$

iii. $\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$

$=L_1+L_2$

iv. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{L_1}{L_2}, L_2\neq 0$

v. $\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a}f(x)\right]^n=L^n$

Example 4: Evaluate: a. $\lim_{x\to 5}\frac{4x+5}{x^2-25}$ b. $\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10}$

Solution:

a.

$\lim_{x\to 5}\frac{4x+5}{x^2-25}$, $\lim_{x\to 5}4x+5=25$,

$\lim_{x\to 5}x^2-25=0$

$$\lim_{x\to 5}\frac{4x+5}{x^2-25}=\frac{25}{0}$$

We can't simplify to remove zero from the denominator, so limit $x\to 5$ doesn't exist.

b.

$\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10}$, $\lim_{x\to -8}2x+10=-6\neq 0$

$=\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10}$ (apply theorem iv)

$$=\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10}=\frac{\lim_{x\to -8}x-\sqrt[3]{x}}{\lim_{x\to -8}2x+10}=\frac{-8-(-8)^{1/3}}{2(-8)+10}$$

$$=\frac{-8-((-2)^3)^{1/3}}{-6}=\frac{-8+2}{-6}=1$$

Exercise 2.1

1. Use a graph to find the given limit, if it exists.

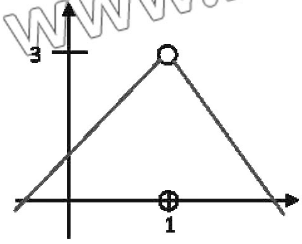
a. $\lim_{x\to 5}\sqrt{x-1}$ b. $\lim_{x\to 1}\frac{x^2-1}{x-1}$ c. $\lim_{x\to 0}\frac{x^2-3x}{x}$ d. $\lim_{x\to 0}\frac{|x|}{x}$

e. $\lim_{x\to 2}f(x)$, where $f(x)=\begin{cases}x & x<2\\ x+1 & x\geq 2\end{cases}$ f. $\lim_{x\to 0}f(x)$, where $f(x)=\begin{cases}x^2 & x<0\\ 2 & x=0\\ \sqrt{x}-1 & x>0\end{cases}$

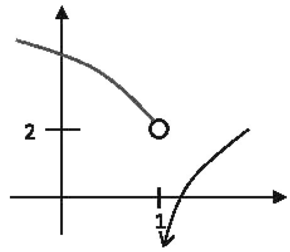
g. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

2. Use the given graph to find each limit $(x \to 1)$, if it exists.

a.



b.



Evaluate the following.

3. $\lim\limits_{x\to -2} \frac{x^3+8}{x+2}$

4. $\lim\limits_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

5. $\lim\limits_{x\to 7} \frac{x^2-21}{x+2}$

6. $\lim\limits_{x\to 0} \frac{x^2-6x}{x^2-7x+6}$

7. $\lim\limits_{y\to 1} \frac{y^3-1}{y-1}$

8. $\lim\limits_{x\to 3^+} \frac{(x+3)^2}{\sqrt{x-3}}$

9. $\lim\limits_{x\to 2}(x-4)^4(x^2-3)^{10}$

10. $\lim\limits_{x\to 0}\left(x-\frac{1}{x-2}\right)$

11. $\lim\limits_{x\to -3} \frac{2x+6}{4x^2-36}$

12. $\lim\limits_{x\to 0} \frac{\tan x}{x}$

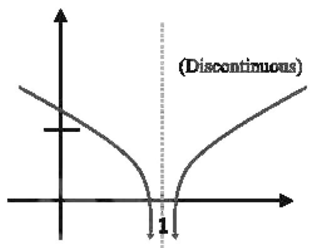
13. $\lim\limits_{x\to 0} \frac{x}{\sin 3x}$

2.2 Continuity

In the case of limit, we have used the phrase "connect the points with smooth curve". The phrase provides the concept of graph that is a nice continuous curve that is, a curve with no gaps or breaks. Indeed, a continuous function is often described as one whose graph can be drawn without lifting pencil from paper.

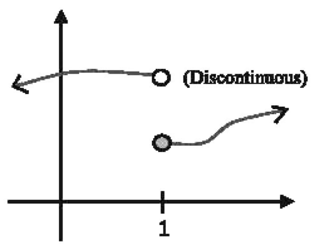
Before moving towards the precise definition of continuity, we demonstrate in figures some intuitive examples of functions that are not continuous or continuous at a number.

Fig (i)



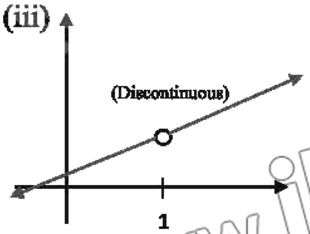
$\lim\limits_{x\to 1} f(x)$ does not exist and $f(1)$ is not defined.

Fig (ii)



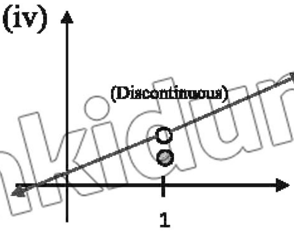
$\lim\limits_{x\to 1} f(x)$ does not exist and $f(1)$ is defined.

Fig (iii)



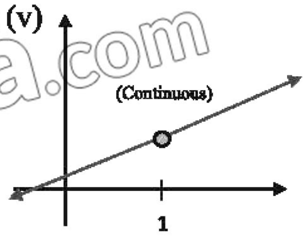
$\lim\limits_{x\to 1} f(x)$ exist but $f(1)$ is not defined.

Fig (iv)



$\lim\limits_{x\to 1} f(x)$ exist and $f(1)$ is defined but $\lim\limits_{x\to 1} f(x) \neq f(1)$

Fig (v)



$\lim\limits_{x\to 1} f(x)$ exist and $f(1)$ is defined and $\lim\limits_{x\to 1} f(x) = f(1)$

2.2.1 Continuity at a Number

Figures (i) - (v), at page 47, suggest the threefold conditions of continuity of a function at a number a (instead of 1 we consider a).

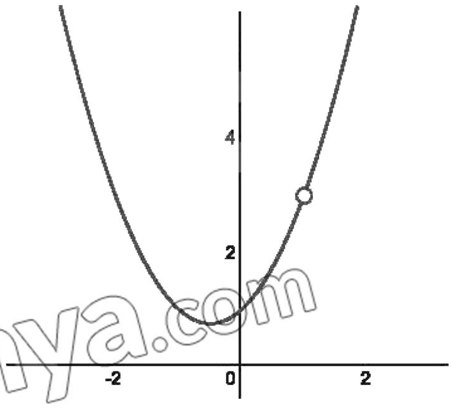
Definition: Continuity

A function is said to be continuous at a number a if

i. $f(a)$ is defined
ii. $\lim_{x\to a} f(x)$ exits, and
iii. $\lim_{x\to a} f(x) = f(a)$

Example 5: The rational function

$$f(x) = \frac{x^3-1}{x-1}$$
$$= \frac{(x-1)(x^2+x+1)}{x-1}$$
$$= x^2+x+1\,, x \neq 1$$

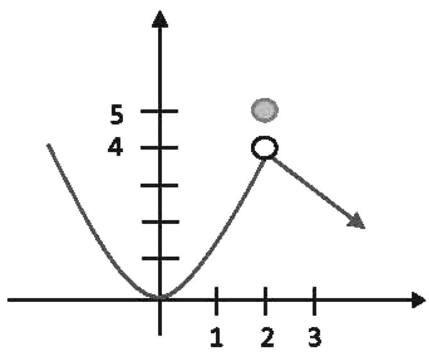


is discontinuous at 1 since $f(1)$ is not define.

From graph, we observe that $\lim_{x\to 1} f(x) = 3$. We can also state that f is continuous at any other number $x \neq 1$.

Example 6: Given figure shows the graph of the piecewise function defined

$$f(x) = \begin{cases} x^2 & x<2 \\ 5 & x=2 \\ -x+6 & x>2 \end{cases}$$



Now $f(2)$ is defined and is equal to 5. Next, we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 4$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -x+6 = 4$$

This implies limit exists: $\lim_{x\to 2} f(x) = 4$.

Since $\lim_{x\to 2} f(x) \neq f(2) = 5$, therefore f is discontinuous at 2.

Example 7: Let $f(x) = \frac{x^2+x-6}{x^2-4}$, for $x \neq 2$. Show how to define $f(2)$ in order to make f continuous function at 2.

Solution: Although $f(2)$ is not defined, if $x \neq 2$, we have

$$f(x) = \frac{x^2+x-6}{x^2-4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}$$

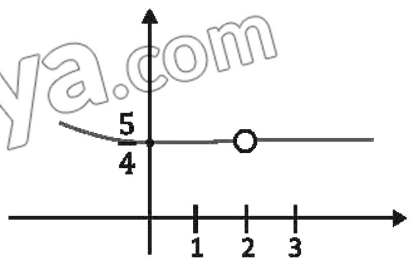


Fig (i)

The function $f(x) = \frac{x+3}{x+2}$ is equal to $f(x)$ for $x \neq 2$, but is also continuous at $x = 2$ having the value of $\frac{5}{4}$. Thus f is the continuous extension of f to $x = 2$ and

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-4} = \lim_{x\to 2}\frac{x+3}{x+2} = \frac{5}{4}$$

The graph of f is shown in figure (i).

The graph of its continuous extension is shown in figure (ii).

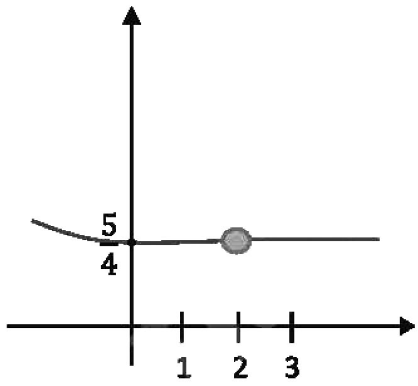


Fig (ii)

$$f(x) = \frac{x+3}{x+2} = \begin{cases} \frac{x^2+x-6}{x^2-4}, & x \neq 2 \\ \frac{5}{4}, & x = 2 \end{cases}$$

We can also observe that $x = 2$ is removable discontinuity for the $f(x) = \frac{x^2+x-6}{x^2-4}$.

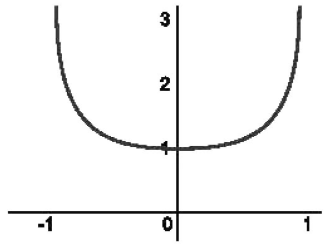
2.2.2 Continuity on an Interval

A function is said to be continuous on an open interval (a, b) if it is continuous at every number in the interval. A function f is continuous on a closed interval $[a, b]$ if it is continuous on $(a, b]$ and in addition, it is continuous on $[a, b)$

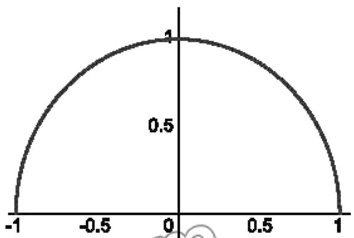
$$\lim_{x\to a^+} f(x) = f(a) \text{ and } \lim_{x\to b^-} f(x) = f(b)$$

Example 8:

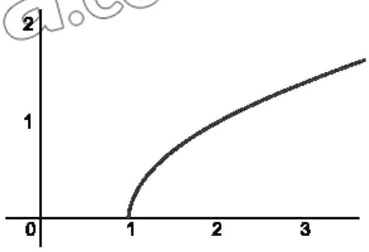
a. $f(x) = \frac{1}{\sqrt{1-x^2}}$ is continuous on the open interval $(-1, 1)$ but is not continuous on the closed interval $[-1, 1]$, since neither $f(-1)$ nor $f(1)$ is defined.



b. $f(x) = \sqrt{1-x^2}$ is continuous on $[-1, 1]$ we can observe from figure that $\lim_{x\to -1^+} f(x) = f(-1) = 0$ and $\lim_{x\to 1^-} f(x) = f(1) = 0$



c. $f(x) = \sqrt{x-1}$ is continuous on $[1, \infty)$ since $\lim_{x\to 1^+} f(x) = f(1) = 0$



Key Facts

Continuity of a Sum, Product and Quotient: If f and g are functions continuous at a number a, then cf(c a constant), $f+g, fg$ and $\frac{f}{g}$, $(g(a)\neq 0)$ are also continues at a.

Removable Discontinuity: If $\lim\limits_{x\to a} f(x)$ exists but f is either not defined at a or $f(a)\neq \lim\limits_{x\to a} f(x)$, then f is said to have a removable discontinuity at a. For example the function $\frac{x^2-1}{x-1}$ is not defined at 1 but $\lim\limits_{x\to 1} f(x)=2$. By definition $f(1)=2$, the new function

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1\\ 2, & x=1\end{cases}$$

is continuous at every number.

Exercise 2.2

In problems 1-8, determine the numbers (if any), at which the given function is discontinuous.

1. $f(x)=x^2-5x+6$
2. $f(x)=\frac{2x}{x^2+5}$
3. $f(x)=\frac{1}{x^2-9x+8}$
4. $f(x)=\frac{x^2-1}{x^4-1}$
5. $f(x)=\frac{x-1}{\sin 2x}$
6. $f(x)=\begin{cases}x, & x<0\\ x^2, & 0\leq x\leq 2\\ x, & x\geq 2\end{cases}$
7. $f(x)=\begin{cases}\frac{\sin x}{x}, & x\neq 0\\ \frac{1}{2}, & x=0\end{cases}$
8. $f(x)=\begin{cases}\frac{x^2-36}{x-6}, & x\neq 6\\ 12, & x=6\end{cases}$

In problems 9-14, determine whether the given function is continuous in the indicated intervals.

9. $f(x)=x^2+1$	a. $[-1,\ 3]$	b. $[3,\ \infty)$
10. $f(x)=\frac{1}{x}$	a. $(-3,\ 3)$	b. $(0,\ 10]$
11. $f(x)=\frac{1}{\sqrt{x}}$	a. $[1,\ 4)$	b. $[1,\ 9]$
12. $f(x)=\sqrt{x^2-9}$	a. $[-3,\ 3]$	b. $[3,\ \infty)$
13. $f(x)=\frac{x}{x^3+8}$	a. $[-4,\ -3]$	b. $[-10,\ 10]$
14. $f(x)=\sin\frac{1}{x}$	a. $[\frac{-1}{\pi},\ 5)$	b. $[\frac{\pi}{2},\frac{3\pi}{2}]$

In problems 15-18, find the values of m and n so that the given function is continuous.

15. $f(x)=\begin{cases}mx, & x<4\\ x^2, & x\geq 4\end{cases}$
16. $f(x)=\begin{cases}\frac{x^2-4}{x-2}, & x\neq 2\\ m, & x=2\end{cases}$
17. $f(x)=\begin{cases}mx, & x<3\\ n, & x=3\\ -2x+9, & x>3\end{cases}$
18. $f(x)=\begin{cases}mx-n, & x<1\\ 5, & x=1\\ 2mx+n, & x>1\end{cases}$

19. Prove that the equation $\frac{x^2+1}{x+3}+\frac{x^4+1}{x-4}=0$ has a solution in the interval $(-3,\ 4)$.

20. Prove that $f(x)=\begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

is discontinuous at every real number. What does the graph of f look like?

2.3 Rate of Change of Functions

2.3.1 Tangent of a Graph

Suppose $y=f(x)$ is a continuous function. In the figure(i), the graph of f possesses a tangent line L at a point P, and then we would like to find its equation.

To do so we need: (i) the coordinates of P and

(ii) the slope m_{tan} of L.

The coordinates of P pose no difficulty since a point on a graph is obtained by specifying a value of x, say $x=a$ in domain of f. The coordinates of point of tangency are $(a, f(a))$.

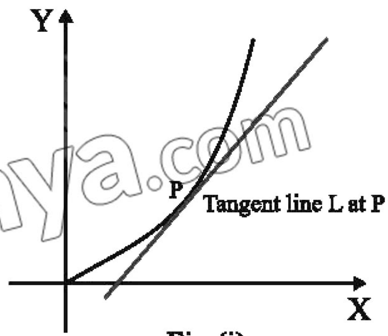


Fig (i)

As a means of approximating the slope m_{tan}, we find the slope of secant lines that pass through the fixed-point P and any other point Q on the graph.

If P has coordinates $(a, f(a))$ and if we let Q have coordinates $(a+\Delta x, f(a+\Delta x))$, then from fig (ii) the slope of the secant line through P and Q is

$$m_{sec}=\frac{\text{change in y-coordinate}}{\text{change in x-coordinate}}$$

$$=\frac{f(a+\Delta x)-f(a)}{(a+\Delta x)-a}=\frac{\Delta y}{\Delta x}$$

Then, $m_{sec}=\frac{\Delta y}{\Delta x}$

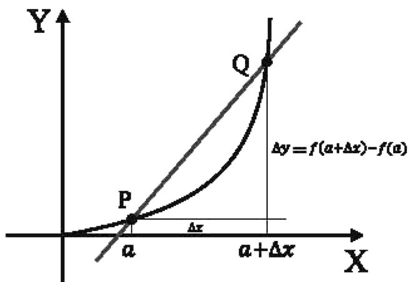


Fig (ii)

When the value of Δx is close to zero either positive or negative, we get points Q and Q' on the graph on each side of P, but close to the point P, we expect that the slopes m_{PQ} and $m_{PQ'}$ are very close to the slope of the tangent line L. See fig (iii)

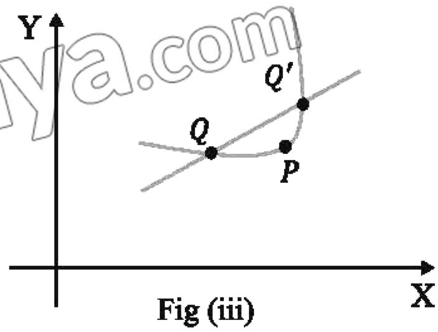


Fig (iii)

Definition: Tangent line

Let $y = f(x)$ be a continuous function. At a point (a, f(a)) the tangent line to the graph is the line that passes through the point with slope.

$$\text{Slope} = m_{tan} = \frac{f(a+\Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

whenever the limit exists.

The slope of the tangent line at $(a, f(a))$ is also called the slope of the curve at the point. The tangent at $(a, f(a))$ is unique since a point and a slope determine a single line.

Example 9: Use definition to find the slope of the tangent line to the graph of $f(x) = x^2$ at $(1, f(1))$.

Solution:

i. $f(1) = 1^2 = 1$ for any $\Delta x \neq 0$

$f(1+\Delta x) = (1+\Delta x)^2 = 1 + 2\Delta x + (\Delta x)^2$

ii. $\Delta y = f(1+\Delta x) - f(1)$

$= 1 + 2\Delta x + (\Delta x)^2 - 1 = 2\Delta x + (\Delta x)^2 = \Delta x(2+\Delta x)$

iii. $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta x(2+\Delta x)}{\Delta x} = 2 + \Delta x$

Slope of the tangent is given by:

iv. $m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 2 + \Delta x = 2$

We summarize the definition into 4 steps:

- Evaluate f at a and $a + \Delta x$: $f(a)$ and $f(a+\Delta x)$
- Find Δy
- Divide Δy by Δx, $\Delta x \neq 0$

$$\frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

- Compute $\lim_{\Delta x \to 0}$

$$m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Example 10: Find the slope of the tangent line to the graph $f(x) = -x^2 + 6x$ at $(4, f(4))$.

Solution:

i. $f(4) = -(4)^2 + 6(4) = 8$, for any $\Delta x \neq 0$

$f(4+\Delta x) = -(4+\Delta x)^2 + 6(4+\Delta x) = 8 - 2\Delta x - (\Delta x)^2$

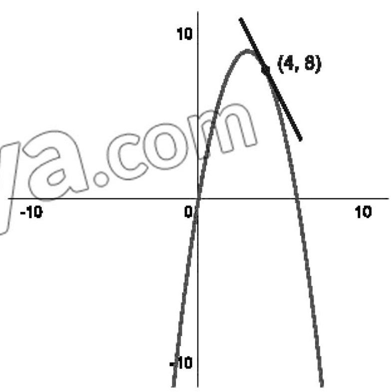
ii. $\Delta y = f(4+\Delta x) - f(4)$

$= 8 - 2\Delta x - (\Delta x)^2 - 8 = -2\Delta x - (\Delta x)^2 = \Delta x(-2-\Delta x)$

iii. $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta x(-2-\Delta x)}{\Delta x} = -2 - \Delta x$

Slope of the tangent is given by:

iv. $m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} -2 - \Delta x = -2$



From graph we observe that the slope of line is -2 at (4, 8).

Key Facts

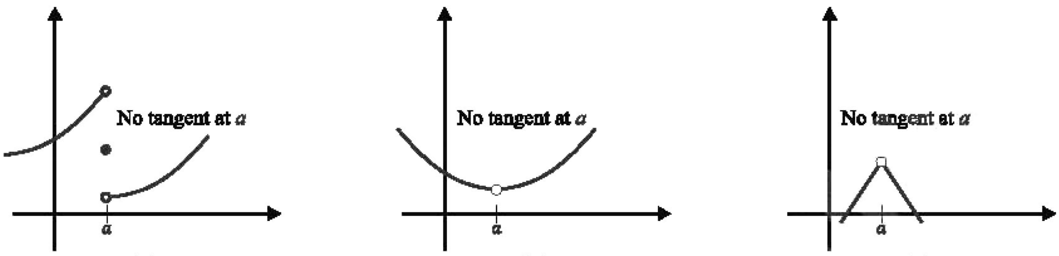
A Tangent May Not Exist: The graph of a function f will not have a tangent line at a point whenever.

i. f is discontinuous at $x = a$, or

ii. The graph of f has corner at $(a, f(a))$.

Moreover, the graph f may not have a tangent line at a point where

iii. The graph has a sharp peak.

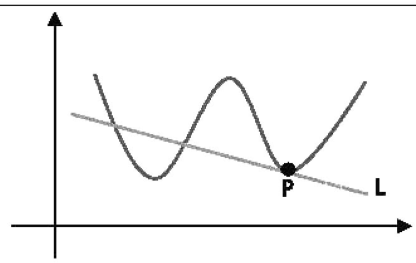


(a) (b) (c)

2.5.2 Rate of Change

The slope $\frac{\Delta y}{\Delta x}$ of a secant through $(a, f(a))$ is also called the average rate of change of f at a. The slope $m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is said to be the instantaneous rate of change of the functions at a, if $m_{tan} = \frac{1}{10}$ at a point $(a, f(a))$, we would not expect the values of f to change drastically for x values near a.

Remark: The line L is tangent at P but intersects the graph of f at three points. but is not tangent to the graph.



2.4 Instantaneous Velocity

Almost everyone has an intuitive notion of speed or velocity as a rate at which a distance is covered in a certain length of time. When, say, a bus travels 60 miles in one hour, the average velocity of the bus must have been 60 mil/hr. Of course, it is difficult to maintain the rate of 60 mil/hr for the entire trip because the bus slow down for towns and speeds up when it passes cars. In other words, the velocity changes with time. If a bus company's schedule demands that the bus travel the 60 miles from one town to another in one hour, the driver knows instinctively that he must compensate for velocities or speeds below 60 mil/hr by travelling at speeds greater than this at other points in journey. Knowing that the average velocity is 60 mil/hr doesn't, however, answer the questions, what is the velocity of the bus at a particular instant?

Average velocity: $V_{ave} = \dfrac{\text{distance travelled}}{\text{time of travel}}$

Consider a runner who finishes a 10 km race in an elapsed time of 1 hour and 15 min (1.25 hr). The runner's average velocity or average speed for the race was

$$V_{ave} = \frac{10}{1.25} = 8km/hr$$

But suppose we now wish to determine velocity at the instant the runner is one half hour into the race. If the distance run in the time interval from 0 hr to 0.5 hr is measured to be 5 km, then

$$V_{ave} = \frac{5}{0.5} = 10\ km/hr$$

Suppose if a runner's completes 5 km in 0.5 hr and 5.7 km in 0.6 hr, however, during the time interval from 0.5 hr to 0.6 hr

$$V_{ave} = \frac{5.7-5}{0.6-0.5} = 7\ km/hr$$

Definition: Instantaneous Velocity

Let $s = f(t)$ be a function that gives the position of an object moving in a straight line. The instantaneous velocity at time t_1 is

$$V(t_1) = \lim_{\Delta t\to 0}\frac{f(t_1+\Delta t)-f(t_1)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$$

whenever the limit exists.

Example 11: The height s above ground of a ball dropped from the top of the tower is given by $s = -4.9t^2 + 192$ where s is measured in meters and t in seconds. Find the instantaneous velocity of the falling ball at $t_1 = 3\ sec$.

Solution: We use the same four step procedure:

Step 1: $f(3) = -4.9(3)^2 + 192 = 147.9$ for any $\Delta t \neq 0$

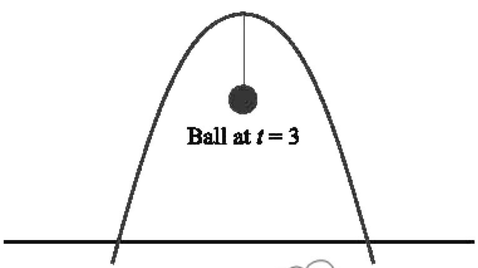
$$f(3+\Delta t) = -4.9(3+\Delta t)^2 + 192$$
$$= -4.9(\Delta t)^2 - 29.4\Delta t + 147.9$$

Step 2: $\Delta s = f(3+\Delta t) - f(3)$

$$= [-4.9(\Delta t)^2 - 29.4\Delta t + 147.9] - 147.9$$
$$= \Delta t[-4.9\Delta t - 29.4]$$

Step 3: $\frac{\Delta s}{\Delta t} = \frac{\Delta t(-4.9\Delta t - 29.4)}{\Delta t}$

$$= -4.9\Delta t - 29.4$$



Step 4: $v(3) = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0}(-4.9\Delta t - 29.4)$

$$= -29.4\ m/sec$$

The minus sign is significant because the ball is moving opposite to the positive or upward direction. The number $f(3) = 147.9\ m$ is the height of the ball above the ground at 3 seconds.

Exercise 2.3

In problem 1-6, find the slope of the tangent line to the graph of the given function at the indicated point.

1. $f(x) = 2x - 1\,;\ (x,\ f(x)) = (4,\ 7)$
2. $f(x) = -\frac{1}{2}x + 3\,;\ (a,\ f(a))$
3. $f(x) = x^2 + 4\,;\ (-1,\ 5)$
4. $f(x) = x^2 - 5x + 4\,;\ (2,\ -2)$
5. $f(x) = x^3\,;\ (1,\ f(1))$
6. $f(x) = \frac{1}{x}\,;\ \left(\frac{1}{3},\ f\left(\frac{1}{3}\right)\right)$

In problem 7-8, find the average rate of change of the given function in the indicated interval.

7. $f(x) = x^3 + 2x^2 - 4x\,;\ [-1,\ 2]$
8. $f(x) = cosx\,;\ [-\pi,\ \pi]$

In problem 9-10, find the instantaneous velocity of the particle at the indicated time.

9. $f(t) = -4t^2 + 10t + 6\,;\ t = 3$
10. $f(t) = t^2 + \frac{1}{5t+1}\,;\ t = 0$
11. The height above ground of a ball dropped from an initial altitude of 122.5 m is given by $s(t) = 122.5 - 4.9t^2$, where s is measured in meters and t in seconds.
    i. What is the instantaneous velocity at $= \frac{1}{2}$?
    ii. At what time does the ball hit the ground?
    iii. What is the impact velocity?
12. The height of a projectile shot from ground level is given by $s(t) = -16t^2 + 256t$, where s is measured in feet and t in seconds.
    i. Determine the height of the projectile at $t = 2,\ t = 6,\ t = 9$ and $t = 10$.
    ii. What is the average velocity of the projectile between $t = 2$ and $t = 5$.
    iii. Show that the average velocity between $t = 7$ and $t = 9$ is zero, interpret physically.
    iv. At what time does the projectile hit the ground?
    v. Determine the instantaneous velocity at time $t = 8$.
    vi. What is the maximum height that the projectile attains?

2.5 The Derivative Functions

In this section we will discuss the concept of a "derivative" which is the primary mathematics tool that is used to calculate and study rates of change.

We have studied a slope of tangent line: $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

For any x, if the limit exists, then it can be interpreted either on the slope of a tangent line to the curve $y = f(x)$ as $x = x_0$ or as the instantaneous rate of change of y with respect to $x = x_0$. This limit is so important that it has special notations.

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x)}{\Delta x}$$

You can think of f'(read "f prime").

Definition: The Derivative Functions

The function f' defined by the formula: $f'(x_0) = \lim\limits_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x)}{\Delta x}$

is called the derivative of f with respect to x. The domain of f' consists of all x in the domain of f for which the limit exists.

Example 12: Find the derivative of $f(x) = x^2$.

Solution: We have: $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - (x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

Example 13: Find the derivative of

$$y = f(x) = -x^2 + 4x + 1$$

Solution: $\Delta y = f(x + \Delta x) - f(x)$

$$= \Delta x[-2x - \Delta x + 4]$$

Therefore $f'(x) = y' = \frac{\Delta y}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[-2x - \Delta x + 4]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [-2x - \Delta x + 4] = -2x + 4$$

Key Point: Notation

Many ways to denote the derivative of a function $y = f(x)$

- y' "y prime"
- $\frac{dy}{dx} = \frac{df}{dx} = \frac{df(x)}{dx} = D_x f = y'$

We also read $\frac{dy}{dx}$ as "the derivative of y with respect of x" and $\frac{df}{dx}$ and $\left(\frac{d}{dx}\right) f(x)$ as "the derivative of f with respect of x".

- y' and f' (used by Newton).
- $\frac{d}{dx}$ (used by Leibniz).

Input

- Function $y = f(x)$. operator $\frac{d}{dx}$

Output

- Derivative $y' = \frac{df}{dx}$
- Process is also called differentiation.

Example 14:

a. Find the derivative of

$y = f(x) = \sqrt{x} f$ or $x = 0$

b. Find the slope of the tangent at $x = 9$.

Solution:

a. $f(x) = \sqrt{x}$, $f(x + \Delta x) = \sqrt{x + \Delta x}$

$$y' = f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} - \sqrt{x}} \text{(rationalise)} = \frac{1}{2\sqrt{x}}$$

b. The slope of the tangent at $x = 9$ is

$$\frac{dy}{dx}\Big|_{x=9} = \frac{1}{2\sqrt{x}}\Big|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

2.6 Rules of Differentiation

2.6.1 Power and Sum Rules

The definition of derivative has the obvious drawback of being rather clumsy and tiresome to apply. For example, to find the derivative of function like $f(x) = 5x^{100} + x^{\frac{7}{5}}$ is a time taking job. Here, we will develop some important theorems that will enable us to calculate derivatives more efficiently.

Theorem 2.1: Power Rule

If n is a positive integer, then: $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$

Proof:

Let $f(x) = x^n$, n a positive integer. By binomial theorem we can write:

$$f(x + \Delta x) = (x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-1}(\Delta x)^2 + \cdots + (\Delta x)^n$$

Thus: $$\frac{d}{dx}[x^n] = f'(x) = \lim_{\Delta x \to 0} \frac{[x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-1}(\Delta x)^2 \cdots + (\Delta x)^n] - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}(\Delta x) + \cdots + (\Delta x)^{n-1}]}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}(\Delta x) + \cdots + (\Delta x)^{n-1}\right)$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

A power rule simply states that differentiate x^n: $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$

Example 15: Find: $\frac{d}{dx}[x^4] = 4x^3$, $\frac{d}{dx}[x^7] = 7x^6$, $\frac{d}{dx}[x^{50}] = 50x^{49}$, $\frac{d}{dx}[x^{200}] = 200x^{199}$,

$$\frac{d}{dx}[x^{31}] = 31x^{30}$$

We can apply this formula for all real numbers like:

$$\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left[x^{\frac{4}{5}}\right] = \frac{4}{5}x^{\frac{4}{5}-1} = \frac{4}{5x^{\frac{1}{5}}}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}x^{-1} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left[\frac{1}{x^{50}}\right] = \frac{d}{dx}x^{-50} = -\frac{50}{x^{51}}$$

$$\frac{d}{dx}\left[10x^{\frac{1}{3}}\right] = 10\frac{d}{dx}x^{\frac{1}{3}} = \frac{10x^{\frac{-2}{3}}}{3} = \frac{10}{3x^{\frac{2}{3}}}$$

Derivative of constant function:

$$\frac{d}{dx}[c] = \frac{d}{dx}[cx^0] = c\frac{d}{dx}[x^0] = c.0x^{0-1} = 0$$

$\frac{d}{dx}[c] = 0$ like $\frac{d}{dx}[10] = 0$

Theorem:
If n is any real number

$$\frac{d}{dx}x^n = nx^{n-1}$$

Theorem:
If c is any real number

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Sum and Difference Rule:

If f and g are differentiable function, then

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

In words, the derivative of a sum equals to the sum of the derivatives and the derivative of difference is equal to the difference of the derivatives.

Example 16:

i. $\frac{d}{dx}[2x^6 + x^{-9}]$

$$= \frac{d}{dx}[2x^6] + \frac{d}{dx}[x^{-9}]$$

$$= 2.\frac{d}{dx}[x^6] + (-9)x^{-9-1}$$

$$= 2(6)x^5 - 9x^{-10}$$

$$= 12x^5 - \frac{9}{x^{10}}$$

ii. $\frac{d}{dx}\left[4x^5 - \frac{1}{2}x^4 + 9x^3 + 7\right]$

$$= 4\frac{d}{dx}[x^5] - \frac{1}{2}\frac{d}{dx}[x^4] + 9\frac{d}{dx}[x^3] + \frac{d}{dx}[7]$$

$$= 4(5)x^4 - \frac{1}{2}(4)x^3 + 9(3)x^2 + 0$$

$$= 20x^4 - 2x^3 + 27x^2$$

iii. $\frac{d}{dx}\left[5x^{-1} - \frac{1}{5}x\right]$

$$= 5\frac{d}{dx}[x^{-1}] - \frac{1}{5}\frac{d}{dx}[x]$$

$$= -5x^{-2} - \frac{1}{5}x^{1-1} = \frac{-5}{x^2} - \frac{1}{5}$$

Example 17: Find the derivative

a. $y=(x+1)^2$ b. $(x+1)(x-2)$ c. $\frac{x^3+x^2}{3x}$

Solution:

a. $y=(x+1)^2=x^2+2x+1$

$$\frac{dy}{dx}=\frac{d}{dx}[x^2+2x+1]=\frac{d}{dx}(x^2)+2\frac{d}{dx}(x)+\frac{d}{dx}(1)=2x+2+0=2x+2$$

b. $y=(x+1)(x-2)=x^2-x-2$

$$\frac{dy}{dx}=\frac{d}{dx}x^2-\frac{d}{dx}x-\frac{d}{dx}2$$

$$=2x-1-0=2x-1$$

c. $y=\frac{x^3+x^2}{3x}=\frac{x^3}{3x}+\frac{x^2}{3x}$

$$y=\frac{x^2}{3}+\frac{x}{3}$$

$$\frac{dy}{dx}=\frac{1}{3}\frac{d}{dx}x^2+\frac{1}{3}\frac{d}{dx}x=\frac{1}{3}(2x)+\frac{1}{3}=\frac{2}{3}x+\frac{1}{3}$$

Note: In the different contents of science, engineering and business functions are often expressed in variable other than x and y. Correspondingly, we must adapt the derivative notation to new symbols, for example:

Function	Derivative	Function	Derivative
$V(t)=4t$	$V'(t)=\frac{dV}{dt}=4$	$H(z)=\frac{1}{4}z^6$	$H'(z)=\frac{dH}{dz}=\frac{3}{2}z^5$
$A(r)=\pi r^2$	$A'(r)=\frac{dA}{dr}=2\pi r$	$r(\theta)=4\theta^3-3\theta$	$r'(\theta)=\frac{dr}{d\theta}=12\theta^2-3$

Exercise 2.4

1. Find the derivative of the functions.

   a. $y=x^9$ b. $f(x)=4x^{\frac{1}{3}}$ c. $f(x)=9$ d. $f(x)=6x^3+3x^2-10$

2. Determine $f'(x)$.

   a. $f(x)=\sqrt{5}$ b. $f(x)=\sqrt{5}x$ c. $f(x)=5\sqrt{x}$ d. $f(x)=\sqrt{5x}$

3. Determine $f'(x)$.

   a. $f(x)=x^2(x^3+5)$ b. $f(x)=(x+9)(x-9)$ c. $f(x)=(x^2+x^3)^3$

   d. $f(x)=-3x^{-8}+2\sqrt{x}$ e. $f(x)=ax^3+bx^2+cx+d$, (a, b, c and d are constants)

   f. $f(x)=x^{24}+2x^{\frac{1}{2}}+3x^8+9x^4$

4. Find $\frac{dy}{dx}$.

a. $y = \frac{x+2x^{\frac{3}{2}}}{\sqrt{x}}$ b. $y = (x^3 - 5)(2x + 3)$ c. $y = (4x^2 - 3)(7x^2 + x)$

5. Find slope of tangent at $x = 1$.

a. $f(x) = x^2 + 3x$ b. $f(x) = x^4 - x^2$

2.7 The Product and Quotient Rules

We will develop techniques for differentiating products and quotients. If functions whose derivative are known.

2.7.1 Derivative of a Product

You might be considered conjecture that the derivative of a product of two functions is the product of their derivatives. However, simple examples will show this not possible.

Consider:

$f(x) = x^2$ and $g(x) = x^3$

The product of their derivative is:

$f'(x)g'(x) = (2x)(3x^2) = 6x^3$

But their product is:

$y = f(x)g(x) = x^5$ and $\frac{dy}{dx} = y' = 5x^4 \neq 6x^3$

Thus, the derivative of the product is not equal to the product of their derivative.

Theorem: Product Rule

If f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

The first function times the derivative of the second function plus the second function times the derivative of first function.

Example 18: Find $\frac{dy}{dx}$ if $y = (4x^2 - 1)(7x^3 + x)$.

Solution: We can use two methods to find $\frac{dy}{dx}$. We can either use the product rule or we can multiply out the factors in y and then differentiate. We provide both methods.

Method I: The Product Rule

$$\frac{dy}{dx} = \frac{d}{dx}[(4x^2 - 1)(7x^3 + x)]$$

$$\frac{dy}{dx} = \overbrace{(4x^2-1)}^{\text{First}}\overbrace{\frac{d}{dx}(7x^3+x)}^{\text{Derivative of second}} + \overbrace{(7x^3+x)}^{\text{Second}}\overbrace{\frac{d}{dx}(4x^2-1)}^{\text{Derivative of first}}$$

$$\frac{dy}{dx} = (4x^2 - 1)(21x^2 + 1) + (7x^3 + x)(8x)$$

$$\frac{dy}{dx} = 140x^4 - 9x^2 - 1$$

Method II: Multiplying First

$$y = (4x^2 - 1)(7x^3 + x) = 28x^5 - 3x^3 - x$$

$$\frac{dy}{dx} = \frac{d}{dx}[28x^5 - 3x^3 - x] = 140x^4 - 9x^2 - 1$$

Both derivatives are same.

Example 19: Find $\frac{dy}{dx}$ if $y=[(1+x^3)\sqrt{x}]$

Solution: Apply the product rule $\frac{dy}{dx}=\frac{d}{dx}[(1+x^3)\sqrt{x}]$

$$=(1+x^3)\frac{d}{dx}\sqrt{x}+\sqrt{x}\frac{d}{dx}(1+x^3)\ =(1+x^3)\frac{1}{2}x^{\frac{1}{2}-1}+\sqrt{x}(3x^2)$$

$$=\frac{(1+x^3)}{2\sqrt{x}}+3x^{\frac{5}{2}}=\frac{1+x^3+6x^3}{2\sqrt{x}}\ =\frac{7x^3+1}{2\sqrt{x}}$$

2.7.2 Derivative of a Quotient

Just as the derivative of a product is not generally the product of derivatives, so the derivative of a quotient is not generally the quotient of the derivatives. The correct relationship/method is given by the following.

Theorem: Quotient Rule

If f and g are differentiable functions and $g(x)\neq 0$, then,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

The denominator times the derivative of numerator minus the numerator times the derivative of denominator all divided by the denominator square.

Example 20: Differentiate $y=\frac{3x^2-1}{2x^3+5x^2+7}$

Solution: Apply the quotient rule:

$$\frac{dy}{dx}=\frac{\overbrace{(2x^3+5x^2+7)}^{\text{Denominator}}\overbrace{\frac{d}{dx}[3x^2-1]}^{\text{Derivative of numerator}}-\overbrace{(3x^2-1)}^{\text{Numerator}}\overbrace{\frac{d}{dx}[2x^3+5x^2+7]}^{\text{Derivative of Denominator}}}{(2x^3+5x^2+7)^2}$$

$$=\frac{(2x^3+5x^2+7)(6x)-(3x^2-1)(6x^2+10x)}{(2x^3+5x^2+7)^2}$$

$$=\frac{-6x^4+6x^2+52x}{(2x^3+5x^2+7)^2}$$

2.8 The Connection Between Derivatives and Continuity

- If a function is differentiable at a point, it is automatically continuous at that point.
- But the reverse is not always true. A function can be continuous at a point and still not be differentiable (like a sharp corner or cusp, for example | x | is continuous but not differentiable).

Exercise 2.5

Find the derivative

Summary of Differentiation Rules:

- $\frac{d}{dx}[c] = 0, \frac{d}{dx}[cf] = cf', \frac{d}{dx}[f \pm g] = f' \pm g'$
- $\frac{d}{dx}[f.g] = fg' + gf'$
- $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$

1. $y = \frac{1}{x}$
2. $y = (x^2 - 7)(x^2 + 4x + 2)$
3. $y = (7x + 1)(x^4 - x^3 - 9x)$
4. $y = \frac{3x+4}{x^2+1}$ 5. $y = \frac{x-2}{x^4+x+1}$

6. $y = \frac{3x^2+5}{3x-1}$ 7. $y = \left(\frac{1}{x} + \frac{1}{x^2}\right)(3x^3 + 27)$ 8. $y = \frac{2-3x}{7-x}$ 9. $y = \frac{x^2-10x+2}{x^3-x}$

10. $y = \frac{x^4+2x^3-1}{x^2}$ 11. $y = \frac{10}{(x^3-10)^9}$ 12. $y = \frac{(x^2+1)^2}{3x-2}$ 13. $y = \frac{(x+1)^2}{(x-1)^2}$

Find the slope of tangent at the indicated point.

14. $y = \frac{4x-1}{x}$, $x = -1$ 15. $y = \frac{54}{x^2+1}$, $x = 2$

16. $y = \frac{2x+5}{x+2}$, $x = 1$ 17. $y = (2\sqrt{x} + 1)(x^3 - 6)$, $x = 0$

2.9 Derivations of Trigonometric Functions

The main objective of this section is to obtain formulas for the derivatives of six basic trigonometric functions. We will assume in this section that the variable x in the trigonometric functions $\sin x, \cos x, \tan x, \cot x, \sec x$ and $\operatorname{cosec} x$ is measured in radians. We also need the limits in results and restated as follows:

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \text{ and } \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$$

We start the problem of differentiating $f(x) = \sin x$. Using the definitions of derivative

$$\frac{d}{dx} f(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{d}{dx} \sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}$$

$= \lim_{\Delta x \to 0}\left[\sin x\left[\frac{\cos\Delta x - 1}{\Delta x}\right] + \cos x\left[\frac{\sin\Delta x}{\Delta x}\right]\right]$

$= \sin x \lim_{\Delta x \to 0}\frac{\cos\Delta x - 1}{\Delta x} + \cos x \lim_{\Delta x \to 0}\frac{\sin\Delta x}{\Delta x}$

$= \sin x(0) + \cos x(1)$

$\sin x$ and $\cos x$ independent of Δx

Thus, we have $\boxed{\frac{d}{dx}\sin x = \cos x}$

In a similar manner it can be shown that $\boxed{\frac{d}{dx}\cos x = -\sin x}$

Example 21: Find $\frac{dy}{dx}$ if $y = x\sin x$

Solution: $\frac{dy}{dx} = \frac{d}{dx}[x\sin x]$

$= x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x$ use product rule

$= x\cos x + \sin x(1) = x\cos x + \sin x$

Example 22: Find $\frac{dy}{dx}$ if $y = \frac{\sin x}{1+\cos x}$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\sin x}{1+\cos x}\right]$

$= \frac{1+\cos x\frac{d}{dx}\sin x - \sin x\frac{d}{dx}[1+\cos x]}{(1+\cos x)^2}$

$= \frac{(1+\cos x)\cos x - \sin x(0-\sin x)}{(1+\cos x)^2}$

$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$

$= \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}$

The other Trigonometric Functions:

Let $y = \tan x$

$\frac{dy}{dx} = \frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$

$= \frac{\cos x\frac{d}{dx}\sin x - \sin x\frac{d}{dx}\cos x}{\cos^2 x}$

$= \frac{(\cos x)\cos x - \sin x(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$

$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$

Similarly, $\boxed{\frac{d}{dx}\cot x = -\mathrm{cosec}^2 x}$

For $y = \sec x$

$\frac{dy}{dx} = \frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x}$

$= \frac{\cos x\frac{d}{dx}(1) - (1)\frac{d}{dx}\cos x}{\cos^2 x}$

$= \frac{\cos x(0) - (-\sin x)}{\cos^2 x}$

$= \frac{\sin x}{\cos^2 x} = \sec x\tan x$

$\boxed{\frac{d}{dx}\sec x = \sec x\tan x}$

Similarly, $\boxed{\frac{d}{dx}\mathrm{cosec}x = \mathrm{cosec}x\cot x}$

Example 23: Find $\frac{dy}{dx}$ if $y = \frac{\cos x}{x-\cot x}$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\cos x}{x-\cot x}\right]$

$= \frac{(x-\cot x)\frac{d}{dx}\cos x - \cos x\frac{d}{dx}(x-\cot x)}{(x-\cot x)^2}$

$= \frac{(x-\cot x)(-\sin x) - \cos x(1-(-\mathrm{cosec}^2 x))}{(x-\cot x)^2}$

$= \frac{-x\sin x + \cot x\sin x - \cos x - \cos x\,\mathrm{cosec}^2 x}{(x-\cot x)^2}$

$= \frac{-x\sin x + \cos x - \cos x - \cos x\,\mathrm{cosec}^2 x}{(x-\cot x)^2}$

$= \frac{-x\sin x - \cos x\,\mathrm{cosec}^2 x}{(x-\cot x)^2}$

Example 24: Find $\frac{dy}{dx}$ if $y = \sin x(2 + \sec x)$

Solution: $\frac{dy}{dx} = \frac{d}{dx}[\sin x(2 + \sec x)]$

$= \sin x \frac{d}{dx}(2 + \sec x) + (2 + \sec x)\frac{d}{dx}(\sin x) = \sin x(0 + \sec x \tan x)) + (2 + \sec x)(\cos x)$

$= \sin x \sec x \tan x + 2\cos x + \sec x \cos x = \sin x \frac{1}{\cos x}\tan x + 2\cos x + \sec x \cos x$

$= \tan^2 x + 2\cos x + 1 = \tan^2 x + 1 + 2\cos x$

$= \sec^2 x + 2\cos x \qquad (1 + \tan^2 x = \sec^2 x)$

2.10 Derivatives of Inverse Trigonometric Functions

The derivative of an inverse trigonometric function can be obtained. Research reveals that the inverse tangent and inverse cotangent are differentiable for all x. However the remaining four inverse trigonometric functions are not differentiable at either $x = -1$ or $x = 1$

Inverse sine function:

For $-1 < x < 1$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$y = \sin^{-1} x$ if and only if $x = \sin y$

Differentiate w.r.t x

$$\frac{dx}{dx} = \frac{d}{dx}\sin y$$

$$1 = \cos y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

Similarly, $\boxed{\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1}$

Inverse tangent function:

For $-\alpha < x < \alpha$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$y = \tan^{-1} x$ if and only if $x = \tan y$

Differentiate w.r.t x

$$\frac{dx}{dx} = \frac{d}{dx}\tan y$$

$$1 = \sec^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$\boxed{\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \text{ for } x \in R}$$

Similarly, $\boxed{\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}, \text{ for } x \in R}$

Inverse secant function:

For $|x| > 1$ and $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$

$y = \sec^{-1} x$ if and only if $x = \sec y$

Differentiate w.r.t x

$\frac{dx}{dx} = \frac{d}{dx}\sec y$

$1 = \sec y \tan y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$

$\therefore 1 + \tan^2 y = \sec^2 y$

$\tan^2 y = \sec^2 y - 1$

$\tan y = \sqrt{\sec^2 y - 1}$

$$= \frac{1}{\sec y\sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2-1}}$$

$$\boxed{\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}\ , \quad \text{for } |x| > 1}$$

Similarly, $\boxed{\frac{d}{dx}\text{cosec}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}, \quad \text{for } |x| > 1}$

Example 25:

Differentiate $y = \sin^{-1}5x$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}5x$

$$= \frac{1}{\sqrt{1-(5x)^2}} \cdot \frac{d}{dx}5x$$

$$\frac{d}{dx}\sin^{-1}5x = \frac{5}{\sqrt{1-25x^2}}$$

Example 26: Differentiate $y = \tan^{-1}12x$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\tan^{-1}12x$

$$= \frac{1}{1+(12x)^2} \cdot \frac{d}{dx}12x$$

$$\frac{d}{dx}\tan^{-1}12x = \frac{12}{1+144x^2}$$

Example 27: Differentiate $y = \sec^{-1}x^2$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\sec^{-1}x^2$

$$= \frac{1}{x^2\sqrt{(x^2)^2-1}} \cdot \frac{d}{dx}x^2$$

$$= \frac{1}{x^2\sqrt{(x^2)^2-1}} \cdot (2x)$$

$$\frac{d}{dx}\text{sect}x^2 = \frac{2x}{x^2\sqrt{x^4-1}}$$

Exercise 2.6

Find the derivative of the given functions

1. $y = x^2 - \cos x$
2. $y = 4x^3 + x + \sin x$
3. $y = 3\cos x - 5\cot x$
4. $y = \sin x \cos x$
5. $y = (x^2 + \sin x)\sec x$
6. $y = \frac{5-\cos x}{5+\sin x}$
7. $y = \frac{\sec x}{1+\tan x}$
8. $y = \frac{\sin x}{x^2+\sin x}$
9. $y = \frac{\cot x}{x+1}$
10. $y = (1+\cos x)(x - \sin x)$

Find the derivative of the given functions

11. $y = \sin^{-1}(5x-1)$
12. $y = 4\cot^{-1}\frac{x}{2}$
13. $y = \frac{\sin^{-1}x}{\sin x}$
14. $y = \frac{\sec^{-1}x}{x}$
15. $y = x\sin^{-1}x + x\cos^{-1}x$
16. $y = \frac{1}{\tan^{-1}x^2}$

2.11 The Chain Rule

In this section, we will derive a formula that expresses the derivative of a composition *fog* in terms of the derivative of *f* and *g*. This formula will enable us to differentiate complicated functions.

Suppose we wish to differentiate:

$$y = (x^5 + 1)^2 \;\ldots\ldots \qquad \text{(i)}$$

We can write $y = (x^5 + 1)(x^5 + 1)$

$$\frac{dy}{dx} = (x^5 + 1)\frac{d}{dx}(x^5 + 1) + (x^5 + 1)\frac{d}{dx}(x^5 + 1)$$

$$= (x^5 + 1)\,(5x^4) + (x^5 + 1)\,(5x^4)$$

$$= 2(x^5 + 1)(5x^4) \ldots\ldots \qquad \text{(ii)}$$

2.11.1 Power Rule for Functions

From (i), $y = (x^5 + 1)^2$

$$\frac{dy}{dx} = 2(x^5 + 1)^{2-1}\frac{d}{dx}(x^5 + 1)$$

$$= 2(x^5 + 1)(5x^4) \ldots\ldots \qquad \text{(iii)}$$

From (ii) and (iii), both expressions are same.

Theorem: Power Rule for Functions

If n is an integer and g is a differentiable function then,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$$

Example 28: Differentiate

a. $y = (2x^3 + 4x + 1)^4$

b. $y = \frac{1}{(7x^5 - x^4 + 2)^{10}}$

Solution:

a. $\frac{dy}{dx} = \frac{d}{dx}(2x^3 + 4x + 1)^4$

$$= 4(2x^3 + 4x + 1)^{4-1}\frac{d}{dx}(2x^3 + 4x + 1)$$

$$= 4(2x^3 + 4x + 1)^3(6x^2 + 4)$$

b. $y = (7x^5 - x^4 + 2)^{-10}$

$$\frac{dy}{dx} = \frac{d}{dx}(7x^5 - x^4 + 2)^{-10}$$

$$= -10(7x^5 - x^4 + 2)^{-10-1}\frac{d}{dx}(7x^5 - x^4 + 2)$$

$$= -10(7x^5 - x^4 + 2)^{-11}(35x^4 - 4x^3)$$

Example 29: Differentiate $y = \frac{(x^2-1)^3}{(5x+1)^8}$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\frac{(x^2-1)^3}{(5x+1)^8}$

$$= \frac{(5x+1)^8\frac{d}{dx}(x^2-1)^3 - (x^2-1)^3\frac{d}{dx}(5x+1)^8}{[(5x+1)^8]^2}$$

$$= \frac{(5x+1)^8 3(x^2-1)^2(2x) - (x^2-1)^3 8(5x+1)^7(5)}{(5x+1)^{16}}$$

$$= \frac{6x(5x+1)^8(x^2-1)^2 - 40(x^2-1)^3(5x+1)^7}{(5x+1)^{16}}$$

$$= \frac{(x^2-1)^2(5x+1)^7[6x(5x+1) - 40(x^2-1)]}{(5x+1)^{16}}$$

$$= \frac{(x^2-1)^2[-10x^2 + 6x + 40]}{(5x+1)^9}$$

2.11.2 Chain Rule: A power of a function can be written as a composite function. If $f(x)=x^n$ and $u=g(x)$, then $f(x)=f(g(x))=[g(x)]^n$ is a special case of the chain rule for differentiating composite function.

Theorem: Chain Rule

If $y=f(x)$ is a differentiable formula of u and $u=g(x)$ is a differentiable function, then $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=f'(g(x)).g'(x)$

Example 30: Differentiate

a. $y=\tan^2 x$ b. $y=(9x^3+1)^2\sin 5x$

Solution:

a. $y=\tan^2 x$

$$\frac{dy}{dx}=2\tan^{2-1}x\frac{d}{dx}\tan x$$
$$=2\tan x\sec^2 x$$

b. $y=(9x^3+1)^2\sin 5x$

$$\frac{dy}{dx}=(9x^3+1)^2\frac{d}{dx}\sin 5x+\sin 5x\frac{d}{dx}(9x^3+1)^2$$
$$=(9x^3+1)^2\cos 5x(5)+\sin 5x.2(9x^3+1)27x^2$$
$$=(9x^3+1)[45x^3\cos 5x+5\cos 5x+54x^2\sin 5x]$$

2.12 Implicit Differentiation

2.12.1 Explicit and Implicit Functions

A function in which the dependent variable is expressed solely in terms of the independent variable x, namely $y=f(x)$ is said to be an explicit functions, for example, $y=\frac{1}{4}x^3-1$ is an explicit function, whereas an equivalent equation $3y-x^3-4=0$ is said to define the function implicitly or y is an implicit of x.

2.12.2 Explicit Differentiation

To illustrate this, let us consider the simple equation:

$$xy=1 \quad \ldots\ldots \quad \text{(i)}$$

One way to find $\frac{dy}{dx}$ is to rewrite this equation as:

$$y=\frac{1}{x}$$

From which it follows that: $\frac{dy}{dx}=-\frac{1}{x^2}\ldots$ (ii)

Another way to obtain this derivative is to differentiate both sides of (i) before solving for y in terms of x.

From (i) $\frac{d}{dx}(xy)=\frac{d}{dx}1$

$$x\frac{d(y)}{dx}+y\frac{d(x)}{dx}=0$$

$$x\frac{dy}{dx}+y=0$$

$$\frac{dy}{dx}=-\frac{y}{x}$$

If we take, $y=\frac{1}{x}$, we get

$$\frac{dy}{dx}=-\frac{1}{x^2}$$

This method of obtaining derivatives is called implicit differentiation.

Example 31: Use implicit differentiation to find $\frac{dy}{dx}$ if $5y^2+\sin y=x^2$

Solution: $\frac{d}{dx}[5y^2+\sin y]=\frac{d}{dx}[x^2]$

$$5\frac{d}{dx}y^2+\frac{d}{dx}\sin y=2x$$

$$5\left(2y\frac{dy}{dx}\right)+\cos y\frac{dy}{dx}=2x$$

$$(10y+\cos y)\frac{dy}{dx}=2x$$

Solving for $\frac{dy}{dx}$ we obtain: $\frac{dy}{dx}=\frac{2x}{10y+\cos y}$

2.13 Derivative of Exponential Functions

The derivative of exponential is: $\boxed{\frac{d}{dx}e^x=e^x}$ like $\frac{d}{dx}e^{3x}=e^{3x}.3$

Example 32: Differentiate $y=x^2e^{5x}$

Solution: $\frac{dy}{dx}=\frac{d}{dx}[x^2e^{5x}]$

$$=x^2\frac{d}{dx}e^{5x}+e^{5x}\frac{d}{dx}x^2 \quad =x^2e^{5x}.5+e^{5x}.2x \quad =5x^2e^{5x}+2xe^{5x} \quad =xe^{5x}(5x+2)$$

2.14 Derivative of Logarithmic Functions

We find the derivative of common logarithmic which is continuous functions.

$\boxed{\frac{d}{dx}\ln x=\frac{1}{x}}$ like $\frac{d}{dx}\ln(x^3+1)=\frac{1}{x^3+1}\frac{d}{dx}(x^3+1)=\frac{3x^2}{x^3+1}$.

Example 33: Differentiate $\ln(4x^3+2x^2+9)$

Solution: $y=\ln(4x^3+2x^2+9)$

$$\frac{dy}{dx}=\frac{1}{4x^3+2x^2+9}\frac{d}{dx}(4x^3+2x^2+9)=\frac{1}{4x^3+2x^2+9}(12x^2+4x)=\frac{4x(3x+1)}{4x^3+2x^2+9}$$

Derivative of $y=a^x$: $\frac{d}{dx}a^x=a^x.\frac{1}{\ln a}$

We will apply the chain rule to find the derivative of parametric equations.

Example 34: Differentiate $y=4^{3x^2+5}$

Solution: Taking $\ln$ both sides

$$\ln y=\ln 4^{3x^2+5}$$

$$\ln y=(3x^2+5).\ln 4$$

$$\frac{1}{y}\frac{dy}{dx}=\ln 4.\frac{d}{dx}(3x^2+5),\quad \frac{dy}{dx}=y\ln 4\,(6x)=\ln 4\left(4^{3x^2+5}\right)6x=6\ln 4\left(4^{3x^2+5}\right)x$$

Example 36: Find $\frac{dy}{dx}$ if $x = \tan t, y = 4t^3 + 1$

Solution: $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\frac{dy}{dt} = \frac{d}{dt}(4t^3 + 1) = 12t^2$$

$$\frac{dx}{dt} = \frac{d}{dt}(\tan t) = \sec^2 t$$

$$\frac{dy}{dx} = \frac{12t^2}{\sec^2 t}$$

Example 37: Find $\frac{dy}{dx}$ if $x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$

Solution: $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$= \frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$= \frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}$$

$$= \frac{-4t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$= \frac{(1+t^2)\frac{d}{dt}(2t) - (2t)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(2) - (2t)(2t)}{(1+t^2)^2}$$

$$= \frac{2 + 2t^2 - 4t^2}{(1+t^2)^2}$$

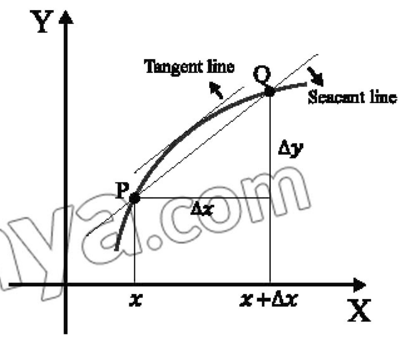
$$= \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{\frac{2(1-t^2)}{(1+t^2)^2}}{\frac{-4t}{(1+t^2)^2}} = \frac{(t^2-1)}{2t}$$

2.15 Differentials

We have already discussed the derivative of finding slope of a tangent line to the graph of a functions $y = f(x)$.

$$m_{\sec} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}$$



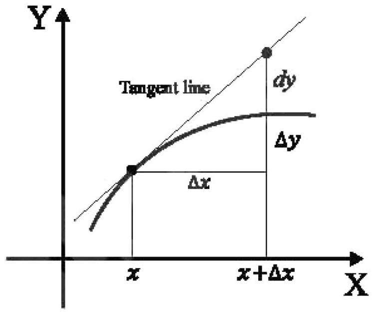
For small values of Δx ,

$$m_{sec} \cong m_{tan} \text{ or } \frac{\Delta y}{\Delta x} = m_{tan} = f'(x)$$

We have: $\frac{\Delta y}{\Delta x} = f'(x)$

$$\Delta y = f'(x)\Delta x$$

Definition: The increment Δx is called the differential of the independent variable x and is denoted by dx. i.e.
The function $f'(x)\Delta x$ is called differential of the dependent variable y and is denoted by dy.
i.e. $dy = f'(x)\Delta x = f'(x)dx$



Since the slope of a tangent to graph is

$$m_{\tan} = \frac{rise}{run} = f'(x) = \frac{f'(x)\Delta x}{\Delta x}, \Delta x \neq 0$$

It follows that the rise of the tangent line can be interrupted in dy

$\Delta y = dy$

Example 38: a) Find Δy and dy for $y = 5x^2 + 4x + 1$

b) Compare the values of Δy and dy for $x = 6$, $\Delta x = dx = 0.02$

Solution:

a) $\Delta y = f(x + \Delta x) - f(x)$

$= [5(x+\Delta x)^2 + 4(x+\Delta x) + 1] - [5x^2 + 4x + 1]$

$= 10x\Delta x + 4\Delta x + 5(\Delta x)^2$

$$\frac{\Delta y}{\Delta x} = \frac{10x\Delta x + 4\Delta x + 5(\Delta x)^2}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{(10x + 4 + 5\Delta x)\Delta x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = 10x + 4$$

$$\frac{dy}{dx} = 10x + 4$$

$$dy = (10x + 4)dx$$

Since $dx = \Delta x$. We observe that

$\Delta y = (10x + 4)\Delta x + 5(\Delta x)^2$ and

$dy = (10x + 4)\Delta x$ differ by the amount $5(\Delta x)^2$.

b) When $x = 6$, $\Delta x = 0.02$

$\Delta y = 10(6)(0.02) + 4(0.02) + 5(0.02)^2$

$= 1.282$

Whereas $dy = (10(6) + 4)(0.02) = 1.28$

The difference in answers is, of course

$5(0.02)^2 = 0.002$

2.16 Approximations

When $\Delta x = 0$, differentials give a means of "predicting" the value of $f(x + \Delta x)$ by knowing the value of the function and its derivative at x. From fig if x is changes by an amount Δx, then the corresponding change in the function is

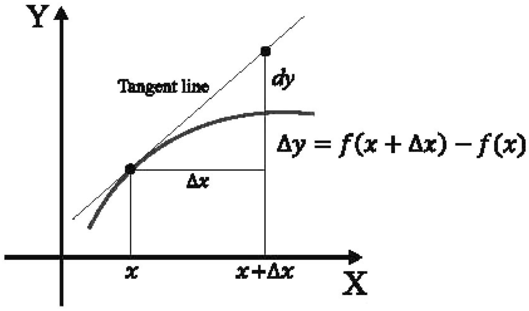
$\Delta y = f(x+\Delta x) - f(x)$ and so

$f(x+\Delta x) = f(x) + \Delta y$

For small change in x, take $\Delta y = dy$

$f(x+\Delta x) = f(x) + dy$

$= f(x) + f'(x)dx$



Example 39: Find an approximation to $\sqrt{25.4}$. Calculate Error.

Solution: First, identify the function $f(x) = \sqrt{x}$

We wish to calculate the approximate value of $f(x+\Delta x) = \sqrt{x+\Delta x}$ when 25 and $\Delta x = 0.4$; $(25.4 = 25 + 0.4)$

Now, $dy = \frac{1}{2}x^{\frac{-1}{2}}dx = \frac{1}{2\sqrt{x}}\Delta x$

We have; $f(x+\Delta x) = f(x) + dy$

$= f(x) + \frac{1}{2\sqrt{x}}\Delta x = \sqrt{x} + \frac{1}{2\sqrt{x}}\Delta x = \sqrt{25} + \frac{1}{2\sqrt{25}}(0.4) = 5.04$

Exercise 2.7

Find the derivative of functions.

1. $y = \left(x - \frac{1}{x^2}\right)^5$
2. $f(x) = \left(\frac{x^2-1}{x^2+1}\right)^2$
3. $y = (3x-1)^4(-2x+9)^5$
4. $f(\theta) = (2\theta+1)^3\tan^2\theta$
5. $y = \sin 2x\cos 3x$
6. $f(x) = (\sec 4x + \tan 2x)^5$
7. $h(t) = \frac{t+\sin 4t}{10+\cos 3t}$
8. $f(x) = \tan\left(\cos\frac{x}{2}\right)$

Use implicit differentiation to find $\frac{dy}{dx}$.

9. $4x^2 + y^2 = 8$
10. $x + xy - y^2 - 20 = 0$
11. $y^4 - y^2 = 10x - 3$
12. $x^3y^2 = 2x^2 + y^2$
13. $xy = \sin x + y$
14. $x + y = \cos xy$
15. $x\sin y - y\cos x = 1$
16. $\sin y = y\cos 2x$

Find $\frac{dy}{dx}$.

17. $y = x^3e^{5x}$
18. $y = e^{4x}(1 + \ln x)$
19. $y = \frac{e^{2x}}{e^{-2x}+1}$
20. $y = \ln(e^x + e^{-x})$
21. $y = \ln\left(x + \sqrt{x^2+1}\right)$
22. $y = e^{-3x}\cos x$

Find $\frac{dy}{dx}$ of the parametric functions.

23. $x = t + \frac{1}{t}, y = t + 1$

24. $x = t^2 + \frac{1}{t^2}, y = t - \frac{1}{t}$

25. $x = \frac{\theta^2-1}{\theta^2+1}, y = \frac{\theta-1}{\theta+1}$

26. $x = \sin 2\theta, y = \cos 4\theta$

Find Δy and dy.

27. $y = x^2 + 1$

28. $y = \sin x$

Use the concept of the differential to find an approximation to the given expressions.

29. $(1.8)^5$

30. $\sqrt{37}$

31. $\sin 31°$

32. $\tan\left(\frac{\pi}{4} + 0.1\right)$

2.17 Higher Order Derivatives

2.17 The Second Derivative

The derivative $f'(x)$ is a function derived from a function $y = f(x)$. By differentiating the first derivative $f'(x)$, we obtain another function called the second derivative, which is denoted by $f''(x)$. In terms of the operation symbol $\frac{d}{dx}$ we define the second derivative with respect to x as the function obtained by differentiating $y = f(x)$ twice is successive.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

The second derivative is commonly denoted by

$$f''(x),\quad y'',\quad \frac{d^2y}{dx^2},\quad D^2y$$

Normally, we shall use one of the first three symbols.

Example 40: Find the second derivative of $y = x^3 - 2x^2$

Solution: The first derivative is: $\frac{dy}{dx} = 3x^2 - 4x$

The second derivative follows from differentiating the first derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 4x) = 6x - 4$$

Example 41: Find the second derivative:

a. $\sin 3x$ b. $(x^3+1)^4$ c. e^{2x}

Solution:

a. The first derivative is: $y' = \frac{dy}{dx} = \frac{d}{dx}(\sin 3x) = 3\cos 3x$

The second derivative is: $y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(3\cos 3x) = -9\sin 3x$

b. The first derivative is:

$$y' = \frac{dy}{dx} = \frac{d}{dx}(x^3+1)^4 = 4(x^3+1)^3\frac{d}{dx}x^3 = 12x^2(x^3+1)^3$$

To find the second derivative, we will use product and power rule

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}[12x^2(x^3+1)^3] = 12\left[x^2\frac{d}{dx}(x^3+1)^3 + (x^3+1)^3\frac{d}{dx}x^2\right]$$

$$= 12\,[x^2 3(x^3+1)^2 3(x^2) + (x^3+1)^3(2x)] = 12x(x^3+1)^2\,[11x^3+2]$$

c. The first derivative is: $y' = \frac{dy}{dx} = \frac{d}{dx}(e^{2x}) = 2e^{2x}$

The second derivative is: $y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(2e^{2x}) = 4e^{2x}$

2.18 Higher Derivative

Assuming all derivatives exist, we can differentiate a function $y = f(x)$ as many times as we want. The third derivative is the derivative of the second derivative. The fourth derivative is the derivative of the third derivative and so on. We denote the third and fourth derivative, by $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$, respectively and define them by:

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)$$

In general, if n is a positive integer, then the nth derivative is denoted by:

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$$

Other notations for the first n derivatives are:

$$f'(x), f''(x), f'''(x), f^{(4)}(x), \dots\dots\dots f^{(n)}(x)$$

$$y',\ y'',\quad y''',\quad y^{(4)}, \dots\dots\dots\ y^{(n)}$$

$$D_xy,\quad D^2{}_xy,\quad D^3{}_xy,\quad D^4{}_xy, \dots\dots\dots D^n{}_xy$$

Example 42: Find the first five derivatives of

$f(x) = 2x^4 - 6x^3 + 7x^2 + 5x - 10$

Solution: We have

$f'(x) = 8x^3 - 18x^2 + 14x + 5$

$f''(x) = 24x^2 - 36x + 14$

$f'''(x) = 48x - 36$

$f^{(4)}(x) = 48$

$f^{(5)}(x) = 0$

Example 43: Find the third derivatives of $y = \frac{1}{x^3}$

Solution: We have $y = \frac{1}{x^3} = x^{-3}$

$\frac{dy}{dx} = -3x^{-4}$

$\frac{d^2y}{dx^2} = (-3)(-4)x^{-5} = 12x^{-5}$

$\frac{d^3y}{dx^3} = (12)(-5)x^{-6} = -60x^{-6} = \frac{-60}{x^6}$

Exercise 2.8

Find the second derivative of the functions.

1. $y = -x^3 + 6x + 9$
2. $f(x) = 30x^2 - x^3$
3. $f(x) = (-5x + 9)^2$
4. $y = 2x^6 + 5x^3 - 6x^2$
5. $y = 20x^{-3}$
6. $y = \frac{2}{x^4}$
7. $f(x) = x^2(3x - 4)^3$
8. $f(x) = (x^2 + 5x - 1)^4$
9. $f(x) = \cos 10x$
10. $f(x) = \tan\frac{x}{2}$
11. $f(\theta) = \sin^2 5\theta$
12. $f(\theta) = \frac{1}{3+2\cos\theta}$
13. $f(x) = e^{2x}(x^2 + 1)$
14. $f(x) = (x^2 + 1)\ln(x^2 + 1)$

Find the indicated derivative.

15. $y = 4x^7 + x^6 - x^4$; $\frac{d^4y}{dx^4}$
16. $y = \frac{2}{x}$; $\frac{d^5y}{dx^5}$
17. $f(x) = \cos\pi x$; $f'''(x)$
18. $f(x) = \frac{1}{\sec(2x+1)}$; $f^{(4)}(x)$
19. Let $f(x) = x^3 + 2x$
    a. Find $f'(x)$ and $f''(x)$
    b. In general; $f''(x) = \lim\limits_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}$

    provided limit exists. Use $f''(x)$ obtained in part (a) and use definitions to find $f''(x)$.
20. Show that $\frac{d^2}{dx^2}(fg) = f''g + 2f'g' + fg''$

    $\frac{d^3}{dx^3}(fg) = f'''g + 3f''g' + 3f'g'' + fg'''$

2.19 Extrema of Functions

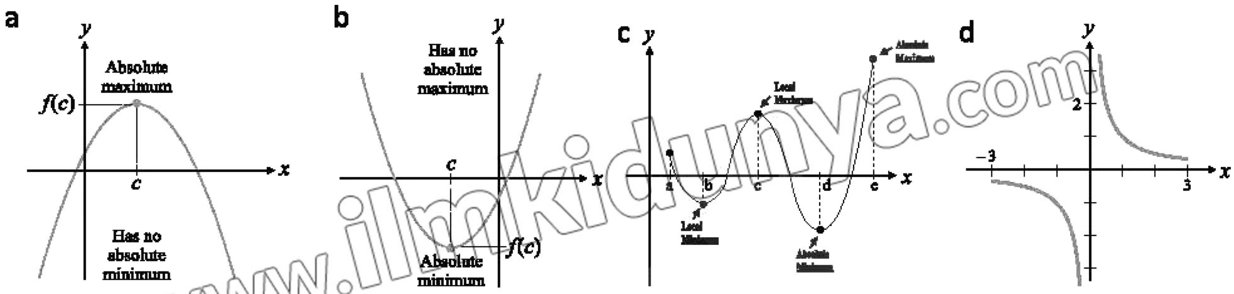
Suppose a function f is defined on an interval I. The maximum and minimum values of f on I (if exist) are said to be extrema of the functions. We have two kinds of extrema.

Definition: Absolute Extrema

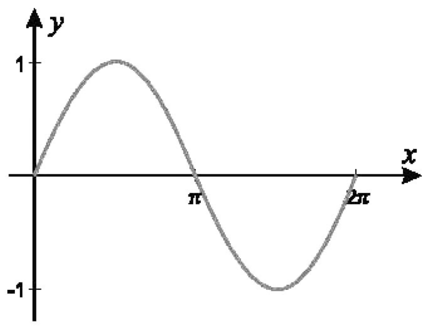
i. A number $f(c)$ is an **Absolute Maximum** of a function f if $f(x) \leq f(c)$ for every x in the domain of f.

ii. A number $f(c)$ is an **Absolute Minimum** of a function f if $f(x) \geq f(c)$ for every x in the domain of f.

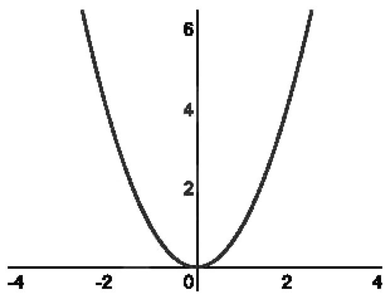
Absolute extrema are called global extrema. Figure shows several possibilities:



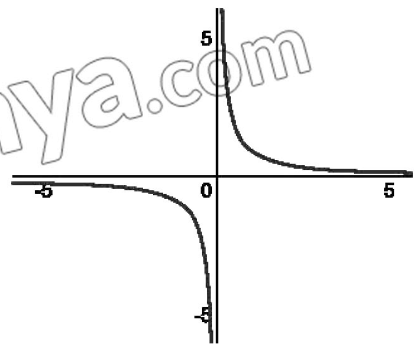
Example 44:



i. For $f(x) = \sin x$, $f\left(\frac{\pi}{2}\right) = 1$ is its absolute maximum and $f\left(\frac{3\pi}{2}\right) = -1$ is its absolute minimum.



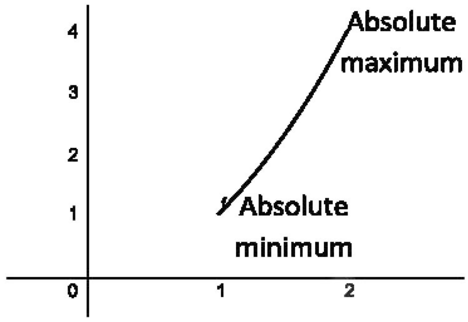
ii. The function $f(x) = x^2$ has the absolute minimum $f(0) = 0$ but has no absolute maximum.



iii. $f(x) = \frac{1}{x}$ has neither an absolute maximum nor an absolute minimum.

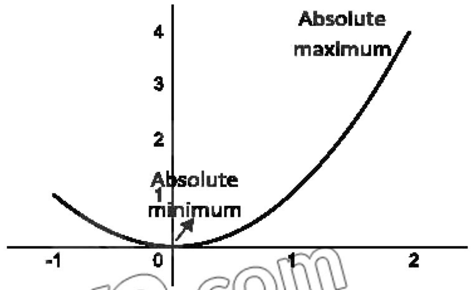
Example 45:

i. $f(x) = x^2$ defined only on the closed interval at $[1, 2]$ has the absolute maximum $f(2) = 4$ and the absolute minimum $f(1) = 1$



ii. On the other hand, if $f(x) = x^2$ is defined on the interval $(1, 2)$. f has no absolute extrema.

iii. $f(x) = x^2$ is defined on the interval $[-1, 2]$. f has absolute maximum $f(2) = 2$ and now the absolute minimum is $f(0) = 0$.



Result: A function f continuous on a closed interval $[a, b]$ always has an absolute maximum and absolute minimum on the interval.

2.19.1 Relative Extrema

The function pictured in fig(a) has no absolute extrema. However, suppose we focus our attention on values of x that are close to, or in a neighborhood of the numbers c_1 and c_2.

As shown in fig(b), $f(c_1)$ is the maximum value of the function in the interval (a_1, b_1) and $f(c_2)$ is a minimum value in the interval (a_2, b_2). These local or relative extrema are defined as follows:

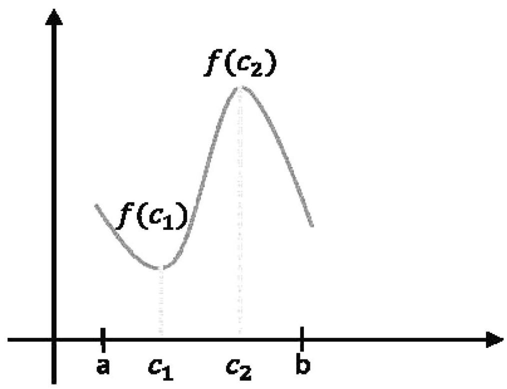
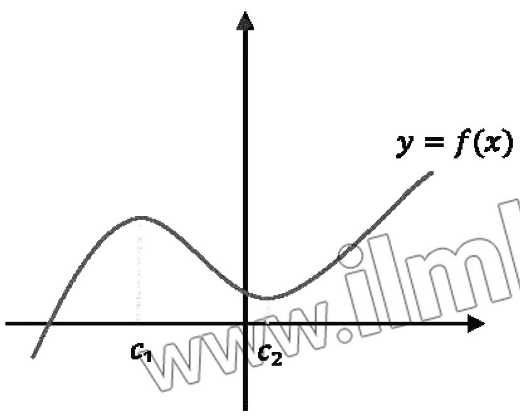


Fig (a)



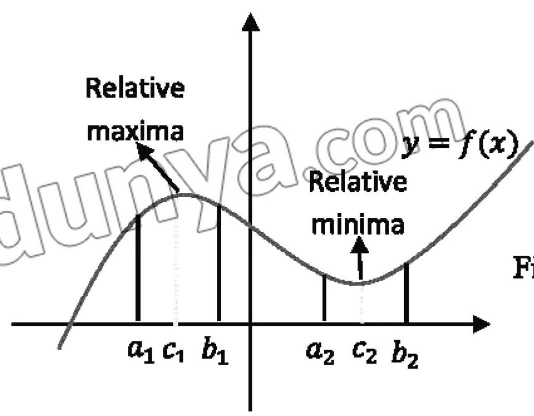


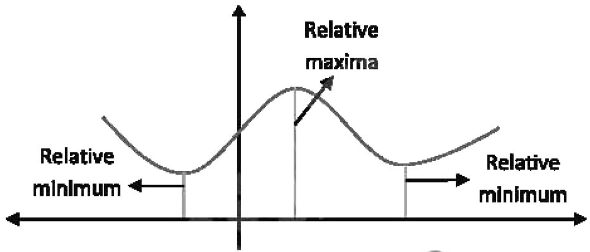
Fig (b)

Definition: Relative Extrema

i. A number $f(c_1)$ is a **Relative Maximum** of a function f if $f(x) \leq f(c_1)$ but every x in some open interval that contains c_1.

ii. A number $f(c_1)$ is a **Relative Minimum** of a function f if $f(x) \geq f(c_1)$ for every x in some open interval that contains c_1.

Result: From fig, we suggest if c is a value at which a function f has a relative extremum, then either $f'(c) = 0$ or $f'(c)$ does not exist.



Critical values: A critical value of a function f is a number in c in its domain for which $f'(c) = 0$ or $f'(c)$ does not exist.

Example 46: Find the critical values of

a. $f(x) = x^3 - 15x + 6$ c. $f(x) = \frac{x^2}{x-1}$ b. $f(x) = (x+4)^{\frac{2}{3}}$

Solution:

a. $f(x) = x^3 - 15x + 6$

$f'(x) = 3x^2 - 15$

$f'(x) = 3(x+\sqrt{5})(x-\sqrt{5})$

The critical values are those number for which $f'(x) = 0$, namely $-\sqrt{5}$ and $\sqrt{5}$.

b. $f(x) = (x+4)^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}(x+4)^{\frac{-1}{3}}$

$f'(x) = \dfrac{2}{3(x+4)^{\frac{1}{3}}}$

We observe that $f'(x)$ doesnot exist, when $x = -4$ since -4 is in the domain of f. We conclude it is a critical value.

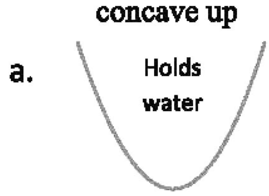
c. $f(x) = \frac{x^2}{x-1}$

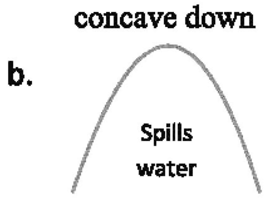
$f'(x) = \frac{x(x-2)}{(x-1)^2}$; by quotient rule

Now $f'(x) = 0$ when $x = 0$ and $x = 2$, whereas $f'(x)$ doesn't exist when $x = 1$. However, inspection of f reveals $x = 1$ is not in its domain and so the any critical values are 0 and 2.

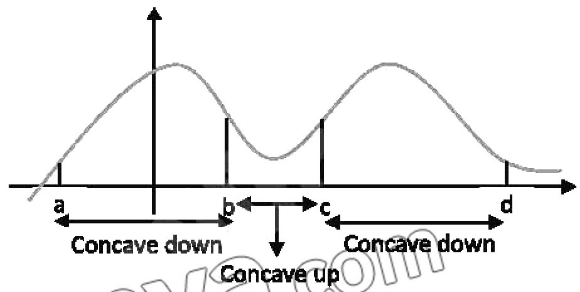
2.20 Second Derivative Test for Relative Extrema

Concavity: We know about the concavity:





The figures (a) and (b) illustrates geometric shapes that are concave upward and concave downward, respectively. Often a shape that is concave upward is said to "hold water" whereas a shape that is concave downward "spills water".



The graph in the fig(c) is concave upward on the interval (b, c) and concave downward on (a, b) and (c, d).

Concavity and The Second Derivative Test

Definition: Test for concavity

Let f be a function for which f'' exists on (a, b).

If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave upward on (a, b).

If $f''(x) < 0$ for all x in (a, b), then the graph of f is concave downward on (a, b).

Example 47: Determine the interval on which the graph of $f(x) = -x^3 + \frac{9}{2}x^2$ is concave upward and the intervals for which the graph is concave downward.

Solution: $f(x) = -x^3 + \frac{9}{2}x^2$

$f'(x) = -3x^2 + 9x$

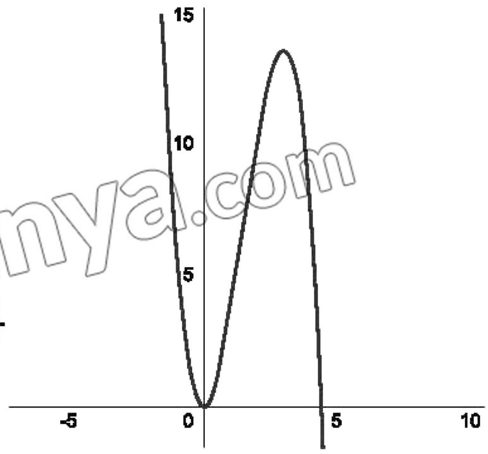
$f''(x) = -6x + 9 = 6\left(-x + \frac{3}{2}\right)$

We observe that $f''(x) < 0$ when $6\left(-x + \frac{3}{2}\right) > 0$ or

$x < \frac{3}{2}$ and that $f''(x) < 0$ when $6\left(-x + \frac{3}{2}\right) < 0$ or $x > \frac{3}{2}$.

If follows that the graph of f is concave upward on

$(-\infty, \frac{3}{2})$ and concave downward on $\left(-\frac{3}{2}, \infty\right)$.



2.21 Point of Inflection

In the example 47 function changes concavity at the point that corresponds to $x = \frac{3}{2}$. As x increases through $\frac{3}{2}$, the graph of f changes from concave upward to concave downward at the point $\left(\frac{3}{2}, \frac{27}{4}\right)$ a point on the graph of a function where the concavity changes from upward or downward or reverse is called a point of inflection.

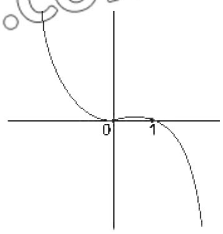
Definition: Point of Inflection

Let f be a continuous at c, a point $(c, f(c))$ is point of inflection if there exists an open interval(a, b) that contains c such that the graph of f is either:

i. Concave upward on (a, c) and concave downward on (c, b) or

ii. Concave downward on (a, c) and concave upward on (c, b).

Example 48: Find points of inflection of $f(x) = -x^3 + x^2$



Solution:

$$f'(x) = -3x^2 + 2x \text{ and } f''(x) = -6x + 2$$

Since $f''(x) = 0$ at $\frac{1}{3}$, the point $\left(\frac{1}{3}, \frac{2}{27}\right)$ is the only possible point of inflection. We have

$$f''(x) = 6\left(-x + \frac{1}{3}\right) > 0 \text{ for } x < \frac{1}{3}$$

$$f''(x) = 6\left(-x + \frac{1}{3}\right) < 0 \text{ for } x > \frac{1}{3}$$

Implies that the graph of f is concave upward on $\left(-\infty, \frac{1}{3}\right)$ and concave downward on $\left(\frac{1}{3}, \infty\right)$.

Thus, $\left(\frac{1}{3}, f\left(\frac{1}{3}\right)\right)$ or $\left(\frac{1}{3}, \frac{2}{27}\right)$ is a point of inflection.

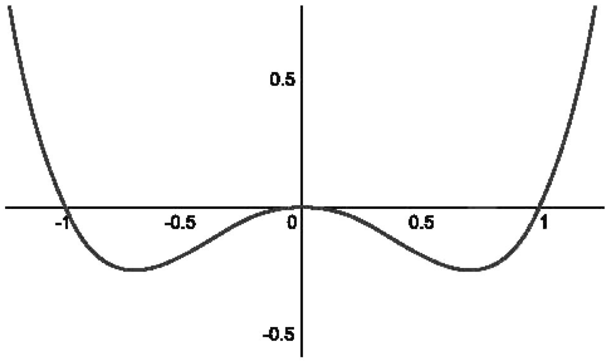
Definition: Second Derivative Test for Relative Extrema

Let f be function for which f'' exists on an interval (a, b) that contains the critical number c.

i. If $f''(c) > 0$, then $f(c)$ is a relative minimum.

ii. If $f''(c) < 0$, then $f(c)$ is a relative maximum.

Example 49: Find the critical point and also relative extrema by second derivative test for $f(x) = x^4 - x^2$.



Solution: $f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$

$$f''(x) = 12x^2 - 2$$

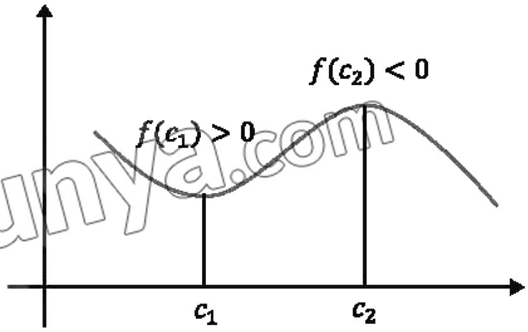
For critical values take $f'(x) = 0$

$$2x(2x^2 - 1) = 0$$

$$x = 0, \frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$$

The second derivative test is summarized as:

x	Sign of $f''(x)$	$f(x)$	Conclusions
0	$f''(x) = -ve$	0	Rel. max
$\frac{\sqrt{2}}{2}$	$f''(x) = +ve$	$\frac{-1}{4}$	Rel. min
$\frac{-\sqrt{2}}{2}$	$f''(x) = +ve$	$\frac{-1}{4}$	Rel. min



Exercise 2.9

1. Find the critical values of the function.

   i. $f(x) = 2x^2 - 6x + 8$ ii. $f(x) = x^3 + x - 2$

   iii. $f(x) = \frac{x}{x^2+2}$ iv. $f(x) = \cos 4x$

   v. $f(x) = (4x-3)^{\frac{1}{3}}$ vi. $f(x) = x^2(x+1)^3$

2. Find the absolute extrema of the function on the indicated interval.

   i. $f(x) = -x^2 + 6x : [1, 4]$ ii. $f(x) = (x-1)^2 : [2, 5]$

   iii. $f(x) = x^{\frac{2}{3}} : [-1, 8]$ iv. $f(x) = x^3 - 6x^2 + 2 : [-3, 2]$

   v. $f(x) = 1 + 5\sin 3x : \left[0, \frac{\pi}{2}\right]$ vi. $f(x) = 2\cos 2x - 4\cos x : [0, 2\pi]$

3. Use the second derivative to determine the intervals on which the function is concave upward and concave downward.

   i. $f(x) = -x^2 + 7x$ ii. $f(x) = -x^3 + 6x^2 + x - 1$

iii. $f(x) = (x + 5)^5$ iv. $f(x) = x(x - 4)^3$

v. $f(x) = x^{\frac{1}{2}} + 2x$ vi. $f(x) = x + \frac{9}{x}$

4. Use the second derivative to locate all points of inflection.

i. $f(x) = x^4 - x^3 + 2x^2 + x - 1$ ii. $f(x) = x^{\frac{5}{3}} + 4x$

iii. $f(x) = \sin x$ iv. $f(x) = \cos x$

v. $f(x) = x - \sin x$ vi. $f(x) = \tan x$

5. Use second derivative test to find the relative extrema of the function.

i. $f(x) = -(-2x - 5)^2$ ii. $f(x) = x^3 + 3x^2 + 3x + 1$

iii. $f(x) = 6x^5 - 10x^2$ iv. $f(x) = x^2 + \frac{1}{x^2}$

v. $f(x) = \cos 3x, [0, 2\pi]$ vi. $f(x) = \cos x + \sin x, [0, 2\pi]$

6. Determine whether the give function has a relative extremum at the indicated points.

i. $f(x) = \cos x \sin x,\ x = \frac{\pi}{4}$ ii. $f(x) = x \sin x,\ x = 0$

iii. $f(x) = \tan^2 x,\ x = \pi$ iv. $f(x) = (1 + \sin x)^3,\ x = \frac{\pi}{8}$

2.22 Applications of Derivatives

Many real world phenomenon involve changing quantities like the speed of the rocket, the inflation of currency, the number in a bacteria in a culture, the stoke intensity of an earth quake, the voltage of an electrical signal and so forth. In this section we will develop the concept of limits, continuity, derivative and extrema of function for use in real world problems. Another important application of the derivative is to find solution of the optimization problems. For example, if time is the main consideration in a problem, we might be interested in finding the quickest way to perform a task and if cost is the main consideration we might be interested in finding the least expensive way to perform a task. Mathematically, optimization problem can be reduced to finding the largest or smallest value of a function on some interval and determining where the largest and smallest values occurs. Using derivatives, we will develop the mathematical tools necessary for solving such problems.

Example 50: A side of a cube is measured to be $30cm$ with the possible error of $\pm 0.02cm$. What is the approximate maximum possible error in the volume of the cube?

Solution: The volume of a cube is $V = x^3$, where x the length of one side. If Δx represents the error in the length of one side, then the corresponding error in the volume is:

$\Delta V = (x + \Delta x)^3 - x^3$

We use differential: $dv = 3x^2 dx = 3x^2 \Delta x$

as an approximate to ΔV. Thus, for $x = 30$ and $\Delta x = \pm 0.02$, the approximate maximum error is:

$dv = 3(30)^2(\pm 0.02) = \pm 54cm^3$

Example 51: A square is expanding with time. What is the rate at which the area increases related to the rate at which a side increases?

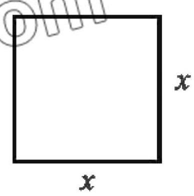
Solution: At any time the area A of a square is a function of length of one side of x:

$$A = x^2$$

Thus, the related rates are derived from the time derivative.

$$\frac{dA}{dt} = 2x\frac{dx}{dt} \text{ (diff w.r.t "t")}$$

is the same as: $\frac{dA}{dt} = 2x\frac{dx}{dt}$



Example 52: Air is being pumped into a spherical baloon at a rate of 20 ft^3/min. At what rate is the radius changing with the radius is 3ft?

Solution: As shown in fig, we denote the radius of the baloon by r and its volume by V. As per statement, air is being pumped at the rate $20ft^3/min$, means we have: $\frac{dV}{dt} = 20ft^3/min$

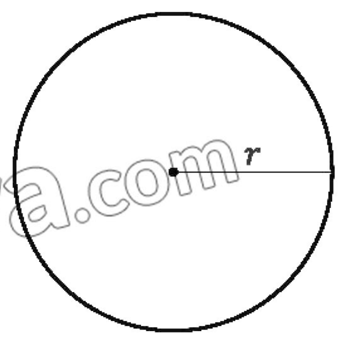
In addition, we require $\left.\frac{dr}{dt}\right|_{r=3}$

We know the relation between V and r is $V = \frac{4}{3}\pi r^3$

Diff w.r.t "t"

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$



But $\frac{dV}{dt} = 20$, therefore $20 = 4\pi r^2\frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{5}{\pi r^2}$$

Thus, $\left.\frac{dr}{dt}\right|_{r=3} = \frac{5}{9\pi}\frac{ft}{min} = 0.18\,ft/min$

Example 53: Find two non-negative numbers whose sum is 15 such that the product of one with the square of other is a maximum.

Solution: Let x and y denote the two non-negative numbers that is, $x \geq 0$ and $y \geq 0$. It is given that: $x + y = 15$ ....(i)

Let p denote the product: $p = x.y^2$ (Product = one number. square of the other)

We can use $y = 15 - x$ to express p in terms of x: $p(x) = x(15 - x)^2$

The function $p(x)$ defined any for $0 \leq x \leq 15$.

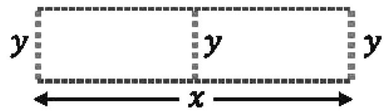
If $x > 15$, then $y = 15 - x$ would be negative.

$p'(x) = x.2(15 - x)(-1) + (15 - x)^2 = (15 - x)(15 - 3x)$

Thus, any critical value is $x = 5$.

Testing the end points of the interval reveal $p(0) = p(15) = 0$, is the minimum value of the product. Hence, $p(5) = 5(10)^2 = 500$ must be the maximum value. The two non-negative numbers are 5 and 10.

Example 54: A rectangular plot of land that contain 1500 m^2 will be fenced and divided into equal portions by any additional fence parallel to two sides. Find the dimensions of the land that require the least amount of fencing.



Solution: Let us introduce variable x and y so that $xy = 1500$. Then the function we wish to minimize is the sum of the lengths of the five portions of the fence.

$L = 2x + 3y$

But $y = \dfrac{1500}{x}$, we have

$L(x) = 2x + \dfrac{4500}{x}$

$L'(x) = 2 - \dfrac{4500}{x^2}$

For critical value, $L'(x) = 0$

$x^2 = 2250$

$x = 15\sqrt{10}$

For 2nd derivative: $L''(x) = \dfrac{13500}{x^3}$

When $x = 15\sqrt{10}$

$L''(15\sqrt{10}) > 0$

Hence $x = 15\sqrt{10}m$, is required minimum amount of fencing.

So,

$L(15\sqrt{10}) = 2(15\sqrt{10}) + \dfrac{4500}{15\sqrt{10}} = 15\sqrt{10}$

$xy = 1500$

$y = \dfrac{1500}{x} = \dfrac{1500}{15\sqrt{10}}$

$y = 10\sqrt{10}m$

Dimension of land:

$xy = 15\sqrt{10} \times 10\sqrt{10}$

Price Growth Model: The price level at time P, considering inflation can be modeled as: $P(t) = P_0e^{rt}$, where, $P(t)$= Price at time t, P_0 = Initial price, continuous annual inflation rate,

t= Time(in years)

To find the rate of change of price with respect to time, take the derivative of P(t) with respect to t. $\frac{d}{dt}P(t) = P_0re^{rt}$, The $\frac{dP(t)}{dt}$, represents the instantaneous rate of change of the price level or how fast prices are increasing at a time t.

Example 55: The price of a product is modeled as: $P(t) = 200e^{0.03t}$, where t is the time in years and P (t) is the price at a time t. Find the rate at which the price is increasing after:

a. 0 years b. 5 years c. 10 ears

Solutions: The price inflation is $P(t) = 200e^{0.03t}$, the derivative gives the rate of price increase:

$$\frac{dP(t)}{dt} = 200(0.03)e^{0.03t} = 6e^{0.03t}$$

a. At t = 0, $\frac{dP(0)}{dt} = 200(0.03)e^{0.03(0)} = 6e^{0} = 6$ units/year

b. At t = 5, $\frac{dP(5)}{dt} = 200(0.03)e^{0.03(5)} = 6e^{0.15} = 6.92$ units/year

c. At t = 10, $\frac{dP(10)}{dt} = 200(0.03)e^{0.03(10)} = 6e^{0.3} = 8.10$ units/year

Using Straight Lines: Derivatives help analyze a line relationship in real life scenarios. Straight lines appear in situations, where variables change at a constant rate and derivatives calculate the rate or optimize related process.

Example 56: Economics, Marginal Cost and Revenue: A company's Revenue R(x) from selling x units is given by: $\text{R}(x) = 50x$ The total cost $\text{C}(x)$ for producing x units is: $\text{C}(x) = 30x + 200$

a. Find the marginal revenue and marginal cost.
b. Determine the break-even point (units sold where revenue equals cost).
c. Interpret the meaning of the straight-line equations and slopes.

Solutions: a. Marginal Revenue and Marginal Cost:

- Marginal Revenue $(\text{R}'(x))$: $\text{R}'(x) = \frac{d}{dx}(50x) = 50$, revenue increases by 50/units.
- Marginal Cost $(\text{C}'(x))$: $C'(x) = \frac{d}{dx}(30x + 200) = 30$, cost increase by 30/units.

b. Break-Even Point: At break even, revenue equal costs:

$R(x) = C(x)$

$50x = 30x + 200$, which gives, $x = 10$.

The company breaks even when 10 units are sold.

c. Interpretation of Slopes:

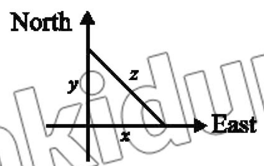
- The slopes of $R(x)$ is 50, showing revenue grows faster than cost.
- The slopes of $C(x)$ is 30, representing slower cost growth.

Exercise 2.10

1. According to Einstein's theory of relativity, the mass m of a body moving with velocity v is $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, where m_0 is the initial mass and c is the speed of light. What happens to m as $v \to c^-$?
2. $f(x) = \begin{cases} kx+1, & x \le 3 \\ 2-kx, & x > 3 \end{cases}$ is continuous at 3. What is K?
3. The volume v of a sphere of radius r is $v = \left(\frac{4\pi}{3}\right)r^3$. Find the surface area s of the sphere if s is the instantaneous rate of change of the volume with respect to the radius.
4. The height S above ground of a projectile time t is given by

   $S(t) = \frac{1}{2}gt^2 + v_0 t + s_0$.

   Where g, v_0 and s_0 are constants. Find the instantaneous rate of change of S with respect to t at $t = 4$.
5. The side of a square is measured to be 10cm with a possible error of ± 0.3 cm. Use differentials to find an approximation to the maximum error in the area. Find the approximate relative error and the approximate error.
6. A woman jogging at a constant rate of 10km/hr crosses a point A heading north. Ten minutes later a man jogging at a constant rate of 9km/hr crosses the same point heading east. How fast is the difference between the joggers hanging 20 minutes after the man crosses A?



7. A plate in the shape of an equilateral triangle is expanding with time. A side increases at a constant rate of 2cm/hr. At what rate is the area increasing when side in 8cm?

8. A rectangle expands with time. The diagonal of the rectangle increases at a rate of 1in/hr and length increases at a rate of $\frac{1}{4}$ min/hr. How fast is its width increasing when the width is 6 in and length is 8 in?
9. The side of a cube increases at a rate of 5cm/hr. At what rate does the diagonal of the cube increases?
10. A particle moves on a graph of $y^2 = x + 1$ so that $\frac{dx}{dt} = 4x + 4$. What is $\frac{dx}{dt}$ when $x = 8$?
11. At 8:00 am ship S_1 is 20km due north of S_2. Ship S_1 sails south at a rate of 9km/hr and S_2 sails west at a rate of 12 km/hr at 9:20 am. At what rate is the distance between the two ships changing?
12. Find two non-negative numbers whose sum is 60 and whose product is a maximum?
13. If the total fence to be used is 8000 m, find the dimensions of the enclosed land in figure that has the greatest area.

14. An open rectangular box is to be constructed with a square base and a volume of 32,000 cm^3. Find the dimensions of box that require the least amount of material.
15. A company determines that for the production of x units of a commodity its remove and cost functions are, respectively, $R(x) = -3x^2 + 970x$ and $G(x) = 2x^2 + 500$. Find the maximum profit and minimum average cost.
16. If the inflation rate is continuously compounded 4% per year and the price of a commodity is $50 today.
    a. Derive the function for the price of the commodity over time.
    b. Find the price after 8 years.
    c. Find the instantaneous rate of price at t=8 years.
17. A company models its operational cost as: $C(t) = 500e^{0.04t} - 100t$, where t is the time in years.
    a. Find the rate of change of cost at any time t.
    b. Determining the rate of increase in cost is minimal.
18. The price of commodity $P(t)$ is given by: $P(t) = 150(1 + 0.05t)^2$, where t is measured in years and $P(t)$ is the price level.
    a. Find the instantaneous rate of change of prices at t= 3years.
    b. Calculate the inflation rate at t=3 years.

19. A ship sails in a straight line. Its distance $d(t)$ (in nautical miles) from port is modeled by: $d(t) = 15t$ where t is time in hours.

a. Find the speed of the ship. b. Calculate the distance after 3 hours. c. Explain the meaning of the slope.

20: A cyclist is traveling along a straight path, and the distance traveled s(t) (in meters) is given by: $s(t) = 5t^2 + 3t$, where, t is the time in seconds.

a. Find the speed at any time t.
b. Determine the speed at t=4 seconds.
c. Interpret the significance of the slope in this context.

Review Exercise

1. Tick the correct options.

i. If $\lim_{x \to a} f(x) = 3$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$:

a. 3 b. 0 c. Exist d. Doesn't exist

ii. If $f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x > 0 \end{cases}$, then $\lim_{x \to 0^-} f(x) = 0$, is:

a. 1 b. -1 c. 0 d. 2

iii. If f and g are continuous at 2, then $\frac{f}{g}$ is continuous at:

a. 0 b. 1 c. 2 d. 3

iv. The function $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$, is continuous at:

a. 0 b. 1 c. -1 d. 0.1

v. If f is differentiable for every value of x, then f is:

a. Discontinuous b. Continuous
c. Finite d. Infinite

vi. If k is a constant and n is positive integer, then $\frac{d}{dx} k^n$ is:

a. nk^{n-1} b. k^{n-1} c. $\ln n . k^n$ d. 0

vii. If $f(2) = 2, g(x) = x^2$, then $\frac{d}{dx}\left[\frac{3g(x)}{f(2)}\right]$ is:

a. $2x$ b. $3x$ c. $\frac{3}{2}x$ d. $\frac{3}{2}x^2$

viii. If $y = f(x)$ is a polynomial function of degree 2, then $\frac{d^3}{dx^3}f(x)$ is:

a. 0 b. 1 c. -1 d. 2

ix. If f is differentiable for every value of x, then f is continuous for:

a. Some value of x b. $[0, \infty]$

c. Every value of x d. $[0, -\infty]$

x. If $f(t) = 2t^3$ is absolute minimum at:

a. 3 b. 0 c. -1 d. 1

2. Evaluate. $\lim_{x\to 0}\frac{x^3}{sin^2 3x}$
3. Find the derivative. $y = \frac{1+\sin x}{x\cos x}$
4. If $f'(0) = -1$ and $g'(0) = 6$, what is $\frac{d^2}{dx^2}[xf(x) + xg(x)]$ at $x = 0$?
5. Find $\frac{d^2y}{dx^2}$, when $x^3 + y^3 = 27$
6. Use differential to find an approximate of $\sqrt{65}$.
7. An oil storage tank in the form of circular cylinder has a height of 5 m. The radius is measured to be 8 m with a possible error of ± 0.25 m. Use differentials to estimate the maximum error in the volume. Find the approximate relative error and the approximate percentage error.
8. A 15 ft ladder is leaning against a wall of a house. The bottom of the ladder is pulled away from the base of the wall at a constant rate of 2 ft/min. At what rate is the top of the ladder sliding down the wall when the bottom of the ladder is 5 ft from the wall?
9. Find the absolute extrema of $f(x) = x^3 - 3x^2 - 24x + 2$,

   a. $[-3, 1]$ b. $[-3, 8]$
10. Graph the function: $f(x) = x + \frac{1}{x}$

UNIT
03

INTEGRATION

After studying this unit, students will be able to:

- Find the general antiderivative of a given function.
- Recognize and use the terms and notations for antiderivatives.
- State the power rule of integrals.
- State and apply the properties of indefinite integrals.
- Integrate functions involving the exponential and logarithmic functions.
- Identify when to use integration by parts to solve integration problems.
- Apply the integration-by-part formula for definite integrals.
- Solve integration problems involving trigonometric substitution.
- Integrate a rational function using the method of partial fraction.
- State the definition of definite integral.
- Explain the terms integrand, limits of integration and value of integration.
- State and apply the properties of definite integrals.
- State and apply fundamental theorem of calculus to evaluate the definite integrals.
- Describe the relation between the definite integral and net area.
- Find the area of a region bounded by a curve and lines parallel to axes, or between a curve and a line or between two curves.
- Find volume of the revolution about one of the axes.
- Demonstrate trapezium rule to estimate the value of a definite integral.
- Apply concept of integration to real world problems such as volume of a container, consumer and producer surplus, growth rate of a population, investment return time period, drug dosage required by integrating the concentration.

There is a lot of applications of integration in various fields. For example, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us. A view of Tarbela dam is shown below.

3.1 Integration

This unit examines the process by which we determine functions from their derivatives. We are already familiar with inverse operations. For example, addition and subtraction are inverse of each other. Similarly, multiplication and division are inverse of each other. In the same way, the inverse operation of differentiation is anti-differentiation or integration.

This unit provides two processes and their relationship to one another. One step is to find function from their derivatives. In the second step, we can determine things like area and volume through successive approximations. This process is called integration. This is very important area in mathematics and was discovered independently by Leibnitz and Newton.

The process of finding a function from one of its known values and its derivative $f(x)$ has two steps:

The first is to find a formula that gives us all the functions that could possibly have $f(x)$ as a derivative. If $f'(x)$ is defined as derivative, then $f(x)$ is called anti-derivative and the formula that gives them all is called the indefinite integral of $f(x)$. The reverse process of derivative or anti-differentiation is the main topic of this unit.

Definition 3.1:

A function $F'(x)$ is called an anti-derivative of another function $f(x)$ is on a given interval if:

$$F'(x) = f(x)$$

For example, the functions:

$$\frac{1}{4}x^4, \quad \frac{1}{4}x^4 + 3, \quad \frac{1}{4}x^4 - \pi, \quad \frac{1}{4}x^4 + c \quad (c \text{ is any constant.})$$

are ani-derivatives of x^3 on the interval $(-\infty, \infty)$ since the derivative of each is x^3.

Above example shows that a function can have many anti-derivatives. In fact, if $F(x)$ is any anti-derivative of $f(x)$ and c is any constant, then $F(x)+c$ is also an anti-derivative of $f(x)$ since:

$$\frac{d}{dx}[F(x)+c] = \frac{d}{dx}[F(x)] + \frac{d}{dx}[c] = f(x) + 0 = f(x)$$

Therefore, if $F(x)$ is any anti-derivative of $f(x)$ on a given interval, then for any value of c, the function $F(x)+c$ is also an anti-derivative of $f(x)$ on that interval.

Symbolically we write:

$$\int f(x)dx = F(x) + c$$

Where the symbol, "$\int$" is called 'integral sign' and $f(x)$ is called integrand. The symbol dx indicates that the integration is performed with respect to the variable x. The arbitrary constant c is called 'constant of integration'.

For Example,

As, $\frac{d}{dx}(x^4) = 4x^3$

Therefore, $\int 4x^3 dx = x^4 + c$

As mentioned above, the constant c is arbitrary constant. Therefore,

$x^4, x^4+1, x^4-\sqrt{2}, x^4+\pi$ etc. all are anti-derivatives of $4x^3$.

Let us derive some basic and common integral formulae with the help of differentiation.

Key Facts

- The variable other than x, can also be used in indefinite integrals.
- A number of indefinite integral formulae are found by reversing derivative formulas.

Formula 3.1: $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$

Derivation: We have,

$$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1} + c\right] = \frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] + \frac{d}{dx}[c] = \frac{(n+1)\,x^n}{n+1} + 0 = x^n \qquad \text{(i)}$$

Integrating both sides of (i) with respect to x, we have:

$$\int \frac{d}{dx}\left[\frac{x^{n+1}}{n+1} + c\right]dx = \int x^n dx$$

$$\frac{x^{n+1}}{n+1} + c = \int x^n dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$$

In general,

$$\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1$$

Formula 3.2: $\int \frac{1}{x} dx = ln\, x + c$

Derivation: We have,

$$\frac{d}{dx}[ln\, x + c] = \frac{1}{x} \qquad \text{(ii)}$$

Integrating both sides of (ii) with respect to x, we have:

$$\int \frac{d}{dx}[ln\, x + c]dx = \int \frac{1}{x} dx$$

$$ln\, x + c = \int \frac{1}{x} dx$$

$$\int \frac{1}{x} dx = ln\, x + c$$

In general,

$$\int \frac{f'(x)}{f(x)} dx = ln[f(x)] + c$$

Formula 3.3: $\int e^x dx = e^x + c$

Derivation: As,

$$\frac{d}{dx}[e^x + c] = e^x \qquad \text{(iii)}$$

Integrating both sides of (iii) with respect to x, we have:

$$\int \frac{d}{dx}[e^x + c]dx = \int e^x dx$$

$$e^x + c = \int e^x dx$$

$$\int e^x dx = e^x + c$$

In general,

$$\int e^{f(x)} f'(x) dx = e^{f(x)} + c$$

Formula 3.4: $\int a^x dx = \frac{1}{\ln a} a^x + c,\ a > 0,\ a \neq 1$

Derivation: As,

$$\frac{d}{dx}\left[\frac{1}{\ln a} a^x + c\right] = a^x \qquad \text{(iv)}$$

Integrating both sides of (iv) with respect to x, we have:

$$\int \frac{d}{dx}\left[\frac{1}{\ln a} a^x + c\right] dx = \int a^x dx$$

$$\frac{1}{\ln a} a^x + c = \int a^x dx$$

$$\int a^x dx = \frac{1}{\ln a} a^x + c$$

In general,

$$\int a^{f(x)} f'(x) dx = \frac{1}{\ln a} a^{f(x)} + c$$

Theorem 3.1:

(i) A constant factor can be moved through an integral sign. That is:

$$\int c\, f(x) dx = c \int f(x) dx$$

(ii) An anti-derivative of a sum is the sum of anti-derivatives. That is:

$$\int [f(x) + g(x)]\, dx = \int f(x) dx + \int g(x) dx$$

(iii) An anti-derivative of a difference is the difference of anti-derivatives. That is:

$$\int [f(x) - g(x)]\, dx = \int f(x) dx - \int g(x) dx$$

(iv) In general, $\int [af(x) \pm bg(x)]\, dx = a \int f(x) dx \pm b \int g(x) dx$

Example 1: Evaluate (i) $\int(4x^7 - 2x^3 + 9x + 3)dx$ (ii) $\int \frac{y^3-2y^6}{y^5}\,dy$

Solution: (i) $\int(4x^7 - 2x^3 + 9x + 3)dx$

$= 4\int x^7 dx - 2\int x^3 dx + 9\int x\,dx + 3\int dx$

Integrating term by term, we get:

$= 4\left(\frac{x^8}{8}\right) - 2\left(\frac{x^4}{4}\right) + 9\left(\frac{x^2}{2}\right) + 3x + c = \frac{x^8}{2} - \frac{x^4}{2} + \frac{9x^2}{2} + 3x + c$

(ii) $\int \frac{y^3-2y^6}{y^5}\,dy = \int\left(\frac{y^3}{y^5} - \frac{2y^6}{y^5}\right)dy = \int\left(\frac{1}{y^2} - 2y\right)dy$

$= \int(y^{-2} - 2y)\,dy = \int y^{-2}\,dy - 2\int y\,dy$

$= \frac{y^{-2+1}}{-2+1} - 2\left(\frac{y^2}{2}\right) + c = -\frac{1}{y} - y^2 + c$

Example 2: Evaluate (i) $\int \frac{ax + \frac{1}{2}b}{ax^2+bx+c}\,dx$ (ii) $\int e^{3x}dx$

Solution: (i) $\int \frac{ax + \frac{1}{2}b}{ax^2+bx+c}\,dx = \frac{1}{2}\int \frac{2ax + b}{ax^2+bx+c}\,dx$

$= \frac{1}{2}\ln\left(ax^2 + bx + c\right) + C$

(ii) $\int e^{3x}dx = \frac{1}{3}\int e^{3x}(3)dx = \frac{1}{3}e^{3x} + c$

Example 3: Evaluate $\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$

Solution: Here, $f(x) = \sin^{-1}x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$

So, by using formula:

$\int e^{f(x)}f'(x)dx = e^{f(x)} + c$

We have:

$\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx = e^{\sin^{-1}x} + c$

Exercise 3.1

Evaluate the following integrals.

1. $\int(x^2 - 3x + 9)dx$
2. $\int\left(y^2 + 8y + \sqrt{2}\right)dy$
3. $\int\left(\sqrt{y} + \frac{1}{y^2}\right)dy$
4. $\int(4 + x^2)^2dx$
5. $\int(1 + x)(1 - x^2)dx$
6. $\int\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)dx$
7. $\int(e^{4x} - e^{-1} + 1)dx$
8. $\int(e^{\frac{9}{2}x} + \frac{1}{x})dx$
9. $\int x\,e^{x^2}dx$
10. $\int 5^x dx$
11. $\int 7^{7y}dy$
12. $\int(x^3 + \frac{1}{2x} - \frac{1}{x^3})dx$

13. $\int \frac{2x+1}{x^2+3} dx$ 14. $\int \frac{e^{tan^{-1}z}}{1+z^2} dz$ 15. $\int \left(x^{\frac{3}{2}} + e^{3x} + x^0\right) dx$

16. $\int (3x^2 + 2x)(x^3 + x^2 + 9)^5 dx$ 17. $\int (5e^{5x} - x^{-3} + 3^{2x}) dx$

18. $\int (z^{\frac{-1}{4}} + \sqrt{3z} + \frac{4}{z} - \frac{1}{e^z}) dz$

3.2 Integration of Trigonometric Functions

While evaluating the integration of trigonometric functions, keep in mind the following formulae.

As, $\frac{d}{dx}(\sin x + c) = \cos x$ therefore, $\int \cos x\, dx = \sin x + c$

Similarly, $\frac{d}{dx}(\cos x + c) = -\sin x$ implies, $\int \sin x\, dx = -\cos x + c$

In the same way, $\frac{d}{dx}(\sin kx + c) = k\cos kx$ implies, $\int \cos kx\, dx = \frac{\sin kx}{k} + c$

And, $\frac{d}{dx}(\cos kx + c) = -k\sin kx$ implies, $\int \sin kx\, dx = -\frac{\cos kx}{k} + c$

Using above pattern, following formulae can be deduced easily.

$\int \sec^2 x\, dx = \tan x + c$ and $\int \text{cosec}^2 x\, dx = -\cot x + c$

$\int \sec x \tan x\, dx = \sec x + c$ and $\int \text{cosec} x \cot x\, dx = -\text{cosec} x + c$

Example 4: Evaluate:

(i) $\int \sin 2x\, dx$ (ii) $\int \cos \frac{3x}{5}\, dx$ (iii) $\int \sec^2 mx\, dx$

(iv) $\int 5\, \text{cosec}^2 \frac{7x}{5}\, dx$ (v) $\int 9 \sec 3x \tan 3x\, dx$

Solution:

(i) $\int \sin 2x\, dx = \int \frac{1}{2} \sin 2x\,(2)\, dx = \frac{1}{2} \int \sin 2x\,(2)\, dx$

$= \frac{1}{2}(-\cos 2x + c) = \frac{-\cos 2x}{2} + \frac{c}{2} = \frac{-\cos 2x}{2} + C$

(ii) $\int \cos \frac{3x}{5}\, dx = \int \frac{5}{3} \cos \frac{3x}{5} \left(\frac{3}{5}\right) dx = \frac{5}{3} \int \cos \frac{3x}{5} \left(\frac{3}{5}\right) dx$

$= \frac{5}{3}\left(\sin \frac{3x}{5} + c\right) = \frac{5}{3} \sin \frac{3x}{5} + \frac{5}{3} c = \frac{5}{3} \sin \frac{3x}{5} + C$

(iii) $\int \sec^2 mx\, dx = \int \frac{1}{m} \sec^2 mx\,(m)\, dx = \frac{1}{m} \int \sec^2 mx\,(m)\, dx$

$= \frac{1}{m}(\tan mx + c) = \frac{1}{m}(\tan mx) + \frac{c}{m} = \frac{1}{m}(\tan mx) + C$

(iv) $\int 5\, \text{cosec}^2 \frac{7x}{5}\, dx = 5 \times \frac{5}{7} \int \text{cosec}^2 \frac{7x}{5} \left(\frac{7}{5}\right) dx = \frac{25}{7}\left(-\cot \frac{7x}{5} + c\right)$

$= -\frac{25}{7}\cot\frac{7x}{5} + \frac{25}{7}c \quad = -\frac{25}{7}\cot\frac{7x}{5} + C$

(v) $\int 9\sec 3x \tan 3x\, dx \quad = \int 3 \times 3 \sec 3x \tan 3x\, dx \quad = 3\int \sec 3x \tan 3x\,(3)dx$

$= 3(\sec 3x + c) \quad = 3\sec 3x + 3c \quad = 3\sec 3x + C$

Example 5: Prove that:

(i) $\int \sec x\, dx = ln|\sec x + \tan x| + c$

(ii) $\int \operatorname{cosec} x\, dx = ln|\operatorname{cosec} x - \cot x| + c$

(iii) $\int \tan x\, dx = -ln(\cos x) + c = ln(\sec x) + c$

Solution:

(i) $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx$ [Multiplying and dividing by $(\sec x + \tan x)$]

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}dx$

$= \int \frac{\frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x}dx = ln|\sec x + \tan x| + c$

$\int \frac{f'(x)}{f(x)}dx = ln[f(x)] + c$

(ii) $\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}dx$

$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x - \cot x}dx = \int \frac{-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x}{\operatorname{cosec} x - \cot x}dx$

$= \int \frac{\frac{d}{dx}(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}dx = ln|\operatorname{cosec} x - \cot x| + c$

(iii) $\int \tan x\, dx \quad = \int \frac{\sin x}{\cos x}dx \quad = -\int \frac{-\sin x}{\cos x}dx \quad = -\int \frac{\frac{d}{dx}(\cos x)}{\cos x}dx$

$= -ln(\cos x) + c \quad = ln(\cos x)^{-1} + c$

$= ln\frac{1}{\cos x} + c \quad = ln(\sec x) + c$

Check Point

Prove that

$\int \cot x\, dx = ln(\sin x) + c$

3.2.1 Integration of $\sin^2 x$ and $\cos^2 x$

Sometimes it is difficult to evaluate integrals directly. Using trigonometric identities, we can easily evaluate integrals. For example, the integrals of $\sin^2 x$ and $\cos^2 x$ cannot be solved directly and can be handled using following relations.

$\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$

Check Point

Evaluate $\int \cos^2 x\, dx$

Example 6: Evaluate $\int \sin^2 x\, dx$

Solution: $\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$

$= \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + c = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

Example 7: Integrate (i) $8\sec 9x - \tan 3x$ (ii) $\cos^2 7x$

Solution:

(i) $\int (8\sec 9x - \tan 3x)dx = \int 8\sec 9x\, dx - \int \tan 3x\, dx$

$= \frac{8}{9}\int \sec 9x\,(9)dx - \frac{1}{3}\int \tan 3x\,(3)dx$

$= \frac{8}{9} ln|\sec 9x + \tan 9x| - ln(\sec 3x) + c$

(ii) $\int \cos^2 7x\, dx = \int \frac{1+\cos 14x}{2}\, dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 14x\, dx$

$= \frac{1}{2}x + \frac{1}{2}\left(\frac{\sin 14x}{14}\right) + c = \frac{1}{2}x + \frac{1}{28}\sin 14x + c$

Exercise 3.2

Evaluate the integrals and recheck your answer by differentiating.

1. $\int (\sin\pi x - 3\sin 3x)dx$
2. $\int -\sec^2\left(\frac{3}{2}y\right)dy$
3. $\int [1 - 8\text{cosec}^2(2x)]dx$
4. $\int \frac{1}{2}(\text{cosec}^2 x - \text{cosec}x\cot x)dx$
5. $\int \frac{\cos^2 z}{7}\,dz$
6. $\int (1 + \tan^2\theta)d\theta$
7. $\int \frac{1+\cos 4t}{2}dt$
8. $\int \sec^2(5x - 1)\,dx$
9. $\int (\tan 5x + \cos 7x)dx$
10. $\int (\cot 9y - 3)\,dy$

Evaluate the integral.

11. $\int (\tan^2 2\theta + \cot^2 2\theta)d\theta$
12. $\int \sin^2\left(\frac{11}{2}y\right)dy$
13. $\int \text{cosec}11x\tan 11x\, dx$
14. $\int \cos\theta(\tan\theta + \sec\theta)d\theta$
15. $\int \text{cosec}^2\left(\frac{x-1}{3}\right)dx$
16. $\int (\cos x)^{\frac{1}{5}}\sin x\, dx$
17. $\int e^y \sin e^y dy$
18. $\int 9\tan(x + 7)dx$

3.3 Integration by Substitution

There are many functions that cannot be integrated by simple techniques and can be integrated easily by using method of substitution. It is an integration technique which involves making a substitution to simplify the integral. In this method any given integral is transformed into a simple form of integral by substituting the independent variable by others. The exact substitution depends on the form of the given integral, as some substitutions are more appropriate for certain problems than others. The choice of substitution is not always immediately obvious. The ability to recognise an appropriate substitution comes from practising many different examples.

Mostly, we substitute trigonometric functions in place of variables to integrate algebraic functions. However, there is no hard and fast rule for selection of trigonometric functions to replace variables as some other substitutions are also used.

Key Facts

Usually, the method of integration by substitution is extremely useful when we make a substitution for a function whose derivative is also present in the integrand. Doing so, the function simplifies and then the basic formulas of integration can be used to integrate the function.

Example 8: Evaluate $\int 3x^2 \cos(x^3)\, dx$

Solution:

In the equation given above the independent variable can be transformed into another variable say t by substituting:

$x^3 = t$ (i)

Differentiation of (i) gives:

$3x^2 dx = dt$ (ii)

Substituting the values of (i) and (ii) in the given integral.

$\int 3x^2 \cos(x^3)\, dx = \int \cos t\, dt = \sin t + c$

Again, substituting back the value of t, we get:

$\int 3x^2 \cos(x^3)\, dx = \sin(x^3) + c$

Key Facts

The method of substitution to find an integral is used when it is set up in the special form.

$\int f(g(x)).g'(x).dx = \int f(t).dt$

where $t = g(x)$

Check Point

Integrate:

$x \sin(x^2 - 3)$ with respect to x.

Example 9: Integrate: $\int \frac{e^{\tan^{-1}x}}{1+x^2} dx$

Solution: Let $u = \tan^{-1}x$ then $du = \frac{1}{1+x^2} dx$

Therefore, $\int \frac{e^{\tan^{-1}x}}{1+x^2} dx = \int e^u du = e^u + c = e^{\tan^{-1}x} + c$

Formula 3.5: $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Derivation: Substituting $x = a\sin\theta$, we have $dx = a\cos\theta\, d\theta$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \int \frac{1}{\sqrt{a^2-(a\sin\theta)^2}} a\cos\theta\, d\theta = \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}} a\cos\theta\, d\theta$$

$$= \int \frac{1}{a\sqrt{1-\sin^2\theta}} a\cos\theta\, d\theta = \int \frac{1}{\cos\theta} \cos\theta\, d\theta$$

$$= \int d\theta = \theta + c = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

Note: We can apply the formula directly too.

Formula 3.6: $\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2} + c$

Derivation: Substituting $x = a\sin\theta$, we have $dx = a\cos\theta\, d\theta$

$$\therefore \int \sqrt{a^2-x^2}\, dx = \int \sqrt{a^2-(a\sin\theta)^2}\, a\cos\theta\, d\theta = \int \sqrt{a^2-a^2\sin^2\theta}\, a\cos\theta\, d\theta$$

$$= \int a\sqrt{1-\sin^2\theta}\, a\cos\theta\, d\theta = \int a\cos\theta\, a\cos\theta\, d\theta = a^2\int \cos^2\theta\, d\theta$$

$$= a^2\int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta = \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\left(\frac{\sin 2\theta}{2}\right) + c = \frac{a^2}{2}\theta + \frac{a^2}{2}\left(\frac{2\sin\theta\cos\theta}{2}\right) + c$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\left(\sin\theta\sqrt{1-\sin^2\theta}\right) + c = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right) + c$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}\right) + c = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2} + c$$

Example 10: Evaluate: $\frac{1}{\sqrt{5-4x-x^2}}$

Solution: $\int \frac{1}{\sqrt{5-4x-x^2}}\,dx = \int \frac{1}{\sqrt{5+4-4-4x-x^2}}\,dx = \int \frac{1}{\sqrt{9-(4+4x+x^2)}}\,dx$

$$= \int \frac{1}{\sqrt{(3)^2-(2+x)^2}}\,dx = \sin^{-1}\left(\frac{2+x}{3}\right) + c \qquad \text{(Using direct formula)}$$

Note: We can also solve by substituting $x + 2 = 3\sin\theta$

Formula 3.7: $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = ln\left(x+\sqrt{x^2-a^2}\right) + C$

Derivation: Substituting $x = a\sec\theta$, we have $dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \int \frac{1}{\sqrt{(a\sec\theta)^2-a^2}}\, a\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{a^2\sec^2\theta-a^2}}\, a\sec\theta\tan\theta\, d\theta = \int \frac{1}{\sqrt{a^2(\sec^2\theta-1)}}\, a\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{1}{a\tan\theta}\, a\sec\theta\tan\theta\, d\theta = \int \sec\theta\, d\theta = ln[\sec\theta+\tan\theta] + c$$

$$= ln\left[\sec\theta+\sqrt{\sec^2\theta-1}\right] + c = ln\left[\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right] + c = ln\left[\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right] + c$$

$$= ln\left[\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right] + c = ln\left[\frac{x+\sqrt{x^2-a^2}}{a}\right] + c = ln\left(x+\sqrt{x^2-a^2}\right) - lna + c$$

$$= ln\left(x+\sqrt{x^2-a^2}\right) + (c - lna) = ln\left(x+\sqrt{x^2-a^2}\right) + C$$

Note: Expression $\frac{1}{\sqrt{x^2-a^2}}$ can also be integrated by making the substitution $x = acosh\theta$.

Formula 3.8: $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = ln\left(x+\sqrt{a^2+x^2}\right) + C$

Derivation: Substituting $x = a\tan\theta$, we have $dx = a\sec^2\theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \int \frac{1}{\sqrt{(a\tan\theta)^2+a^2}}\, a\sec^2\theta\, d\theta = \int \frac{1}{\sqrt{a^2\tan^2\theta+a^2}}\, a\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{a^2(\tan^2\theta+1)}}\, a\sec^2\theta\, d\theta = \int \frac{1}{a\sec\theta}\, a\sec^2\theta\, d\theta = \int \sec\theta\, d\theta$$

$= ln[\sec\theta + \tan\theta] + c = ln[\tan\theta + \sec\theta] + c = ln\left[\tan\theta + \sqrt{1+\tan^2\theta}\right] + c$

$= ln\left[\frac{x}{a} + \sqrt{1+\frac{x^2}{a^2}}\right] + c = ln\left[\frac{x}{a} + \sqrt{\frac{a^2+x^2}{a^2}}\right] + c = ln\left[\frac{x}{a} + \frac{\sqrt{a^2+x^2}}{a}\right] + c$

$= ln\left[\frac{x+\sqrt{a^2+x^2}}{a}\right] + c = ln\left(x+\sqrt{a^2+x^2}\right) - lna + c$

$= ln\left(x+\sqrt{a^2+x^2}\right) + (c - lna) = ln\left(x+\sqrt{a^2+x^2}\right) + C$

Formula 3.9: $\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

This formula can easily be proved by substituting $x = a\tan\theta$.

Example 11: Evaluate: $\int \frac{1}{x^2+4x+5}\, dx$

Solution: $\int \frac{1}{x^2+4x+5}\, dx = \int \frac{1}{x^2+4x+4+1}\, dx = \int \frac{1}{(x+2)^2+(1)^2}\, dx$

$= \frac{1}{1}\tan^{-1}\left(\frac{x+2}{1}\right) + c = \tan^{-1}(x+2) + c$ (Using direct formula)

Note: We can also solve by substituting $x + 2 = \tan\theta$

Example 12: Evaluate: $\int x(x^2 - a^2)^{\frac{3}{2}}\, dx$

Solution: Putting $x^2 - a^2 = u$

$\Rightarrow 2xdx = du \Rightarrow xdx = \frac{du}{2}$

$\therefore \int x(x^2-a^2)^{\frac{3}{2}}\, dx = \int (u)^{\frac{3}{2}}\frac{du}{2} = \frac{1}{2}\int (u)^{\frac{3}{2}}\, du$

$= \frac{1}{2}\frac{(u)^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \frac{1}{2}\times\frac{u^{\frac{5}{2}}}{\frac{5}{2}} = \frac{1}{5}(x^2-a^2)^{\frac{5}{2}} + c$

Exercise 3.3

Use suitable substitution, to evaluate the integrals.

1. $\int \frac{dx}{x^2+9}$
2. $\int \frac{dx}{\sqrt{5-x^2}}$
3. $\int (2x+7)(x^2+7x+3)^{\frac{4}{5}}\, dx$
4. $\int \frac{x^2}{x^3+1}\, dx$
5. $\int \frac{dy}{y^2+8y+20}$
6. $\int \frac{dx}{\sqrt{20-x^2-4x}}$
7. $\int \frac{x\, dx}{(4x^2+1)^3}$
8. $\int x^4\sqrt{3x^5-5}\, dx$
9. $\int \frac{2ax+b}{ax^2+bx+c}\, dx$
10. $\int \frac{dx}{(1-3x)^2}$
11. $\int \frac{z^3}{1+z^4}\, dz$
12. $\int \frac{\cot^{-1}x}{1+x^2}\, dx$

3.4 Integration by Parts

Integration by parts is a special method of integration that is very helpful technique to evaluate a wide variety of integrals that sometimes do not fit any of the basic integration formula. This method is used to find the integrals by reducing them into standard forms.

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]\,dx \qquad (1)$$

Formula (1) is called the formula for integration by parts. Using this formula, we integrate the product of two functions. The important thing to use this formula is the selection of given functions given in the product as a first or second function. The function whose integration can easily be found is considered as the second function while the first function is chosen whose derivative could be easily found. In formula (1), $f(x)$ is treated as first function while $g(x)$ as a second function.

Key Facts

- Integration by parts is not applicable for functions such as $\int \sqrt{x}\,\sin x\,dx$.
- We do not add any constant while finding the integral of the second function.
- Usually, if any function is a power of x or a polynomial in x, then we take it as the first function. However, if the other function is an inverse trigonometric function or logarithmic function, then we take them as first function.
- If the product of functions contains exponential and trigonometric functions, then we can select any one of the two as a first function.

Example 13: Evaluate the integral: $\int xe^x\,dx$

Solution: In the integral $\int xe^x\,dx$, we take 'x' as a first function as its derivative will reduce it and 'e^x' as second function.

$$\therefore \quad \int xe^x\,dx = x\int e^x dx - \int\left[\frac{d}{dx}(x)\int e^x dx\right]dx$$

$$= xe^x - \int 1.e^x dx = xe^x - e^x + c$$

Example 14: Evaluate: (i) $\int x^2\, lnx\,dx$ (ii) $\int x\tan^{-1}x\,dx$

Solution:

(i) In the integral $\int x^2\, lnx\,dx$, we take 'lnx' as first function and 'x^2' as second function.

$$\therefore \int x^2\, lnx\,dx = \int(lnx)\,(x^2)dx = lnx\int x^2dx - \int\left[\frac{d}{dx}(lnx)\int x^2dx\right]dx$$

$$= lnx\,.\frac{x^3}{3} - \int\frac{1}{x}\,.\frac{x^3}{3}dx = \frac{x^3\,lnx}{3} - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3\,lnx}{3} - \frac{1}{3}\,.\frac{x^3}{3} + c \quad = \frac{x^3\,lnx}{3} - \frac{x^3}{9} + c$$

(ii) In the integral $\int x\tan^{-1}x\,dx$, we take '$\tan^{-1}x$' as first function and 'x' as second function.

$$\therefore \int x\tan^{-1}x\,dx = \int(\tan^{-1}x)\,(x)\,dx$$

$$= \tan^{-1}x \int x dx - \int \left[\frac{d}{dx}(\tan^{-1}x) \int x dx\right] dx$$

$$= \tan^{-1}x \,.\frac{x^2}{2} - \int \frac{1}{x^2+1} \,.\frac{x^2}{2} dx = \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\int \frac{x^2}{x^2+1} dx$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right) dx \;=\; \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}(x - \tan^{-1}x) + c$$

Example 15: Apply integration by parts to evaluate:

(i) $\int \sqrt{a^2 - x^2}\; dx$ (ii) $\int \sqrt{a^2 + x^2}\; dx$

Solution:

(i) $\int \sqrt{a^2 - x^2}\; dx = \int \sqrt{a^2 - x^2}\; (1)\, dx,$

Here, we take '$\sqrt{a^2 - x^2}$' as first function and '1' as second function.

$\therefore \int \sqrt{a^2 - x^2}\; dx = \int \sqrt{a^2 - x^2}\,(1) dx$

$$= \sqrt{a^2 - x^2} \int 1 dx - \int \left[\frac{d}{dx}(\sqrt{a^2 - x^2}) \int 1 dx\right] dx$$

$$= \sqrt{a^2 - x^2}\,(x) - \int \frac{-2x}{2\sqrt{a^2-x^2}}(x) dx = x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}} dx + \int \frac{a^2}{\sqrt{a^2-x^2}} dx$$

$$\int \sqrt{a^2 - x^2}\; dx = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2 - x^2}\; dx + \int \sqrt{a^2 - x^2}\; dx = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$2\int \sqrt{a^2 - x^2}\; dx = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2 - x^2}\; dx \;= \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{c}{2} = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

(ii) $\int \sqrt{a^2 + x^2}\; dx = \int \sqrt{a^2 + x^2}\; (1)\, dx,$

Here, we take '$\sqrt{a^2 + x^2}$' as first function and '1' as second function.

$\therefore \int \sqrt{a^2 + x^2}\; dx = \int \sqrt{a^2 + x^2}\,(1) dx$

$$= \sqrt{a^2 + x^2} \int 1 dx - \int \left[\frac{d}{dx}(\sqrt{a^2 + x^2}) \int 1 dx\right] dx$$

$$= \sqrt{a^2 + x^2}\,(x) - \int \frac{2x}{2\sqrt{a^2 + x^2}}(x) dx = x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}} dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2 - a^2}{\sqrt{a^2 + x^2}} dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2}{\sqrt{a^2 + x^2}} dx + \int \frac{a^2}{\sqrt{a^2 + x^2}} dx$$

$$\int \sqrt{a^2 + x^2}\; dx = x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\, dx + a^2 \times \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2+x^2}\,dx + \int \sqrt{a^2+x^2}\,dx = x\sqrt{a^2+x^2} + a\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$2\int \sqrt{a^2+x^2}\,dx = x\sqrt{a^2+x^2} + a\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a}{2}\tan^{-1}\left(\frac{x}{a}\right) + \frac{c}{2} = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a}{2}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Check Point

Using integration by parts, prove that:

$$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + c$$

Example 16: Apply integration by parts to evaluate:

$$\int e^{ax}\sin bx\,dx$$

Solution: Let, $I = \int e^{ax}\sin bx\,dx = \int (\sin bx)(e^{ax})\,dx$

$$= \sin bx \int e^{ax}dx - \int \left[\frac{d}{dx}(\sin bx)\int e^{ax}dx\right]dx$$

$$= \sin bx\left(\frac{e^{ax}}{a}\right) - \int \left[(b\cos bx)\left(\frac{e^{ax}}{a}\right)\right]dx$$

$$= \sin bx\left(\frac{e^{ax}}{a}\right) - \frac{b}{a}\int [(\cos bx)(e^{ax})]\,dx$$

$$= \sin bx\left(\frac{e^{ax}}{a}\right) - \frac{b}{a}\left[\cos bx\int e^{ax}dx - \int\left\{\frac{d}{dx}(\cos bx)\int e^{ax}dx\right\}dx\right]$$

$$I = \sin bx\left(\frac{e^{ax}}{a}\right) - \frac{b}{a}\left[\cos bx\left(\frac{e^{ax}}{a}\right) - \int\left\{(-b\sin bx)\left(\frac{e^{ax}}{a}\right)\right\}dx\right]$$

$$I = \sin bx\left(\frac{e^{ax}}{a}\right) - \frac{b}{a}\cos bx\left(\frac{e^{ax}}{a}\right) - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx + c$$

$$I = \sin bx\left(\frac{e^{ax}}{a}\right) - \frac{b}{a}\cos bx\left(\frac{e^{ax}}{a}\right) - \frac{b^2}{a^2}I + c$$

$$I + \frac{b^2}{a^2}I = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx + c$$

$$\left(\frac{a^2+b^2}{a^2}\right)I = e^{ax}\left[\frac{1}{a}\sin bx - \frac{b}{a^2}\cos bx\right] + c$$

$$I = e^{ax}\left[\frac{1}{a}\times\frac{a^2}{a^2+b^2}\sin bx - \frac{b}{a^2}\times\frac{a^2}{a^2+b^2}\cos bx\right] + c\times\frac{a^2}{a^2+b^2}$$

$$I = e^{ax}\left[\frac{a}{a^2+b^2}\sin bx - \frac{b}{a^2+b^2}\cos bx\right] + C$$

Exercise 3.4

Evaluate the integrals using integration by parts.

1. $\int \ln x\,dx$
2. $\int (\ln x)^2\,dx$
3. $\int \sin(\ln x)\,dx$
4. $\int x^3\ln x\,dx$
5. $\int y\sin 2y\,dy$
6. $\int e^x\cos x\,dx$
7. $\int x\sec^{-1}x\,dx$
8. $\int \ln(2x+3)\,dx$
9. $\int x^2e^x dx$
10. $\int x\cos x\,dx$
11. $\int \cos^{-1}x\,dx$
12. $\int \tan^{-1}x\,dx$
13. $\int x\sec^2x\,dx$
14. $\int x^2\sin^{-1}x\,dx$
15. $\int \ln\left[x+\sqrt{1+x^2}\right]dx$
16. $\int x^3e^{x^2}dx$
17. $\int x^2\sin x\,dx$
18. $\int \frac{\ln x}{\sqrt{x}}\,dx$

3.5 Integration by Partial Fraction

When the terms in the sum:

$$\frac{3}{x+4}+\frac{4}{x+2} \qquad \text{(i)}$$

are combined by means of a common denominator, we obtain a single rational expression:

$$\frac{7x+22}{(x+4)(x+2)} \qquad \text{(ii)}$$

Suppose that we are faced with the problem of evaluating the integral:

$$\int \frac{7x+22}{(x+4)(x+2)}dx$$

From (i) and (ii), we have:

$$\int \frac{7x+22}{(x+4)(x+2)}dx = \int\left[\frac{3}{(x+4)}+\frac{4}{(x+2)}\right]dx = \int\frac{3}{(x+4)}dx+\int\frac{4}{(x+2)}dx$$

$$= 3\int\frac{1}{(x+4)}dx+4\int\frac{1}{(x+2)}dx = 3\ln(x+4)+4\ln(x+2)+c$$

This example illustrates a procedure for integrating certain rational fractions $\frac{P(x)}{Q(x)}$, where the degree of $P(x)$ is less than the degree of $Q(x)$. This method, known as partial fractions consists of decomposing such rational fractions into simplest component fractions and then evaluating the integral term by term.

Example 17: Evaluate: $\int\frac{x^3-2x}{x^2+3x+2}dx$

Solution: We observe that degree of numerator is greater than that of denominator.

$$\therefore \quad \int\frac{x^3-2x}{x^2+3x+2}dx = \int\left[x-3+\frac{5x+6}{x^2+3x+2}\right]dx \qquad \text{(i)}$$

Now, $\frac{5x+6}{x^2+3x+2}=\frac{5x+6}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$

Check Point

Evaluate $\int\frac{2x+1}{(x-1)(x+3)}dx$

By equating numerator, we get:

$$5x+6 = A(x+2)+B(x+1) \qquad \text{(ii)}$$

If we set $x=-2$ and $x=-1$, we get $B=4$ and $A=1$, repectively.

$$\therefore \quad \int\frac{x^3-2x}{x^2+3x+2}dx = \int\left[x-3+\frac{1}{x+1}+\frac{4}{x+2}\right]dx = \int x\,dx-3\int dx+\int\frac{1}{x+1}dx+4\int\frac{1}{x+2}dx$$

$$=\frac{x^2}{2}-3x+ln(x+1)+4\,ln(x+2)+c$$

Example 18: Evaluate: $\int\frac{x^2+2x+4}{(x+1)^3}dx$

Solution: Given fraction can be written as:

$$\frac{x^2+2x+4}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

By equating numerator, we get:

$x^2 + 2x + 4 = A(x+1)^2 + B(x+1) + C$

$x^2 + 2x + 4 = Ax^2 + (2A+B)x + (A+B+C)$

Comparing coefficients of like powers of x from both sides, we gat:

$A = 1, 2A + B = 2$ and $A + B + C = 4$

Solving the equations, we have:

$A = 1, B = 0$ and $C = 3$

$$\therefore \quad \int \frac{x^2+2x+4}{(x+1)^3}dx = \int \left[\frac{1}{x+1} + \frac{0}{(x+1)^2} + \frac{3}{(x+1)^3}\right]dx = \int \frac{1}{x+1}dx + 3\int \frac{1}{(x+1)^3}dx$$

$$= \int \frac{1}{x+1}dx + 3\int (x+1)^{-3}dx = ln(x+1) - \frac{3}{2}(x+1)^{-2} + c$$

$$= ln(x+1) - \frac{3}{2(x+1)^2} + c$$

Example 19: Evaluate: $\int \frac{3x^2+5x+3}{(x+2)(x^2+1)}dx$

Solution: Given fraction can be written as:

$$\frac{3x^2+5x+3}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$3x^2 + 5x + 3 = A(x^2+1) + (Bx+C)(x+2)$

$3x^2 + 5x + 3 = (A+B)x^2 + (2B+C)x + (A+2C)$

Equating coefficients:

$$A + B, \quad 2B + C = 5, \quad A + 2C = 3$$

Solving the equations, we have:

$$A = 1, B = 2, C = 1$$

$$\therefore \quad \int \frac{3x^2+5x+3}{(x+2)(x^2+1)}dx = \int \left(\frac{1}{x+2} + \frac{2x+1}{x^2+1}\right)dx$$

$$= \int \frac{1}{x+2}dx + \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$

$$= ln(x+1) + ln(x^2+1) + \tan^{-1}x + c$$

Exercise 3.5

Evaluate the integrals using partial fractions.

1. $\int \frac{3x+7}{(x+2)(x+3)}\,dx$
2. $\int \frac{4x+9}{x^2+x-12}\,dx$
3. $\int \frac{21-8x}{x^2+x-6}\,dx$
4. $\int \frac{3x+7}{(x+2)^2}\,dx$
5. $\int \frac{5x^2-5x+2}{(x+1)(x-1)^2}\,dx$
6. $\int \frac{9x^2+3x+29}{(x+1)(x^2+4)}\,dx$
7. $\int \frac{7x^2+7x+4}{(2x+1)(x^2+x+1)}\,dx$
8. $\int \frac{x^3+4x^2+9x+14}{x^2+4x+3}\,dx$
9. $\int \frac{1}{x^2-9}\,dx$
10. $\int \frac{1}{x^3+2x^2+x}\,dx$
11. $\int \frac{e^x}{(e^x+1)^2(e^x-2)}\,dx$
12. $\int \frac{x}{(x+1)^2(x^2+1)}\,dx$

3.6 The Definite Integral

In this section, we will introduce the concept of a definite integral. Which will link the concept of area to other important concepts such as length, volume, density, probability work.

3.6.1 Partition of the Interval

A partition of the interval $[a, b]$ is a collection of points:

$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$

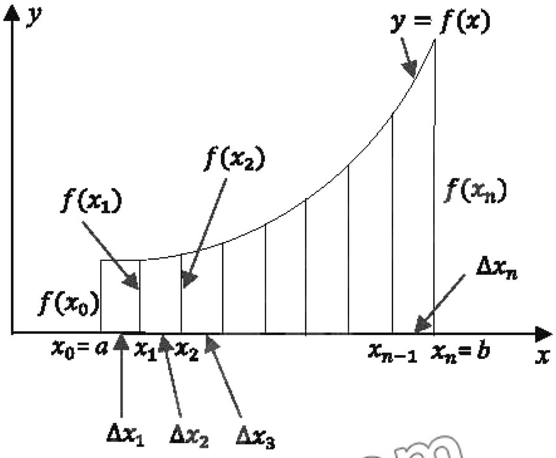
that divides $[a, b]$ into n subintervals of lengths:

$\Delta x_1 = x_1 - x_0, \quad \Delta x_2 = x_2 - x_1,$

$\Delta x_3 = x_3 - x_2, \ldots, \Delta x_n = x_n - x_{n-1}$

The partition is said to be regular provided all subintervals have the same length:

$$\Delta x = \Delta x_1 = \frac{b-a}{n}$$



In the figure, each partition looks like a rectangle. For a regular partition, widths of the rectangles approach to zero as n is made large.

Area of first (left most) rectangle = length × width = $f(x_1) \times \Delta x_1$

Area under the curve = sum of areas of n rectangles

$= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + \cdots + f(x_n)\Delta x_n$

$$= \sum_{k=1}^{n} f(x_k)\Delta x_k \quad \ldots\ldots\ldots \ (i)$$

Expression (i) represents approximation of sum of areas of n rectangles.

Based on our inductive concept, the area under the curve and between the interval $[a, b]$ is:

$$A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k \quad \ldots\ldots\ldots (ii)$$

Expression (ii) provides the fundamental concept of integral calculus and form the basis of the following definition.

Definition 3.2: A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit:

$$\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

exists and does not depend upon the choice of partitions or on the choice of the points x_k in the subintervals. In the such case, we denote the limit by the symbol:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k \ \ldots\ldots (iii)$$

Expression (iii) is called the definite integral of f from a to b. The numbers a and b are called lower limit and upper limit of integration respectively and $f(x)$ is called the integrand.

Theorem 3.2: If a function f is continuous on an interval $[a, b]$ then f is integrable on $[a, b]$ and the net signed area under the curve between the interval $[a, b]$ is:

$$A = \int_a^b f(x)dx$$

In the simplest cases, definite integrals of continuous functions can be calculated using formulas from plane geometry to compute the shaded area.

Example 20:

Sketch the region where area is represented by the definite integral and evaluate the integral using an appropriate formula from geometry.

(i) $\int_1^5 3dx$ (ii) $\int_{-2}^2 (x+3)dx$ (iii) $\int_0^1 \sqrt{1-x^2}dx$

Solution:

(i) Graph of the integral is the horizontal line $y = 3$.
So, the region is a rectangle of height 3 drawn over the interval from1 to 5.
From figure (1), we have:
$\int_1^5 3dx$ = area of rectangle = 4 × 3 = 12 sq. units

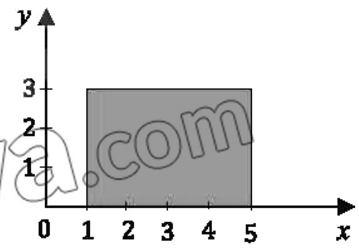


Fig. (1)

(ii) Graph of the integral is the line $y = x + 3$.
When $x = -2$, $y = -2 + 3 = 1$
When $x = 2$, $y = 2 + 3 = 5$
So, the region is trapezoid where base ranges from $x = -2$ to $x = 2$.
From figure (2), we have:
$\int_{-2}^2 (x+3)dx$ = area of trapezoid
$= \frac{1}{2}(1+5)(4)$ = 12 sq. units

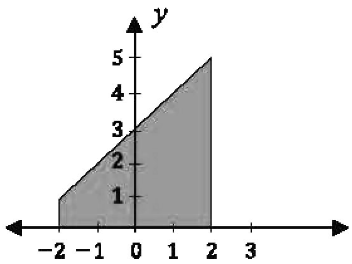


Fig. (2)

(iii) Graph of the function $y = \sqrt{1-x^2}$ is the upper semi-circle of radius 1 centred at the origin.
So, the region is upper right quarter-circle of radius 1 centred at origin.
From figure (3), we have:
$\int_0^1 \sqrt{1-x^2}dx$ = area of quarter circle
$= \frac{1}{4} \times \pi(1)^2 = \frac{\pi}{4}$ sq. units

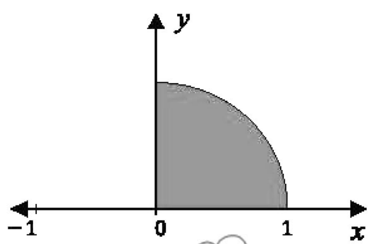


Fig. (3)

Example 21: Evaluate the following.

(i) $\int_0^1 (x-1)dx$ (ii) $\int_0^2 (x-1)dx$

Solution:

(i) The graph of the integral is the line $y = x - 1$.
When $x = 0,\ y = 0 - 1 = -1$
When $x = 1,\ y = 1 - 1 = 0$
The region is a triangle from $x = 0$ to $x = 1$.
From figure (4), we get:
$\int_0^1 (x-1)dx$ = area of triangle $= \frac{1}{2}(1)(1) = \frac{1}{2}$ sq. units

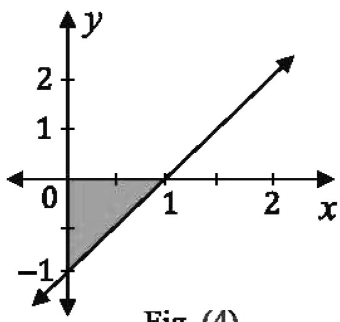


Fig. (4)

(ii) The graph of the integral is the line $y = x - 1$.
When $x = 0,\ y = 0 - 1 = -1$
When $x = 1,\ y = 1 - 1 = 0$
When $x = 2,\ y = 2 - 1 = 1$
The regions are two triangles from $x = -1$ to $x = 0$ and $x = 1$ to $x = 2$. From figure (5), we get:
$\int_0^2 (x-1)dx = \int_0^1 (x-1)dx + \int_1^2 (x-1)dx$
= area of triangle A_1 + Area of triangle A_2
$= \frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) = 1$ sq. units

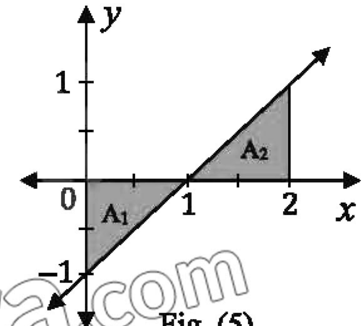


Fig. (5)

Note: In the figure (5), the area of triangle A_1 is below the x-axis and the area of triangle A_2 is above x-axis, therefore:
$A_1 = -\frac{1}{2}$ and $A_2 = \frac{1}{2}$ which implies $A_1 + A_2 = -\frac{1}{2} + \frac{1}{2} = 0$
But area cannot be negative, therefore in such cases, we take net area as:

$$A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = 1$$

3.7 Properties of The Definite Integral

In the finite closed interval [*a, b*], when upper limit of integration in the definite integral is greater than the lower limit of integration ($a < b$), the following facts are true.

(i) If lower and upper limits of integration are equal, then area is zero. i.e.,

$$\int_a^a f(x)dx = 0$$

For example,

$$\int_2^2 x\,dx = 0$$

(ii) If the lower limit of integration is greater than the upper limit of integration, then:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Which states that interchanging the limits of integral reverses the sign of integral.
For example,

$$\int_1^0 (x-1)dx = -\int_0^1 (x-1)dx = \frac{1}{2}$$

(iii) If c is the point between a and b then:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b dx$$

For example, in figure (5), we have:

$$\int_0^2 (x-1)dx = \int_0^1 (x-1)dx + \int_1^2 (x-1)dx$$

Theorem 3.3:

If f and g are integrable on $[a, b]$ and c is a constant, then cf, $f+g$ and $f-g$ are integrable on $[a, b]$ and the following statements are true.

(i) $\int_a^b c\, f(x)dx = c\int_a^b f(x)dx$ (The constant has no effects of limits on it.)

(ii) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Showing that the limit of a sum or difference is the sum or difference of the limits.

(iii) $\int_a^b [c\, f(x) \pm d\, g(x)]dx = c\int_a^b f(x)dx \pm d\int_a^b g(x)dx$

Example 22: Find:

(i) $\int_{-1}^4 [2f(x) + 5g(x)]dx$ if $\int_{-1}^4 f(x)dx = 2$ and $\int_{-1}^4 g(x)dx = 4$

(ii) $\int_{-1}^3 4f(x)dx$ if $\int_{-1}^2 f(x)dx = 3$ and $\int_2^3 f(x)dx = 1$

Solution:

(i) $\int_{-1}^4 [2f(x) + 5g(x)]dx = \int_{-1}^4 2f(x)dx + \int_{-1}^4 5g(x)dx = 2\int_{-1}^4 f(x)dx + 5\int_{-1}^4 g(x)dx$

$$= 2(2) + 5(4) = 24$$

(ii) $\int_{-1}^3 4f(x)dx = 4\int_{-1}^3 f(x)dx = 4\left[\int_{-1}^2 f(x)dx + \int_2^3 f(x)dx\right]$

$$= 4(3+1) = 4 \times 4 = 16$$

Exercise 3.6

1. Sketch the region where area is represented by the definite integral and evaluate the integral using an appropriate formula from geometry.

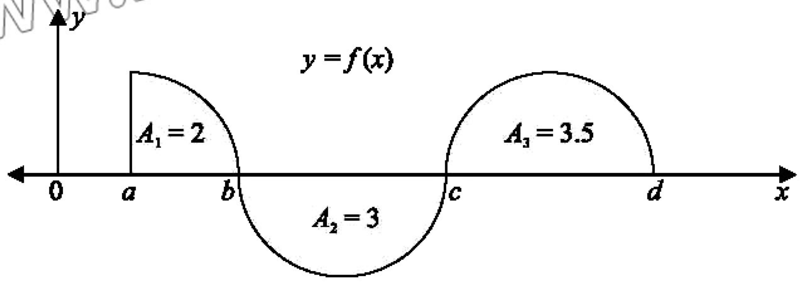
(i) $\int_0^4 x dx$ (ii) $\int_{-3}^0 x dx$ (iii) $\int_0^2 (x-1)dx$

(iv) $\int_0^2 (x+1)dx$ (v) $\int_{-3}^3 2dx$ (vi) $\int_0^2 \sqrt{1-x^2}\, dx$

2. Evaluate the integrals in each part when $f(x) = \begin{cases} x; & x \le 1 \\ 3; & x > 1 \end{cases}$.

(i) $\int_0^1 f(x)dx$ (ii) $\int_{-1}^1 f(x)dx$ (iii) $\int_1^4 f(x)dx$ (iv) $\int_{-1}^2 f(x)dx$

3. Using the area shown below in the figure, evaluate the integrals.



(i) $\int_a^b f(x)dx$ (ii) $\int_b^c f(x)dx$ (iii) $\int_c^d f(x)dx$

(iv) $\int_a^c f(x)dx$ (v) $\int_b^d f(x)dx$ (vi) $\int_a^d f(x)dx$

4. Find:
$\int_1^5 [3f(x) - 2g(x)]dx$ if $\int_1^5 f(x)dx = 4$ and $\int_1^5 g(x)dx = 5$

5. Find:
$\int_1^4 f(x)dx$ if $\int_1^2 f(x)dx = 1$ and $\int_2^4 f(x)dx = 2$

6. Find:
$\int_3^{-2} f(x)dx$ if $\int_{-2}^1 f(x)dx = 1$ and $\int_1^3 f(x)dx = -5$

7. Use appropriate formula from geometry to evaluate integrals.
(i) $\int_{-1}^4 (3 - x)dx$ (ii) $\int_0^2 [2 + \sqrt{1 - x^2}\,]dx$ (iii) $\int_{-2}^{-2} \sqrt{x^3 - 4}\, dx$

3.8 Fundamental Theorem of Calculus

In this section, we will establish two basic relationships between definite and indefinite integrals that together constitute a result called the 'Fundamental Theorem of Calculus'. We will provide a powerful method for evaluating definite integrals using anti-derivatives.

We consider a non-negative and continuous function f on an interval $[a, b]$. The area A under the graph f over the interval $[a, b]$ is represented by the definite integral:

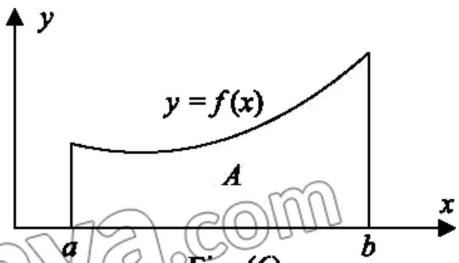


Fig. (6)

$$A = \int_a^b f(x)dx \;\ldots\ldots\ldots (i)$$

From (i), we have:

$A(a) = 0$ [The area under the curve from a to a is the area above the single point a and hence is zero.]

Similarly, $A(b) = A$ [The area under the curve from a to b is A.]

The formula $A'(x) = f(x)$ provides that $A(x)$ is an anti-derivative of $f(x)$ which implies that every other anti-derivative of $f(x)$ on $[a, b]$ can be obtained by adding a constant to $A(x)$.
By definition of anti-derivative, suppose:
$F(x) = A(x) + c \ldots\ldots\ldots (ii)$

We check what happens when we subtract $F(a)$ from $F(b)$. From (ii):
$F(a) = A(a) + c \ldots\ldots\ldots (iii)$ and $F(b) = A(b) + c \ldots\ldots\ldots (iv)$
Subtracting (iii) from (iv):

$$F(b) - F(a) = [A(b) + c] - [A(a) + c] = A(b) - A(a) = A - 0 = A$$

Therefore, from (i), we have:

$$A = \int_a^b f(x)dx = F(b) - F(a) \ldots\ldots\ldots (v)$$

Statement: The Fundamental Theorem of Calculus states that if f is continuous on $[a, b]$ ad F is antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

This can be written as:

$$\int_a^b f(x)dx = |F(x)|_{x=a}^{x=b} = F(b) - F(a)$$

We can emphasise that a and b are values for the variable x.

> Thus, the definite integral can be evaluated by finding any anti-derivative of the integral and then subtracting the value of this anti-derivative at the lower limit of integration from its value at the upper limit of integration.

Example 23: Evaluate: $\int_1^3 x\,dx$

Solution: $\int_1^3 x\,dx = \left|\frac{x^2}{2}\right|_1^3 = \frac{3^2}{2} - \frac{1^2}{2}$ → First, we apply upper limit and then lower limit.

$$= \frac{9}{2} - \frac{1}{2} = \frac{8}{2} = 4$$

Example 24: Evaluate: $\int_{-2}^2 (3x^2 - x + 1)dx$

Solution: $\int_{-2}^2 (3x^2 - x + 1)dx = \left|x^3 - \frac{x^2}{2} + x\right|_{-2}^2 = \frac{3^2}{2} - \frac{1^2}{2}$

$$= \left(2^3 - \frac{2^2}{2} + 2\right) - \left((-2)^3 - \frac{(-2)^2}{2} + (-2)\right)$$

$$= (8 - 2 + 2) - (-8 - 2 - 2) = 8 + 12 = 20$$

Example 25: Evaluate: $\int_0^2 \sqrt{2x^2+1}\, xdx$

Check Point

Evaluate: $\int_{\frac{\pi}{6}}^{\pi} \cos x\, dx$

Solution: We can apply two methods.

Method-1: By substitution but without changing the limits.

Let $u = 2x^2 + 1$ which implies $du = 4x\, dx$

Thus, $\int_0^2 \sqrt{2x^2+1}\, xdx = \frac{1}{4}\int_0^2 \sqrt{2x^2+1} \times 4xdx$

$= \frac{1}{4}\int_0^2 \sqrt{u} \times du = \frac{1}{4} \times \left|\frac{2}{3}u^{\frac{3}{2}}\right|_0^2$ (Substituting for u)

$= \left|\frac{1}{6}(2x^2+1)^{\frac{3}{2}}\right|_0^2$ (Resubstituting for x)

Applying limits, we get:

$= \frac{1}{6}[2(2)^2+1]^{\frac{3}{2}} - \frac{1}{6}[2(0)^2+1]^{\frac{3}{2}} = \frac{1}{6}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$

$= \frac{1}{6}(27-1) = \frac{26}{6} = \frac{13}{3}$

Method-2: By substitution with changing the limits.

Let $u = 2x^2 + 1$ which implies $du = 4x\, dx$

When $x = 0, u = 2(0)^2 + 1 = 1$ and when $x = 2, u = 2(2)^2 + 1 = 9$

Thus, $\int_0^2 \sqrt{2x^2+1}\, xdx = \frac{1}{4}\int_0^2 \sqrt{2x^2+1} \times 4xdx$

$= \frac{1}{4}\int_1^9 \sqrt{u} \times du = \frac{1}{4} \times \left|\frac{2}{3}u^{\frac{3}{2}}\right|_1^9$ (Substituting for u)

$= \frac{1}{6}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{1}{6}(27-1) = \frac{26}{6} = \frac{13}{3}$

Example 26: Evaluate: $\int_a^b \frac{1}{1-\cos x}\, dx$ when $a = \frac{\pi}{4}, b = \frac{\pi}{3}$

Solution: $\int_a^b \frac{1}{1-\cos x}\, dx = \int_a^b \frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x}\, dx = \int_a^b \frac{1+\cos x}{1-\cos^2 x}\, dx$

$= \int_a^b \frac{1+\cos x}{\sin^2 x}\, dx = \int_a^b \left[\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\right] dx$

$= \int_a^b [\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x] dx$

$= |-\cot x|_a^b + |-\operatorname{cosec} x|_a^b$

Check Point

Evaluate:

$\int_0^1 \sin^{-1} x\, dx$

Applying limits and substituting values of a and b, we get:

$\int_a^b \frac{1}{1-\cos x}\, dx = -\left(\cot\frac{\pi}{3} - \cot\frac{\pi}{4}\right) - \left(\operatorname{cosec}\frac{\pi}{3} - \operatorname{cosec}\frac{\pi}{4}\right)$

$= -\left(\frac{1}{\sqrt{3}} - 1\right) - \left(\frac{2}{\sqrt{3}} - \sqrt{2}\right) = -\frac{1}{\sqrt{3}} + 1 - \frac{2}{\sqrt{3}} + \sqrt{2}$

$= 1 + \sqrt{2} - \sqrt{3}$

Example 27: Evaluate: $\int_1^e x\, lnx\, dx$

Solution: Taking lnx as first function and integrating by parts, we get:

$$\int_1^e x\, lnx\, dx = \int_1^e (lnx)(x)\, dx = \left|lnx \times \frac{x^2}{2}\right|_1^e - \int_1^e \frac{1}{x} \times \frac{x^2}{2} dx$$

$$= \left|lnx \times \frac{x^2}{2}\right|_1^e - \frac{1}{2}\int_1^e x\, dx = \left|lnx \times \frac{x^2}{2}\right|_1^e - \frac{1}{2} \times \left|\frac{x^2}{2}\right|_1^e$$

$$= \left(lne \times \frac{e^2}{2} - ln1 \times \frac{1^2}{2}\right) - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1^2}{2}\right) = \left(1 \times \frac{e^2}{2} - 0 \times \frac{1}{2}\right) - \frac{e^2}{4} + \frac{1}{4}$$

$$= \frac{e^2}{2} - 0 - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}$$

Exercise 3.7

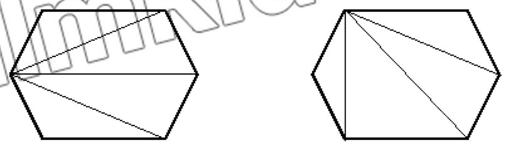
Evaluate the definite integrals.

1. $\int_{-1}^{2}(2x+3)\, dx$
2. $\int_{-4}^{12}\sqrt{y+4}\, dy$
3. $\int_0^{\frac{1}{2}}(2x+1)^{-\frac{1}{3}} dx$
4. $\int_0^3(6x^2-4x+5)\, dx$
5. $\int_{-2}^{1}(12x^5-36)\, dx$
6. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} \cos\theta\, d\theta$
7. $\int_0^{\frac{\pi}{4}} \sec^2 2\theta\, d\theta$
8. $\int_2^4 \frac{x^2+8}{x^2} dx$
9. $\int_{-\frac{1}{2}}^{\frac{3}{2}} x - \cos\pi x\, dx$
10. $\int_1^4 \frac{\cos\sqrt{x}}{2\sqrt{x}} dx$
11. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x \cos x\, dx$
12. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+\cos\theta}{(\theta+\sin\theta)^2} d\theta$
13. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec x + \tan x)^2\, dx$
14. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\, dx$
15. $\int_1^3 lnx\, dx$
16. $\int_2^4\left(e^{\frac{x}{2}} - e^{\frac{x}{4}}\right) dx$
17. $\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} dx$
18. $\int_0^{\frac{\pi}{4}} \tan^{-1} y\, dy$
19. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(2+\cos x)(5+\cos x)} dx$
20. $\int_2^5 \frac{1}{x(x+1)} dx$

3.9 Area and Volume

The definite integrals have applications that extend far beyond the area problems. In this section, we will also apply definite integrals for finding the volume. We have an inductive idea of what is meant by the area of certain geometrical figures. It is a number that in same way measures the size of the region enclosed by the figure. The area of a rectangle is the product of its length and width likewise the area of a triangle is half the product of lengths of the base and the altitude.

The area of a polygon may be defined as the sum of the areas of triangles into which it is decomposed and it can be proved that the area thus obtained is independent of how the polygon is decomposed into triangles.



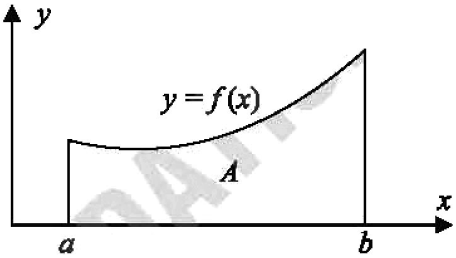
However, how do we define the area of a region in a plane if the region is bounded by a curve? We even certain that such a region has an area? In the same way volume of solids can be found by using definite integration.

3.10 Area of Bounded Region

3.10.1 Area Between a Curve and the X-axis

If f is a non-negative continuous function on $[a, b]$, then the area under the graph of f from a to b is:

$$A = \int_a^b f(x)dx$$



Example 28: Find the area of the region bounded by the line $2y + x = 8$, the x-axis and, the lines $x = 2$ and $x = 4$.

Solution: In the graph, CD is the given line.

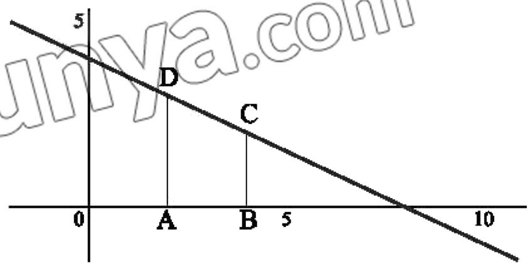
$2y + x = 8 \Rightarrow y = \frac{8-x}{2} \Rightarrow y = 4 - \frac{x}{2}$

Required area = area of trapezium ABCD

= area between line CD and x-axis from $x = 2$ to $x = 4$

$= \int_2^4 ydx = \int_2^4 \left(4 - \frac{x}{2}\right) dx = \left|4x - \frac{x^2}{4}\right|_2^4$

$= \left[4(4) - \frac{4^2}{4}\right] - \left[4(2) - \frac{2^2}{4}\right] = (16 - 4) - (8 - 1) = 5$ sq. units



3.10.2 Area Between Curves

If the function $f(x)$ is greater than the function $g(x)$ for all x between a and b, then the area under the graph of $f(x)$ minus the area under the graph of $g(x)$ is the area between the curves. Thus, the area between the curves $f(x)$ and $g(x)$ is:

$$A = \int_a^b [f(x) - g(x)]dx \; ; \; f(x) > g(x)$$

Example 29: Find the area of the region bounded by graphs of:

$$f(x) = (x-1)^2 \text{ and } g(x) = 3 - x$$

Solution: To find the limits of integration, we find common points of both functions by solving

$f(x) = g(x) \Rightarrow (x-1)^2 = 3 - x$

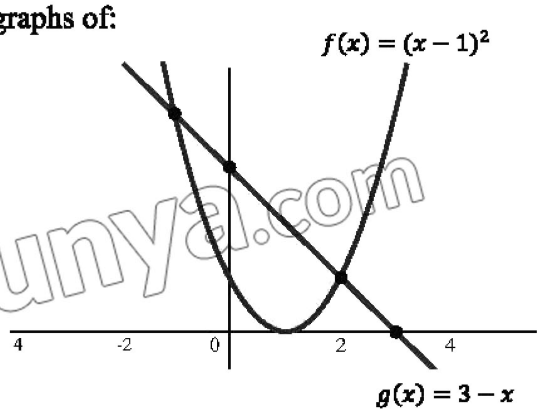
$\Rightarrow x^2 - x - 2 = 0$

After solving, we get:

$x = -1$ and $x = 2$

For $-1 < x < 2$, $g(x) > f(x)$

(Also clear from the graph of both curves.)



Thus, the area of region bounded is:

$$A = \int_{-1}^{2}[g(x) - f(x)]dx = \int_{-1}^{2}[(3-x)-(x-1)^2]dx \int_{-1}^{2}(2+x-x^2)dx$$

$$= \left|2x + \frac{x^2}{2} - \frac{x^3}{3}\right|_{-1}^{2} = \left[2(2) + \frac{2^2}{2} - \frac{2^3}{3}\right] - \left[2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right]$$

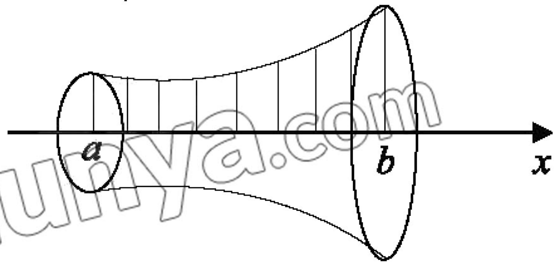
$$= \left[4 + 2 - \frac{8}{3}\right] - \left[-2 + \frac{1}{2} + \frac{1}{3}\right] = 6 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3} = 8 - 3 - 0.5 = 4.5 \text{ sq. units}$$

3.11 Volume of Solids of Revolution

3.11.1 Disc Method

Consider a region bounded by the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$ that is rotated about x-axis. If $a = x_0 < x_1 < x_2 \ldots < x_n = b$ is partition of the interval $[a, b]$, the volume V of the resulting 3-D region can approximated by the sum of volumes of discs obtained after rotation.

The radius and height of discs D_i are $f(x)_i$ and Δx_i Respectively. Thus:



$$V = \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x_i$$

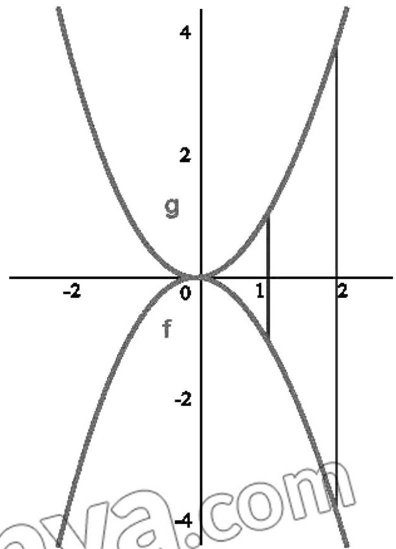
Volume of disc = area of base × height = $(\pi r^2)(h)$

Letting $\Delta x_i \to 0$, we have:

$$V = \pi \int_a^b [f(x)]^2 dx$$

Example 30:

Find the volume of the solid obtained by rotating the graph $y = x^2$ between $x = 1$ and $x = 2$ about x-axis.



Solution:

$$V = \pi \int_1^2 [f(x)]^2 dx$$

$$V = \pi \int_1^2 (x^2)^2 dx = \pi \int_1^2 x^4 dx$$

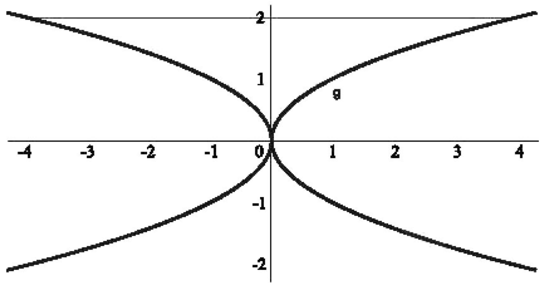
$$V = \pi \left|\frac{x^5}{5}\right|_1^2 = \frac{\pi}{5}(2^5 - 1^5) = \frac{31\pi}{5} \text{ cu. units}$$

Note: If a solid is obtained by rotating the regions bounded by the graph $x = g(y)$ about y-axis, we can also use the disc method to find the volume as follows.

$$V = \pi \int_a^b [g(x)]^2 dx$$

Example 31:

Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}, y = 0$ and $y = 2$ is revolved about the y-axis.



Solution:

First sketch the region and the solid. The cross section taken perpendicular to the y-axis and disk suggests that we can rewrite $y = \sqrt{x}$ as $x = y^2$. Thus, $g(y) = y^2$ and the volume is:

$$V = \pi \int_a^b [g(y)]^2 dy = \pi \int_0^2 (y^2)^2 dy = \pi \int_0^2 y^4 dy$$

$$V = \pi \left| \frac{y^5}{5} \right|_0^2 = \frac{\pi}{5}(2^5 - 0^5) = \frac{32\pi}{5} \text{ cu. units}$$

3.12 Applications

3.12.1 Consumer and Producer Surpluses

Economists use the definite integral to define the concept of consumer and producer surpluses.

The demand for a commodity by consumers as well as the amount supplied to the market by the manufacturers can often be expressed as a function of the per unit price. Let $D(x)$ and $S(x)$ be the number of units demanded and the number of units supplied, respectively, when the commodity sells at a price x per unit.

If the demand equals the supply:

$$D(x) = S(x)$$

The market is said to be in equilibrium and the corresponding price of the commodity is called the equilibrium price. If p is the equilibrium price and b is the price at which the demand of the commodity is zero ($b(s) = 0$), the integral:

$$Cs = \int_p^b D(x)dx$$

is called the consumer surplus. Similarly, the integral:

$$Ps = \int_c^P S(x)dx$$

where $S(c) = 0$, is called the producer surplus.

Example 32:

Suppose the demand and supply of a commodity selling for x dollars a unit and $D(x) = 1000 - 20x$ and $S(x) = x^2 + 10x$, respectively. Find the consumer and producer surplus.

Solution: From the graph it is clear that $D(x) = 0$ when $b = 50$, $S(x) = 0$ when $c = 0$ and $D(s) = S(x)$ for $p = 20$. Cs represents the area under the graph of $D(x)$ on the interval [20, 50] and Ps is the area under the graph of $S(x)$ on [0, 20]. We have:

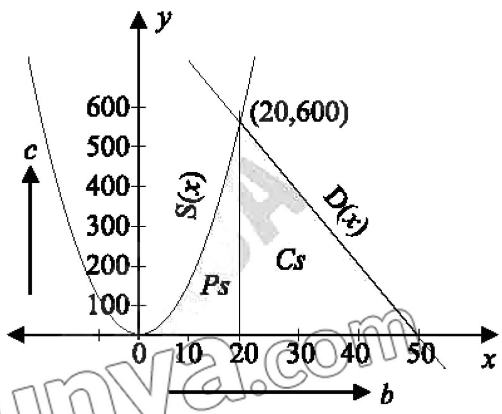
$$Cs = \int_p^b D(x)dx = \int_{20}^{50} (1000 - 20x)dx$$

$$Cs = \left|1000x - \frac{20}{2}x^2\right|_{20}^{50} = \$\,9000$$

And,

$$Ps = \int_c^p S(x)dx = \int_0^{20} (x^2 + 10x)dx$$

$$Ps = \left|\frac{1}{3}x^3 + 5x^2\right|_0^{20} = \$\,4666.67$$



3.12.2 Rectilinear Motion

If $f(t)$ is the position function of an object moving in the straight line, then we have:

velocity = $v(t) = \frac{dS}{dt}$ and acceleration = $a(t) = \frac{dv}{dt}$

By using the definition of anti-derivative, the quantities S and v can be written as indefinite integrals.

$S(t) = \int v(t)dt$ and $v(t) = \int a(t)dt$

By knowing the initial position $S(0)$ and the initial velocity $v(0)$, we can find specific values of the constants of integration.

Key Facts

(i) For upward motion:

$S(0) = 0,\quad v(0) > 0,\quad a = g = -98m/s^2 = -32ft/s^2$

(ii) For downward motion:

$S(0) = h,\quad v(0) = 0,\quad a = g = 98m/s^2 = 32ft/s^2$

Example 33:

The position function of an object that moves on a coordinate line is $S(t) = t^2 - 6t$. Where S is measured in centimetres and t in seconds. Find the distance travelled in the time interval [3, 9].

Solution: The velocity function:

$$v(t) = \frac{dS}{dt} = 2t - 6$$

implies that $v \geq 0$ for $3 \leq t \leq 9$. Hence the distance travelled is:

$$S(t) = \int_3^9 v(t)dt = \int_3^9 (2t - 6)dt$$

$$= |t^2 - 6t|_3^9 = (81 - 54) - (9 - 18) = 4\ cm$$

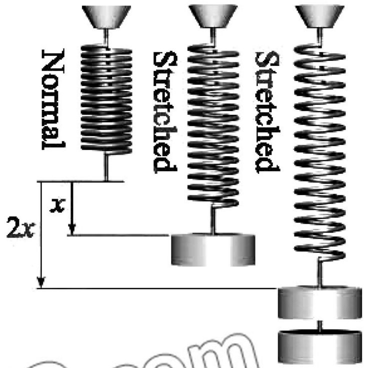
3.12.3 Work

In physics when a constant force F moves an object a distance d in the same direction, the work done is defined as $W = Fd$.

Definition: Let $F(x)$ be a continuous force acting at a point in the interval $[a, b]$, then the work done W by the force on moving an object from a to b is:

$$W = \int_a^b F(x)dx$$

3.12.4 Motion of Spring



Hook's law states that "when a spring is stretched (or compressed) beyond its natural length, the restoring force exerted by the spring is directly proportional to the amount of elongation (or compression)". Thus, in order to stretch a spring, x units beyond its natural length, we need to apply the force:

$F(x) = kx$; k is spring constant.

Example 34:

A force of 130 N is required to stretch a spring 50 cm. Find the work done in stretching the spring 20 cm beyond its natural (unstretched) length.

Solution:

$x = 50\ cm = 0.5m$ and $F = 130N$

Substituting values of x and F in $F = kx$, we have:

$130 = k \times 0.5 \Rightarrow k = 260\ N/m$

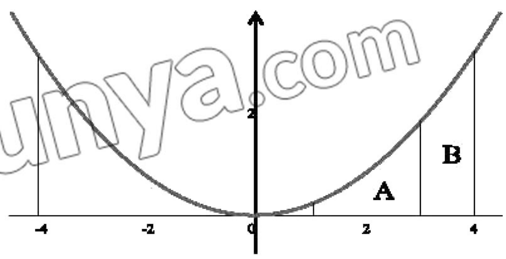
Thus, $F = kx \Rightarrow F = 260\,x$

Now, $x = 20\ cm = 0.2m$, so that the work done in stretching the spring by this amount is:

$$W = \int_0^{\frac{1}{5}} 260x\,dx = |130x^2|_0^{\frac{1}{5}} = \frac{26}{5} = 5.2\,J$$

Exercise 3.8

1. Find the area of region bounded by the curve $y = x^2$, the x-axis, lines $x = 1$ and $x = 3$.
2. Find the area under the curve $y = \sqrt{6x+4}$ (above x-axis) from $x = 0$ to $x = 2$.
3. Find the area of region bounded by the curve $y^2 = 4x$ and line $x = 3$.
4. In the figure, a sketch of the function $y = \frac{1}{2}(0.2x^2 + x)$ is shown. Find:
   (i) the area of region A.
   (ii) the area of region B.
   (iii) area of the region from $x = 1$ to $x = 4$.
   (iv) area of the region from $x = -1$ to $x = -4$.

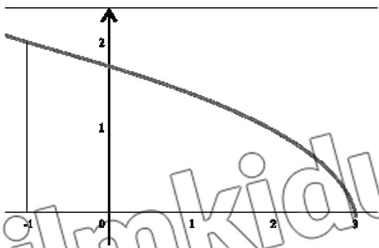


5. Find the area bunded by the graph:

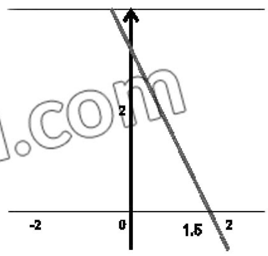
   (i) $y = 1 + \cos x$; $[0, 3\pi]$ (ii) $y = -1 + \sin x$; $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

6. Find the area of the region bounded by the graphs of $y = x, y = -2x$ and $x = 3$.
7. Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$ and bounded on the sides by the lines $x = 0$ and $x = 2$.
8. Find the area bounded by the curve $y = x^3 + 1$, the x-axis and the line $x = 1$.
9. Find the area of the region enclosed by $x = y^2$ and $y = x - 2$ integrating with respect to y.
10. Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval [1, 4] is revolved about the x-axis.
11. Find the volume of the sloid that results when the shaded region is revolved about the indicated axis.

    (i) $y = \sqrt{3 - x}$ about x-axis



    (ii) $y = 3 - 2x$ about y-axis



12. An object moves in a straight line according to the position function given below. If f is measured in centimetres, find the distance travelled by the object in the indicated time interval:

    (i) $S(t) = t^2 - 2t$; [0, 5] (ii) $S(t) = t^3 - 3t^2 - 9t$; [0, 4]

    (iii) $S(t) = 6 \sin \pi t$; [1, 3]

13. It takes a force of 50 N to stretch a spring of 0.5m. Find the work done in stretching the spring 0.6m beyond its natural length.
14. A force $F = \frac{3}{2}x\ lb$ is needed to stretch a 10 inch spring an additional x inch. Find the work done in stretching the spring 16 inch.
15. Find the consumer and producer surpluses, when:

    (i) $S(x) = 24, \quad D(x) = 100 - 2x$

    (ii) $S(x) = x^2 - 4, \quad D(x) = -x + 8$

    (iii) $S(x) = 2x^2 + 3x, \quad D(x) = 36 - x^2$

16. Find the total revenue obtained in 4 years if the rate of increase in dollars per year is:

$$f(t) = 200(t - 5)^2$$

17. Find the total revenue obtained in 8 years if the rate of increase in dollars per year is:

$$f(t) = 600\sqrt{1 + 3t}$$

18. Find the area bounded by the curve $f(x) = x^3 - 2x^2 + 1$ and the x-axis in the first quadrant.

Review Exercise

1. Select the correct option in the following.
   (i) If f is integrable, then it is:
   (a) discontinuous (b) unbounded (c) continuous (d) linear
   (ii) If $f'(x) = 3x^2 + 2x$, then $f(x)$ is:
   (a) $6x + 2 + c$ (b) $x^3 + x^2 + c$ (c) $3x^3 + 2x^2 + c$ (d) $1.5x^3 + x^2 + c$
   (iii) $\int \frac{d}{dx}(x^2)dx$ is equal to:
   (a) $x^2 + c$ (b) $2x + c$ (c) $\frac{x^3}{3} + c$ (d) $2x + c$
   (iv) $\int \sin 2x dx$ is:
   (a) $\frac{\cos 2x}{2} + c$ (b) $2\cos 2x + c$ (c) $-\frac{\sin 2x}{2} + c$ (d) $-\frac{\cos 2x}{2} + c$
   (v) $\int_3^7 dx$ is:
   (a) 3 (b) 4 (c) 5 (d) 6
   (vi) $\int_{\frac{\pi}{6}}^{\pi} \cos x dx$ is:
   (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$
   (vii) $\frac{d}{dx}\int_{-2}^{x} t^3 dt$ is equal to:
   (a) t^4 (b) t^3 (c) x^3 (d) $x^3 - 16$
   (viii) What is relation between $\int_1^2 xdx$ and $\int_1^2 tdt$?
   (a) $\int_1^2 xdx < \int_1^2 tdt$ (b) $\int_1^2 xdx > \int_1^2 tdt$
   (c) $\int_1^2 xdx \neq \int_1^2 tdt$ (d) $\int_1^2 xdx = \int_1^2 tdt$
   (ix) Area under the graph of $f(x) = 4$; [2, 5] is:
   (a) 2 (b) 4 (c) 5 (d) 12
   (x) $\int \sqrt{x}dx$ is:
   (a) $x^{\frac{3}{2}} + c$ (b) $\frac{2}{3}x^{\frac{3}{2}} + c$ (c) $\frac{3}{2}x^{\frac{3}{2}} + c$ (d) $x^{\frac{1}{2}} + c$
2. Evaluate:
   (i) $\int \frac{4x+2}{x^2+x+1}dx$ (ii) $\int x(x^2+1)^4 dx$ (iii) $\int \cos^2 3x\, dx$
   (iv) $\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}dx$ (v) $\int \sin^{-1}x dx$ (vi) $\int 2x \sin 3x\, dx$
   (vii) $\int x^2 e^x dx$ (viii) $\int_0^{\frac{\pi}{4}}(\sin 2x - 5\cos 4x)dx$ (ix) $\int_1^4 \frac{\cos\sqrt{x}}{2\sqrt{x}}\, dx$
3. Use the substitution $u = 2x + 1$ to evaluate $\int_0^1 \frac{x^2}{\sqrt{2x+1}}\, dx$.
4. A model rocket is launched upward from ground level with an initial speed of 60m/s.
   (a) How long does it take for the rocket to reach its highest point?
   (b) How high does the rocket go?
5. Suppose that a parachute moves with a velocity $V(t) = \cos \pi t$ m/s along a coordinate line. Assuming that the parachute has the coordinate $S = 4m$ at time $t = 0$ sec, find its position.

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