GENERAL MATHEMATICS

10

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UNIT

1

ALGEBRAIC FORMULAS AND APPLICATIONS

- **Algebraic Expressions**
- **Algebraic Formulas**
- **Surds and their Applications**
- **Rationalization**

After completion of this unit, the students will be able to:

- know that a rational expression behaves like a rational number.
- define a rational expression as the quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$ where $q(x)$ is not the zero polynomial.
- examine whether a given algebraic expression is a
  - Polynomial or not.
  - Rational expression or not.
- define $\frac{p(x)}{q(x)}$ as a rational expression in its lowest terms if $p(x)$ and $q(x)$ are polynomials with integral coefficients and having no common factor.
- examine whether a given rational algebraic expression is in lowest form or not.
- reduce a given rational expression to its lowest terms.
- find the sum, difference and product of rational expressions.
- divide a rational expression with another and express the result in its lowest terms.
- find value of algebraic expression at some particular real number.
- know the formulas

  $(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$ $\qquad$ $(a+b)^2 - (a-b)^2 = 4ab$
  - Find the value of $a^2 + b^2$ and of ab when the values of $a+b$ and $a-b$ are known.
- know the formula

  $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca.$
  - Find the value of $a^2+b^2+c^2$ when the values of $a+b+c$ and $ab+bc+ca$ are given.
  - Find the value of $a+b+c$ when the values of $a^2+b^2+c^2$ and $ab+bc+ca$ are given.
  - Find the value of $ab+bc+ca$ when the values of $a^2+b^2+c^2$ and $a+b+c$ are given.
- know the formulas

  $(a \pm b)^3 = a^3 \pm 3ab(a \pm b) \pm b^3$, $\quad$ $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2).$
  - Find the value of $a^3 \pm b^3$ when the values of $a \pm b$ and ab are given.
  - Find the continued product of $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2).$
- recognize the surds and their applications.
- explain the surds of second order. Use basic operations on surds of second order to rationalize the denominators and evaluate it.
- explain rationalization (with precise meaning) of real numbers of the types $\frac{1}{a+b\sqrt{x}}$, $\frac{1}{\sqrt{x}+\sqrt{y}}$ and their combinations where x and y are natural numbers and a, b are integers.

1.1 ALGEBRAIC EXPRESSIONS

Algebra is an extension of arithmetic. In algebra, we use alphabets such as *a, b, c* to stand for constants and *x, y, z* to stand for any numerical value we choose.

An algebraic expression involves numbers and letters together with operational signs such as $+, -, \times, \div$. The signs + and − separate an algebraic expression into terms.

Example:

$ax + by$	consists of *2* terms
$3x - 2y$	consists of *2* terms
$9x^2 - 7xy + 7y^2$	consists of *3* terms
$5xy$	consists of *1* term

The numbers *a, b, 3, 2, 9, 7, 5* in these expressions are called coefficients, while the letters *x, y* are known as variables.

An algebraic expression is of three types.

(i) Polynomial *(ii) Rational* *(iii) Irrational*

A polynomial of degree *n* in variable *'x'* is defined as:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_3 x^3 + a_2 x^2 + a_1 x + a_0,$$

where *'n'* is a non-negative integer and $a_n, a_{n-1}, a_{n-2}, \ldots, a_3, a_2, a_1, a_0$ are real numbers, where as $a_n \neq 0$.

As the highest power of the variable *'x'* in this polynomial is *'n'*, therefore this polynomial is of degree *'n'*.

1.1.1 Rational Expression

We know that a number of the form $\frac{p}{q}, q \neq 0$, $p, q \in Z$, is called a rational number.

An expression which can be written in the form $\frac{P(x)}{Q(x)}, Q(x) \neq 0$, where $P(x)$ and $Q(x)$ are polynomials in 'x' is called a rational expression.

For example:

(i) $\frac{x^2+1}{x^3+x^2+3}$ (ii) $\frac{x^3+8}{x+1}$ (iii) $\frac{2x^2+3x+3}{x^2+x+2}$ (iv) $\frac{x+1}{x^2+2x+3}$

are all rational expressions. The rational expressions can also be added, subtracted, multiplied and divided like rational numbers.

Rational expressions are of two types.

(i) Proper Rational Expression

(ii) Improper Rational Expression

Proper Rational Expression:-

A rational expression $\frac{P(x)}{Q(x)}, Q(x) \neq 0$, in which the degree of $P(x)$ is less than the degree of $Q(x)$ is called a proper rational expression.

For example:

$$\frac{x+1}{x^2+3x+7}, \frac{3x^3+4x^2+5}{2x^4+1}$$

Improper Rational Expression

A rational expression $\frac{P(x)}{Q(x)}, Q(x) \neq 0$, in which the degree of $P(x)$ is either equal or greater than the degree of $Q(x)$ is called an improper rational expression. For example:

$$\frac{x^2+2x+4}{x+1}, \frac{x^2+4x+9}{x^2+1}, \frac{x^3+1}{x^2-x+4}, \frac{x+5}{x-1}$$

1.1.3 Examine a Given Algebraic Expression

Let us consider the following:

(i) $2x^2+3x+9$ (ii) $x+5$ (iii) $\sqrt{x}+\frac{1}{\sqrt{x}}+1$ (iv) $\frac{-4}{x^3}$

(i) and *(ii)* are Polynomials , but *(iii)* and *(iv)* are not polynomials, because in *(iii)* and *(iv)* the powers of the variables are negative and rational numbers.

Consider the following as well:

(i) $\frac{x+1}{x^3+x^2+3}$ (ii) $\frac{x^3+1}{x-1}$ (iii) $\sqrt[3]{x}+\frac{1}{\sqrt[3]{x}}+1$

(iv) $2\sqrt{y}+\frac{3}{\sqrt{x}}+1$ (v) $\frac{\sqrt{y}+3}{x^{2/3}}$

(i) and *(ii)* are rational expressions, but *(iii)*, *(iv)* and *(v)* are not rational expressions, because the powers of the variables are not integers.

1.1.4 Rational Expression in its Lowest Terms

If A,B and C are polynomials where $B,C\neq 0$, then $\frac{AC}{BC}=\frac{A}{B}$;

(which is the fundamental principle of fractions)

This is used to reduce a rational fraction to its lowest terms.
A rational expression is in its lowest terms, when the numerator and denominator have no common factors other than 1 and -1.

To examine whether the given rational expression is in its lowest terms or not, let us consider the following example.

EXAMPLE *Find the lowest term of* $\frac{8x^3y^2}{12xy^5}$.

SOLUTION:

$$\frac{8x^3y^2}{12xy^5} = \frac{2x^2.4xy^2}{3y^3.4xy^2}$$

$$= \frac{2x^2}{3y^3}$$

Thus to examine a rational expression in lowest terms, we first write the numerator and denominator in factored form and then use the fundamental principle of fractions to obtain,

$$\frac{b^2-a^2}{b^3-a^3} = \frac{(b-a)(b+a)}{(b-a)(b^2+ab+a^2)}$$

$$= \frac{b+a}{b^2+ab+a^2}$$

1.1.5 Reduce a Rational Expression to its Lowest Terms

EXAMPLE *Reduce to lowest terms:*

(i) $\frac{32x^5x^7}{-4x^2y^9}$ (ii) $\frac{2-x}{3x^2-5x-2}$

SOLUTION: (i) $\frac{32x^5y^7}{-4x^2y^9}$

$$= -\frac{8x^3.4x^2y^7}{y^2.4x^2y^7}$$

$$= -\frac{8x^3}{y^2}$$

(ii) $\frac{2-x}{3x^2-5x-2}$

$$= \frac{2-x}{3x^2-6x+x-2}$$

$$= \frac{2-x}{3x(x-2)+1(x-2)}$$

$$= \frac{2-x}{(3x+1)(x-2)}$$

$$= \frac{(-1)(x-2)}{(3x+1)(x-2)}$$

$$= \frac{-1}{3x+1}$$

1.1.6 Sum, Difference and Product of Rational Expressions

We find the sum, difference and product of rational expression with the help of following examples:

EXAMPLE-1

Solve:

(i) $\frac{x+1}{x^2-3x+2}+\frac{x+2}{x^2-4x+3}$

(ii) $\frac{x+2}{x^3+1}+\frac{x}{x^2-1}$

SOLUTION: (i) $\frac{x+1}{x^2-3x+2}+\frac{x+2}{x^2-4x+3}$

$$=\frac{x+1}{x^2-2x-x+2}+\frac{x+2}{x^2-3x-x+3}$$

$$=\frac{x+1}{x(x-2)-1(x-2)}+\frac{x+2}{x(x-3)-1(x-3)}$$

$$=\frac{x+1}{(x-2)\,(x-1)}+\frac{x+2}{(x-3)\,(x-1)}$$

$$=\frac{(x+1)(x-3)+(x+2)(x-2)}{(x-1)\,(x-2)\,(x-3)}$$

$$=\frac{x^2-3x+x-3+x^2-2x+2x-4}{(x^2-2x-x+2)\,(x-3)}$$

$$=\frac{2x^2-2x-7}{(x^2-3x+2)\,(x-3)}$$

$$=\frac{2x^2-2x-7}{x^3-6x^2+11x-6}$$

(ii) $\dfrac{x+2}{x^3+1}+\dfrac{x}{x^2-1}$

$$=\frac{x+2}{(x+1)\ (x^2-x+1)}+\frac{x}{(x-1)\ (x+1)}$$

$$=\frac{(x+2)\ (x-1)\ +\ x(x^2-x+1)}{(x+1)\ (x-1)\ (x^2-x+1)}$$

$$=\frac{x^2+2x-x-2+x^3-x^2+x}{(x^2-1)\ (x^2-x+1)}$$

$$=\frac{x^3+2x-2}{x^4-x^3+x^2-x^2+x-1}$$

$$=\frac{x^3+2x-2}{x^4-x^3+x-1}$$

EXAMPLE-2

Solve:

(i) $\dfrac{x+3}{x^2-4}-\dfrac{x-1}{x+2}$

(ii) $\dfrac{x+5}{x^2-6x}-\dfrac{x}{x-6}$

SOLUTION: *(i)* $\dfrac{x+3}{x^2-4}-\dfrac{x-1}{x+2}$

$$=\frac{x+3}{(x-2)\ (x+2)}-\frac{x-1}{x+2}$$

$$=\frac{(x+3)(1)\ -\ (x-1)(x-2)}{(x-2)\ (x+2)}$$

$$= \frac{x+3-(x^2-2x-x+2)}{x^2-4}$$

$$= \frac{x+3-x^2+3x-2}{x^2-4}$$

$$= \frac{4x-x^2+1}{x^2-4}$$

$$= \frac{1+4x-x^2}{x^2-4}$$

(ii) $\frac{x+5}{x^2-6x} - \frac{x}{x-6}$

$$= \frac{x+5}{x(x-6)} - \frac{x}{x-6}$$

$$= \frac{x+5-x.x}{x(x-6)}$$

$$= \frac{x+5-x^2}{x^2-6x}$$

$$= \frac{5+x-x^2}{x^2-6x}$$

EXAMPLE-3

Simplify:

(i) $\frac{x^2+x}{x^2-x} \times \frac{x-1}{x^3+1}$

(ii) $\frac{2x^2}{2x-1} \times \frac{2x-1}{6x+1}$

SOLUTION: (i) $\frac{x^2+x}{x^2-x} \times \frac{x-1}{x^3+1}$

$$= \frac{x(x+1)}{x(x-1)} \times \frac{x-1}{(x+1)(x^2-x+1)}$$

$$= \frac{x(x+1)(x-1)}{x(x-1)(x+1)(x^2-x+1)}$$

$$= \frac{1}{x^2-x+1}$$

(ii) $\dfrac{2x^2}{2x-1} \times \dfrac{2x-1}{6x+1}$

$$= \frac{2x^2(2x-1)}{(2x-1)(6x+1)}$$

$$= \frac{2x^2}{6x+1}$$

1.1.7 Division of a Rational Expression

The rule of division of rational expression is first factorize the expression and then cancel the same expressions in numerator and denominator.

EXAMPLE

Simplify:

(i) $\dfrac{x^2-2x}{x+1} \div \dfrac{x^2-4}{x^2+2x+1}$

(ii) $\dfrac{3x-1}{1+x} \div \dfrac{1-3x}{x^2+2x+1}$

SOLUTION: (i) $\dfrac{x^2-2x}{x+1} \div \dfrac{x^2-4}{x^2+2x+1}$

$$= \frac{x(x-2)}{x+1} \div \frac{(x-2)(x+2)}{(x+1)^2}$$

$$x + 2$$

(ii) $\dfrac{3x-1}{1+x} \div \dfrac{1-3x}{x^2+2x+1}$

$$= \frac{3x-1}{1+x} \div \frac{1-3x}{(x+1)^2}$$

$$= \frac{3x-1}{1+x} \times \frac{(x+1)\,(x+1)}{1-3x}$$

$$= \frac{(3x-1)\,(x+1)}{(1-3x)}$$

$$= \frac{(3x-1)\,(x+1)}{-(3x-1)}$$

$$= -(x+1)$$

1.1.8 Value of an Algebraic Expression

If we put a real number against a variable "x" in a polynomial $P(x)$, we get a real number. This real number is called value of $P(x)$. For $x = a, a \in R$, $P(x)$ will have the value $P(a)$.

For example:

If $P(x) = 4x^3 + 3x^2 + 5x + 1$, then find $P(x)$, for (i) $x = 1$, (ii) $x = 2$.

$$P(x) = 4x^3 + 3x^2 + 5x + 1$$

(i) $P(1) = 4(1)^3 + 3(1)^2 + 5(1) + 1$

$$= 4 + 3 + 5 + 1$$

$$= 13$$

Thus $P(1) = 13$ and

(ii) $P(2) = 4(2)^3 + 3(2)^2 + 5(2) + 1$

$$= 32 + 12 + 10 + 1 = 55$$

Thus $P(2) = 55$

EXAMPLE-1

If $P(x) = 4x^4 + 3x^2 - 5x + 1$, then find $P(-1)$

SOLUTION: Given: $P(x) = 4x^4 + 3x^2 - 5x + 1$

$$P(-1) = 4(-1)^4 + 3(-1)^2 - 5(-1) + 1$$

$$= 4 + 3 + 5 + 1$$

$$= 13$$

EXAMPLE-2

If $P(x) = \dfrac{x^2 - 5x + 6}{x^3 + 8}$, then find $P(1)$

SOLUTION:

$$P(x) = \frac{x^2 - 5x + 6}{x^3 + 8}$$

$$P(1) = \frac{1^2 - 5(1) + 6}{1^3 + 8} = \frac{1 - 5 + 6}{1 + 8}$$

$$= \frac{2}{9}$$

EXERCISE – 1.1

Solve:

1- If $P(x) = x^4 + 3x^2 - 5x + 9$, then find $P(x)$, for $x = 0, x = 1$.

2- If $P(x) = 2x^3 + 2x^2 + x - 1$, then find $P(-2)$.

3- If $P(y) = 3y^2 + \dfrac{y}{4} + 9$, then find $P(0)$.

4- If $P(x) = 9x^3 - 2x^2 + 3x + 1$, then find $P(1)$ and $P(2)$.

5- If $P(x) = \dfrac{x^2 - 5x + 6}{x + 1}$, then find $P(1)$ and $P(2)$.

6- If $P(r) = 2\pi r$, then find $P(r)$, for $r = 3$ and $\pi = \dfrac{22}{7}$.

7- If $P(r) = 4\pi r^2$, then find $P(r)$, for $r = 8$ and $\pi = \dfrac{22}{7}$.

8- If $P(y) = y^4 + \dfrac{3y^3}{2} - y^2 + 1$, then find $P(y)$, for $y = 2$ and $y = -2$.

Reduce the given rational expressions to lowest terms.

9- $\dfrac{8x^2y^2}{12x^4y}$

10- $\dfrac{25a^3b^2}{14a^2b^4}$

11- $\dfrac{16a^6b^7}{12a^3b^5+20a^5b^4}$

12- $\dfrac{18m^5x^3}{27m^4x^8-36m^6x^6}$

13- $\dfrac{5c-5d}{c^2-d^2}$

14- $\dfrac{x^2-y^2}{3y-3x}$

Simplify:

15- $\dfrac{x}{x-y}+\dfrac{x^2}{x^2+y^2}$

16- $\dfrac{x^2+2x}{x^2+x-2}+\dfrac{3x}{x+1}$

17- $\dfrac{x+2}{x^2+3x+2}-\dfrac{x-5}{x^2-x-6}$

18- $\dfrac{8x^2+18y^2}{4x^2-9y^2}-\dfrac{2x+3y}{2x-3y}$

19- $\dfrac{x}{x^2+xy}-\dfrac{y}{x^2-y^2}$

20- $\dfrac{x+y}{xy+y^2}-\dfrac{x}{x^2-xy}$

21- $\dfrac{(x+1)^2}{x^2-1}-\dfrac{x^2+1}{x^2+1}$

22- $\dfrac{5x}{x-9}+\dfrac{x^2-2x+1}{x^2-12x+27}-\dfrac{6x}{x-3}$

23- $\dfrac{x^2-4x+4}{x^2-4}\div\dfrac{x}{x-2}$

24- $\dfrac{x^2-36}{x^2-1}\div\dfrac{x-6}{1-x}$

25- $\dfrac{x^2-5x}{x-1}\div\dfrac{x^2-25}{x^2+x+20}$

26- $\dfrac{2x^2-5x-12}{4x^2+4x-3}\div\dfrac{2x^2-7x-4}{6x^2+5x-4}$

27- $\dfrac{x(2x-1)^2}{2x^2-1}\div\dfrac{4x^2-1}{4x^2+4x+1}$

28- $\dfrac{x^2+x}{x^2-1}\times\dfrac{x+1}{x^3+1}$

29- $\dfrac{x^2-9}{x^2-6x+9}\times\dfrac{x}{3x+9}$

30- $\dfrac{x+5}{x^2+6x}\times\dfrac{x^3+6x^2}{x+5}$

31- $\dfrac{x^2-2x+1}{x^2-1}\times\dfrac{x+1}{x-1}$

32- $\dfrac{x^2+4x+3}{x+3}\times\dfrac{x^2-2x+1}{x^2-1}$

1.2. FORMULAE:

A formula expresses a rule in algebraic terms, its plural is formulae.

1.2.1 Formula 1

$$(a+b)^2+(a-b)^2 = 2(a^2+b^2)$$

Proof: L.H.S $= (a+b)^2+(a-b)^2$

$= a^2+2ab+b^2+a^2-2ab+b^2$

$= 2a^2+2b^2$

$= 2(a^2+b^2)$

$=$ R.H.S

Formula 2

$$(a+b)^2-(a-b)^2 = 4ab$$

Proof: L.H.S $= (a+b)^2-(a-b)^2$

$= (a^2+2ab+b^2)-(a^2-2ab+b^2)$

$= a^2+2ab+b^2-a^2+2ab-b^2$

$= 4ab$

$=$ R.H.S

EXAMPLE-1

Find the value of a^2+b^2 when $a+b = 8$ and $ab = 12$

SOLUTION: Given $a+b = 8$

$(a+b)^2 = 8^2$ — Squaring both the sides

$a^2+2ab+b^2 = 64$

$a^2+b^2 = 64-2ab$

$= 64-2(12) \quad \because ab = 12$

$= 64-24$

$a^2+b^2 = 40$

The symbol "$\because$" stands for "because"

EXAMPLE-2

Find the value of ab when $a+b=9$ and $a-b=3$

SOLUTION: We have

$$(a+b)^2-(a-b)^2 = 4ab$$
$$(9)^2-(3)^2 = 4ab$$
$$81-9 = 4ab$$
$$4ab = 72$$
$$ab = \frac{72}{4}$$
$$ab = 18$$

1.2.2 Formula 3

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

Proof: Put $p = a+b$

$$\begin{aligned} L.H.S &= (a+b+c)^2 = (p+c)^2 \\ &= p^2+2pc+c^2 \\ &= (a+b)^2+2(a+b)c+c^2 \quad \text{(where } p=a+b) \\ &= a^2+2ab+b^2+2ac+2bc+c^2 \\ &= a^2+b^2+c^2+2ab+2bc+2ca \\ &= R.H.S \end{aligned}$$

EXAMPLE-3

Find the value of $a^2+b^2+c^2$ when $a+b+c=12$ and $ab+bc+ca=8$

SOLUTION: We have

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$$
$$= a^2+b^2+c^2+2(ab+bc+ca)$$
$$(12)^2 = a^2+b^2+c^2+2(8)$$
$$144 = a^2+b^2+c^2+16$$
$$a^2+b^2+c^2 = 128$$

EXAMPLE-4

Find the value of $a+b+c$ when $a^2+b^2+c^2 = 100$ and $ab+bc+ca = 22$

SOLUTION: We have

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$$

$$= (a^2+b^2+c^2)+2(ab+bc+ca)$$

$$= 100+2(22)$$

$$= 100+44$$

$$(a+b+c)^2 = 144$$

$$(a+b+c)^2 = (12)^2$$

$$a+b+c = \pm 12$$

RESULTS

(i) $x^2 = a^2$

$x = \pm a$

(ii) $x^2 = a$

$x = \pm\sqrt{a}$

EXAMPLE-5

Find the value of $ab+bc+ca$ when $a^2+b^2+c^2 = 36$ and $a+b+c = 8$

SOLUTION: We have

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$$

$$8^2 = 36+2(ab+bc+ca)$$

$$64-36 = 2(ab+bc+ca)$$

$$2(ab+bc+ca) = 28$$

$$ab+bc+ca = \frac{28}{2}$$ (Dividing by 2 on both sides)

$$ab+bc+ca = 14$$

1.2.3 Formula 4

$$(a+b)^3 = a^3 + 3ab(a+b) + b^3$$

Proof: $L.H.S = (a+b)^3$

$= (a+b)^2\ (a+b)$

$= (a^2 + 2ab + b^2)\ (a+b)$

$= a^3 + a^2b + 2a^2b + 2ab^2 + b^2a + b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

$= a^3 + 3ab(a+b) + b^3$

$= R.H.S$

Formula 5

$$(a-b)^3 = a^3 - 3ab(a-b) - b^3$$

Proof: $L.H.S = (a-b)^3$

$= (a-b)^2\ (a-b)$

$= (a^2 - 2ab + b^2)\ (a-b)$

$= a^3 - a^2b - 2a^2b + 2ab^2 + b^2a - b^3$

$= a^3 - 3a^2b + 3ab^2 - b^3 \qquad (\because ab^2 = b^2a)$

$= a^3 - 3ab(a-b) - b^3$

$= R.H.S$

Formula 6

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Proof: $R.H.S = (a+b)(a^2 - ab + b^2)$

$$= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$$

$$= a^3 + b^3$$

$$= L.H.S$$

Formula 7

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Proof: $R.H.S = (a-b)(a^2 + ab + b^2)$

$$= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$$

$$= a^3 - b^3$$

$$= L.H.S$$

EXAMPLE-6

Find the value of $x^3 + y^3$ when $xy = 8$ and $x + y = 5$

SOLUTION: $x + y = 5$ (Given)

$(x+y)^3 = (5)^3$ (Takeing cube of both the sides)

$$x^3 + y^3 + 3xy(x+y) = 125$$

$x^3 + y^3 + 3(8)(5) = 125$ (Putting $x + y = 5$ and $xy = 8$)

$$x^3 + y^3 + 120 = 125$$

$$x^3 + y^3 = 125 - 120$$

$$\therefore x^3 + y^3 = 5$$

EXAMPLE-7

Find the value of $a^3 - b^3$ when the values of $a - b = 6$ and $ab = 7$

SOLUTION: $a - b = 6$ (Given)

$$(a-b)^3 = (6)^3 \quad \text{(Taking cube of both the sides)}$$

$$a^3 - b^3 - 3ab(a-b) = 216$$

$$a^3 - b^3 - 3(7)(6) = 216$$

$$a^3 - b^3 - 126 = 216$$

$$a^3 - b^3 = 216 + 126$$

$$a^3 - b^3 = 342$$

EXAMPLE-8

Resolve into factors $x^3p^2 - 8y^3p^2 - 4x^3q^2 + 32y^3q^2$

SOLUTION: $x^3p^2 - 8y^3p^2 - 4x^3q^2 + 32y^3q^2$ (Rearranging the terms)

$$= p^2(x^3 - 8y^3) - 4q^2(x^3 - 8y^3)$$

$$= (p^2 - 4q^2)(x^3 - 8y^3)$$

$$= \left[(p)^2 - (2q)^2\right]\left[(x)^3 - (2y)^3\right]$$

$$= (p - 2q)\ (p + 2q)\ (x - 2y)(x^2 + 2xy + 4y^2)$$

EXAMPLE-9

Factorize $64x^6 - 729y^6$

SOLUTION: $64x^6 - 729y^6 = 2^6x^6 - 3^6y^6$

$$= (2x)^6 - (3y)^6$$

$$= \left[(2x)^3\right]^2 - \left[(3y)^3\right]^2$$

$$= \left[(2x)^3 - (3y)^3\right]\left[(2x)^3 + (3y)^3\right]$$

$$= (2x - 3y)\left[4x^2 + 6xy + 9y^2\right](2x + 3y)\left[4x^2 - 6xy + 9y^2\right]$$

The symbol "$\therefore$" stands for "therefore"

EXAMPLE-10

Resolve into factors. $(x+y)^3+64$

SOLUTION: $(x+y)^3+64$

$$= (x+y)^3+(4)^3$$

$$= (x+y+4)\left[(x+y)^2-(x+y)4+(4)^2\right]$$

$$= (x+y+4)\left[x^2+y^2+2xy-4x-4y+16\right]$$

EXAMPLE-11

Find the continued product for x^6-y^6.

SOLUTION: x^6-y^6

$$= (x^3)^2-(y^3)^2$$

$$= (x^3+y^3)\,(x^3-y^3)$$

$$= (x+y)(x^2-xy+y^2)\,(x-y)(x^2+xy+y^2)$$

$$= (x+y)\,(x-y)\,(x^2-xy+y^2)\,(x^2+xy+y^2)$$

EXERCISE - 1.2

Solve the Following Questions Using Formulas.

1. $(x+2y)^2+(x-2y)^2$
2. $(5x+3y)^2+(5x-3y)^2$
3. $(3l+2m)^2-(3l-2m)^2$
4. $(l+m)(l-m)\,(l^2+m^2)(l^4+m^4)$

5. $(ab-\frac{1}{ab})^3$

6. $(2x+3y+2)^2$

7. $(2p+q)^3$

8. $(3p+q+r)^2$

9. $(2x+3y)^3$

10. $(x+y)^3-1$

11. $(x-y)^3+64$

12. $8x^3+27y^3$

13. x^6-729y^6

14. $64a^6-b^6$

15. Find the value of a^3-b^3 when $a-b=4$ and $ab=-5$.

16. Show that $\left(z+\frac{1}{z}\right)^2-\left(z-\frac{1}{z}\right)^2=4$.

17. Find the value of a^2+b^2 and ab when $a+b=5$ and $a-b=3$.

18. Find the value of $a^2+b^2+c^2$ if $ab+bc+ca=11$ and $a+b+c=6$.

19. Find the value of x^3+y^3 if $xy=10$ and $x+y=7$.

20. Find the value of $(x-y)^2$ if $x^2+y^2=86$ and $xy=-16$.

21. Find the value of $ab+bc+ca$ when the values of $a^2+b^2+c^2=81$, $a+b+c=11$.

22. Find the value of $(a+b+c)^2$ when the values of $a^2+b^2+c^2=32$ and $ab+bc+ca=7$.

1.3 SURDS AND THEIR APPLICATIONS

1.3.1 Surds

Rational Numbers:

A number which can be expressed in the form $\left(\frac{p}{q}\right)$, where 'p' and 'q' are integers and $q \neq 0$ is called a rational number.

e.g. $\frac{3}{4}, \frac{2}{1}, \frac{8}{7}, \frac{-2}{5}$ are all rational numbers.

Irrational Numbers:

A real number which is not a rational number, is called an irrational number. For example:

$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}$ etc. are irrational numbers.

Clearly, an irrational number cannot be expressed in the form $\left(\frac{p}{q}\right)$, where p and q are integers and $q \neq 0$.

Real Numbers:

The set IR of all real numbers is the union of two disjoint subsets, namely the set Q of all rational numbers and the set Q' of all irrational numbers.

Surds of Radicals:

A surd is an irrational number that contains a radical signs.

e.g. $\sqrt{2}, 2\sqrt{3}, 4+3\sqrt{5}, 10-4\sqrt{6}, \frac{\sqrt{2}}{5}, \frac{9}{\sqrt{7}}$ are all surds.

EXAMPLE

(i) $\sqrt{3} = 3^{\frac{1}{2}}$ *is a surd of order 2, i.e. it is a quadratic surd.*

(ii) $\sqrt[3]{4} = 4^{\frac{1}{3}}$ *is a surd of order 3, i.e. it is a cubic surd.*

(iii) $\sqrt[n]{a} = a^{\frac{1}{n}}$ *is called a surd of radical of order 'n' and 'a' is called the radicand.*

The symbol " i.e " stands for " That is "

Laws of Radicals:

As the surd can be expressed with rational exponents, the laws of indices, are therefore, applicable in surds also.
Thus for any positive integer 'n' and positive rational numbers 'a and b', we have the following laws:

Laws of Radicals	Laws of Indices
(i) $(\sqrt[n]{a})^n = a$	(i) $\left(a^{\frac{1}{n}}\right)^n = a$
(ii) $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$	(ii) $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}\,b^{\frac{1}{n}}$
(iii) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	(iii) $\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$
(iv) $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$	(iv) $\left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$

Pure Surds:

A surd which has unity only as rational factor, the other factor being irrational, is called a pure surd.

Example: $\sqrt{2}$, $\sqrt{11}$, $\sqrt[4]{3}$, are pure surds.

Mixed Surds:

A surd which has rational factor other than unity, the other factor being irrational, is called a mixed surd.

Example: $2\sqrt{3}$, $5\sqrt{7}$, are mixed surds.

1.3.2 Surds of Second Order:

$\sqrt{a} = a^{\frac{1}{2}}$ is a surd of order 2, i.e. a quadratic surd.

Remark:

The symbol $\sqrt{\ }$ is called the radical sign of index 2.

Similar Surds:

Surds having the same irrational factor are called similar or like surds.
For example, $\sqrt{3},\ 5\sqrt{3},\ \frac{1}{7}\sqrt{3}$ are similar surds.
Surds having no common irrational factor are known as **unlike surds**.

Example: $\sqrt{2},\ 3\sqrt{5},\ 2\sqrt{3}$ are unlike surds.

Addition And Subtraction of Surds:

Similar surds can be added and subtracted

Example: (i) $6\sqrt{3}+5\sqrt{3} = (6+5)\sqrt{5} = 11\sqrt{3}$

(ii) $12\sqrt{5}+4\sqrt{5}-6\sqrt{5} = (12+4-6)\sqrt{5} = 10\sqrt{5}$

Multiplication and division of two surds:

Surds of the same order can be multiplied and divided according to following laws:

For any natural numbers 'm' and 'n'

(i) $\sqrt{m}\times\sqrt{n} = \sqrt{mn}$ (ii) $\dfrac{\sqrt{m}}{\sqrt{n}} = \sqrt{\dfrac{m}{n}}$

EXAMPLE-1

Simplify: $\sqrt{8}\times\sqrt{2}$

SOLUTION: *We use the rule* $\sqrt{m}\times\sqrt{n} = \sqrt{mn}$

$$\sqrt{8}\times\sqrt{2} = \sqrt{8\times 2} = \sqrt{16} = 4$$

EXAMPLE-2

Simplify: $\sqrt{180} \div \sqrt{24}$

SOLUTION: $\sqrt{180} \div \sqrt{24} = \dfrac{\sqrt{180}}{\sqrt{24}} = \sqrt{\dfrac{180}{24}}$ $\left[\text{using } \dfrac{\sqrt{m}}{\sqrt{m}} = \sqrt{\dfrac{m}{n}}\right]$

$$= \sqrt{\frac{\not{2} \times \not{2} \times \not{3} \times 3 \times 5}{\not{2} \times \not{2} \times 2 \times \not{3}}}$$

$$= \sqrt{\frac{15}{2}}$$

Rationalizing the Denominator:

We can simplify a fraction by removing a square root from the denominator.

We can do this by multiplying the numerator and denominator by the same square root.

This process is called rationalizing the denominator.

EXAMPLE-1

Simplify these $(a)\ \dfrac{2}{\sqrt{3}}$ $(b)\ \dfrac{5}{7\sqrt{2}}$

SOLUTION: (a) *Multiply by* $\dfrac{\sqrt{3}}{\sqrt{3}}$

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{(\sqrt{3})^2} = \frac{2\sqrt{3}}{3}$$

(b) *Multiply by* $\dfrac{\sqrt{2}}{\sqrt{2}}$

$$\frac{5}{7\sqrt{2}} = \frac{5}{7\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{7 \times 2} = \frac{5\sqrt{2}}{14}$$

Multiply: $(2+\sqrt{3})\ (5-\sqrt{3})$

SOLUTION: $(2+\sqrt{3})\ (5-\sqrt{3})$

$$= 2\times 5+2\times(-\sqrt{3})+5\times\sqrt{3}+\sqrt{3}(-\sqrt{3})$$

$$= 10-2\sqrt{3}+5\sqrt{3}-3$$

$$= 7+3\sqrt{3}$$

EXAMPLE-3

Multiply: $(3\sqrt{5}-5\sqrt{2})(4\sqrt{5}+3\sqrt{2})$

SOLUTION: $(3\sqrt{5}-5\sqrt{2})\ (4\sqrt{5}+3\sqrt{2})$

$$= 12(\sqrt{5})^2+9\sqrt{5}\sqrt{2}-20\sqrt{2}\sqrt{5}-15(\sqrt{2})^2$$

$$= 12\times 5+9\sqrt{10}-20\sqrt{10}-15\times(2)$$

$$= 60-30-11\sqrt{10}$$

$$= 30-11\sqrt{10}$$

EXAMPLE-4

Express in the simplest form

(i) $\sqrt{288}$ (ii) $\sqrt{147}$ (iii) $\sqrt{36a^3}$

SOLUTION: (i) $\sqrt{288}$

$= \sqrt{2\times2\times2\times2\times2\times3\times3}$

$= \sqrt{2\times2}\times\sqrt{2\times2}\times\sqrt{3\times3}\times\sqrt{2}$

$= 2\times2\times3\times\sqrt{2}$

$= 12\sqrt{2}$

2	288
2	144
2	72
2	36
2	18
3	9
3	3
	1

(ii) $\sqrt{147}$

$= \sqrt{7\times7\times3}$

$= \sqrt{7\times7}\times\sqrt{3}$

$= 7\sqrt{3}$

7	147
7	21
3	3
	1

(iii) $\sqrt{36a^3}$

$= \sqrt{6\times6\times a\times a\times a}$

$= \sqrt{6\times6}\times\sqrt{a\times a}\times\sqrt{a}$

$= 6\times a\times\sqrt{a}$

$= 6a\sqrt{a}$

1.4 RATIONALIZATION:

Binomial Surd:

An expression is called a binomial surd if it consists of two terms in which at least one term is a surd. For example:

$a+b\sqrt{x}$, $\sqrt{x}+\sqrt{y}$ are binomial surds.

Conjugate of Binomial Surds:

(i) $a+b\sqrt{x}$ and $a-b\sqrt{x}$

(ii) $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$

are surds whose product is a rational number. The pair of such surds is called conjugate binomial surds. Each of these two surds is a conjugate of the other. For example:

(i) $2+3\sqrt{5}$ is conjugate binomial surd of $2-3\sqrt{5}$.

(ii) $\sqrt{3}+\sqrt{7}$ is conjugate binomial surd of $\sqrt{3}-\sqrt{7}$.

Remember that:

Conjugate binomial surds are rationalizing factors of each other.

Rationalizing Factor:

When the product of two surds is rational, then each one of them is called the rationalizing factor of the other.

EXAMPLE

(i) $2\sqrt{3}\times\sqrt{3} = 6$, which is rational.

So $\sqrt{3}$ is rationalizing factor of $2\sqrt{3}$.

(ii) $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 3-2 = 1$ which is rational.

So $(\sqrt{3}+\sqrt{2})$ is rationalizing factor of $(\sqrt{3}-\sqrt{2})$.

Rationalization of Surds:

The process of converting a surd to a rational number by multiplying it with a suitable rationalizing factor, is called the rationalization of the surds.

EXAMPLE-1

Express $\frac{1}{5+2\sqrt{3}}$ *with rational denominator.*

SOLUTION: $\frac{1}{5+2\sqrt{3}}$

$$= \frac{1}{5+2\sqrt{3}} \times \frac{5-2\sqrt{3}}{5-2\sqrt{3}}$$

$$= \frac{5-2\sqrt{3}}{5^2-(2\sqrt{3})^2}$$

$$= \frac{5-2\sqrt{3}}{25-12} = \frac{5-2\sqrt{3}}{13}$$

EXAMPLE-2

Express $\frac{1}{\sqrt{5}+\sqrt{3}}$ *with rational denominator.*

SOLUTION: $\frac{1}{\sqrt{5}+\sqrt{3}}$

$$= \frac{1}{\sqrt{5}+\sqrt{3}} \times \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$$

$$= \frac{\sqrt{5}-\sqrt{3}}{(\sqrt{5})^2-(\sqrt{3})^2}$$

$$= \frac{\sqrt{5}-\sqrt{3}}{5-3} = \frac{\sqrt{5}-\sqrt{3}}{2}$$

(i) $\frac{1}{x}$ (ii) $x+\frac{1}{x}$ (iii) $x-\frac{1}{x}$ (iv) $\left(x+\frac{1}{x}\right)^2$

(v) $\left(x-\frac{1}{x}\right)^2$ (vi) $x^2+\frac{1}{x^2}$ (vii) $x^2-\frac{1}{x^2}$

SOLUTION: $x = 3+\sqrt{8}$

(i) $\frac{1}{x} = \frac{1}{3+\sqrt{8}}$

$= \frac{1}{3+\sqrt{8}} \times \frac{3-\sqrt{8}}{3-\sqrt{8}}$

$= \frac{3-\sqrt{8}}{(3)^2-(\sqrt{8})^2} = \frac{3-\sqrt{8}}{9-8}$

$= 3-\sqrt{8}$

(ii) $x+\frac{1}{x}$

$= (3+\sqrt{8})+(3-\sqrt{8})$

$= 3+\sqrt{8}+3-\sqrt{8}$

$= 6$

(iii) $x-\frac{1}{x}$

$= (3+\sqrt{8})-(3-\sqrt{8})$

$= 3+\sqrt{8}-3+\sqrt{8}$

$= 2\sqrt{8}$

(iv) $(x+\frac{1}{x})^2$

$(x+\frac{1}{x})^2 = 6^2$ (from (ii))

$(x+\frac{1}{x})^2 = 36$

(v) $(x-\frac{1}{x})^2$

$(x-\frac{1}{x})^2 = (2\sqrt{8})^2$ (from (iii))

$(x-\frac{1}{x})^2 = 32$

(vi) $x^2+\frac{1}{x^2}$

$= x^2+\frac{1}{x^2}+2-2$

$= (x^2+\frac{1}{x^2}+2)-2$

$= (x+\frac{1}{x})^2-2$

$= 36-2$ (from (iv))

$= 34$

(vii) $x^2-\frac{1}{x^2}$

$= (x+\frac{1}{x})(x-\frac{1}{x}) = 6(2\sqrt{8})$ (from (ii), (iii))

$= 12\sqrt{8}$

EXERCISE – 1.3

1. **Remove the radical sign from the denominator:**

   (i) $\frac{1}{\sqrt{5}}$ (ii) $\frac{2}{\sqrt{2}}\cdot\frac{7}{\sqrt{3}}$ (iii) $\frac{\sqrt{6}}{\sqrt{7}}$

2. **Simplify the following expressions:**

   (i) $\sqrt{2}+\sqrt{8}$ (ii) $4\sqrt{50}+\sqrt{200}+\sqrt{50}$

   (iii) $(\sqrt{12}-\sqrt{2})(\sqrt{20}-3\sqrt{2})$ (iv) $(6+\sqrt{2})(5-\sqrt{5})$

   (v) $(\sqrt{3}-2)(5-\sqrt{5})$ (vi) $(7+\sqrt{3})(5+\sqrt{2})$

3. **Rationalize the denominators of the following :**

   (i) $\frac{1}{\sqrt{3}+2}$ (ii) $\frac{1}{4-\sqrt{5}}$ (iii) $\frac{4\sqrt{3}}{\sqrt{7}+\sqrt{5}}$ (iv) $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

   (v) $\frac{5\sqrt{7}}{2+3\sqrt{7}}$ (vi) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ (vii) $\frac{29}{11+3\sqrt{5}}$

   (viii) $\frac{17}{3\sqrt{7}+2\sqrt{3}}$

4. If $x=\sqrt{5}+2$, then find the values of (i) $x+\frac{1}{x}$ and (ii) $x^2+\frac{1}{x^2}$

5. If $x=2+\sqrt{3}$, then find the values of (i) $x-\frac{1}{x}$ and (ii) $x^2+\frac{1}{x^2}$

6. If $x=\sqrt{3}-\sqrt{2}$, then find the values of (i) $x-\frac{1}{x}$ and (ii) $x^2+\frac{1}{x^2}$

7. If $\frac{1}{x}=3-\sqrt{2}$, then evaluate (i) $x+\frac{1}{x}$ (ii) $x-\frac{1}{x}$

8. If $\frac{1}{p}=\sqrt{10}+3$, then evaluate (i) $(p+\frac{1}{p})^2$ (ii) $(p-\frac{1}{p})^2$

9. Rationalize (i) $\frac{b+\sqrt{b^2-a^2}}{b-\sqrt{b^2-a^2}}$ (ii) $\frac{\sqrt{a+3}-\sqrt{a-3}}{\sqrt{a+3}+\sqrt{a-3}}$

Review Exercise-1

I- **Encircle the Correct Answer.**

1. An algebraic expression of the form $\frac{P(x)}{Q(x)}, Q(x) \neq 0$, $P(x)$ and $Q(x)$ are polynomials, is called a:

   (a) *rational number* (b) *rational expression*

   (c) *surd* (d) *mixed surd*

2. $(a+b)^2-(a-b)^2=?$

   (a) $2(a^2+b^2)$ (b) $4ab$

   (c) $-4ab$ (d) a^2+b^2

3. $(a+b)^2+(a-b)^2=?$

   (a) $-4ab$ (b) a^2+b^2

   (c) $4ab$ (d) $2(a^2+b^2)$

4. $(a-b)(a^2+ab+b^2)=?$

   (a) $(a-b)^3$ (b) $(a+b)^3$

   (c) a^3-b^3 (d) a^3+b^3

5. $(a+b)(a^2-ab+b^2)=?$

   (a) a^3-b^3 (b) $(a+b)^3$

   (c) $(a-b)^3$ (d) a^3+b^3

6. $a^3+3ab(a+b)+b^3=?$

   (a) $(a+b)^3$ (b) $(a-b)^3$

   (c) a^3+b^3 (d) a^3-b^3

7. $a^3-3ab(a-b)-b^3=?$

   (a) a^3+b^3 (b) $(a+b)^3$

   (c) a^3-b^3 (d) $(a-b)^3$

8. An irrational number that contains radical signs is called a:

   (a) *mixed surd* (b) *surd*

   (c) *…ional number* (d) *natural number*

9. $\sqrt{a} = a^{1/2}$ is a surd of order:

(a) *zero* (b) *1*

(c) *2* (d) *½*

10. Surds can be multiplied, if they are of the

(a) *same order* (b) *order 2*

(c) *different order* (d) *order n*

II- **Fill in the blanks.**

1. A number of the form $\frac{p}{q}$, $q \neq 0, p, q \in Z$ is called a ____________.

2. An expression of the form $\frac{P(x)}{Q(x)}, Q(x) \neq 0, P(x), Q(x)$ are polynomials is called ____________.

3. $(a+b)^2 - (a-b)^2 =$ ____________

4. $(a+b)^2 + (a-b)^2 =$ ____________

5. $a^3 + 3ab(a+b) + b^3 =$ ____________

6. $a^3 - 3ab(a-b) - b^3 =$ ____________

7. $(a-b)(a^2+ab+b^2) =$ ____________

8. $(a+b)(a^2-ab+b^2) =$ ____________

9. An irrational number that contains radical signs is called a ____________.

10. $\sqrt{a} = a^{1/2}$ is a surd of order ____________.

SUMMARY

Formulae:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$

$$(a+b)^2 - (a-b)^2 = 4ab$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$$

$$(a \pm b)^3 = a^3 \pm 3ab(a \pm b) \pm b^3$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Surd: A surd is an irrational number that contains radical signs.

Pure Surd: A surd which has unity only as rational factor, the other factor being irrational is called a pure surd.

Mixed surd: A surd which has rational factor other than unity, the other factor being irrational, is called mixed surd.

Similar surd: Surds having the same irrational factor are called similar or like surds.

Unlike surd: Surds having no common irrational factor are known as unlike surds.

Rationalizing Factor: When the product of two surds is rational, then each one of them is called the rationalizing factor of the other.