# UNIT

# 2

# FACTORIZATION

- **Factorization**
- **Remainder Theorem and Factor Theorem**
- **Factorization of Cubic Polynomial**

***After completion of this unit, the students will be able to:***

- factorize the expressions of following types.
  - Type I: $kx + ky + kz$,
  - Type II: $ax + ay + bx + by$,
  - Type III: $a^2 \pm 2ab + b^2$,
  - Type IV: $a^2 - b^2$,
  - Type V: $(a^2 \pm 2ab + b^2) - c^2$,
  - Type VI: $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$,
  - Type VII: $x^2 + px + q$,
  - Type VIII: $ax^2 + bx + c$,
  - Type IX: $\begin{cases} a^3+3a^2b+3ab^2+b^3, \\ a^3-3a^2b+3ab^2-b^3, \end{cases}$
  - Type X: $a^3 \pm b^3$,
- state and apply remainder theorem.
- find remainder (without dividing) when a polynomial is divided by a linear polynomial.
- define zeros of a polynomial.
- state factor theorem and explain through examples.
- use factor theorem to factorize a cubic polynomial.

## 2.1 FACTORIZATION OF EXPRESSIONS

### Linear Polynomial :-

A polynomial of degree *'1'* is called a linear polynomial.
For example: $x + 3$, $2x - 5$ etc. The general form of linear polynomials is $ax + b$ where $a$ , $b$ are real numbers and $a \neq 0$ .

### Quadratic Polynomial :-

A polynomial of degree *'2'* is called quadratic polynomial e.g. $3x^2 + 5x - 2$, $4x^2 - 3x + 1$ etc. The general form of a quadratic polynomial is $ax^2 + bx + c$, where $a$, $b$, $c$ are real numbers and $a \neq 0$.

### Cubic Polynomial :-

A polynomial of degree *'3'* is called a cubic polynomial. e.g. $x^3 - 3x^2 + 5x + 2$, $4x^3 + 5x^2 - 2$ etc. The general form of cubic polynomial is $ax^3 + bx^2 + cx + d$ where $a$, $b$, $c$, $d$ are real numbers and $a \neq 0$.

Let $P(x)$ be any polynomial and let $a$, $b$, $c$ be any real numbers such that $P(x) = (x-a)\ (x-b)\ (x-c)$. Then, clearly each one of $(x-a)$, $(x-b)$, $(x-c)$ is a linear factor of $P(x)$.

To express a given polynomial as the product of linear factors or factors of degree less than that of the given polynomial, is known as factorization.

We see that in $15 = 3 \times 5$, $3$ and $5$ are factors of $15$. Similarly, in $ax + ay = a(x + y)$, $a$ and $(x + y)$ are factors of $ax + ay$ and in $ax + ay + az = a(x + y + z)$, $a$ and $(x + y + z)$ are factors of $ax + ay + az$.

The process of writing an expression as a product of two or more factors is called factorization.

We factorize the expressions of different types.

Following examples will explain the factorization of the expression.

### EXAMPLE-1

Factorize the following

(i) $3x + 12y$  (ii) $x^2 + xy$  (iii) $ad + dc + df$

(iv) $2pq + 6p^2q - 4p^3q$

**SOLUTION:**

(i) $3x + 12y = 3(x + 4y)$

(ii) $x^2 + xy = x(x + y)$

(iii) $ad + dc + df = d(a + c + f)$

(iv) $2pq + 6p^2q - 4p^3q = 2pq(1 + 3p - 2p^2)$

### Factorization of the expression of the form:

$$ax + ay + bx + by$$

Following examples will explain the factorization of the expression.

### EXAMPLE-2

Factorize the following expressions:

(i) $2ax + bx + 6ay + 3by$  (ii) $2yx + 18y^2 - 3zx + 27zy$

(iii) $5ym + 15yn + 2zm + 6zn$

**SOLUTION:**

(i) $2ax + bx + 6ay + 3by$

$= x(2a + b) + 3y(2a + b)$

$= (2a + b)(x + 3y)$

Now check $(2a + b)(x + 3y) = 2ax + bx + 6ay + 3by$

(ii) $2yx + 18y^2 + 3zx + 27zy$

$= 2y(x + 9y) + 3z(x + 9y)$

$= (2y + 3z)(x + 9y)$

(iii) $5ym + 15yn + 2zm + 6zn$

$= 5y(m + 3n) + 2z(m + 3n)$

$= (5y + 2z)(m + 3n)$

### Factorization of the expression of the form:

$$a^2 \pm 2ab + b^2$$

We know that: (i) $a^2 + 2ab + b^2 = (a+b)^2$

(ii) $a^2 - 2ab + b^2 = (a-b)^2$

Expressions which have the pattern of the left hand side of *(i)* and *(ii)* are called perfect squares. These identities are useful in helping us to factorize certain expressions. Following examples will explain the factorization of the expressions.

### EXAMPLE-3

*Factorize the following.*

(i) $x^2 + 6x + 9$ (ii) $t^2 - 12t + 36$

**SOLUTION:** (i) $x^2 + 6x + 9 = x^2 + 2(3)(x) + 3^2$

$$= (x+3)^2$$

(ii) $t^2 - 12t + 36 = t^2 - 2(6)(t) + 6^2$

$$= (t-6)^2$$

### Factorization of the expression of the form:

$$a^2 - b^2$$

This is called difference of two squares. $a^2 - b^2 = (a-b)(a+b)$

Following examples will explain the factorization of the expression.

### EXAMPLE-4

*Factorize the following.*

(i) $k^2 - 81$ (ii) $9a^2 - (b+c)^2$

**SOLUTION:**

(i) $k^2 - 81 = k^2 - 9^2$

$$= (k+9)(k-9)$$

(ii) $9a^2 - (b+c)^2 = (3a)^2 - (b+c)^2$

$$= \left[3a + (b+c)\right]\left[3a - (b+c)\right]$$

$$= \left[3a + b + c\right]\left[3a - b - c\right]$$

**EXAMPLE-5** Factorize $36d^2 - 1$

**SOLUTION:** $36d^2 - 1 = (6d)^2 - (1)^2$

$= (6d + 1)(6d - 1)$

## EXERCISE – 2.1

**Factorize:**

1- $3a(x+y) - 7b(x+y)$

2- $ax + ay - x^2 - xy$

3- $a^3 + a - 3a^2 - 3$

4- $x^3 + y - xy - x$

5- $3ax + 6ay - 8by - 4bx$

6- $2a^2 - bc - 2ab + ac$

7- $a(a - b + c) - bc$

8- $8 - 4a - 2a^3 + a^4$

9- $16x^2 - 24xa + 9a^2$

10- $1 - 14x + 49x^2$

11- $20x^2 + 5 - 20x$

12- $2a^3b + 2ab^3 - 4a^2b^2$

13- $x^2 + x + \frac{1}{4}$

14- $x^2 + \frac{1}{x^2} - 2$

15- $5x^3 - 30x^2 + 45x$

16- $a^2 + b^2 + 2ab + 2bc + 2ac$

Factorization of the expression of the form:

(i) $(a^2 + 2ab + b^2) - c^2$

(ii) $(a^2 - 2ab + b^2) - c^2$

*Following examples will explain the factorization of the expressions.*

**EXAMPLE-1**

Resolve into factors:

$x^2 + 2xy + y^2 - 4z^2$

**SOLUTION:** $(x^2 + 2xy + y^2) - 4z^2$

$= (x+y)^2 - (2z)^2$

$= (x + y - 2z)(x + y + 2z)$

## EXAMPLE-2

*Resolve into factors:*

$$c^2 + 6bc + 9b^2 - 16x^2$$

**SOLUTION:**

$$\begin{aligned}(c^2 + 6bc + 9b^2) - 16x^2 & \\ &= (c + 3b)^2 - (4x)^2 \\ &= (c + 3b + 4x) \quad (c + 3b - 4x)\end{aligned}$$

## EXAMPLE-3

*Factorize:*

(i) $a^2 - 2ab + b^2 - 9c^2$

(ii) $x^2 - 6xy + 9y^2 - 4z^2$

**SOLUTION:**

$$\begin{aligned}\text{(i)}\quad a^2 - 2ab + b^2 - 9c^2 &= a^2 - 2ab + b^2 - 9c^2 \\ &= (a - b)^2 - (3c)^2 \\ &= (a - b - 3c)(a - b + 3c)\end{aligned}$$

$$\begin{aligned}\text{(ii)}\quad x^2 - 6xy + 9y^2 - 4z^2 &= x^2 - 2(3)xy + 9y^2 - 4z^2 \\ &= (x - 3y)^2 - (2z)^2 \\ &= (x - 3y - 2z)(x - 3y + 2z)\end{aligned}$$

**Factorization of the expression of the form:**

(i) $a^4 + a^2b^2 + b^4$

(ii) $a^4 + 4b^4$

## EXAMPLE-4

*Factorize:* $x^4 + x^2 + 1$

**SOLUTION:**

$$\begin{aligned}x^4 + x^2 + 1 &= x^4 + x^2 + 1 + x^2 - x^2 \\ &= (x^4 + 2x^2 + 1) - x^2 \\ &= (x^2 + 1)^2 - x^2 \\ &= (x^2 + x + 1)(x^2 - x + 1)\end{aligned}$$

**SOLUTION:** $x^4 + 64 = (x^2)^2 + 8^2 + 2(8)x^2 - 2(8)x^2$

$$= (x^2 + 8)^2 - 16x^2$$

$$= (x^2 + 8)^2 - (4x)^2$$

$$= (x^2 + 8 + 4x)(x^2 + 8 - 4x)$$

## EXAMPLE-6

*Resolve into factors:*

$$x^4 + x^2y^2 + y^4$$

**SOLUTION:** $x^4 + x^2y^2 + y^4 = (x^4 + 2x^2y^2 + y^4) - x^2y^2$

$$= (x^2 + y^2)^2 - (xy)^2$$

$$= (x^2 + y^2 + xy)(x^2 + y^2 - xy)$$

## EXERCISE – 2.2

**Resolve into Factors:**

**1.** $x^2 + 2xy + y^2 - a^2$

**2.** $4a^2 + 4ab + b^2 - 9c^2$

**3.** $x^2 + 6ax + 9a^2 - 16b^2$

**4.** $y^2 - c^2 + 2cx - x^2$

**5.** $x^2 + y^2 + 2xy - 4x^2y^2$

**6.** $a^2 - 4ab + 4b^2 - 9a^2c^2$

**7.** $x^2 - 2xy + y^2 - a^2 + 2ab - b^2$

**8.** $y^4 + 4$

**9.** $z^4 + 64y^4$

**10.** $x^4 + 324$

**11.** $z^4 - z^2 + 16$

**12.** $4x^4 - 5x^2y^2 + y^4$

## Factorization of the expression of the form:

$$x^2 + px + q$$

Let $x^2 + px + q = (x + r)(x + s)$

Then $x^2 + px + q = x^2 + (r + s)x + rs$

comparing coefficients of the like terms on both sides, we get

$$r + s = p \quad \text{and} \quad rs = q$$

Thus, in order to factorize $x^2 + px + q$, we have to find two numbers '$r$' and '$s$' such that $r + s = p$ and $rs = q$

*Following examples will explain the factorization of the expression.*

**EXAMPLE**

*Factorize*

*(i)* $x^2 + 7x + 12$ *(ii)* $x^2 + 4x - 21$ *(iii)* $x^2 - 5x - 14$

**SOLUTION:**

*(i) In order to factorize* $x^2 + 7x + 12$, *we must find two numbers '$r$' and '$s$' such that*

$$r + s = 7 \quad \text{and} \quad rs = 12$$

Clearly $4 + 3 = 7$ and $4 \times 3 = 12$

$$\begin{aligned} \therefore \quad x^2 + 7x + 12 &= x^2 + 4x + 3x + 12 \\ &= x(x + 4) + 3(x + 4) \\ &= (x + 4)(x + 3) \end{aligned}$$

*(ii) In order to factorize* $x^2 + 4x - 21$, *we must find two numbers '$r$' and '$s$' such that*

$$r + s = 4 \quad \text{and} \quad rs = -21$$

Clearly $7 + (-3) = 4$ and $7(-3) = -21$

$$\begin{aligned} \therefore \quad x^2 + 4x - 21 &= x^2 + 7x - 3x - 21 \\ &= x(x + 7) - 3(x + 7) \\ &= (x + 7)(x - 3) \end{aligned}$$

Clearly $-7+2=-5$ and $-7\times 2=-14$

$$\begin{aligned}\therefore\quad x^2-5x-14 &= x^2-7x+2x-14\\ &= x(x-7)+2(x-7)\\ &= (x-7)\,(x+2)\end{aligned}$$

## Factorization of the expression of the form:

## $ax^2+bx+c,\ a\neq 0$

To factorize the expression of the form $ax^2+bx+c$, we find numbers $p$ and $q$ such that $p+q=b$ and $pq=ac$ in the given expression, where $a,b,c$ are constants and $a\neq 0$.

*Following examples will explain the factorization of the expression.*

### EXAMPLE

Factorize: (i) $6x^2+7x-3$ (ii) $\sqrt{3}x^2+11x+6\sqrt{3}$

**SOLUTION:**

(i) *The given expression* $6x^2+7x-3$,
*is of the form* $ax^2+bx+c$, $ac=6\times(-3)=-18$

$$\begin{aligned}\therefore\quad 6x^2+7x-3 &= 6x^2+9x-2x-3\\ &= 3x(2x+3)-1(2x+3)\\ &= (2x+3)\,(3x-1)\end{aligned}$$

$6\times(-3)=-18$

**Possible Pairs**

$18\times(-1)=-18$

$(-18)\times(1)=-18$

$6\times(-3)=-18$

$-6\times 3=-18$

$-9\times 2=-18$

$9\times(-2)=-18$

**Selected Pair**

$9\times(-2)=-18$

(ii) $\sqrt{3}x^2+11x+6\sqrt{3}$; $ac=\sqrt{3}\times 6\sqrt{3}=18$

Clearly $9+2=11$

$$\begin{aligned}\therefore\quad \sqrt{3}x^2+11x+6\sqrt{3} &= \sqrt{3}x^2+9x+2x+6\sqrt{3}\\ &= \sqrt{3}x\left[x+3\sqrt{3}\right]+2\left[x+3\sqrt{3}\right]\\ &= (\sqrt{3}x+2)\,(x+3\sqrt{3})\end{aligned}$$

## EXERCISE – 2.3

**Factorize:**

**1.** $x^2 + 9x + 20$

**2.** $x^2 + 5x - 14$

**3.** $x^2 + 5x - 6$

**4.** $x^2 - 7x + 12$

**5.** $x^2 - x - 156$

**6.** $x^2 - x - 2$

**7.** $x^2 - 9x - 90$

**8.** $a^2 - 12a - 85$

**9.** $98 - 7x - x^2$

**10.** $y^2 - 11y - 152$

**11.** $2x^2 + 3x + 1$

**12.** $3x^2 + 5x + 2$

**13.** $2x^2 - x - 1$

**14.** $6x^2 + 7x - 3$

**15.** $2 - 3x - 2x^2$

**16.** $8 + 6x - 5x^2$

**17.** $3u^2 - 10u + 8$

**18.** $10x^2 - 7x - 12$

**19.** $5x^2 - 32x + 12$

**20.** $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$

### Factorization of the expression of the form:

$$\left\{\begin{matrix} a^3 + 3a^2b + 3ab^2 + b^3 \\ a^3 - 3a^2b + 3ab^2 - b^3 \end{matrix}\right\}$$

*We know that:*

(i) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

(ii) $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

*Following examples will explain the factorization of the expression.*

**EXAMPLE** Factorize: (i) $x^3 + 6x^2 + 12x + 8$ (ii) $x^3 - 6x^2 + 12x - 8$

**SOLUTION:**

(i) $x^3 + 6x^2 + 12x + 8 = (x)^3 + 3(2)(x)^2 + 3(2)^2x + (2)^3$

$= (x + 2)^3$

(ii) $x^3 - 6x^2 + 12x - 8 = (x)^3 - 3(2)(x)^2 + 3(2)^2x - (2)^3$

$= (x - 2)^3$

$a^3 \pm b^3$

We know that

(i) $a^3 + b^3 = (a+b)\ (a^2 - ab + b^2)$

(ii) $a^3 - b^3 = (a-b)\ (a^2 + ab + b^2)$

*Following examples will explain the factorization of the expression.*

## EXAMPLE-1

Factorize

(i) $x^3 + 27$ (ii) $8a^3 - 125b^3$ (iii) $x^6 - y^6$ (iv) $a^3 - b^3 - a + b$

**SOLUTION:**

(i) $x^3 + 27 = x^3 + 3^3$

$= (x+3)\ (x^2 - 3x + 9)$

(ii) $8a^3 - 125b^3 = (2a)^3 - (5b)^3$

$= (2a - 5b)\left[(2a)^2 + (2a) \times (5b) + (5b)^2\right]$

$= (2a - 5b)\left[4a^2 + 10ab + 25b^2\right]$

(iii) $x^6 - y^6 = (x^3)^2 - (y^3)^2$

$= (x^3 + y^3)\ (x^3 - y^3)$

$= (x+y)\ (x^2 - xy + y^2)\ (x-y)\ (x^2 + xy + y^2)$

$= (x+y)\ (x-y)\ (x^2 - xy + y^2)\ (x^2 + xy + y^2)$

(iv) $a^3 - b^3 - a + b = (a^3 - b^3) - (a - b)$

$= (a-b)\ (a^2 + ab + b^2) - (a-b)$

$= (a-b)\left[a^2 + ab + b^2 - 1\right]$

**Remember that:**

(i) $a^2 + 2ab + b^2 = (a+b)^2$

(ii) $a^2 - 2ab + b^2 = (a-b)^2$

(iii) $a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3$

(iv) $a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$

(v) $(x+y)(x^2 - xy + y^2) = x^3 + y^3$

(vi) $(x-y)(x^2 + xy + y^2) = x^3 - y^3$

## EXERCISE – 2.4

**Factorize:**

**1.** $8x^3 - y^3$

**2.** $27x^3 + 1$

**3.** $1 - 343x^3$

**4.** $a^3b^3 + 512$

**5.** $27 - 1000y^3$

**6.** $27x^3 - 64y^3$

**7.** $x^3y^3 + z^3$

**8.** $216P^3 - 343$

**9.** $8x^3 - \frac{1}{27}$

**10.** $a^3 + b^3 + a + b$

**11.** $a - b - a^3 + b^3$

**12.** $x - 8xy^3$

**13.** $x^{12} - y^{12}$

**14.** $1 - \frac{64p^3}{q^3}$

**15.** $1 + 64U^3$

**16.** $8x^3 - 6x - 9y + 27y^3$

**17.** $z^3 + 125$

**18.** $x^9 + y^9$

**19.** $m^6 - n^6$

**20.** $64x^7 - xa^6$

**21.** $x^3 - 27a^3$

**22.** $x^3 + 27a^3$

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots\dots\dots + a_1 x + a_0 \ , \ a_n \neq 0$$

where '$n$' is a non-negative integer and the coefficients are constants, is called a polynomial function of degree '$n$'.

For example:

(i) $P(x) = a_1 x + a_0$ (is a polynomial function of degree one), $a_1 \neq 0$

(ii) $P(x) = 3x^2 + 5x + 11$ (is a polynomial function of degree two)

(iii) $P(x) = 7x^5 + 2x^4 + 4x^3 + 7x^2 + 5x + 6$ (is a polynomial function of degree 5)

(iv) $P(x) = 5x^5 + \frac{7}{x} + 6 = 5x^2 + 7x^{-1} + 6$ (is not a polynomial function)

## EXAMPLE

Divide $P(x) = 2x^4 + 3x^3 - x - 5$ by $x + 2$

**SOLUTION:**

$$\begin{array}{rl}
 & 2x^3 - x^2 + 2x - 5 \longleftarrow \boxed{\text{quotient}} \\
\boxed{\text{divisor}} \longrightarrow x+2 & \overline{)\,2x^4 + 3x^3 - x - 5} \longleftarrow \boxed{\text{dividend}} \\
 & \underline{\pm 2x^4 \pm 4x^3} \\
 & \quad -x^3 - x - 5 \\
 & \quad \underline{\mp x^3 \qquad \mp 2x^2} \\
 & \qquad 2x^2 - x - 5 \\
 & \qquad \underline{\pm 2x^2 \pm 4x} \\
 & \qquad\quad -5x - 5 \\
 & \qquad\quad \underline{\mp 5x \mp 10} \\
 & \qquad\qquad 5 \longleftarrow \boxed{\text{remainder}}
\end{array}$$

**Remember that:**

$$\text{Dividend} = \text{Divisor} \times \text{quotient} + \text{remainder}$$

### 2.2.1 The Remainder Theorem

If $R$ is the remainder after dividing the polynomial $P(x)$ by $x - a$, then

$$P(a) = R$$

or

If a polynomial $P(x)$ of degree $n \geq 1$ is divided by polynomial $x - a$, where $a$ is any constant, then remainder is $P(a)$, provided there is no term left containing '$x$'.

**EXAMPLE-1**

*If $P(x) = 4x^4 + 10x^3 + 19x + 5$, is divided by $x+3$, then find the remainder.*

**SOLUTION:** $P(x) = 4x^4 + 10x^3 + 19x + 5$

$$x - a = x + 3 \quad \Rightarrow \quad a = -3$$

Therefore $P(-3) = 4(-3)^4 + 10(-3)^3 + 19(-3) + 5$

$$= 4 \times 81 - 10 \times 27 - 57 + 5$$

$$= 324 - 270 - 57 + 5$$

$$= -3 + 5$$

Thus $P(-3) = 2$

$$\boxed{R = 2}$$

**EXAMPLE-2**

*If $P(x) = 5x^4 + 14x^3 + 3x^2 - 5x - 3$ is divided by $x - 1$, find the remainder.*

**SOLUTION:** $P(x) = 5x^4 + 14x^3 + 3x^2 - 5x - 3$

$$x - a = x - 1 \quad \Rightarrow \quad a = 1$$

Therefore $P(1) = 5(1)^4 + 14(1)^3 + 3(1)^2 - 5(1) - 3$

$$= 5 + 14 + 3 - 5 - 3$$

$$= 14$$

Thus $P(1) = 14$

$$\boxed{R = 14}$$

## 2.2.2 Finding Remainder Without Dividing

In the following examples, we learn to find the remainder without division, when a polynomial is divided by a linear polynomial.

### EXAMPLE-1

*Use the remainder theorem to find the remainder when the first polynomial is divided by the second polynomial.*

(i) $x^2 + 3x + 7,\ x + 1$ (ii) $x^3 - 2x^2 + 3x + 3,\ x - 3$

**SOLUTION:** (i) Let $P(x) = x^2 + 3x + 7$

Since the divisor $= x + 1$

Therefore $x - a = x + 1 \Rightarrow a = -1.$

By the remainder theorem

$$R = P(-1)$$

$$P(-1) = (-1)^2 + 3(-1) + 7$$

Now $= 1 - 3 + 7$

$$R = 5$$

(ii) Let $P(x) = x^3 - 2x^2 + 3x + 3$

$$x - a = x - 3 \Rightarrow a = 3$$

$$R = P(3)$$

Now $P(3) = (3)^3 - 2(3)^2 + 3(3) + 3$

$$= 27 - 18 + 9 + 3$$

$$\boxed{R = 21}$$

### EXAMPLE-2

When $x^4 + 2x^3 + kx^2 + 3$ is divided by $x - 2$ the remainder is $1$. Find the value of '$k$'.

**SOLUTION:** Let $P(x) = x^4 + 2x^3 + kx^2 + 3$

Since the divisor $= x - 2$, therefore $x - a = x - 2 \Rightarrow a = 2$

Now $P(2) = (2)^4 + 2(2)^3 + k(2)^2 + 3$

$= 16 + 16 + 4k + 3 \quad = 35 + 4k$

$P(2) = 1\,(\text{given})$

$$1 = 35 + 4k \quad \Rightarrow \quad 4k = -34 \quad \Rightarrow \quad k = \frac{-17}{2}$$

## 2.2.3 Zeros of a Polynomial

If $P(x) = x - a_1$ and $Q(x) = x - a_2$ are any first degree polynomials such that $P(a_1) = 0$ and $Q(a_2) = 0$ for polynomials $P(x)$ and $Q(x)$. Then $a_1$ , $a_2$ are called zeros of $P(x)$ and $Q(x)$.

## 2.2.4 The Factor Theorem

If a polynomial $P(x)$ is divided by $x - a$ such that $P(a) = 0$, then $x - a$ is a factor of $P(x)$;
conversely, if $x - a$ is a factor of $P(x)$, then '$a$' is zero of $P(x)$.

### EXAMPLE-1

*Use the factor theorem to determine if the first polynomial is a factor of the second polynomial.*

$x - 1, \; x^2 + 4x - 5$

**SOLUTION:** Let $P(x) = x^2 + 4x - 5$

$x - a = x - 1$

$\Rightarrow \quad a = 1$

$P(1) = 1^2 + 4(1) - 5$

$= 1 + 4 - 5$

$= 5 - 5$

$= 0$

Since $P(1) = 0$

*Thus by factor theorem,* $x - 1$ *is a factor of* $x^2 + 4x - 5$

## EXAMPLE-2

Use the factor theorem to show that $x + 1$ is a factor of $P(x) = x^{25} + 1$

**SOLUTION:** By direct substitution we see that $-1$ is a zero of $P(x)$

$$
\begin{aligned}
P(x) &= x^{25} + 1 \\
P(-1) &= (-1)^{25} + 1 \qquad \therefore (-1)^{odd} = -1 \\
&= -1 + 1 \\
&= 0
\end{aligned}
$$

Since $-1$ is a zero of $P(x) = x^{25} + 1$,
The linear polynomial $x - (-1) = x + 1$ is,
by the factor theorem, a factor of $x^{25} + 1$.

## EXAMPLE-3

Use the factor theorem to show that $x - 1$ is not a factor of $4x^7 - 2x^6 + x^2 + 2x + 5$?

**SOLUTION:** Let $P(x) = 4x^7 - 2x^6 + x^2 + 2x + 5$

$$
\begin{aligned}
x - a &= x - 1 \quad \Rightarrow \quad a = 1 \\
P(1) &= 4(1)^7 - 2(1)^6 + 1^2 + 2(1) + 5 \\
&= 4 - \cancel{2} + 1 + \cancel{2} + 5 \\
&= 10 \neq 0
\end{aligned}
$$

Then by factor theorem $x - 1$ is not a factor of $4x^7 - 2x^6 + x^2 + 2x + 5$

## EXAMPLE-4

Use the factor theorem to show that $x+1$ is not a factor of $2x^5 - 5x^2 - x + 4$

**SOLUTION:** Let $P(x) = 2x^5 - 5x^2 - x + 4$

$$
\begin{aligned}
x - a &= x + 1 \quad \Rightarrow \quad a = -1 \\
P(-1) &= 2(-1)^5 - 5(-1)^2 - (-1) + 4 \\
&= -2 - 5 + 1 + 4 \\
P(-1) &= -2 \neq 0
\end{aligned}
$$

$x + 1$ is not a factor of $2x^5 - 5x^2 - x + 4$.

## 2.3 FACTORIZING A CUBIC POLYNOMIAL

In order to factorize a cubic polynomial, we see the following examples:

**EXAMPLE-1**

*Factorize the following*

$x^3 - x^2 - 10x + 10;\ x - 1$

**SOLUTION:** $P(x) = x^3 - x^2 - 10(x) + 10;\ x - 1$ .

$$x - a = x - 1 \Rightarrow a = 1$$

$$P(1) = 1^3 - 1^2 - 10 + 10$$

$$= 0, \text{therefore } x - 1 \text{ is a factor of } P(x)$$

Now

$$\begin{array}{r|l} & x^2 - 10 \\ \hline x - 1 & x^3 - x^2 - 10x - 10 \\ & \underline{-x^3 \mp x^2} \\ & \quad -10x + 10 \\ & \quad \underline{\mp 10x \pm 10} \\ & \qquad 0 \end{array}$$

Now $P(x) = \text{quotient} \times \text{divisor}$

Thus $x^3 - x^2 - 10x + 10 = (x - 1)\ (x^2 - 10)$

**EXAMPLE-2**

Factorize the following $x^3 - 8;\ x - 2$

**SOLUTION:** $P(x) = x^3 - 8$ , $x - a = x - 2 \Rightarrow a = 2$

$$P(2) = 2^3 - 8 = 8 - 8$$

$$= 0, \text{therefore } x - 2 \text{ is a factor of } P(x)$$

Now

$$\begin{array}{r|l} & x^2 + 2x + 4 \\ \hline x - 2 & x^3 - 8 \\ & \underline{-x^3 \mp 2x^2} \\ & \quad 2x^2 - 8 \\ & \quad \underline{-2x^2 \mp 4x} \\ & \qquad 4x - 8 \\ & \qquad \underline{-4x \mp 8} \\ & \qquad 0 \end{array}$$

Now $P(x) = x^3 - 8$

$= (x - 2)\ (x^2 + 2x + 4)$

## EXERCISE – 2.5

**I- Evaluate each of the polynomials for the value indicated.**

**1.** $P(x) = 2x^3 - 5x^2 + 7x - 7;\ P(2)$

**2.** $P(x) = x^4 - 10x^2 + 25x - 2;\ P(-4)$

**3.** $P(x) = x^4 + 5x^3 - 13x^2 - 30;\ P(-1)$

**4.** $P(x) = x^5 - 10x^3 + 7x + 6;\ P(3)$

**5.** $P(x) = x^4 + 4x^3 - 9x^2 + 19x + 6;\ P(-2)$

**II- Determine whether the second polynomial is a factor of the first polynomial without dividing *(Hint: evaluate directly and use the factor theorem)*.**

**6.** $x^{18} - 1;\ x + 1$

**7.** $x^{18} - 1;\ x - 1$

**8.** $x^9 - 2^9;\ x + 2$

**9.** $x^9 + 2^9;\ x - 2$

**10.** $3x^4 - 2x^3 + 5x - 6;\ x - 1$

**11.** $5x^6 - 7x^3 - 6x + x;\ x - 1$

**12.** $3x^3 - 7x^2 - 8x + 2;\ x + 1$

**13.** $5x^8 - 2x^5 + 3x^3 + 6x + 2;\ x + 1$

**14.** $6x^3 + 2x^2 - x + 9;\ x - 1$

**15.** $4x^3 - 3x^2 - 8x + 4;\ x - 2$

**16.** $5x^3 + 3x^2 - x + 1;\ x + 1$

**17.** $2y^3 - 8y^2 + y - 4;\ y - 4$

**18.** $z^3 - 5z^2 - 4z - 4;\ z + 2$

**III- Solve.**

**19.** *If* $P(x) = x^3 - kx^2 + 3x + 5$ *is divided by* $x - 1$*, find* $k$*, if remainder is 8.*

**20.** *If* $P(x) = 3x^3 + kx - 26$ *is divided by* $x - 2$*, find* $k$*, if remainder is* $0$*.*

## Review Exercise-2

### I- Encircle the Correct Answer.

**1.** A linear polynomial is of degree =

(a) $0$ (b) $1$

(c) $2$ (d) $3$

**2.** A quadratic polynomial is of degree =

(a) $0$ (b) $1$

(c) $2$ (d) $3$

**3.** A cubic polynomial is of degree =

(a) $0$ (b) $1$

(c) $2$ (d) $3$

**4.** Factorization of $(x+3)^2-4$ is

(a) $(x+1)(x+5)$ (b) $(x-1)(x+5)$

(c) $(x+1)(x-5)$ (d) $(x-1)(x-5)$

**5.** Factorization of $x^4-16$ is

(a) $(x-2)(x+2)$ (b) $(x-4)(x+4)$

(c) $(x-2)(x+2)(x^2+4)$ (d) $(x-2)(x+4)$

**6.** Factorization of $x^3-y^3$ is =

(a) $(x-y)(x^2+y^2)$ (b) $(x-y)(x^2+xy+y^2)$

(c) $(x-y)(x^2-xy+y^2)$ (d) $(x+y)(x^2+xy+y^2)$

**7.** Factorization of $a^4-1$ is =

(a) $(a-1)(a+1)(a^2+1)$ (b) $(a-1)(a^2+1)$

(c) $(a+1)(a^2-1)$ (d) $(a^2+1)(a+1)$

**8.** If a polynomial $P(x)$ of degree $n \geq 1$ is divided by polynomial '$x-a$', where $a$ is any constant, then $P(a)$ is

(a) *remainder* (b) *zero*

$1$ (d) $a$

**9.** If $x - a$ is a factor of $P(x)$, then $P(a) =$

(a) $0$ (b) $1$

(c) $-a$ (d) $a$

**10.** If $P(x) = x^3 - 2x^2 + 5x + 1$, then $P(1) =$

(a) $5$ (b) $-5$

(c) $-7$ (d) $0$

## *II*- Fill in the blanks.

**1.** A linear polynomial is of degree ______________ .

**2.** A quadratic polynomial is of degree ______________ .

**3.** A cubic polynomial is of degree ______________ .

**4.** Factorization of $x^2 - 9$ is ______________ .

**5.** Factorization of $(x+2)^2 - 1$ is ______________ .

**6.** Factorization of $x^3 + 8$ is ______________ .

**7.** Factorization of $x^3 - 8$ is ______________ .

**8.** If $P(x) = x^4 + 3x^2 - 2x + 1$ is divided by $x - 1$ , then $P(1) =$ __________ .

**9.** If $P(x) = x^3 + 3x^2 - 3x + 1$ is divided by $x + 2$ ,then $P(-2) =$ __________ .

**10.** If $P(x) = x^3 - a^3$ is divided by $x - a$ ,then $P(a) =$ ______________ .

## SUMMARY

**Linear Polynomial:** A polynomial of degree *"1"* is called linear polynomial.

**Quadratic Polynomial:** A polynomial of degree *"2"* is called quadratic polynomial.

**Cubic Polynomial:** A polynomial of degree *"3"* is called cubic polynomial.

**Factorization of following types of polynomials:**

$kx + ky + kz,\ ax + ay + bx + by,\ a^2 \pm 2ab + b^2$

$a^2 - b^2$, $(a^2 \pm 2ab + b^2) - c^2$, $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$,

$x^2 + px + q,\ ax^2 + bx + c,$

$a^3 + 3a^2bx + 3ab^2 + b^3$, $a^3 - 3a^2b + 3ab^2 - b^3$,

$a^3 \pm b^3$.

**Remainder Theorem:** If a polynomial $P(x)$ of degree $n \geq 1$ is divided by a polynomial '$x-a$' where '$a$' is any constant, then remainder is $P(a)$.

**Factor Theorem:** If a polynomial $P(x)$ is divided by '$x-a$' such that $P(a) = 0$, then '$x-a$' is a factor of $P(x)$.