UNIT

3 ALGEBRAIC MANIPULATION

- H.C.F and L.C.M
- Basic Operations on Algebraic Fractions
- Square Roots of Algebraic Fractions

After completion of this unit, the students will be able to:

- find highest common factor (HCF) and least common multiple (LCM) of algebraic expressions.
- use factor or division method to determine HCF and LCM.
- know the relationship between HCF and LCM.
- use HCF and LCM to reduce fractional expressions involving $+, -, \times, \div$.
- find square root of an algebraic expression by factorization and division.

3.1 HIGHEST COMMON FACTOR (H.C.F) AND LEAST COMMON MULTIPLE (L.C.M)

3.1.1 Highest Common Factor (H.C.F)

The highest common factor of two or more algebraic expressions is the expression of highest degree which divides each of them without remainder.

The abbreviation of the words **highest common factor** is **H.C.F**.

We can find the *H.C.F* of two or more than two algebraic expressions by the following two methods:

(i) Factorization

(ii) Division

H.C.F BY FACTORIZATION METHOD:

Method of finding highest common factor by factorization is explained by the following examples:

EXAMPLE-1

Find the H.C.F of $12p^3q^2$, $8p^2qr^3$ *and* $4p^2q^3r$

SOLUTION:

Factorization of $12p^3q^2 = 2 \times 2 \times 3 \times p \times p \times p \times q \times q$

Factorization of $8p^2qr^3 = 2 \times 2 \times 2 \times p \times p \times q \times r \times r \times r$

Factorization of $4p^2q^3r = 2 \times 2 \times p \times p \times q \times q \times q \times r$

Common factors are: $2 \times 2 \times p \times p \times q$

Thus $H.C.F = 4p^2q$

EXAMPLE-2

Find H.C.F of $2x^2 + 3x + 1$, $2x^2 + 5x + 2$ and $2x^2 - x - 1$

SOLUTION:

Factorization of $2x^2 + 3x + 1 = 2x^2 + 2x + x + 1$
$= 2x(x+1) + 1(x+1)$
$= (2x+1)(x+1)$

Factorization of $2x^2 + 5x + 2 = 2x^2 + 4x + x + 2$
$= 2x(x+2) + 1(x+2)$
$= (2x+1)(x+2)$

Factorization of $2x^2 - x - 1 = 2x^2 - 2x + x - 1$
$= 2x(x-1) + 1(x-1)$
$= (2x+1)(x-1)$

Common factor $= 2x+1$

Thus H.C.F $= 2x+1$

EXAMPLE-3

Find H.C.F of $24(6x^4 - x^3 - 2x^2)$ and $20(2x^6 + 3x^5 + x^4)$

SOLUTION: Let $P(x) = 24(6x^4 - x^3 - 2x^2)$
$= 24x^2(6x^2 - x - 2)$
$= 24x^2\left[6x^2 - 4x + 3x - 2\right]$
$= 24x^2\left[2x(3x-2) + 1(3x-2)\right]$

$P(x) = 24x^2(2x+1)(3x-2) = 2^2 \times 2 \times 3 \times x^2(2x+1)(3x-2)$

Also let $Q(x) = 20(2x^6 + 3x^5 + x^4)$
$= 20x^4\left[2x^2 + 3x + 1\right]$
$= 20x^4(2x^2 + 2x + x + 1)$
$= 20x^4\left[2x(x+1) + 1(x+1)\right]$
$= 20x^4(x+1)(2x+1)$
$= 2^2 \times 5 \times x^2 \times x^2(x+1)(2x+1)$

Common factors $= 2^2 \times x^2 \times (2x+1)$

Thus H.C.F $= 4x^2(2x+1)$

EXAMPLE-4

Find H.C.F of x^2-4, $x^2-7x+10$ and x^2+x-6

SOLUTION: Factorization of $x^2-4 = (x-2)(x+2)$

Factorization of $x^2-7x+10$

$= x^2-5x-2x+10$

$= x(x-5)-2(x-5)$

$= (x-5)(x-2)$

Factorization of x^2+x-6

$= x^2+3x-2x-6$

$= x(x+3)-2(x+3)$

$= (x+3)(x-2)$

Common factor $= x-2$

Thus H.C.F $= x-2$

EXERCISE – 3.1

Find H.C.F by Factorization.

1. $abxy$, a^2bc
2. $6pqr$, $15qrs$
3. $8xy^2z^3$, $12x^2y^2z^2$
4. $14a^2bc$, $21ab^2$
5. $3x^5y^2$, $12x^2y^4$, $15x^3y^2$
6. $4abc^3$, $8a^3bc$, $6ab^3c$
7. x^3+64, x^2-16
8. x^2-y^2, x^4-y^4, x^6-y^6
9. t^2-9, $(t+3)^2$, t^2+t-6
10. x^2-x-2, x^2+x-6, x^2-3x+2
11. $1-x^2$, x^3+1, $1-x-2x^2$
12. x^3-8, $x^2-7x+10$
13. x^2+3x+2, x^2+4x+3, x^2+5x+4
14. $x^4+x^3-6x^2$, x^4-9x^2, x^3+x^2-6x
15. $35a^2c^3b$, $45a^3cb^2$, $30ac^2b^3$

H.C.F BY DIVISION METHOD

In order to find the H.C.F by division method, arrange the given expressions in descending powers of the common variable.
Divide the larger degree polynomial by another one. Get the remainder.
Take the previous divisor as the dividend and this remainder as the divisor. Divide and get the remainder.
Go on repeating the process till we get zero as the remainder. The last divisor is the required H.C.F.

EXAMPLE-1

Find the H.C.F of $(x^3 - x^2 + x - 1)$ and $(x^3 - x^2 - 3x + 3)$ by division method.

SOLUTION:

$$\begin{array}{r|l} & 1 \\ x^3 - x^2 + x - 1 & x^3 - x^2 - 3x + 3 \\ & -x^3 \mp x^2 \pm x \mp 1 \\ \hline & -4x + 4 = -4(x-1) \end{array}$$

Now dividing $-4x + 4$ *by* -4, *we get* $x - 1$

$$\begin{array}{r|l} & x^2 + 1 \\ x - 1 & x^3 - x^2 + x - 1 \\ & -x^3 \mp x^2 \\ \hline & x - 1 \\ & -x \mp 1 \\ \hline & 0 \end{array}$$

Thus $H.C.F = x - 1$

Remember that:

H.C.F is not affected by multiplying or dividing the polynomials with any number during the process of finding H.C.F.

EXAMPLE-2

Find H.C.F of $2x^3+6x^2+5x+2$, $5x^3+10x^2-3x-6$ and $3x^3+6x^2+2x+4$

SOLUTION:

$$\begin{array}{r|l} & 5 \\ 2x^3+6x^2+5x+2 & 5x^3+10x^2-3x-6 \\ & \times 2 \\ \hline & 10x^3+20x^2-6x-12 \\ & \pm 10x^3 \pm 30x^2 \pm 25x \pm 10 \\ \hline & -10x^2-31x-22 \end{array}$$

Now dividing $-10x^2-31x-22$ *by '-1', we get* $10x^2+31x+22$

$$\begin{array}{r|l} & x-1 \\ 10x^2+31x+22 & 2x^3+6x^2+5x+2 \\ & \times 5 \\ \hline & 10x^3+30x^2+25x+10 \\ & \pm 10x^3 \pm 31x^2 \pm 22x \\ \hline & -x^2+3x+10 \\ & \times 10 \\ \hline & -10x^2+30x+100 \\ & \mp 10x^2 \mp 31x \mp 22 \\ \hline & 61x+122 \end{array}$$

Now dividing $61x+122$ *by 61, we get* $x+2$

$$\begin{array}{r|l} & 10x+11 \\ x+2 & 10x^2+31x+22 \\ & \pm 10x^2 \pm 20x \\ \hline & 11x+22 \\ & \pm 11x \pm 22 \\ \hline & 0 \end{array}$$

Now

$$\begin{array}{r|l} & 3x^2+2 \\ \hline x+2 & 3x^3+6x^2+2x+4 \\ & \pm 3x^3 \pm 6x^2 \\ \hline & 2x+4 \\ & \pm 2x \pm 4 \\ \hline & 0 \end{array}$$

Thus $H.C.F = x+2$

EXAMPLE-3

If $x-a$ is the H.C.F. of x^2-x-6 and $x^2+3x-18$ then find the value of a.

SOLUTION: Clearly, $(x-a)$ divides both x^2-x-6 and $x^2+3x-18$, so $x=a$ makes both polynomials zero.

i.e. $a^2-a-6=0$ and $a^2+3a-18=0$

$$a^2-a-6 = a^2+3a-18$$
$$4a = 12$$
$$a = 3$$

Divisor

A polynomial $D(x)$ is called a divisor of a polynomial $P(x)$, if

$$P(x) = D(x).\ Q(x) \text{ for some polynomial } Q(x).$$

For example:

Let $P(x) = (x-2)(x+3)$ and $D(x) = x-2$,

then clearly $D(x)$ is a divisor of $P(x)$.

Since $P(x) = (x-2)(x+3)$

$= D(x).Q(x)$, where $Q(x) = x+3$

EXERCISE – 3.2

Find the H.C.F by Division Method.

1. $x^4+x^2+1\ ,\ x^4+x^3+x+1$
2. $6x^3+7x^2-9x+2\ ,\ 8x^4+6x^3-15x^2+9x-2$
3. $4x^3+2x^2-6x\ ,\ 4x^3-8x+4$
4. $x^3+7x^2+12x\ ,\ x^3-2x^2-15x$
5. $x^3-x^2-x+1\ ,\ x^4-2x^3+2x-1$
6. $x^3-x^2-x-2\ ,\ x^3+3x^2-6x-8$
7. $x^2+3x-4\ ,\ x^3-2x^2-2x+3$
8. $3x^3-14x^2+9x+10\ ,\ 15x^3-34x^2+21x-10$
9. $2x^4+x^3+4x+2\ ,\ 6x^3+5x^2+x\ ,\ 2x^4+3x^3+x^2+2x+1$
10. $x^3+x^2-5x+3\ ,\ x^3-7x+6\ ,\ x^3+2x^2-2x+3$

3.1.2 Least Common Multiple (L.C.M)

The least common multiple of two or more algebraic expressions is the expression of lowest degree which is divisible by each of them without remainder.

The abbreviation of the words **least common multiple** is **L.C.M**.

We can find the L.C.M by factorization method:

L.C.M BY FACTORIZATION:

To find L.C.M by factorization, consider the following examples:

EXAMPLE-1

Find L.C.M of $12p^3q^2$, $8p^2qr^3$ and $4p^2q^3r$.

SOLUTION:

Factorization of $12p^3q^2 = 2 \times 2 \times 3 \times p \times p \times p \times q \times q$

Factorization of $8p^2qr^3 = 2 \times 2 \times 2 \times p \times p \times q \times r \times r \times r$

Factorization of $4p^2q^3r = 2 \times 2 \times p \times p \times q \times q \times q \times r$

L.C.M = Product of common factors × product of uncommon factors

$= (2^2 \times p^2 \times q^2 \times r) \times (2 \times 3 \times p \times q \times r^2)$

$= 4p^2q^2r \times 6pqr^2$

$= 4 \times 6 \times p^2 \times p \times q^2 \times q \times r \times r^2$

L.C.M $= 24p^3q^3r^3$

> **Remember that:**
>
> Common factors are not repeated while taking product of common factors.

EXAMPLE-2

Find L.C.M of $18ab^2c^3$, $6ab^2c^3$ and $24ab^2c^2$.

SOLUTION:

Factorization of $18ab^2c^3 = 2 \times 3 \times 3 \times a \times b \times b \times c \times c \times c$

Factorization of $6a^2bc^3 = 2 \times 3 \times a \times a \times b \times c \times c \times c$

Factorization of $24ab^2c^2 = 2 \times 2 \times 2 \times 3 \times a \times b \times b \times c \times c$

Thus L.C.M = Product of common factors × product of uncommon factors

$= (2 \times 3 \times a \times b^2 \times c^3) \times (2 \times 2 \times 3 \times a)$

$= (6ab^2c^3) \times (12a)$

L.C.M $= 72a^2b^2c^3$

EXAMPLE-3

Find L.C.M of $x^2 - 49$ and $x^2 - 4x - 21$

SOLUTION:

$$x^2 - 49 = x^2 - 7^2$$
$$= (x-7)(x+7)$$

and
$$x^2 - 4x - 21 = x^2 - 7x + 3x - 21$$
$$= x(x-7) + 3(x-7)$$
$$= (x-7)(x+3)$$

Common factor $= x - 7$

Product of uncommon factors $= (x+7)(x+3)$

L.C.M = Product of common factors × product of uncommon factors

$$= (x-7)\times(x+7)(x+3)$$
$$= (x^2 - 7^2)(x+3)$$
$$= (x^2 - 49)(x+3)$$

$$L.C.M = x^3 + 3x^2 - 49x - 147$$

EXERCISE – 3.3

Find L.C.M by Factorization.

1. $21a^4x^3y$, $35a^2x^4y$, $28a^3xy^4$
2. $3a^4b^2c^3$, $5a^2b^3c^5$
3. $2ab$, $3ab$, $4ca$
4. x^2yz , xy^2z , xyz^2
5. $p^3q - pq^3$, $p^5q^2 - p^2q^5$
6. $x^3 + 64$, $x^2 - 16$
7. $x^2 - x - 2$, $x^2 + x - 6$, $x^2 - 3x + 2$
8. $y^2 - 9$, $(y+3)^2$, $y^2 + y - 6$
9. $1 - y^2$, $y^3 + 1$, $1 - y - 2y^2$
10. $x^2 - y^2$, $x^4 - y^4$, $x^6 - y^6$
11. $x^3 + 1$, $x^4 + x^2 + 1$, $(x^2 + x + 1)^2$
12. $x^3 + y^3$, $x^4 - y^4$, $x^6 + y^6$
13. $2x^2 + 5x + 3$, $x^2 + 2x + 1$, $2x^2 + 9x + 9$
14. $x^4 + x^3 - 6x^2$, $x^4 - 9x^2$, $x^3 + x^2 - 6x$
15. $x^2 + 4xy + 4y^2$, $x^2 + 3xy + 2y^2$, $x^2 + 2xy + y^2$

3.1.3 Relationship between HCF and LCM

If A and B are two algebraic expressions and **H.C.F.** and **L.C.M** of these is represented by H and L respectively, then the relation among them can be expressed as:

$$\boxed{A \times B = H \times L}$$

It is called a formula between **L.C.M.** and **H.C.F.**

PROOF: Suppose that

$$\frac{A}{H} = x \quad \text{and} \quad \frac{B}{H} = y$$

$$A = Hx \quad \ldots\ldots\ldots\ldots \quad (i)$$

$$B = Hy \quad \ldots\ldots\ldots\ldots \quad (ii)$$

Since there is no common factor between x and y.

Therefore $L = H.x.y$

$HL = H(H.x.y)$ *(Multiplying both the sides by H)*

$= (Hx).(Hy)$

$HL = A.B.$

Important results:

(i) $L = \dfrac{A \times B}{H}$

(ii) $H = \dfrac{A \times B}{L}$

(iii) $A = \dfrac{H \times L}{B}$

Note: *If A and B are two algebraic expressions, then we find **H.C.F** first, before finding the **L.C.M**.*

*If **H.C.F** of two algebraic expressions is given, then we can find **L.C.M**.*

EXAMPLE-1

L.C.M and H.C.F of two algebraic expressions is $(2x+1)(x^2-1)$ and $(2x+1)$ respectively. If one expression is $(x-1)(2x+1)$, then find the other.

SOLUTION: $L = (2x+1)(x^2-1)$

$H = 2x+1$

$A = (x-1)(2x+1)$

$B = ?$

We have that $A \times B = H \times L$

$$B = \frac{H \times L}{A}$$

$$= \frac{(2x+1)(x^2-1)(2x+1)}{(x-1)(2x+1)}$$

$$= \frac{(2x+1)(x+1)(x-1)(2x+1)}{(x-1)(2x+1)}$$

$$B = (2x+1)(x+1)$$

EXAMPLE-2

The H.C.F of two polynomials is $(x+3)$ and their L.C.M is x^3-7x+6. If one of the polynomials is (x^2+2x-3), then find the other.

SOLUTION: Let the required polynomial be B. Then:

$$A \times B = H \times L$$

$$B = \frac{H \times L}{A}$$

$$= \frac{(x+3)(x^3-7x+6)}{x^2+2x-3}$$

$$= (x+3)(x-2)$$

$$B = x^2+x-6$$

$$\begin{array}{r|l} & x-2 \\ x^2+2x-3 & x^3-7x+6 \\ & -x^3 \mp 3x \quad \pm 2x^2 \\ \hline & -2x^2-4x+6 \\ & \mp 2x^2 \mp 4x \pm 6 \\ \hline & 0 \end{array}$$

EXAMPLE-3

Product of two expressions is $x^4 + 3x^3 - 12x^2 - 20x + 48$ and their L.C.M is $x^3 + 5x^2 - 2x - 24$. Find their H.C.F.

SOLUTION: Given that $A \times B = x^4 + 3x^3 - 12x^2 - 20x + 48$

$$L = x^3 + 5x^2 - 2x - 24$$

$$H = ?$$

$$L \times H = A \times B$$

$$H = \frac{A \times B}{L}$$

$$H = \frac{x^4 + 3x^3 - 12x^2 - 20x + 48}{x^3 + 5x^2 - 2x - 24}$$

$$\begin{array}{r|l} & x - 2 \\ x^3 + 5x^2 - 2x - 24 & x^4 + 3x^3 - 12x^2 - 20x + 48 \\ & \pm x^4 \pm 5x^3 \mp 2x^2 \mp 24x \\ \hline & -2x^3 - 10x^2 + 4x + 48 \\ & \mp 2x^3 \mp 10x^2 \pm 4x \pm 48 \\ \hline & 0 \end{array}$$

$$\therefore \; H.C.F = x - 2$$

Remember that:

The product of two algebraic expressions $= L.C.M \times H.C.F$

$$L.C.M = \frac{\text{product of two algebraic expressions}}{H.C.F}$$

$$H.C.F = \frac{\text{product of two algebraic expressions}}{L.C.M}$$

EXERCISE – 3.4

Find the H.C.F and L.C.M of the Following.

1. $x^3 + x^2 + x + 1\ ,\ x^3 - x^2 + x - 1$
2. $x^3 - 3x^2 - 4x + 12\ ,\ x^3 - x^2 - 4x + 4$
3. $2x^3 + 2x^2 + x + 1\ ,\ 2x^3 - 2x^2 + x - 1$
4. $6x^3 + 7x^2 - 9x + 2\ ,\ 8x^4 + 6x^3 - 15x^2 + 9x - 2$
5. $3x^4 + 17x^3 + 27x^2 + 7x - 6\ ,\ 6x^4 + 7x^3 - 27x^2 + 17x - 3$
6. $2x^4 + 3x^3 - 13x^2 - 7x + 15\ ,\ 2x^4 + x^3 - 20x^2 - 7x + 24$
7. $x^4 - x^3 - x + 1\ ,\ x^4 + x^3 - x - 1$
8. $x^4 + x^3 + x + 1\ ,\ x^4 + x^3 - x - 1$

Find the Required Polynomial.

9. $A = x^2 - 5x - 14\ ,\ H = x - 7\ ,\ L = x^3 - 10x^2 + 11x + 70\ ,\ B = ?$
10. $B = 3x^2 + 14x + 8\ ,\ H = 3x + 2\ ,\ L = 6x^3 + 25x^2 + 2x - 8\ ,\ A = ?$
11. The product of two polynomials and their $L.C.M.$ are $x^4 + 6x^3 - 3x^2 - 56x - 48$ and $x^3 + 2x^2 - 11x - 12$ respectively. Find their $H.C.F.$
12. The product of two polynomials and their $L.C.M.$ are $x^4 + 5x^3 - x^2 - 17x + 12$ and $x^3 + 6x^2 + 5x - 12$ respectively. Find their $H.C.F.$
13. The product of two polynomials and their $H.C.F.$ are $x^4 - 12x^3 + 53x^2 - 102x + 72$ and $x - 3$ respectively. Find $L.C.M.$
14. The product of two polynomials and their $H.C.F.$ is $x^4 - 5x^3 + 2x^2 + 20x - 24$ and $x + 2$ respectively. Find their $L.C.M.$
15. One algebraic expression is $x^3 + 3x^2 - 4x - 12$ and other one is $x^3 + 5x^2 - 4x - 20$. Their $H.C.F$ is $x^2 - 4$. Find their $L.C.M.$
16. One alegbraic expression is $x^3 - x^2 + 2x - 2$ and other one is $x^3 - x^2 - 2x + 2$. Their $H.C.F$ is $x - 1$. Find their $L.C.M.$
17. Prove that $H^3 + L^3 = A^3 + B^3$ where $H + L = A + B$
'H' and 'L' stand for $H.C.F$ and $L.C.M$ respectively and 'A,B' represent two polynomials.

3.2 BASIC OPERATIONS ON THE ALGEBRAIC FRACTIONS

3.2.1 Addition and Subtraction of the Algebraic Fractions

Addition and subtraction of the algebraic fractions are explained in the following examples.

EXAMPLE-1

Simplify $\dfrac{x^2+3x+2}{x^2-2x-8}+\dfrac{x^2-5x+6}{x^2-7x+12}-\dfrac{x^2+x-6}{x^2-6x+8}$

SOLUTION: $\dfrac{x^2+3x+2}{x^2-2x-8}+\dfrac{x^2-5x+6}{x^2-7x+12}-\dfrac{x^2+x-6}{x^2-6x+8}$

$$=\frac{x^2+2x+x+2}{x^2-4x+2x-8}+\frac{x^2-3x-2x+6}{x^2-4x-3x+12}-\frac{x^2+3x-2x-6}{x^2-4x-2x+8}$$

$$=\frac{(x+2)(x+1)}{(x-4)(x+2)}+\frac{(x-3)(x-2)}{(x-4)(x-3)}-\frac{(x+3)(x-2)}{(x-4)(x-2)}$$

$$=\frac{x+1}{x-4}+\frac{x-2}{x-4}-\frac{x+3}{x-4}$$

$$=\frac{x+1+x-2-x-3}{x-4}$$

$$=\frac{x-4}{x-4}=1$$

Remember that:

(i) In the algebraic fractions, the numerators and denominators are polynomials.

(ii) When we add or subtract these fractions, we reduce this to lowest terms.

EXAMPLE-2

Simplify $\dfrac{a+b}{a^2+ab+b^2}+\dfrac{1}{a-b}-\dfrac{ab}{a^3-b^3}$

SOLUTION: $\dfrac{a+b}{a^2+ab+b^2}+\dfrac{1}{a-b}-\dfrac{ab}{a^3-b^3}$

$$=\frac{(a-b)(a+b)+1(a^2+ab+b^2)-ab}{a^3-b^3}$$

$$=\frac{a^2-b^2+a^2+ab+b^2-ab}{a^3-b^3}$$

$$=\frac{2a^2}{a^3-b^3}$$

3.2.2 Multiplication and Division of the Algebraic Fractions

If P, Q, R and S are algebraic expressions, then $\dfrac{P}{Q}$ and $\dfrac{R}{S}$ are called algebraic fractions, where $Q \neq 0$, $S \neq 0$.

Multiplication of algebraic fractions:

$$\frac{P}{Q}\times\frac{R}{S}=\frac{PR}{QS} \qquad \text{where} \quad Q\neq 0,\ S\neq 0.$$

Division of algebraic fractions.

$$\frac{P}{Q}\div\frac{R}{S}=\frac{P}{Q}\times\frac{S}{R}$$

$$=\frac{PS}{QR} \qquad \text{where} \quad Q\neq 0,\ S\neq 0.$$

EXAMPLE-1

Simplify $\dfrac{b^2-c^2-a^2+2ac}{c^2+a^2-b^2+2ac} \times \dfrac{b^2+c^2-a^2-2bc}{a^2-b^2+c^2-2ac}$

SOLUTION: $\dfrac{b^2-c^2-a^2+2ac}{c^2+a^2-b^2+2ac} \times \dfrac{b^2+c^2-a^2-2bc}{a^2-b^2+c^2-2ac}$

$$= \frac{b^2-(c^2+a^2-2ac)}{(c^2+a^2+2ac)-b^2} \times \frac{(b^2+c^2-2bc)-a^2}{(a^2+c^2-2ac)-b^2}$$

$$= \frac{b^2-(a-c)^2}{(a+c)^2-b^2} \times \frac{(b-c)^2-a^2}{(a-c)^2-b^2}$$

$$= \frac{\left[b^2-(a-c)^2\right]}{(a+c-b)(a+c+b)} \times \frac{(b-c-a)(b-c+a)}{(-1)\left[b^2-(a-c)^2\right]}$$

$$= \frac{-(b-c-a)(b-c+a)}{(a+c-b)(a+b+c)}$$

$$= \frac{(a+c-b)(a+b-c)}{(a+c-b)(a+b+c)} = \frac{a+b-c}{a+b+c}$$

EXAMPLE-2

Simplify $\dfrac{a^3-b^3}{a^4-b^4} \div \dfrac{a^2+ab+b^2}{a^2+b^2}$

SOLUTION: $\dfrac{a^3-b^3}{a^4-b^4} \div \dfrac{a^2+ab+b^2}{a^2+b^2}$

$$= \frac{a^3-b^3}{a^4-b^4} \times \frac{a^2+b^2}{a^2+ab+b^2}$$

$$= \frac{(a-b)(a^2+ab+b^2)}{(a^2+b^2)(a+b)(a-b)} \times \frac{a^2+b^2}{a^2+ab+b^2}$$

$$= \frac{1}{a+b}$$

EXERCISE – 3.5

Simplify:

1. $\frac{1}{a}+\frac{2}{a+1}-\frac{3}{a+2}$

2. $\frac{2a}{(x-2a)}-\frac{x-a}{x^2-5ax+6a^2}+\frac{2}{x-3a}$

3. $\frac{1}{a^2+1}-\frac{a^4}{a^2+1}+\frac{a^6}{a^2-1}-\frac{1}{a^2-1}$

4. $\frac{1}{x^2+x+1}-\frac{1}{x^2-x+1}+\frac{2x+1}{x^4+x^2+1}$

5. $\frac{a^2(b-c)}{(a+b)(a+c)}-\frac{b^2(c-a)}{(b+c)(b+a)}+\frac{c^2(a-b)}{(c+a)(c+b)}$

6. $\frac{1}{x-1}+\frac{1}{x+1}-\frac{x+2}{x^2+x+1}-\frac{x-2}{x^2-x+1}$

7. $\frac{a^2+ab+b^2}{a+b}+\frac{a^2-ab+b^2}{a-b}$

8. $\frac{x^4-y^4}{x^2-2xy+y^2}\times\frac{x-y}{x(x+y)}\div\frac{x^2+y^2}{x}$

9. $\frac{x^2-1}{x^2+x-2}\times\frac{x^3+8}{x^4+4x^2+16}\div\frac{x^2+x}{x^3+2x^2+4x}$

10. $\frac{a^3+64b^3}{a^2+20ab+64b^2}\div\frac{a^2-4ab+16b^2}{a^2+4ab+16b^2}\times\frac{a^2+12ab-64b^2}{a^3-64b^3}$

11. $\frac{a}{(a+b)^2-2ab}\times\frac{a^4-b^4}{(a+b)^3-3ab(a+b)}\div\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$

12. $\frac{a^2-1}{a^2-a-2}\div\frac{a^2+5a+6}{a^2-5a+6}\div\frac{a^2-4a+3}{a^2+4a+3}$

3.3 SQUARE ROOT OF AN ALGEBRAIC EXPRESSION

We can find the square root of an algebraic expression by

(i) FACTORIZATION

(ii) DIVISION

3.3.1 Square Root by Factorization Method

By this method we find the square root of the expressions which can be expressed as a complete square.

For example:

$$x^2 \pm 2xy + y^2 = (x \pm y)^2$$

or $$x^2 \pm 2xy + y^2 = [\pm(x \pm y)]^2$$

or $$\sqrt{x^2 \pm 2xy + y^2} = \pm(x \pm y)$$

Therefore, the square root of an algebraic expression consists of two expressions which are additive inverses to each other.

EXAMPLE-1

Find the square root of $49x^2 + 112xy + 64y^2$ by factorization.

SOLUTION: $49x^2 + 112xy + 64y^2$

$$= (7x)^2 + 2(7x)(8y) + (8y)^2$$

$$= (7x + 8y)^2$$

$$49x^2 + 112xy + 64y^2 = [\pm(7x + 8y)]^2$$

Taking square root of both the sides, we have

$$\sqrt{49x^2 + 112xy + 64y^2} = \pm(7x + 8y)$$

EXAMPLE-2

Find square root of $(x^2+\frac{1}{x^2})+10(x+\frac{1}{x})+27$

SOLUTION: Let $x+\frac{1}{x} = z$,

$$(x+\frac{1}{x})^2 = z^2 \quad \text{(Squaring both sides)}$$

$$x^2+\frac{1}{x^2}+2 = z^2$$

$$x^2+\frac{1}{x^2} = z^2-2$$

$$\therefore (x^2+\frac{1}{x^2})+10(x+\frac{1}{x})+27 = z^2-2+10z+27$$

$$= z^2+10z+25$$

$$= (z)^2+2(z)5+(5)^2$$

$$= (z+5)^2 \quad \left[\text{Putting } z = x+\frac{1}{x}\right]$$

$$= (x+\frac{1}{x}+5)^2$$

$$(x^2+\frac{1}{x^2})+10(x+\frac{1}{x})+27 = \left[\pm(x+\frac{1}{x}+5)\right]^2$$

Taking square root of both the sides, we get

$$\sqrt{(x^2+\frac{1}{x^2})+10(x+\frac{1}{x})+27} = \pm(x+\frac{1}{x}+5)$$

EXAMPLE-3

Find square root of $x(x-1)(x-2)(x-3)+1$

SOLUTION: $x(x-1)(x-2)(x-3)+1$

$$= [x(x-3)]\,[(x-1)(x-2)]+1$$

$$= \left[x^2-3x\right]\left[x^2-3x+2\right]+1$$

Put $x^2-3x = z$

$$x(x-1)(x-2)(x-3)+1 = z(z+2)+1$$

$$= z^2+2z+1$$

$$= (z+1)^2$$

Now put $z = x^2-3x$

$$x(x-1)(x-2)(x-3)+1 = (x^2-3x+1)^2$$

$$= \left[\pm(x^2-3x+1)\right]^2$$

Taking square root of both the sides, we get

$$\sqrt{x(x-1)(x-2)(x-3)+1} = \pm(x^2-3x+1)$$

EXAMPLE-4

Find square root of $(\frac{x}{y}+\frac{y}{x})^2-4(\frac{x}{y}-\frac{y}{x}),(x\neq o\,,y\neq o)$

SOLUTION: *Let* $\frac{x}{y}-\frac{y}{x}=z$

$$(\frac{x}{y}-\frac{y}{x})^2=z^2 \quad \text{(Squaring both the sides)}$$

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}-2=z^2$$

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}=z^2+2$$

$$(\frac{x}{y}+\frac{y}{x})^2-4(\frac{x}{y}-\frac{y}{x})=(\frac{x^2}{y^2}+\frac{y^2}{x^2}+2)-4(\frac{x}{y}-\frac{y}{x})$$

$$=z^2+2+2-4z$$

$$=z^2-4z+4$$

$$=(z-2)^2$$

$$=\left[\pm(z-2)\right]^2 \quad \left[\text{Putting } z=\frac{x}{y}-\frac{y}{x}\right]$$

$$(\frac{x}{y}+\frac{y}{x})^2-4(\frac{x}{y}-\frac{y}{x})=\left[\pm(\frac{x}{y}-\frac{y}{x}-2)\right]^2$$

Taking square root of both the sides

$$\sqrt{(\frac{x}{y}+\frac{y}{x})^2-4(\frac{x}{y}-\frac{y}{x})}=\pm(\frac{x}{y}-\frac{y}{x}-2)$$

3.3.2 Square Root by Division Method

We explain the method of finding the square root by division method in the following examples.

EXAMPLE-1

Find the square root of $x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$

SOLUTION:

$$
\begin{array}{r|l}
 & x + y + z \\
\hline
x & x^2 + 2xy + 2xz + 2yz + y^2 + z^2 \\
 & \pm x^2 \\
\hline
2x + y & 2xy + 2xz + 2yz + y^2 + z^2 \\
 & \pm 2xy \qquad\qquad \pm y^2 \\
\hline
2x + 2y + z & 2xz + 2yz + z^2 \\
 & \pm 2xz \pm 2yz \pm z^2 \\
\hline
 & 0
\end{array}
$$

Required square roots are $\pm(x + y + z)$.

(i) Write the given expression in descending order.
Take square root x of the *1st* term x^2.
On subtraction, remainder is $2xy + 2xz + 2yz + y^2 + z^2$

(ii) Multiply 2 times the quotient x by y, which is equal to the *1st* term of the remainder. Therefore by dividing the remainder with $2x + y$, we get the new remainder $2xz + 2yz + z^2$ and $x + y$ as quotient, which are the *1st* two terms of the square root.

(iii) Divide this remainder by sum of 2 times the quotient and z i.e $2x + 2y + z$.
We get the quotient $x + y + z$ and remainder zero.
Thus $\pm(x + y + z)$ are the required square roots.

EXAMPLE-2

Find square root of $(x^2-\frac{1}{x^2})^2-12(x^2-\frac{1}{x^2})+36$

SOLUTION: $(x^2-\frac{1}{x^2})^2-12(x^2-\frac{1}{x^2})+36$

$$= x^4+\frac{1}{x^4}-2-12x^2+\frac{12}{x^2}+36$$

$$= x^4-12x^2+34+\frac{12}{x^2}+\frac{1}{x^4} \qquad \text{(Writing in descending order)}$$

$$\begin{array}{r|l}
 & x^2-6-\frac{1}{x^2} \\ \hline
x^2 & x^4-12x^2+34+\frac{12}{x^2}+\frac{1}{x^4} \\
 & \pm x^4 \\ \hline
2x^2-6 & -12x^2+34+\frac{12}{x^2}+\frac{1}{x^4} \\
 & \mp 12x^2 \pm 36 \\ \hline
2x^2-12-\frac{1}{x^2} & -2+\frac{12}{x^2}+\frac{1}{x^4} \\
 & \mp 2 \pm \frac{12}{x^2} \pm \frac{1}{x^4} \\ \hline
 & 0
\end{array}$$

Thus $\pm(x^2-6-\frac{1}{x^2})$ *is the required square root.*

$x^2 = a \Rightarrow x = \pm a$ and $x = \pm\sqrt{a} \Rightarrow x^2 = a$

EXAMPLE-3

For making $x^4 - 12x^3 + 217x + 320$ a complete square,

(i) What should be added ? (ii) What should be subtracted ?

(iii) What should be the value of x ?

SOLUTION:

$$
\begin{array}{r|l}
 & x^2 - 6x - 18 \\
\hline
x^2 & x^4 - 12x^3 + 0x^2 + 217x + 320 \\
 & \pm x^4 \\
\hline
2x^2 - 6x & -12x^3 + 0x^2 + 217x + 320 \\
 & \mp 12x^3 \pm 36x^2 \\
\hline
2x^2 - 12x - 18 & -36x^2 + 217x + 320 \\
 & \mp 36x^2 \pm 216x \pm 324 \\
\hline
 & x - 4
\end{array}
$$

(i) By adding $-x + 4$, the expression will be a complete square.

(ii) By subtracting $x - 4$, the expression will be a complete square.

(iii) If $x - 4 = 0$ i.e $x = 4$ then the expression will be a complete square.

EXAMPLE-4

For what value of ℓ and m the expression

$4x^4 - 12x^3 + 25x^2 - \ell x + m$ is a complete square, where $x \neq 0$

SOLUTION:

$$
\begin{array}{r|l}
 & 2x^2 - 3x + 4 \\
\hline
2x^2 & 4x^4 - 12x^3 + 25x^2 - \ell x + m \\
 & \pm 4x^4 \\
\hline
4x^2 - 3x & -12x^3 + 25x^2 \\
 & \mp 12x^3 \pm 9x^2 \\
\hline
4x^2 - 6x + 4 & 16x^2 - \ell x + m \\
 & -16x^2 \mp 24x \pm 16 \\
\hline
 & (-\ell + 24)x + (m - 16) = \text{Remainder}
\end{array}
$$

The given expression will be a complete square, if for each value of ℓ and m, the given expression $(-\ell + 24)x + (m - 16)$ is zero.
It will be possible only if:

$$-\ell + 24 = 0 \quad \text{and} \quad m - 16 = 0$$

$$\ell = 24 \quad \text{and} \quad m = 16$$

Thus for $\ell = 24$ and $m = 16$, the expression will be a complete square.

EXERCISE – 3.6

Find the Square Root of the Following.

1. $16x^2 + 24xy + 9y^2$
2. $(x^2 - 7x + 12)(x^2 - 9x + 20)(x^2 - 8x + 15)$
3. $(x^2 + 8x + 7)(2x^2 - x - 3)(2x^2 + 11x - 21)$
4. $x(x + 2)(x + 4)(x + 6) + 16$
5. $(2x + 1)(2x + 3)(2x + 5)(2x + 7) + 16$
6. $(x^2 + \frac{1}{x^2}) - 10(x + \frac{1}{x}) + 27, x \neq 0$
7. $(t - \frac{1}{t})^2 - 4(t + \frac{1}{t}) + 8, (t \neq 0)$
8. $(x^2 + \frac{1}{x^2})^2 - 4(x + \frac{1}{x})^2 + 12, x \neq 0$
9. $4x^4 + 12x^3 + 25x^2 + 24x + 16$
10. $\frac{9x^2}{4y^2} - \frac{3x}{2y} - \frac{7}{4} + \frac{2y}{3x} + \frac{4x^2}{9y^2}, (x \neq 0, y \neq 0)$
11. For what value of x, $x^4 + 4x^2 + x + \frac{8}{x^2} + \frac{4}{x^4}$ is a complete square, where $x \neq 0$
12. If $x^4 + \ell x^3 + mx^2 + 12x + 9$ is a complete square then find the values of ℓ and m.

Review Exercise-3

I- Encircle the Correct Answer.

1. $\dfrac{\text{Product of two expressions}}{L.C.M} = ?$
   (a) $H.C.F$ (b) $L.C.M$
   (c) $L.C.M \times H.C.F$ (d) $L.C.M + H.C.F$

2. The number of methods to find L.C.M are:
   (a) 0 (b) 1 (c) 2 (d) 3

3. The number of methods to find the H.C.F are:
   (a) 4 (b) 1 (c) 2 (d) 3

4. H.C.F of $12pq,\ 8p^2q$ is:
   (a) $4pq$ (b) $4p^2q^2$ (c) $4pq^2$ (d) $4p^2q$

5. H.C.F of $2x^2+3x+1,\ 2x^2-x-1$ is:
   (a) $2x-1$ (b) $2x+1$ (c) $x+1$ (d) $x-1$

6. H.C.F of $6pqr,\ 15qrs$ is:
   (a) $3qr$ (b) $3pqr$ (c) $3pqrs$ (d) $15pqrs$

7. L.C.M of $12p^3q^2,\ 8p^2$ is:
   (a) $24pq^2$ (b) $24p^3q$ (c) $24p^3q^2$ (d) $12p^2q$

8. Product of two expressions =
   (a) $H.C.F$ (b) $L.C.M$
   (c) $H.C.F \times L.C.M$ (d) $H.C.F + L.C.M$

9. $\dfrac{\text{Product of two expressions}}{H.C.F} =$
   (a) $L.C.M$ (b) $H.C.F$
   (c) 0 (d) $L.C.M \times H.C.F$

10. $\dfrac{L.C.M \times H.C.F}{\text{First Expression}} =$
    (a) second expression (b) 1 (c) $H.C.F$ (d) $L.C.M$

SUMMARY

H.C.F:

The *H.C.F* of two or more algebraic expressions is the expression of highest degree which divides each of them without remainder.

L.C.M:

The least common multiple of two or more algebraic expressions is the expression of lowest degree which is divisible by each of them without reminder.