UNIT 6

MATRICES AND DETERMINANTS

- Introduction to Matrices
- Types of Matrices
- Addition and Subtraction of Matrices
- Multiplication of Matrices
- Multiplicative Inverse of a Matrix
- Solution of Simultaneous Linear Equations

After completion of this unit, the students will be able to:

- define
  - A matrix with real entries and relate its rectangular layout (formation) with real life.
  - Rows and columns of a matrix. • The order of matrix. • Equality of two matrices.
- define and identify row matrix, column matrix, rectangular matrix, square matrix, zero/null matrix, identity matrix, scalar matrix, diagonal matrix, transpose of a matrix, symmetric and skew-symmetric matrices.
- know whether the given matrices are conformable for addition/subtraction.
- add and subtract matrices.
- multiply a matrix by a real number.
- verify commutative and associative laws under addition.
- define additive identity of a matrix.
- find additive inverse of a matrix.
- know whether the given matrices are conformable for multiplication.
- multiplication of two (or three) matrices.
- verify associative law under multiplication.
- verify distributive laws.
- show with an example that commutative law under multiplication does not hold in general (i.e., $AB \neq BA$).
- define multiplicative identity of a matrix.
- verify the result $(AB)^t = B^t A^t$.
- define the determinant of a square matrix.
- evaluate determinant of a matrix.
- define singular and non-singular matrices.
- define adjoint of a matrix.
- find multiplicative inverse of a non-singular matrix A and verify that $AA^{-1} = I = A^{-1}A$, where I is the identity matrix.
- use adjoint method to calculate inverse of a non-singular matrix.
- verify the result $(AB)^{-1} = B^{-1}A^{-1}$.
- solve a system of two linear equations and related real life problems in two unknown using.
  - Matrix inversion method, • Cramer's rule.

6.1 INTRODUCTION

In this chapter we will introduce a new mathematical form, called a matrix, that will enable us to represent a number of different quantities as a single unit.

The idea of matrices was introduced by a famous mathematician Arther Cayley in 1857. Matrices are widely used in both the physical and the social sciences.

A matrix is a square or a rectangular array of numbers written within square brackets or parentheses in a definite order, in rows and columns.

Generally, the matrices *(plural of the matrix)* are denoted by capital letters *A, B, C*.............. etc. while the elements of a matrix are denoted by small letters *a, b, c*.............. and numbers *1, 2, 3*.............. For example:

$$A=\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix},\quad B=\begin{bmatrix}7\\ 2\end{bmatrix},\quad C=\begin{bmatrix}4 & 5\end{bmatrix},\quad D=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

Look at another example:

A company that manufactures shirts makes a standard model and a competition model. The labour *(in hours)* required for each model is conveniently represented by the 2×3 matrix.

	Fabricating	Finishing	Packaging and handling	

$$M=\begin{bmatrix}5 & 1 & 0.2\\ 7 & 2 & 0.2\end{bmatrix}\ \text{Standard Shirts}$$

The weekly production can be represented by the row matrix

Standard Shirts	Competition Shirts

$$\begin{bmatrix}100 & 10\end{bmatrix}$$

Each matrix consists of horizontally and vertically arranged elements.

$$A = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 3 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 5 \end{matrix}} \end{bmatrix} \qquad A = \begin{bmatrix} \boxed{1 \quad 2} \\ \boxed{3 \quad 5} \end{bmatrix}$$

Vertically arranged elements. | Horizontally arranged elements

Rows: Horizontally arranged elements are said to form rows.

Columns: Vertically arranged elements are said to form columns.

The number of rows and columns in matrices may be equal or different. However, the number of elements in different rows are same and similar is the case in the columns of a matrix that remains the same.

Generally, rows and columns are denoted by R and C respectively. For example:

Column 1 or C_1 ↓   *Column 2* or C_2 ↓

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{matrix} \longleftarrow \text{Row 1 or } R_1 \\ \longleftarrow \text{Row 2 or } R_2 \end{matrix}$$

Matrix A has two, rows and two columns whereas a, b, c, d are its elements. The number of rows and the number of columns are denoted by m and n respectively.

In the above example $m = 2$ and $n = 2$

Order of a Matrix:-

If a matrix A has 'm' number of rows and 'n' number of columns, then order of the matrix A is $m \times n$ (read as "m–by–n matrix")

EXAMPLE-1 *Find the order of* $P = [\,3\,]$

SOLUTION: *Matrix P has only one row and one column. So order of P is 1×1 matrix.*

EXAMPLE-2 *Find the order of* $Q = [\,4 \quad 7\,]$

SOLUTION: *Matrix Q has one row and two columns. So order of Q is 1×2 matrix.*

EXAMPLE-3 *Find the order of* $R = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$

SOLUTION: *In matrix R, the number of rows is two i.e $m = 2$ and the number of columns is two i.e $n = 2$. The order of R is 2×2 matrix.*

EXAMPLE-4 *Find the order of* $A = \begin{bmatrix} 3 & 4 & 9 \\ 5 & 7 & 2 \\ 1 & 2 & 5 \end{bmatrix}$

SOLUTION: *In matrix A, the number of rows is three i.e $m = 3$ and the number of columns is three, i.e $n = 3$. The order of A is 3×3 matrix.*

Remember that:

- We denote order of a matrix $m \times n$ instead of m by n.
- It is important to remember that in the order of a matrix, the number of rows are mentioned first.

Equal Matrices:-

Two matrices A and B are said to be equal if and only if they have the same order and their corresponding elements are equal.
Their equality is denoted by $A = B$.

For example: $\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Leftrightarrow \begin{cases} w = a \\ x = b \\ y = c \\ z = d \end{cases}$

EXAMPLE

Which of the following matrices are equal and which of them are not equal ?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ B = \begin{bmatrix} 3-2 & \frac{4}{2} \\ 3 & 2\times 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix},\ D = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 1 & 3 \\ 4 & 2 & 5 \end{bmatrix},\ E = \begin{bmatrix} 1 & \frac{6}{2} & 7 \\ \frac{6}{3} & 1 & 3 \\ 2\times 2 & 2 & \frac{10}{2} \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix}$$

SOLUTION: *Matrix B, can be written as*

$$B = \begin{bmatrix} 3-2 & \frac{4}{2} \\ 3 & 2\times 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

(i) Order of A and B is same 2 by2, and corresponding elements are equal, so $A = B$.

(ii) Order of A, B and C is same 2 by2, but corresponding elements are not equal, so $A = B \neq C$.

(iii) Matrix E can be written as:
Order of D and E is same, i.e 3by3 and corresponding elements are equal, so $D = E$.

$$E = \begin{bmatrix} 1 & \frac{6}{2} & 7 \\ \frac{6}{3} & 1 & 3 \\ 2\times 2 & 2 & \frac{10}{2} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 1 & 3 \\ 4 & 2 & 5 \end{bmatrix} = D$$

(iv) Order of the matrix F is 2 by3 , so $F \neq D$ and $E \neq F$.

EXERCISE – 6.1

With the help of the given matrices answer the questions from 1 to 3.

$$A = \begin{bmatrix} 2 & -2 \\ -5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -2 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix},$$

$$D = \begin{bmatrix} -3 & 2 & 0 \\ 0 & 1 & 5 \\ 4 & -2 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} -3 & 2 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} -3 & 4 \\ 0 & 5 \\ 3 & -1 \end{bmatrix}$$

1- What are the orders of matrices A, C and F?

2- What are the orders of matrices B, D and E?

3- What element is in the second row and third column of matrix D?

4- Which of the following matrices are equal and which of them are not?

$$A = \begin{bmatrix} 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 6 \\ 9 \end{bmatrix}, \quad D = \begin{bmatrix} 2+2 \end{bmatrix}$$

$$E = \begin{bmatrix} 3+3 \\ 8+1 \end{bmatrix}, F = \begin{bmatrix} \frac{5}{5} & \frac{4}{2} \end{bmatrix}, G = \begin{bmatrix} 1 & 3 \\ 6 & 8 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 4 \\ 2 & 6 & 3 \end{bmatrix}, I = \begin{bmatrix} 1 & 3 \\ 6 & 7 \end{bmatrix}, J = \begin{bmatrix} 1 & 3 \\ 6 & 16/2 \end{bmatrix},$$

$$K = \begin{bmatrix} 1 & 2 & 3+2 \\ 0 & 3 & 4 \\ 2 & 4+2 & 3 \end{bmatrix}, L = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 4 \\ 2 & 6 & 3 \end{bmatrix},$$

6.2 TYPES OF MATRICES

(I) Row Matrix:-

A matrix with only one row is called a row matrix.

For example: $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ is of order 1×2.

$B = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ is of order 1×3.

(II) Column Matrix:-

A matrix with only one column is called a column matrix.

For example: $C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is of order 2×1.

$D = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ is of order 3×1.

(III) Rectangular Matrix:-

If in a matrix, the number of rows and the number of columns are not equal, then the matrix is called a rectangular matrix.

For example: $A = \begin{bmatrix} 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$,

$C = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 4 & 5 \end{bmatrix}$

are rectangular matrices of order 1×2, 2×1 and 2×3 respectively.

(IV) Square Matrix:-

If a matrix has equal number of rows and columns, it is called a square matrix.

For example: $P = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$

are square matrices of order 2×2 and 3×3 respectively.

(V) Zero or Null Matrix:-

If all the elements in a matrix are zeros, it is called a zero matrix or null matrix. A null matrix is denoted by the letter O.

For example: $O = [0]$ is of order 1×1.

$O = [0 \quad 0]$ is of order 1×2.

$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is of order 2×2.

$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is of order 3×3.

(VI) Diagonal Matrix:-

A square matrix in which all the elements except at least the one element in the diagonal are zeros is called a diagonal matrix. Some elements of the diagonal in a matrix may be zero but not all.

For example: $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

are all diagonal matrices.

(VII) Scalar Matrix:-

A diagonal matrix having equal elements is called a scalar matrix.

For example:

$A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$, $C = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ are scalar matrices.

(VIII) Unit Matrix or Identity Matrix:-

A scalar matrix having each element equal to 1 is called a unit or identity matrix. Identity or unit matrix is generally denoted by I.

For example:

$$I = [1] \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are identity matrices of different orders.

(IX) Transpose of a Matrix:-

If A is a matrix of order $(m \times n)$, then a matrix $(n \times m)$ obtained by interchanging the rows and columns of A is called the transpose of A. It is denoted by A^t.

For example: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $B^t = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

If A^t and B^t are transposes of A and B respectively, and if k is scalar. Then:

(a) $(A^t)^t = A$ (b) $(kA)^t = kA^t$

(c) $(A+B)^t = A^t + B^t$ (d) $(AB)^t = B^t A^t$

(X) Symmetric Matrix:- A square matrix A is called symmetric if $A^t = A$

For example: $A = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$ and $A^t = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$

Since $A^t = A$

A is symmetric matrix.

$B = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 0 & 4 \\ 3 & 4 & 2 \end{bmatrix} \qquad B^t = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 0 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

Since $B^t = B$

Matrix B is symmetric.

(XI) Skew-Symmetric Matrix:-

A square matrix A is called skew symmetric (or anti-symmetric) if $A^t = -A$.

For example: $A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

$$A^t = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix} = -A$$

$A^t = -A$ Hence A is skew symmetric.

EXERCISE – 6.2

1- Identify row matrices, column matrices, square matrices, and rectangular matrices in the following matrices.

$$A = [3 \quad 1 \quad 1 \quad 1],\ B = \begin{bmatrix} 5+2 & 4 \\ 2 & 6 \end{bmatrix},\ C = \begin{bmatrix} a+x \\ b+y \end{bmatrix},\ D = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix},$$

$$E = \begin{bmatrix} x & -2 \\ b & 5 \end{bmatrix},\ F = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 5 \\ 1 & -5 & 0 \end{bmatrix},\ G = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 7 & 8 \end{bmatrix},\ H = [0]$$

2- Identify, diagonal matrices, scalar matrices, identity matrices.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix},\ F = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix},\ G = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

3- Find transpose of the following matrices.

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 4 \end{bmatrix}, B = \begin{bmatrix} -3 & -2 \\ -1 & 4 \end{bmatrix}, C = \begin{bmatrix} a & -b \\ c & d \end{bmatrix}, D = \begin{bmatrix} \ell & m & n \\ p & q & r \\ a & b & c \end{bmatrix}$$

4- Identify all row matrices, if:

$$A = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 3 \\ 4 & 6 \end{bmatrix}, \; C = \begin{bmatrix} e & f & g \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & 7 & 5 \\ 4 & 6 & 2 \\ 1 & 9 & 8 \end{bmatrix}, \; F = \begin{bmatrix} 1 & 4 & 6 \\ 3 & 7 & 3 \end{bmatrix}$$

5- Identify all column matrices, if:

$$A = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 3 \\ 6 & 5 \\ 4 & 7 \end{bmatrix}, \; C = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \; D = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 6 & 5 \\ -2 & 3 & 4 \end{bmatrix},$$

$$E = \begin{bmatrix} 5 \\ 7 \\ -4 \end{bmatrix}, \; F = \begin{bmatrix} 9 & 7 & 1 \end{bmatrix}$$

6- Identify all column matrices, if:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 6 & 5 \\ 7 & 3 & 4 \end{bmatrix}, \; C = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}, \; D = \begin{bmatrix} 7 & 8 \\ 6 & 5 \end{bmatrix},$$

$$E = \begin{bmatrix} 3 & 5 & 7 \end{bmatrix}, \; F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

7- Identify all 3×3 square matrices, if:

$$A = \begin{bmatrix} 2 & -3 & 6 \\ 1 & 5 & 4 \\ 3 & 6 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \; C = \begin{bmatrix} 7 & 3 & 4 \end{bmatrix}$$

6.3 ADDITION AND SUBTRACTION OF MATRICES

Two matrices A and B are said to be conformable for addition $A+B$, if they are of the same order and their sum is obtained by adding their corresponding elements.

Order of matrix $A+B$ will be the same as the order of matrices A and B.

6.3.1 Add and Subtract Matrices

Addition of Matrices:

When two matrices are conformable for addition, we find addition by adding their corresponding elements.

For example:

(i) $$\begin{bmatrix} w & x \\ y & z \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} w+a & x+b \\ y+c & z+d \end{bmatrix}$$

(ii) $$\begin{bmatrix} -3 & 0 & 1 \\ 5 & 2 & -4 \end{bmatrix} + \begin{bmatrix} 5 & 3 & -2 \\ -2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$

(iii) $$\begin{bmatrix} 1 & 2 & 4 \\ 2 & -3 & 5 \\ 3 & 4 & 7 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -5 \\ -5 & 2 & 1 \\ -1 & -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & 2+(-2) & 4+(-5) \\ 2+(-5) & -3+2 & 5+1 \\ 3+(-1) & 4+(-3) & 7+(-2) \end{bmatrix} = \begin{bmatrix} 4 & 0 & -1 \\ -3 & -1 & 6 \\ 2 & 1 & 5 \end{bmatrix}$$

Subtraction of Matrices:

If two matrices A and B are of the same order then their difference can be written as $A-B$.

The difference $A-B$ is obtained by subtracting the elements of B from the corresponding elements of matrix A.

EXAMPLE-1 If $A = \begin{bmatrix} 1 & x \\ y & 4 \end{bmatrix}$ $B = \begin{bmatrix} a & 2 \\ 3 & b \end{bmatrix}$ then find $A-B$

SOLUTION: $A-B = \begin{bmatrix} 1 & x \\ y & 4 \end{bmatrix} - \begin{bmatrix} a & 2 \\ 3 & b \end{bmatrix}$

$$A-B = \begin{bmatrix} 1-a & x-2 \\ y-3 & 4-b \end{bmatrix}$$

EXAMPLE-2

If $A = \begin{bmatrix} 2 & 7 & 3 \\ -1 & 3 & 4 \\ 0 & 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 6 \\ -1 & 8 & 3 \end{bmatrix}$ then find $A-B$

SOLUTION: $A-B = \begin{bmatrix} 2 & 7 & 3 \\ -1 & 3 & 4 \\ 0 & 4 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 6 \\ -1 & 8 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2-1 & 7-3 & 3-5 \\ -1-2 & 3-1 & 4-6 \\ 0-(-1) & 4-8 & -2-3 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 1 & 4 & -2 \\ -3 & 2 & -2 \\ 1 & -4 & -5 \end{bmatrix}$$

EXAMPLE-3 *Add the matrices A,B and C where*

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -6 \\ -4 & 1 \end{bmatrix}$$

SOLUTION: *Since A , B and C matrices have the same order, so they are conformable for addition*

$$A+B+C = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} + \begin{bmatrix} 2 & -6 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1+2 & 1+3-6 \\ 3-2-4 & 4+5+1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 \\ -3 & 10 \end{bmatrix}$$

EXAMPLE-4

Subtract matrix B from matrix A.

$$A = \begin{bmatrix} 2 & 4 & 7 \\ 1 & 3 & -2 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 11 & -5 & 2 \\ 2 & 4 & -6 \\ 3 & 6 & -1 \end{bmatrix}$$

SOLUTION: *Since A and B have the same order, so they are conformable for subtraction.*

$$A-B = \begin{bmatrix} 2 & 4 & 7 \\ 1 & 3 & -2 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 11 & -5 & 2 \\ 2 & 4 & -6 \\ 3 & 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-11 & 4+5 & 7-2 \\ 1-2 & 3-4 & -2+6 \\ 4-3 & 5-6 & 6+1 \end{bmatrix}$$

$$A-B = \begin{bmatrix} -9 & 9 & 5 \\ -1 & -1 & 4 \\ 1 & -1 & 7 \end{bmatrix}$$

A Scalar Multiplication

Any element from the set of real numbers is also called a scalar. We define the product of a matrix A and a scalar k, denoted by kA, to be the matrix formed by multiplying each element of $A \times k$.

For example:

(i) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

(ii) If $A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 6 & -1 \\ -3 & 4 & 7 \end{bmatrix}$, then $3A = \begin{bmatrix} 12 & 9 & 6 \\ 6 & 18 & -3 \\ -9 & 12 & 21 \end{bmatrix}$

6.3.2 Laws of Addition of Matrices

Commutative Law:

For any two matrices A and B of the same order

$$A + B = B + A$$

This law is called commutative law of matrices with respect to addition.

EXAMPLE-1

If $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & 2 \\ -6 & 1 \end{bmatrix}$

then show that $A + B = B + A$.

SOLUTION: *Matrices A,B have same order, so they are conformable for addition.*

$$A+B = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -6 & 1 \end{bmatrix} = \begin{bmatrix} 1+4 & 3+2 \\ 4-6 & 5+1 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 5 & 5 \\ -2 & 6 \end{bmatrix}$$

and

$$B+A = \begin{bmatrix} 4 & 2 \\ -6 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4+1 & 2+3 \\ -6+4 & 1+5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 \\ -2 & 6 \end{bmatrix}$$

Thus $A + B = B + A$

Associative Law:

For three matrices A , B and C of same order,

$$(A + B) + C = A + (B + C)$$

This law is called associative law of matrices with respect to addition.

EXAMPLE *If* $A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ -5 & -4 \end{bmatrix}$

then verify the associative law of matrices with respect to addition.

SOLUTION:

$$(A+B)+C = \left(\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\right) + \begin{bmatrix} 2 & 3 \\ -5 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ 0 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -5 & -4 \end{bmatrix}$$

$$(A+B)+C = \begin{bmatrix} 7 & 5 \\ -5 & 3 \end{bmatrix} \ldots\ldots\ldots\ldots (i)$$

$$A+(B+C) = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} + \left(\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -5 & -4 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 3 \\ -5 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ -5 & 3 \end{bmatrix} \ldots\ldots\ldots\ldots (ii)$$

From (i) and (ii), we have $(A + B) + C = A + (B + C)$

6.3.3 Additive Identity of Matrices

In real numbers, zero is the additive identity i.e. the sum of a real number and zero is equal to the real number e.g, $5 + 0 = 0 + 5 = 5$. Similarly, a zero matrix O of order m - by - n is called the additive identity matrix such that

$$A + O = O + A = A$$

For example: $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$

Consider, $A+O = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A+O = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} = A$$

and, $O + A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} = A$

Thus $A + O = O + A = A$ *so 'O' is additive identity of matrix A.*

Remember that: The order of 'A' and 'O' is same.

6.3.4 Additive Inverse of a Matrix

If two matrices A and B are such that their sum $(A + B)$ is a zero matrix, then A and B are called additive inverse of each other.

For example:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ and } B = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

Consider $$A+B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Therefore A and B are inverse of each other.

EXAMPLE

If $$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & -4 & 5 \\ -2 & -1 & 7 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 2 & 1 & -7 \end{bmatrix}$$

then

$$A+B = \begin{bmatrix} -1 & 2 & -3 \\ 2 & -4 & 5 \\ -2 & -1 & 7 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 2 & 1 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} -1+1 & 2-2 & -3+3 \\ 2-2 & -4+4 & 5-5 \\ -2+2 & -1+1 & 7-7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A + B = O$

A and B are inverse of each other.

EXERCISE – 6.3

1- If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 5 \\ 4 & 9 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 5 \\ 2 & 3 & 6 \\ 1 & 4 & -2 \end{bmatrix}$

Find (i) $A+B$ (ii) $A-B$ (iii) $B-A$
(iv) $2A+3B$ (v) $3A-4B$ (vi) $A-2B$

2- Find the additive inverses of the following matrices.

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} \sqrt{2} & 3 \\ 4 & \sqrt{3} \end{bmatrix}, C = \begin{bmatrix} 1 \\ -7 \\ 4 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 4 \\ 2 & -1 & -3 \end{bmatrix}, E = \begin{bmatrix} 2 & 5 & -3 \end{bmatrix}$$

3- If $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 7 \\ 4 & 6 \end{bmatrix}$ then show that

(i) $4A - 3A = A$ (ii) $3B - 3A = 3(B - A)$

4- Find x and y if $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

5- If $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 6 \\ 3 & -2 \end{bmatrix}$ then prove that,

(i) $A+B = B+A$ (ii) $A+(B+C) = (A+B)+C$

6- Solve the matrix equation for X.

$3X - 2A = B$ if $A = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$

7- Find a,b,c,d,e and f such that

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} 3 & -2 & 1 \\ 5 & 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \end{bmatrix}$$

8- Find w,x,y,z such that

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & -3 \end{bmatrix}$$

9- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then what is the additive inverse of A?

10- Given that $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ verify that $A^2 - 4A + 5I = 0$.

11- If $A = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ 4 & 6 \end{bmatrix}$, then verify that $(A+B)^t = A^t + B^t$.

12- If $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}, B = \begin{bmatrix} 2 & -7 \\ 5 & 8 \end{bmatrix}, C = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$ then show that

$$A + B - C = \begin{bmatrix} 2 & -10 \\ 8 & 2 \end{bmatrix}$$

6.4.1 MULTIPLICATION OF MATRICES

Two matrices A and B are said to be conformable for the product AB, if the number of columns in A is equal to the number of rows in B.

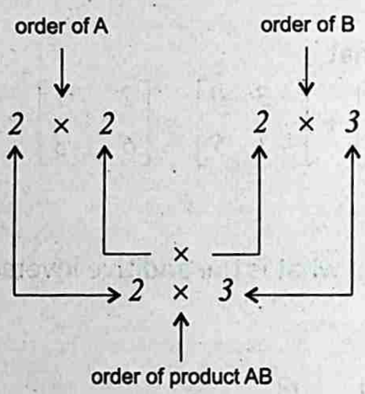
For example: $\begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\times 4 + 3\times 2 \end{bmatrix}$

$= \begin{bmatrix} 8+6 \end{bmatrix}$

$= \begin{bmatrix} 14 \end{bmatrix}$

EXAMPLE-1

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$



The product of AB shall contain the elements like

$$AB = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \text{ where}$$

$c_{11} = a_{11}\, b_{11} + a_{12}\, b_{21}$ (Multiplication of the elements of 1st row of A with elements of 1st column of B)

$c_{12} = a_{11}\, b_{12} + a_{12}\, b_{22}$ (Multiplication of the elements of 1st row of A with elements of 2nd column of B)

$c_{13} = a_{11}\, b_{13} + a_{12}\, b_{23}$ (Multiplication of the elements of 1st row of A with elements of 3rd column of B)

$c_{21} = a_{21}\, b_{11} + a_{22}\, b_{21}$

$c_{22} = a_{21}\, b_{12} + a_{22}\, b_{22}$

$c_{23} = a_{21}\, b_{13} + a_{22}\, b_{23}$, thus

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} & a_{11}\, b_{12} + a_{12}\, b_{22} & a_{11}\, b_{13} + a_{12}\, b_{23} \\ a_{21}\, b_{11} + a_{22}\, b_{21} & a_{21}\, b_{12} + a_{22}\, b_{22} & a_{21}\, b_{13} + a_{22}\, b_{23} \end{bmatrix}$$

EXAMPLE-2

If $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then find AB.

SOLUTION: order of $A = 2\times 2$
order of $B = 2\times 1$
order of $AB = 2\times 1$

Because number of columns in A = number of rows in $B = 2$

$$AB = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5\times 2 + 2\times 3 \\ 3\times 2 + 4\times 3 \end{bmatrix} = \begin{bmatrix} 10 + 6 \\ 6 + 12 \end{bmatrix} = \begin{bmatrix} 16 \\ 18 \end{bmatrix}$$

Result:

If A is a square matrix then $A^2 = A.A$

$$A^3 = A.A.A = AA^2 = A^2A$$

Finally $A^n = A.A.A \ldots\ldots\ldots\ldots n \text{ times.}$

Remember that:

For multiplication AB of two matrices A and B the following points should be kept in mind.

(i) The number of columns in A = number of rows in B.
(ii) The product of matrices A and B is denoted by $A\times B$ or AB.
(iii) If A is a $m\times p$ matrix and B is a $p\times n$ matrix then AB is $m\times n$ matrix.

6.4.3 Associative Law of Matrices with respect to Multiplication

If three matrices A, B and C are conformable for multiplication, then

$$A(BC) = (AB)C$$

is called associative law with respect to multiplication.

EXAMPLE

If $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$

then verify associative law under multiplication.

SOLUTION:

Consider $A(BC) = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}\right)$

$$= \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1\times4 + 1\times3 & 1\times2 + 1\times1 \\ 2\times4 + 3\times3 & 2\times2 + 3\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4+3 & 2+1 \\ 8+9 & 4+3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 17 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times7 + 1\times17 & 2\times3 + 1\times7 \\ 3\times7 + 1\times17 & 3\times3 + 1\times7 \end{bmatrix} = \begin{bmatrix} 14+17 & 6+7 \\ 21+17 & 9+7 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 31 & 13 \\ 38 & 16 \end{bmatrix} \text{ ....(i)}$$

Consider $(AB)C = \left(\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\right)\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2\times1 + 1\times2 & 2\times1 + 1\times3 \\ 3\times1 + 1\times2 & 3\times1 + 1\times3 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

$$ABC = \begin{bmatrix} 2+2 & 2+3 \\ 3+2 & 3+3 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times4 + 5\times3 & 4\times2 + 5\times1 \\ 5\times4 + 6\times3 & 5\times2 + 6\times1 \end{bmatrix} = \begin{bmatrix} 16+15 & 8+5 \\ 20+18 & 10+6 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 31 & 13 \\ 38 & 16 \end{bmatrix} \text{............(ii)}$$

From equation (i) and (ii) , A(BC) = (AB)C
Associative law holds in multiplication of matrices.

6.4.4 Distributive Laws

If the matrices A, B and C are conformable for addition and multiplication, then

(i) $A(B + C) = AB + AC$ *(left distributive law for matrices).*

(ii) $(A + B)C = AC + BC$ *(right distributive law for matrices)*

(i) and *(ii)* are called distributive laws.

EXAMPLE

If $A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ *and* $C = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$
then verify left and right distributive laws.

SOLUTION: *(i) Left distributive law* $A(B + C) = AB + AC$

$$B + C = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$$

$$B+C = \begin{bmatrix} 2-1 & 3-4 \\ -1+3 & -2+6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Consider $A(B+C) = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 4+6 & -4+12 \\ 2+4 & -2+8 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 8 \\ 6 & 6 \end{bmatrix} \text{............(i)}$$

Consider $AB + AC = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 8-3 & 12-6 \\ 4-2 & 6-4 \end{bmatrix} + \begin{bmatrix} -4+9 & -16+18 \\ -2+6 & -8+12 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 8 \\ 6 & 6 \end{bmatrix} \text{............(ii)}$$

From equations (i) and (ii)
$A\,(B + C) = AB + AC.$

(ii) Right distributive law $(A + B)\,C = AC + BC$

$$A+B=\begin{bmatrix}4 & 3\\2 & 2\end{bmatrix}+\begin{bmatrix}2 & 3\\-1 & -2\end{bmatrix}=\begin{bmatrix}6 & 6\\1 & 0\end{bmatrix}$$

Consider $(A+B)\,C=\begin{bmatrix}6 & 6\\1 & 0\end{bmatrix}\begin{bmatrix}-1 & -4\\3 & 6\end{bmatrix}$

$$=\begin{bmatrix}-6+18 & -24+36\\-1+0 & -4+0\end{bmatrix}$$

$$=\begin{bmatrix}12 & 12\\-1 & -4\end{bmatrix}\ \ldots\ldots\ldots\ldots(i)$$

Consider $AC+BC=\begin{bmatrix}4 & 3\\2 & 2\end{bmatrix}\begin{bmatrix}-1 & -4\\3 & 6\end{bmatrix}+\begin{bmatrix}2 & 3\\-1 & -2\end{bmatrix}\begin{bmatrix}-1 & -4\\3 & 6\end{bmatrix}$

$$=\begin{bmatrix}-4+9 & -16+18\\-2+6 & -8+12\end{bmatrix}+\begin{bmatrix}-2+9 & -8+18\\1-6 & 4-12\end{bmatrix}$$

$$=\begin{bmatrix}5 & 2\\4 & 4\end{bmatrix}+\begin{bmatrix}7 & 10\\-5 & -8\end{bmatrix}$$

$$=\begin{bmatrix}12 & 12\\-1 & -4\end{bmatrix}\ \ldots\ldots\ldots\ldots(ii)$$

From equations *(i)* and *(ii)*

$(A + B)\,C = AC + BC.$

6.4.5 Commutative Law

Commutative law does not hold in multiplication of matrices in general i.e. $AB \neq BA$

EXAMPLE-1 $A = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$ verify $AB \neq BA$.

SOLUTION: *Given matrices A and B are conformable for multiplication AB and BA.*

Consider $AB = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -1+8 & 3+4 \\ -3-8 & 9-4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & 7 \\ -11 & 5 \end{bmatrix} \text{............}(i)$$

Consider $BA = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -1+9 & -2-6 \\ 4+6 & 8-4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 & -8 \\ 10 & 4 \end{bmatrix} \text{............}(ii)$$

From equations (i) and (ii)

$AB \neq BA$

Hence commutative law does not hold in multiplication of matrices, in general.

EXAMPLE-2

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then verify $I_2A = AI_2 = A$

SOLUTION:

Consider $I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$= \begin{bmatrix} 1\times a + 0\times c & 1\times b + 0\times d \\ 0\times a + 1\times c & 0\times b + 1\times d \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$I_2A = A \quad \text{...........(i)}$$

Consider $AI_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} a\times 1 + b\times 0 & a\times 0 + b\times 1 \\ c\times 1 + d\times 0 & c\times 0 + d\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$AI_2 = A \quad \text{...........(ii)}$$

From equations (i) and (ii)

$$I_2A = AI_2 = A$$

6.4.7 Theorem

$(AB)^t = B^t A^t$ where A and B are two matrices.

EXAMPLE

If $A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$, *then*

show that $(AB)^t = B^t A^t$

SOLUTION:

Consider $AB = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -2+6 & 6+3 \\ -1-4 & 3-2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 9 \\ -5 & 1 \end{bmatrix}$$

$$L.H.S = (AB)^t = \begin{bmatrix} 4 & 9 \\ -5 & 1 \end{bmatrix}^t = \begin{bmatrix} 4 & -5 \\ 9 & 1 \end{bmatrix} \text{............}(i)$$

Now $A^t = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$, $B^t = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$

Consider $R.H.S = B^t A^t = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2+6 & -1-4 \\ 6+3 & 3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -5 \\ 9 & 1 \end{bmatrix} \text{............}(ii)$$

From (i) and (ii)

$(AB)^t = B^t A^t$

EXERCISE – 6.4

In Problems 1 to 8 Verify Each Statement, Using

$$A = \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix},\ C = \begin{bmatrix} -1 & 2 \\ 4 & 2 \end{bmatrix}$$

1. $(AB)C = A(BC)$
2. $AB \neq BA$
3. $A(B+C) = AB + AC$
4. $(B+C)A = BA + CA$
5. $(B+C)(B-C) \neq B^2 - C^2$
6. $(BC)^t = C^t B^t$
7. $BI = B$
8. $BC \neq CB$

Find the Matrix Products.

9. $\begin{bmatrix} 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

11. $\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$

12. $\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ -1 & 3 \end{bmatrix}$

13. $\begin{bmatrix} -5 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}$

14. $\begin{bmatrix} -2 & 4 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -5 & -5 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -0 \end{bmatrix}$

15- If $\begin{bmatrix} 1 & 5 \\ 3 & a \end{bmatrix}\begin{bmatrix} b \\ 7 \end{bmatrix} = \begin{bmatrix} 35 \\ 10 \end{bmatrix}$, then find the values of a and b.

16. If $A = \begin{bmatrix} 2 & 6 \\ 7 & 8 \end{bmatrix},\ B = \begin{bmatrix} -1 & -3 \\ 2 & 0 \end{bmatrix}$, then verify $(AB)^t = B^t A^t$.

6.5 MULTIPLICATIVE INVERSE OF A MATRIX

6.5.1 Determinant Function

In this section, we are going to define a new function, called a determinant of a square matrix. Its domain is the set of all square matrices with real elements, and its range is the set of all real numbers.

If A is a square matrix, then $det\ A$ or $|A|$ read "The determinant of A" is used to denote the unique real number.

The determinant of a matrix of order 2 is defined as follows.

$$det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

6.5.2 Evaluate Determinant of a Matrix

EXAMPLE-1

If $A = \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$, then evaluate $det\ A$.

SOLUTION: $|A| = \begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} = (-1)\times(-4) - (-3)\times 2$

$= 4 + 6 = 10$

EXAMPLE-2

If $A = \begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix}$, then evaluate $det\ A$.

SOLUTION: $det\ A = \begin{vmatrix} 6 & 2 \\ 4 & 2 \end{vmatrix} = 12 - 8 = 4$

EXAMPLE-3

If $A = \begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}$, then evaluate $det\ A$.

SOLUTION: $det\ A = \begin{vmatrix} 5 & 2 \\ 10 & 4 \end{vmatrix} = 20 - 20 = 0$

6.5.3 Singular and Non-Singular Matrices

Singular Matrix:

A square matrix A is called a singular matrix. If $det\ A = 0$

EXAMPLE

If $A = \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}$

$det\ A = \begin{vmatrix} 12 & 6 \\ 6 & 3 \end{vmatrix} = 36 - 36$

$det\ A = 0$. *Hence matrix A is singular.*

Non-Singular Matrix:

A square matrix A is called non-singular matrix, if $det\ A \neq 0$.

EXAMPLE

If $A = \begin{bmatrix} 2 & 5 \\ 6 & 8 \end{bmatrix}$

$det\ A = \begin{vmatrix} 2 & 5 \\ 6 & 8 \end{vmatrix} = 16 - 30$

$det\ A = -14 \neq 0$. *Hence matrix A is non-singular.*

6.5.4 Adjoint of a Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square matrix of order 2×2 .Then the matrix obtained by interchanging the elements of the diagonal (i.e a and d) and by changing the signs of the other elements b and c is called the adjoint of the matrix A.

The adjoint of the matrix A is denoted by $adj\ A$. For example:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Look at another example:

If $P = \begin{bmatrix} -6 & -2 \\ 3 & 4 \end{bmatrix}$, then $adj\ P = \begin{bmatrix} 4 & 2 \\ -3 & -6 \end{bmatrix}$

6.5.5 Multiplicative Inverse

In the set of real numbers, we know that for each real number a *(except zero)* there exists a real number a^{-1} such that $a\,a^{-1} = 1$. The number a^{-1} is called the multiplicative inverse of a.

Similarly, each square matrix A has a multiplicative inverse A^{-1} such that $AA^{-1} = A^{-1}A = I$, provided $det\ A \neq 0$.

Multiplicative inverse A^{-1} of any non-singular matrix A is given by

$$A^{-1} = \frac{adj\ A}{|A|}, \quad |A| \neq 0$$

If A is a singular matrix then the multiplicative inverse of A does not exist.

Remember That:

(i) Inverse of square matrix A is denoted by A^{-1}.

(ii) Only non-singular matrices have inverses.

(iii) Inverse of square matrix A is always unique.

(iv) Non-square matrices cannot possess inverses.

(v) $A^{-1} = \frac{adj\ A}{|A|}$

EXAMPLE

If $A = \begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}$*, then verify* $AA^{-1} = A^{-1}A = I$

where I is the identity matrix.

SOLUTION:

$$A = \begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}; \quad adj\, A = \begin{bmatrix} 4 & -2 \\ -5 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix}$$

$$= 16 - 10$$

$$|A| = 6 \neq 0$$

We have $\quad A^{-1} = \dfrac{1}{|A|}\, adj\, A$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & -2 \\ -5 & 4 \end{bmatrix}$$

Consider $\quad AA^{-1} = \dfrac{1}{6}\begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -5 & 4 \end{bmatrix}$

$$= \frac{1}{6}\begin{bmatrix} 16-10 & -8+8 \\ 20-20 & -10+16 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^{-1} = I \quad ....(i)$$

Now

Consider $\quad A^{-1}A = \dfrac{1}{6}\begin{bmatrix} 4 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}$

$$A^{-1}A = \frac{1}{6}\begin{bmatrix} 16-10 & 8-8 \\ -20+20 & -10+16 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = I \quad ....(ii)$$

From equation *(i)* and *(ii)* $AA^{-1} = I = A^{-1}A$.

6.5.6 Inverse of a Non-Singular Matrix

The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has the inverse $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, provided, $ad-bc \neq 0$.

EXAMPLE-1

If $A = \begin{bmatrix} 7 & 3 \\ 14 & 9 \end{bmatrix}$, then find inverse of matrix A.

SOLUTION: $A = \begin{bmatrix} 7 & 3 \\ 14 & 9 \end{bmatrix}$, then $adj\ A = \begin{bmatrix} 9 & -3 \\ -14 & 7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 7 & 3 \\ 14 & 9 \end{vmatrix}$$

$$|A| = 63-42 = 21 \neq 0$$

We know that $A^{-1} = \frac{1}{|A|} adj\ A$

$$= \frac{1}{21}\begin{bmatrix} 9 & -3 \\ -14 & 7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{9}{21} & \frac{-3}{21} \\ \frac{-14}{21} & \frac{7}{21} \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{-1}{7} \\ \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

EXAMPLE-2

If $B = \begin{bmatrix} 3 & -4 \\ -3 & -2 \end{bmatrix}$, then find B^{-1}.

SOLUTION: $B = \begin{bmatrix} 3 & -4 \\ -3 & -2 \end{bmatrix}$ $adj\ B = \begin{bmatrix} -2 & 4 \\ 3 & 3 \end{bmatrix}$

$$|B| = \begin{vmatrix} 3 & -4 \\ -3 & -2 \end{vmatrix}$$

$$|B| = -6 - 12 = -18 \neq 0$$

We know that $B^{-1} = \dfrac{1}{|B|} \, adj\, B$

$$= \frac{1}{-18}\begin{bmatrix} -2 & 4 \\ 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-2}{-18} & \dfrac{4}{-18} \\ \dfrac{3}{-18} & \dfrac{3}{-18} \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \dfrac{1}{9} & \dfrac{-2}{9} \\ \dfrac{-1}{6} & \dfrac{-1}{6} \end{bmatrix}$$

EXAMPLE-3

If $P = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$, then find P^{-1} if possible.

SOLUTION:

$$P = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}, \quad adj\, P = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$$

$$|P| = \begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix}$$

$$= 3 \times 4 - 2 \times 6 = 12 - 12 = 0$$

Since $|P| = 0$

The inverse of P is not defined,

because $\dfrac{1}{0}$ is not defined.

6.5.7 Verify $(AB)^{-1} = B^{-1}A^{-1}$

We verify this with the help of following example.

EXAMPLE

If $A = \begin{bmatrix} 3 & 6 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}$, *then verify* $(AB)^{-1} = B^{-1}A^{-1}$.

SOLUTION:

$$AB = \begin{bmatrix} 3 & 6 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 12+30 & 6+24 \\ 8+5 & 4+4 \end{bmatrix} = \begin{bmatrix} 42 & 30 \\ 13 & 8 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 42 & 30 \\ 13 & 8 \end{vmatrix} = 42\times 8 - 13\times 30$$

$$= 336 - 390$$

$$= -54 \neq 0$$

Consider $L.H.S = (AB)^{-1} = \dfrac{1}{|AB|}adj(AB)$

$$(AB)^{-1} = \frac{1}{-54}\begin{bmatrix} 8 & -30 \\ -13 & 42 \end{bmatrix} \;\;\ldots\ldots\ldots\ldots(i)$$

$$A = \begin{bmatrix} 3 & 6 \\ 2 & 1 \end{bmatrix}, \quad adj\ A = \begin{bmatrix} 1 & -6 \\ -2 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 6 \\ 2 & 1 \end{vmatrix} = 3-12$$

$$= -9 \neq 0$$

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$A^{-1} = \frac{1}{-9}\begin{bmatrix} 1 & -6 \\ -2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}, \quad adj\ B = \begin{bmatrix} 4 & -2 \\ -5 & 4 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix} = 16-10 = 6$$

Now $$B^{-1} = \frac{adj\ B}{|B|}$$

$$B^{-1} = \frac{1}{6}\begin{bmatrix} 4 & -2 \\ -5 & 4 \end{bmatrix}$$

Consider $$B^{-1}A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & -2 \\ -5 & 4 \end{bmatrix}\left(\frac{1}{-9}\right)\begin{bmatrix} 1 & -6 \\ -2 & 3 \end{bmatrix}$$

$$= -\frac{1}{54}\begin{bmatrix} 4 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 1 & -6 \\ -2 & 3 \end{bmatrix}$$

$$= -\frac{1}{54}\begin{bmatrix} 4+4 & -24-6 \\ -5-8 & 30+12 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{-1}{54}\begin{bmatrix} 8 & -30 \\ -13 & 42 \end{bmatrix} \text{ ....(ii)}$$

From (i) and (ii)

$(AB)^{-1} = B^{-1}A^{-1}$

EXERCISE – 6.5

1- Find the determinants of the following matrices.

(i) $\begin{bmatrix} u & v \\ x & y \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 5 \\ 1 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} -8 & -4 \\ -4 & -2 \end{bmatrix}$ (iv) $\begin{bmatrix} \frac{1}{1} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$

2. Identify the singular and non-singular matrices.

(i) $\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 8 \\ 4 & 9 \end{bmatrix}$ (iii) $\begin{bmatrix} -a & b \\ a & b \end{bmatrix}$

3. Find the inverse of each matrix A and show that $A^{-1}A = I$.
If the inverse does not exist, give reason.

(i) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}$ (vi) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(vii) $\begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

4. Let $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(a) Find M^{-1}

(b) Verify that $M^{-1}M = MM^{-1}$

5. If $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

6.6 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

To determine the value of two variables, we need a pair of equations. Such a pair of equations is called a system of simultaneous linear equations.

1- *The technique of solving a pair of simultaneous equations by*
   (a) *Matrix Inversion Method*
   (b) *Cramer's Rule*

2- *To apply the technique to solve some practical problems.*

6.6.1 Matrix Inversion Method

Let $a_1x + a_2y = b_1$ ....*(i)*
and $a_3x + a_4y = b_2$ ....*(ii)*

be the two simultaneous linear equations. These equations can be written in matrix form as:

$$\begin{bmatrix} a_1x + a_2y \\ a_3x + a_4y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

i.e $$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

or $AX = B$ ....*(iii)*

where $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

To find values of the variable x and y, the equation *(iii)* is solved by the following method.

$$AX = B$$

If A has an inverse A^{-1},

then $A^{-1}AX = A^{-1}B \qquad (\because A^{-1}A = I)$

$$IX = A^{-1}B \qquad (\because IX = X)$$

$$X = \frac{adj\ A}{|A|}B \qquad \text{provided } |A| \neq 0$$

In case A is singular $(|A| = 0)$, then it is not possible to find the solution of the given equations.

EXAMPLE

Solve the following set of equations using the matrix inversion method. $3x - 4y = 7, \quad 5x - 7y = 12$

SOLUTION:

The given simultaneous equations may be written in matrix form as:

$$\begin{bmatrix} 3 & -4 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

$$AX = B$$

Here $A = \begin{bmatrix} 3 & -4 \\ 5 & -7 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -4 \\ 5 & -7 \end{vmatrix} = -21 + 20 = -1$$

$$|A| = -1 \neq 0$$

As A is non-singular matrix, so the equations can be solved.

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1} \begin{bmatrix} -7 & 4 \\ -5 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7 & -4 \\ 5 & -3 \end{bmatrix}$$

But $X = A^{-1}B$

$$X = \begin{bmatrix} 7 & -4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

$$X = \begin{bmatrix} 49 - 48 \\ 35 - 36 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus $x = 1$ and $y = -1$.

$\therefore$ *Solution set* $= \{(1, -1)\}$

Cramer's Rule:

Simultaneous linear equations can be solved by Cramer's rule. The method to solve linear equations by Cramer's rule is explained below. Consider the linear equations.

$$a_1x + a_2y = b_1$$
$$a_3x + a_4y = b_2$$

In matrix form

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$AX = B$$

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}, \text{ provided} |A| \neq 0$$

$$|D_1| = \begin{vmatrix} b_1 & a_2 \\ b_2 & a_4 \end{vmatrix}$$

$$|D_2| = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_2 \end{vmatrix}$$

Now $x = \dfrac{|D_1|}{|A|}$ and $y = \dfrac{|D_2|}{|A|}$

EXAMPLE-1

Use Cramer's rule to solve the following linear equations.
$x + 3y = 6$, $2x + y = 4$

SOLUTION: $x + 3y = 6$, $2x + y = 4$ *in matrix form:*

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$AX = B$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1-6 = -5 \neq 0$$

Consider $$|D_1| = \begin{vmatrix} 6 & 3 \\ 4 & 1 \end{vmatrix} = 6-12 = -6$$

$$|D_2| = \begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} = 4-12 = -8$$

$$\therefore\ x = \frac{|D_1|}{|A|} = \frac{-6}{-5} \quad and \quad y = \frac{|D_2|}{|A|} = \frac{-8}{-5}$$

$$x = \frac{6}{5} \quad , \quad y = \frac{8}{5}$$ $\therefore$ Solution set $= \left\{\left(\frac{6}{5}, \frac{8}{5}\right)\right\}$

EXAMPLE-2

7 apples and 4 pears cost Rs. 11 while the 5 apples and 2 pears cost Rs. 7. How much each apple and pear cost ?

SOLUTION: We denote apple by 'x' and pear by 'y'

then $7x + 4y = 11$

$5x + 2y = 7$

In matrix form

$$\begin{bmatrix} 7 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$$

$$AX = B$$

$$|A| = \begin{vmatrix} 7 & 4 \\ 5 & 2 \end{vmatrix} = 14-20 = -6 \neq 0$$

$$|D_1| = \begin{vmatrix} 11 & 4 \\ 7 & 2 \end{vmatrix} = 22-28 = -6$$

$$|D_2| = \begin{vmatrix} 7 & 11 \\ 5 & 7 \end{vmatrix} = 49-55 = -6$$

$$x = \frac{|D_1|}{|A|} = \frac{-6}{-6} = 1$$, one apple costs Rs 1.

$$y = \frac{|D_2|}{|A|} = \frac{-6}{-6} = 1$$,one pear costs Rs 1.

EXAMPLE-3

Find two numbers whose sum is 67 and difference is 3.

SOLUTION:

Let x and y be two numbers and also $x > y$

$x + y = 67$ ________ (i)

$x - y = 3$ ________ (ii)

In matrix form: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 67 \\ 3 \end{bmatrix}$

$$AX = B$$

$$|A| = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0$$

$$|D_1| = \begin{vmatrix} 67 & 1 \\ 3 & -1 \end{vmatrix} = -67 - 3 = -70$$

$$|D_2| = \begin{vmatrix} 1 & 67 \\ 1 & 3 \end{vmatrix} = 3 - 67 = -64$$

$$x = \frac{|D_1|}{|A|} = \frac{-70}{-2} = 35$$

$$y = \frac{|D_2|}{|A|} = \frac{-64}{-2} = 32$$

$$x = 35 \qquad y = 32$$

$\therefore$ the required numbers are 35 and 32.

EXAMPLE-4

A belt and a wallet cost Rs. 42, while 7 belts and 4 wallets cost Rs. 213. Calculate the cost of each item.

SOLUTION:

Let the cost of a belt and wallet is denoted by x and y respectively

$x + y = 42$ ________ (i)

$7x + 4y = 213$ ________ (ii)

In matrix form: $\begin{bmatrix} 1 & 1 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 42 \\ 213 \end{bmatrix}$

$$AX = B$$

$$|A| = \begin{vmatrix} 1 & 1 \\ 7 & 4 \end{vmatrix} = 4-7 = -3 \neq 0$$

$$adj\ A = \begin{bmatrix} 4 & -1 \\ -7 & 1 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$X = \left(\frac{adj\ A}{|A|}\right) B$$

$$= \frac{1}{-3}\begin{bmatrix} 4 & -1 \\ -7 & 1 \end{bmatrix}\begin{bmatrix} 42 \\ 213 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 168 - 213 \\ -294 + 213 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -45 \\ -81 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 27 \end{bmatrix}$$

$x = 15$, $y = 27$

∴ the cost of a belt and wallet is Rs. 15 and Rs. 27 respectively.

EXERCISE – 6.6

1- Write the equation $2x + ky = 7$ and $4x - 9y = 4$ in matrix form. Also find the value of k if the matrix of the coefficients is singular.

2. Solve the simultaneous equations by the matrix inversion method where possible. Where there is no solution, explain why this is so.

(i) $2x - 5y = 1$, $3x - 7y = 2$

(ii) $3x + 2y = 10$, $2y - 3x = -4$

(iii) $4x + 5y = 0$, $2x + 5y = 1$

(iv) $5x + 6y = 25$, $3x + 4y = 17$

(v) $x + y = 2$, $y = 2 + x$

(vi) $\frac{x}{2} + \frac{y}{3} = 1$, $-4x + y = 14$

3. Solve, using matrix inversion method

$$3x - y = 10$$
$$2x + 3y = 3$$

4. Use Cramer's rule to solve the simultaneous equations. Give the reason where solution is not possible.

(i) $x + 2y = 3$, $x + 3y = 5$

(ii) $2x + y = 1$, $5x + 3y = 2$

(iii) $x + 3y = 1$, $2x + 8y = 0$

(iv) $-2x + 6y = 5$, $x - 3y = -7$

(v) $x - 3y = 5$, $2x - 5y = 9$

(vi) $5x + 2y = 13$, $2x + 5y = 17$

5. Write the following matrices in the form of linear equations.

(i) $\begin{bmatrix} 2 & -1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(ii) $\begin{bmatrix} -5 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(iii) $\begin{bmatrix} -4 & 1 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(iv) $\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Review Exercise-6

circle the Correct Answer.

he number of rows and columns in a matrix determine its:

order (b) rows

columns (d) determinant

matrix consisting of one row is called a:

row matrix (b) column matrix

identity matrix (d) scalar matrix

o matrices are conformable for addition, if they are of

the same order (b) the different order

the order 2×2 (d) order 3×3

uare matrix the number of rows and columns is:

(b) 3×2

(d) 2×1

the same order and equal corresponding

(b) diagonal matrices

(d) unequal matrices

s are:

9. In matrices $(A\,B)^t = ?$

(a) A (b) B

(c) $B^t\ A^t$ (d) $A^t\ B^t$

10. In matrices $(A\,B)^{-1} = ?$

(a) A^{-1} (b) B^{-1}

(c) $B^{-1}\ A^{-1}$ (d) $A^{-1}\ B^{-1}$

II- **Fill in the blanks.**

1. The number of rows and columns in a matrix determine its ________ .

2. A matrix consisting of one row only is called a ____________ .

3. Two matrices are conformable for addition, if they are of the __________ .

4. In a square matrix the number of rows and columns is__________ .

5. Two matrices of the same order are ____________ if their corresponding elements are same.

6. In a unit matrix the diagonal elements are____________ .

7. $(AB)C = A(BC)$, where A,B and C are matrices is called ____________ under multiplication.

8. If $A^t = -A$, then the matrix A is said to be ____________ .

9. In matrices $(AB)^t =$ ____________ .

10. In matrices $(AB)^{-1} =$ ____________ .

SUMMARY

Matrix: A rectangular array of numbers, enclosed by a pair of brackets and subject to certain rules is called a matrix.

Order of a matrix: The number of rows and columns in a matrix determine its order.

Row matrix: A matrix consisting of one row only is called a row matrix.

Column matrix: A matrix consisting of one column only is called a column matrix.

Square matrix: In a square matrix, the number of rows and columns are equal.

Rectangular matrix: In a rectangular matrix, number of rows and columns are not same.

Zero or null matrix: If all elements in a matrix are zero, the matrix is called a zero or null matrix.

Unit or Identity matrix: In an identity matrix, the diagonal elements are unity and off diagonal elements are all zero.

Transpose of a matrix: A matrix obtained by interchanging rows into columns is called transpose of a matrix.

Symmetric matrix: A matrix A is said to be symmetric, if $A' = A$.

Skew-Symmetric matrix : A matrix A is said to be skew-symmetric, if $A' = -A$.

Determinant: A real number associated with a square matrix is called determinant of a square matrix.

Singular matrix: If the determinant of a square matrix is zero, it is called a singular matrix, otherwise non-singular matrix.

Adjoint of a square matrix of order 2×2
In the adjoint of a square matrix of order 2×2 the diagonal elements are interchanged, whereas the sign of other diagonal elements are changed.
Multiplicative inverse of a square matrix,
A matrix B is said to be multiplicative inverse of 'A', if $AB = I$.