CHAPTER

1 Number Systems

Animation 1.1: Complex Plane
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1.1 Introduction

In the very beginning, human life was simple. An early ancient herdsman compared sheep (or cattle) of his herd with a pile of stones when the herd left for grazing and again on its return for missing animals. In the earliest systems probably the vertical strokes or bars such as I, II, III, llll etc.. were used for the numbers 1, 2, 3, 4 etc. The symbol "lllll" was used by many people including the ancient Egyptians for the number of fingers of one hand.

Around 5000 B.C, the Egyptians had a number system based on 10. The symbol 𓎆 for 10 and 𓍢 for 100 were used by them. A symbol was repeated as many times as it was needed. For example, the numbers 13 and 324 were symbolized as 𓎆III and 𓍢𓍢𓍢𓎆𓎆𓏺𓏺𓏺𓏺 respectively. The symbol 𓍢𓍢𓍢𓎆𓎆𓏺𓏺𓏺𓏺 was interpreted as 100 + 100 +100+10+10+1+1+1 +1. Different people invented their own symbols for numbers. But these systems of notations proved to be inadequate with advancement of societies and were discarded. Ultimately the set {1, 2, 3, 4, ...} with base 10 was adopted as the counting set (also called the set of natural numbers). The solution of the equation $x + 2 = 2$ was not possible in the set of natural numbers, So the natural number system was extended to the set of whole numbers. No number in the set of whole numbers W could satisfy the equation $x + 4 = 2$ or $x + a = b$, if $a > b$, and $a, b, \in W$. The negative integers –1, –2, –3, ... were introduced to form the set of integers Z = {0, ±1, ±2 ,...).

Again the equation of the type $2x = 3$ or $bx = a$ where $a,b,\in Z$ and $b \neq 0$ had no solution in the set Z, so the numbers of the form $\frac{a}{b}$ where $a,b,\in Z$ and $b \neq 0$, were invented to remove such difficulties. The set $Q = \{\frac{a}{b} \mid a,b,\in Z \wedge b \neq 0\}$ was named as the set of rational numbers. Still the solution of equations such as $x^2 = 2$ or $x^2 = a$ (where a is not a perfect square) was not possible in the set Q. So the irrational numbers of the type $\pm\sqrt{2}$ or $\pm\sqrt{a}$ where a is not a perfect square were introduced. This process of enlargement of the number system ultimately led to the set of real numbers $\Re = Q \cup Q'$ (Q' is the set of irrational numbers) which is used most frequently in everyday life.

1.2 Rational Numbers and Irrational Numbers

We know that a rational number is a number which can be put in the form $\frac{p}{q}$ where $p, q \in Z \wedge q \neq 0$. The numbers $\sqrt{16}$, 3.7, 4 etc., are rational numbers. $\sqrt{16}$ can be reduced to the form $\frac{p}{q}$ where $p, q \in Z$, and $q \neq 0$ because $\sqrt{16} = 4 = \frac{4}{1}$.

Irrational numbers are those numbers which cannot be put into the form $\frac{p}{q}$ where $p, q \in Z$ and $q \neq 0$. The numbers $\sqrt{2}$, $\sqrt{3}$, $\frac{7}{\sqrt{5}}$, $\sqrt{\frac{5}{16}}$ are irrational numbers.

1.2.1 Decimal Representation of Rational and Irrational Numbers

1) Terminating decimals: A decimal which has only a finite number of digits in its decimal part, is called a terminating decimal. Thus 202.04, 0.0000415, 100000.41237895 are examples of terminating decimals.

Since a terminating decimal can be converted into a common fraction, so every terminating decimal represents a **rational number**.

2) Recurring Decimals: This is another type of **rational numbers**. In general, a recurring or periodic decimal is a decimal in which one or more digits repeat indefinitely.

It will be shown (in the chapter on sequences and series) that a recurring decimal can be converted into a common fraction. So **every recurring decimal represents a rational number:**

A non-terminating, non-recurring decimal is a decimal which neither terminates nor it is recurring. It is not possible to convert such a decimal into a common fraction. Thus a **non-terminating, non-recurring decimal represents an irrational number.**

Example 1:

i) .25 $\left(=\frac{25}{100}\right)$ is a rational number.

ii) .333...$\left(=\frac{1}{3}\right)$ is a recurring decimal, it is a rational number.

iii) $2.\overline{3}(=2.333...)$ is a rational number.

iv) 0.142857142857... $\left(=\frac{1}{7}\right)$ is a rational number.

v) 0.01001000100001 ... is a non-terminating, non-periodic decimal, so it is an irrational number.

vi) 214.121122111222 1111 2222 ... is also an irrational number.

vii) 1.4142135 ... is an irrational number.

viii) 7.3205080 ... is an irrational number.

ix) 1.709975947 ... is an irrational number.

x) 3.141592654... is an important irrational number called it π(Pi) which denotes the constant ratio of the circumference of any circle to the length of its diameter i.e.,

$$\pi = \frac{\text{circumference of any circle}}{\text{length of its diameter.}}$$

An approximate value of π is $\frac{22}{7}$,a better approximation is $\frac{355}{113}$ and a still better approximation is 3.14159. The value of π correct to 5 lac decimal places has been determined with the help of computer.

Example 2: Prove $\sqrt{2}$ is an irrational number.

Solution: Suppose, if possible, $\sqrt{2}$ is rational so that it can be written in the form p/q where $p,q \in$ Z and $q \neq 0$. Suppose further that p/q is in its lowest form.

Then $\sqrt{2} = p/q$, $(q \neq 0)$

Squaring both sides we get;

$$2 = \frac{p^2}{q^2} \text{ or } p^2 = 2q^2 \qquad (1)$$

The R.H.S. of this equation has a factor 2. Its L.H.S. must have the same factor.

Now a prime number can be a factor of a square only if it occurs at least twice in the square. Therefore, p^2 should be of the form $4p'^2$ so that equation (1) takes the form:

$$4p'^2 = 2q^2 \qquad ....(2)$$

i.e., $$2p'^2 = q^2 \qquad ....(3)$$

In the last equation, 2 is a factor of the L.H.S. Therefore, q^2 should be of the form $4q'^2$ so that equation 3 takes the form

$$2p'^2 = 4q'^2 \quad \text{i.e., } p'^2 = 2q'^2 \qquad ....(4)$$

From equations (1) and (2),

$$p = 2p'$$

and from equations (3) and (4)

$$q = 2q'$$

$$\therefore \quad \frac{p}{q} = \frac{2p'}{2q'}$$

This contradicts the hypothesis that $\frac{p}{q}$ is in its lowest form. Hence $\sqrt{2}$ is irrational.

Example 3: Prove $\sqrt{3}$ is an irrational number.

Solution: Suppose, if possible $\sqrt{3}$ is rational so that it can be written in the form p/q when $p,q \in Z$ and $q \neq 0$. Suppose further that p/q is in its lowest form,

then $\sqrt{3} = p/q$, $(q \neq 0)$

Squaring this equation we get;

$$3 = \frac{p^2}{q^2} \qquad \text{or } p^2 = 3q^2 \qquad ....(1)$$

The R.H.S. of this equation has a factor 3. Its L.H.S. must have the same factor.

Now a prime number can be a factor of a square only if it occurs at least twice in the square. Therefore, p^2 should be of the form $9p'^2$ so that equation (1) takes the form:

$$9p'^2 = 3q^2 \qquad (2)$$

i.e., $$3p'^2 = q^2 \qquad (3)$$

In the last equation, 3 is a factor of the L.H.S. Therefore, q^2 should be of the form $9q'^2$ so that equation (3) takes the form

$$3p'^2 = 9q'^2 \text{ i.e., } p'^2 = 3q'^2 \qquad (4)$$

From equations (1) and (2),

$$P = 3P'$$

and from equations (3) and (4)

$$q = 3q'$$

$$\therefore \quad \frac{p}{q} = \frac{3p'}{3q'}$$

This contradicts the hypothesis that $\frac{p}{q}$ is in its lowest form.

Hence $\sqrt{3}$ is irrational.

Note: Using the same method we can prove the irrationality of $\sqrt{5}, \sqrt{7}, \ldots, \sqrt{n}$ where n is any prime number.

1.3 Properties of Real Numbers

We are already familiar with the set of real numbers and most of their properties. We now state them in a unified and systematic manner. Before stating them we give a preliminary definition.

Binary Operation: A binary operation may be defined as a function from A $\times$ A into A, but for the present discussion, the following definition would serve the purpose. A *binary operation* in a set A is a rule usually denoted by * that assigns to any pair of elements of A, taken in a definite order, another element of A.

Two important binary operations are addition and multiplication in the set of real numbers. Similarly, union and intersection are binary operations on sets which are subsets of the

same Universal set.

$\Re$ usually denotes the set of real numbers. We assume that two binary operations addition (+) and multiplication (. or x) are defined in $\Re$. Following are the properties or laws for real numbers.

1. Addition Laws: -

i) Closure Law of Addition

$\forall \; a, b \in \Re, \; a + b \in \Re$ ($\forall$ stands for "for all")

ii) Associative Law of Addition

$\forall \; a, b, c \in \Re, \; a + (b + c) = (a + b) + c$

iii) Additive Identity

$\forall \; a \in \Re, \; \exists \; 0 \in \Re$ such that $a + 0 = 0 + a = a$

($\exists$ stands for "there exists").

0(read as zero) is called the identity element of addition.

iv) Additive Inverse

$\forall \; a \in \Re, \; \exists \; (-a) \in \Re$ such that

$a + (-a) = 0 = (-a) + a$

v) Commutative Law for Addition

$\forall \; a, b \in \Re, \; a + b = b + a$

2. Multiplication Laws

vi) Closure I.aw of Multiplication

$\forall \; a, b \in \Re, \; a.b \in \Re$ (a,b is usually written as ab).

vii) Associative Law for Multiplication

$\forall \; a, b, c \in \Re, \; a(bc) = (ab)c$

viii) Multiplicative Identity

$\forall \; a \in \Re, \; \exists \; 1 \in \Re$ such that $a.1 = 1.a = a$

1 is called the multiplicative identity of real numbers.

ix) Multiplicative Inverse

$\forall \; a (\neq 0) \in \Re, \; \exists \; a^{-1} \in \Re$ such that $a.a^{-1} = a^{-1}.a = 1$ (a^{-1} is also written as $\frac{1}{a}$).

x) Commutative Law of multiplication

$\forall \; a, b \in \Re, \; ab = ba$

3. Multiplication – Addition Law

xi) $\forall$ ***a, b, c*** $\in \Re$,

$a(b + c) = ab + ac$ (Distrihutivity of multiplication over addition).

$(a + b)c = ac + bc$

In addition to the above properties $\Re$ possesses the following properties.

i) Order Properties (described below).

ii) Completeness axiom which will be explained in higher classes.

The above properties characterizes $\Re$ i.e., only $\Re$ possesses all these properties.

Before stating the order axioms we state the properties of equality of numbers.

4. Properties of Equality

Equality of numbers denoted by "=" possesses the following properties:-

i) Reflexive property $\forall\ a\in\Re,\ a = a$

ii) Symmetric Property $\forall\ a,b\in\Re,\ a = b \Rightarrow b = a$.

iii) Transitive Property $\forall\ a,b,c\in\Re,\ a = b \wedge b = c \Rightarrow a = c$

iv) Additive Property $\forall\ a,b,c\in\Re,\ a = b \Rightarrow a + c = b + c$

v) Multiplicative Property $\forall\ a,b,c\in\Re,\ a = b \Rightarrow ac = bc \wedge ca = cb$.

vi) Cancellation Property w.r.t. addition

$\forall\ a,b,c\in\Re,\ a + c = b + c \Rightarrow a = b$

vii) Cancellation Property w.r.t. Multiplication:

$\forall\ a,b,c\in\Re,\ ac = bc \Rightarrow a = b,\ c \neq 0$

5. Properties of Ineualities (Order properties)

1) Trichotomy Property $\forall\ a,b\in\Re$

either $a = b$ or $a > b$ or $a < b$

2) Transitive Property $\forall\ a,b,c\in\Re$

i) $a > b \wedge b > c \Rightarrow a > c$ ii) $a < b \wedge b < c \Rightarrow a < c$

3) Additive Property: $\forall\ a,b,c\in\Re$

a) i) $a > b \Rightarrow a + c > b + c$ b) i) $a > b \wedge c > d \Rightarrow a+c > b+d$

ii) $a < b \Rightarrow a + c < b + c$ ii) $a < b \wedge c < d \Rightarrow a+c < b+d$

4) Multiplicative Properties:

a) $\forall\ a,b,c\in\Re$ and $c > 0$

i) $a > b \Rightarrow ac > bc$ ii) $a < b \Rightarrow ac < bc$.

b) $\forall\ a,b,c\in\Re$ and $c < 0$.

i) $a > b \Rightarrow ac < bc$ ii) $a < b \Rightarrow ac > bc$

c) $\forall\ a,b,c,d\in\Re$ and a,b,c,d are all positive.

i) $a > b \wedge c > d \Rightarrow ac > bd$. ii) $a < b \wedge c < d \Rightarrow ac < bd$

Note That:

1. Any set possessing all the above 11 properties is called a field.
2. From the multiplicative properties of inequality we conclude that: - If both the sides of an inequality are multiplied by a +ve number, its direction does not change, but multiplication of the two sides by -ve number reverses the direction of the inequality.
3. a and $(-a)$ are additive inverses of each other. Since by definition inverse of $-a$ is a,

$$\therefore\ -(-a) = a$$

4. The left hand member of the above equation should be read as negative of 'negative a' and not 'minus minus a'.
5. a and $\frac{1}{a}$ are the multiplicative inverses of each other. Since by definition inverse of $\frac{1}{a}$ is a (i.e., inverse of a^{-1} is a), $a\neq 0$

$$\therefore\ (a^{-1})^{-1} = a \text{ or } \frac{1}{\frac{1}{a}} = a$$

Example 4: Prove that for any real numbers a, b

i) $a.0 = 0$ ii) $ab = 0 \Rightarrow a = 0 \vee b = 0$ [$\vee$ stands for "or"]

Solution: i) $a.0 = a[1+(-1)]$ (Property of additive inverse)

$= a(1-1)$ (Def. of subtraction)

$= a.1 - a.1$ (Distributive Law)

$= a - a$ (Property of multiplicative identity)

$= a + (-a)$ (Def. of subtraction)

$= 0$ (Property of additive inverse)

Thus $a.0=0$.

ii) Given that $ab = 0$ (1)

Suppose $a \neq 0$, then exists

(1) gives: $\frac{1}{a}(ab) = \frac{1}{a}.0$ (Multiplicative property of equality)

$\Rightarrow (\frac{1}{a}.a)b = \frac{1}{a}.0$ (Assoc. law of $\times$)

$\Rightarrow 1.b = 0$ (Property of multiplicative inverse).

$\Rightarrow b = 0$ (Property of multiplicative identity).

Thus if $ab = 0$ and $a \neq 0$, then $b = 0$

Similarly it may be shown that

if $ab = 0$ and $b \neq 0$, then $a = 0$.

Hence $ab = 0 \Rightarrow a = 0$ or $b = 0$.

Example 5: For real numbers a,b show the following by stating the properties used.

i) $(-a)\,b = a\,(-b) = -ab$ ii) $(-a)\,(-b) = ab$

Solution: i) $(-a)(b) + ab = (-a + a)b$ (Distributive law)

$= 0.b = 0.$ (Property of additive inverse)

$\therefore\ (-a)b + ab = 0$

i.e.. $(-a)b$ and ab are additive inverse of each other.

$\therefore (-a)b = -(ab) = -ab$ (Θ $-(ab)$ is written as $-ab$)

ii) $(-a)\,(-b) - ab = (-a)(-b) + (-ab)$

$= (-a)(-b) + (-a)(b)$ (By (i))

$= (-a)(-b + b)$ (Distributive law)

$= (-a)\,.0 = 0.$ (Property of additive inverse)

$(-a)(-b) = ab$

Example 6: Prove that

i) $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$ (Principle for equality of fractions

ii) $\frac{1}{a}\cdot\frac{1}{b} = \frac{1}{ab}$

iii) $\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$ (Rule for product of fractions).

iv) $\frac{a}{b} = \frac{ka}{kb}, (k \neq 0)$ (Golden rule of fractions)

v) $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$ (Rule for quotient of fractions).

The symbol $\Leftrightarrow$ stands for iff i.e.. if and only if.

Solution:

i) $\frac{a}{b} \Rightarrow \frac{c}{d} \quad \frac{a}{b}(bd) \quad \frac{c}{d}(bd)$

$\Rightarrow \frac{a.1}{b}(bd) = \frac{c.1}{d}(bd)$

$\Rightarrow a.(\frac{1}{b}.b).d = c.(\frac{1}{d}.bd)$

$= c(bd.\frac{1}{d})$

$\Rightarrow ad = cb$

$\therefore\ ad = bc$

Again $ad = bc \Rightarrow (ad) \times \frac{1}{b}.\frac{1}{d} = b.c.\frac{1}{b}.\frac{1}{d}$

$\Rightarrow a.\frac{1}{b}.d\frac{1}{d} = b.\frac{1}{b}\ .\ c.\frac{1}{d}$

$\Rightarrow \frac{a}{b} = \frac{c}{d}.$

ii) $(ab).\frac{1}{a}.\frac{1}{b} = (a.\frac{1}{a}).(b\frac{1}{b}) = 1.1 = 1$

Thus ab and $\frac{1}{a}.\frac{1}{b}$ are the multiplicative inverse of each other. But multiplicative inverse of ab is $\frac{1}{ab}$

$\therefore \frac{1}{ab} = \frac{1}{a}.\frac{1}{b}.$

iii) $\frac{a}{b}.\frac{c}{d} = (a.\frac{1}{b}).(c.\frac{1}{d})$

$$=(ac)(\frac{1}{b}.\frac{1}{d})$$ (Using commutative and associative laws of multiplication)

$$=ac.\frac{1}{bd}=\frac{ac}{bd}.$$
$$=\frac{a}{b}.\frac{c}{d}=\left|\frac{ac}{bd}\right|$$

iv) $\frac{a}{b}=\frac{a}{b}.1=\frac{a}{b}.\frac{k}{k}=\frac{ak}{ak}$

$\therefore \frac{a}{b}=\frac{ak}{bk}.$

v) $\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{\frac{a}{b}(bd)}{\frac{c}{d}(bd)}=\frac{ad(\frac{1}{b}.b)}{cb(\frac{1}{d}.d)}=\frac{ad}{bc}.$

Example 7: Does the set {1, –1 } possess closure property with respect to
i) addition ii) multiplication?

Solution: i) 1 + 1 = 2, 1 + (–1) = 0 = –1 + 1
–1 + (–1) = –2

But 2, 0, –2 do not belong to the given set. That is, all the sums do not belong to the given set. So it does not possess closure property w.r.t. addition.

ii) 1.1= 1, 1. (–1) = –1, (–1) .1 = –1, (–1). (–1) = 1

Since all the products belong to the given set, it is closed w.r.t multiplication.

Exercise 1.1

1. Which of the following sets have closure property w.r.t. addition and multiplication?

i) {0} ii) {1} iii) (0 , – 1) iv) { 1, –1 }

2. Name the properties used in the following equations.
(Letters, where used, represent real numbers).

i) 4 + 9 = 9 + 4 ii) $(a+1)+\frac{3}{4}=a+(1+\frac{3}{4})$

iii) $(\sqrt{3}+\sqrt{5})+\sqrt{7}=\sqrt{3}+(\sqrt{5}+\sqrt{7})$ iv) 100 + 0 = 100

v) $1000 \times 1 = 1000$ vi) 4.1 + (– 4 .1) = 0

vii) $a - a = 0$ viii)

ix) $a(b - c) = ab - ac$ x) $(x - y)z = xz - yz$

xi) $4\times(5 \times 8) = (4 \times 5) \times 8$ xii) $a(b + c - d) = ab + ac - ad$.

3. Name the properties used in the following inequalities:

i) $-3 < -2 \Rightarrow 0 < 1$ ii) $-5 < -4 \Rightarrow 20 > 16$

iii) $1 > -1 \Rightarrow -3 > -5$ iv) $a < 0 \Rightarrow -a > 0$

v) $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$ vi) $a > b \Rightarrow -a < -b$

4. Prove the following rules of addition: -

i) $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ ii) $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

5. Prove that $-\frac{7}{12}-\frac{5}{18}=\frac{-21-10}{36}$

6. Simplify by justifying each step: -

i) $\frac{4+16x}{4}$ ii) $\frac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}$

iii) $\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}$ iv) $\dfrac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{a}.\frac{1}{b}}$

1.4 Complex Numbers

The history of mathematics shows that man has been developing and enlarging his concept of **number** according to the saying that "Necessity is the mother of invention". In the remote past they stared with the set of counting numbers and invented, by stages, the negative numbers, rational numbers, irrational numbers. Since square of a positive as well as negative number is a positive number, the square root of a negative number does not exist in the realm of real numbers. Therefore, square roots of negative numbers were given no attention for centuries together. However, recently, properties of numbers involving square roots of negative numbers have also been discussed in detail and such numbers have been found useful and have been applied in many branches o f pure and applied mathematics. The numbers of the form $x + iy$, where $x, y \in \Re$, and i = ,are called **complex numbers,** here x is called **real part** and y is called **imaginary part** of the complex

number. For example, 3 + 4i, 2 – i etc. are complex numbers.

Note: Every real number is a complex number with 0 as its imaginary part.

Let us start with considering the equation.

$$x^2 + 1 = 0 \qquad (1)$$

$$\Rightarrow \quad x^2 = -1$$

$$\Rightarrow \quad x = \pm\sqrt{-1}$$

$\sqrt{-1}$ does not belong to the set of real numbers. We, therefore, for convenience call it **imaginary number** and denote it by i (read as iota).

The product of a real number and i is also an **imaginary number**

Thus $2i$, $-3i$, $\sqrt{5i}$, $-\frac{11}{2}i$ are all imaginary numbers, i which may be written $1.i$ is also an imaginary number.

Powers of i :

$$i^2 = -1 \text{ (by defination)}$$

$$i^3 = i^2.i = -1.i = -i$$

$$i^4 = i^2 \times i^2 = (-1)(-1) = 1$$

Thus any power of i must be equal to 1, i,–1 or $-i$. For instance,

$$i^{13} = (i^2)^6.^2 = (-1)^6.i = i$$

$$i^6 = (i^2)^3 = (-1)^3 = -1 \text{ etc.}$$

1.4.1 Operations on Complex Numbers

With a view to develop algebra of **complex numbers**, we state a few definitions. The symbols a,b,c,d,k, where used, represent real numbers.

1) $a + bi = c + di \Rightarrow a = c \;\; b = d$.

2) Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$

3) $k(a + bi) = ka + kbi$

4) $(a + bi) - (c + di) = (a + bi) + [-(c + di)]$
$= a + bi + (-c - di)$
$= (a - c) + (b - d)i$

5) $(a + bi).(c + di) = ac + adi + bci + bdi = (ac - bd) + (ad + bc)i$.

6) Conjugate Complex Numbers: Complex numbers of the form (a + bi) and (a – bi) which have the same real parts and whose imaginary parts differ in sign only, are called conjugates of each other. Thus 5 + 4i and 5 – 4i, –2 + 3i and – 2 – 3i,– $\sqrt{5}\,i$ and $\sqrt{5}\,i$ are three pairs of conjugate numbers.

Note: A real number is self-conjugate.

1.4.2 Complex Numbers as Ordered Pairs of Real Numbers

We can define complex numbers also by using ordered pairs. Let C be the set of ordered pairs belonging to $\Re \times \Re$ which are subject to the following properties: -

i) $(a,b)=(c,d) \Leftrightarrow a=c \wedge b=d.$

ii) $(a, b) + (c, d) = (a + c, b + d)$

iii) If k is any real number, then $k(a, b) = (ka, kb)$

iv) $(a, b)(c, d) = (ac - bd, ad + bc)$

Then C is called the set of *complex numbers*. It is easy to sec that $(a, b) - (c, d) = (a - c,\ b - d)$

Properties (1), (2) and (4) respectively define equality, sum and product of two complex numbers. Property (3) defines the product of a real number and a complex number.

Example 1: Find the sum, difference and product of the complex numbers (8, 9) and (5, –6)

Solution: Sum = (8 + 5, 9 – 6) = (13, 3)

Difference = (8 – 5, 9 – (–6)) = (3, 15)

Product = (8.5 – (9)(–6), 9.5 + (–6) 8)

= (40 + 54, 45 – 48)

= (94, –3)

1.4.3 Properties of the Fundamental Operations on Complex Numbers

It can be easily verified that the set C satisfies all the field axioms i.e., it possesses the properties 1(i to v), 2(vi to x) and 3(xi) of Art. 1.3.

By way of explanation of some points we observe as follows:-

i) The additive identity in C is (0, 0).

ii) Every complex number (a, b) has the additive inverse $(-a, -b)$ i.e., $(a, b) + (-a, -b) = (0, 0)$.

iii) The multiplicative identity is (1, 0) i.e.,
$(a, b).(1, 0) = (a.1 - b.0, b.1 + a.0) = (a, b)$.
$= (1, 0)(a, b)$

iv) Every non-zero complex number {i.e., number not equal to (0, 0)} has a multiplicative inverse.

The multiplicative inverse of (a, b) is $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

$$(a,b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1, 0), \text{ the identity element}$$

$$= \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)(a,b)$$

v) $(a,\ b)[(c,\ d) \pm (e,\ f)] = (a,\ b)(c,\ d) \pm (a,\ b)(e,\ f)$

Note: The set C of complex numbers does not satisfy the order axioms. In fact there is no sense in saying that one complex number is greater or less than another.

1.4.4 A Special Subset of *C*

We consider a subset of C whose elements are of the form $(a, 0)$ i.e., second component of each element is zero.

Let $(a, 0)$, (c, 0) be two elements of this subset. Then

i) $(a, 0) + (c, 0) = (a + c, 0)$ ii) $k(a, 0) = (ka, 0)$

iii) $(a, 0) \times (c, 0) = (ac, 0)$

iv) Multiplicative inverse of $(a, 0)$ is $\left(\frac{1}{a},\ 0\right),\ a \neq 0.$

Notice that the results are the same as we should have obtained if we had operated on the real numbers a and c ignoring the second component of each ordered pair i.e., 0 which has played no part in the above calculations.

On account of this special feature wc identify the complex number $(a, 0)$ with the real number a i.e., we postulate:

$(a, 0) = a$ (1)

Now consider (0, 1)

(0, 1) . (0, 1) = (–1, 0)

= –1 (by (1) above).

If we set $(0, 1) = i$ (2)

then $(0, 1)^2 = (0,1)(0,1) = i.i = i^2 = -1$

We are now in a position to write every complex number given as an ordered pair, in terms of i. For example

$(a, b) = (a, 0) + (0, b)$ (def. of addition)

$= a(1, 0) + b(0, 1)$ (by (1) and (2) above)

$= a.1 + bi$

$= a + ib$

Thus $(a, b) = a + ib$ where $i^2 = -1$

This result enables us to convert any Complex number given in one notation into the other.

Exercise 1.2

1. Verify the addition properties of complex numbers.

2. Verify the multiplication properties of the complex numbers.

3. Verify the distributive law of complex numbers.

$(a, b)[(c, d) + (e, f)] = (a, b)(c, d) + (a, b)(e, f)$

(**Hint:** Simplify each side separately)

4. Simplify' the following:

i) i^9 ii) i^{14} iii) $(-i)^{19}$ iv) $(-\frac{21}{2})$

5. Write in terms of i

i) $\sqrt{-1b}$ ii) $\sqrt{-5}$ iii) $\sqrt{\frac{-16}{25}}$ iv) $\sqrt{\frac{1}{-4}}$

Simplify the following:

6. $(7, 9) + (3, -5)$ **7.** $(8, -5) - (-7, 4)$ **8.** $(2, 6)(3, 7)$

9. $(5, -4)(-3, -2)$ **10.** $(0, 3)(0, 5)$ **11.** $(2, 6) \div (3, 7)$.

12. $(5, -4) \div (-3, -8)$ $\left(\text{Hint for 11: } \frac{(2,6)}{(3,7)} = \frac{2+6i}{3+7i} \times \frac{3-7i}{3-7i} \text{ etc.}\right)$

13. Prove that the sum as well as the product of any two conjugate complex numbers is a real number.

14. Find the multiplicative inverse of each of the following numbers:

i) $(-4, 7)$ ii) $\left(\sqrt{2}, -\sqrt{5}\right)$ iii) $(1, 0)$

15. Factorize the following:

i) $a^2 + 4b^2$ ii) $9a^2 + 16b^2$ iii) $3x^2 + 3y^2$

16. Separate into real and imaginary parts (write as a simple complex number): -

i) $\frac{2-7i}{4+5i}$ ii) $\frac{(-2+3i)^2}{(1+i)}$ iii) $\frac{i}{1+i}$

1.5 The Real Line

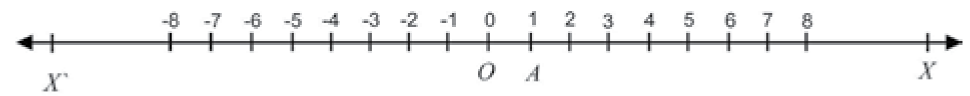

Fig. (1)

In Fig.(1), let $\overleftrightarrow{X'X}$ be a line. We represent the number 0 by a point *O* (called the origin) of the line. Let $|OA|$ represents a unit length. According to this unit, positive numbers are represented on this line by points to the right of *O* and negative numbers by points to the left of *O*. It is easy to visualize that all +ve and –ve rational numbers are represented on this line. What about the irrational numbers?

The fact is that all the irrational numbers are also represented by points of the line. Therefore, we postulate: -

Postulate: A (1 – 1) correspondence can be established between the points of a line ℓ and the real numbers in such a way that:-

i) The number 0 corresponds to a point *O* of the line.

ii) The number 1 corresponds to a point *A* of the line.

iii) If x_1, x_2 are the numbers corresponding to two points P_1, P_2, then the distance between P_1 and P_2 will be $|x_1 - x_2|$.

It is evident that the above correspondence will be such that corresponding to any real number there will be one and only one point on the line and vice versa.

When a (1 – 1) correspondence between the points of a line $x'x$ and the real numbers has been established in the manner described above, the line is called the **real line** and the real number, say *x*, corresponding to any point *P* of the line is called the **coordinate** of the point.

1.5.1 The Real Plane or The Coordinate Plane

We know that the *cartesian product* of two non-empty sets *A* and *B*, denoted by $A \times B$, is the set: $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$

The members of a *cartesian* product are **ordered pairs.**

The cartesian product $\Re\times\Re$ where $\Re$ is the set of real numbers is called the **cartesian plane**.

By taking two perpendicular lines $x'ox$ and $y'oy$ as coordinate axes on a geometrical plane and choosing a convenient unit of distance, elements of $\Re\times\Re$ can be represented on the plane in such a way that there is *a* (1–1) correspondence between the elements of $\Re\times\Re$ and points of the plane.

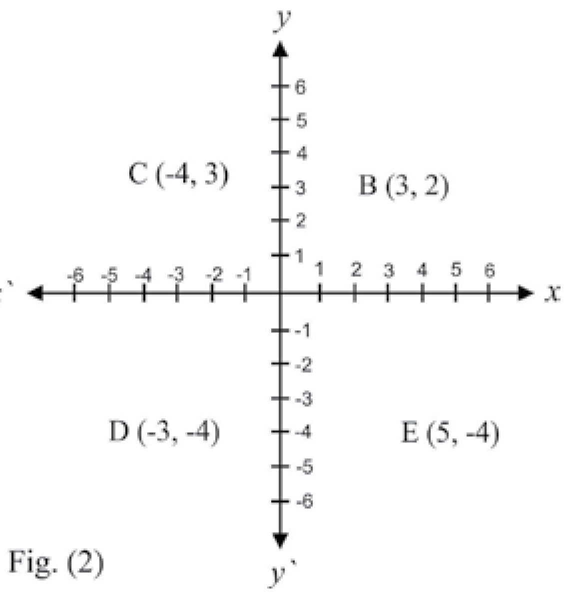

Fig. (2)

The geometrical plane on which coordinate system has been specified is called the **real plane** or the **coordinate plane**.

Ordinarily we do not distinguish between the Cartesian plane $\Re\times\Re$ and the coordinate plane whose points correspond to or represent the elements of $\Re\times\Re$.

If a point *A* of the coordinate plane corresponds to the ordered pair (*a*, *b*) then *a*, *b* are called the **coordinates** of *A*. *a* is called the *x* - coordinate or **abscissa** and *b* is called the y - coordinate or **ordinate**.

In the figure shown above, the coordinates of the points*B, C, D* and *E* are (3, 2), (–4, 3), (–3, –4) and (5, –4) respectively.

Corresponding to every ordered pair (*a, b*) $\in\Re\times\Re$ there is one and only one point in the plane and corresponding to every point in the plane there is one and only one ordered pair (*a, b*) in $\Re\times\Re$.

There is thus a (1 – 1) correspondence between $\Re\times\Re$ and the plane.

1.6 Geometrical Representation of Complex Numbers The Complex Plane

We have seen that there is a (1–1) correspondence between the elements (ordered pairs) of the Cartesian plane $\Re\times\Re$ and the complex numbers. Therefore, there is a (1– 1) correspondence between the points of the coordinate plane and the complex numbers. We can, therefore, represent complex numbers by points of the coordinate plane. In this representation every complex number will be represented by one and only one point of

the coordinate plane and every point of the plane will represent one and only one complex number. The components of the complex number will be the coordinates of the point representing it. In this representation the ***x*-axis** is called the real axis and the ***y*-axis** is called the **imaginary axis**. The coordinate plane itself is called the **complex plane** or **z – plane**.

By way of illustration a number of complex numbers have been shown in figure 3.

The figure representing one or more complex numbers on the *complex plane* is called an **Argand diagram**. Points on the **x-axis** represent **real numbers** whereas the points on the **y-axis** represent **imaginary numbers.**

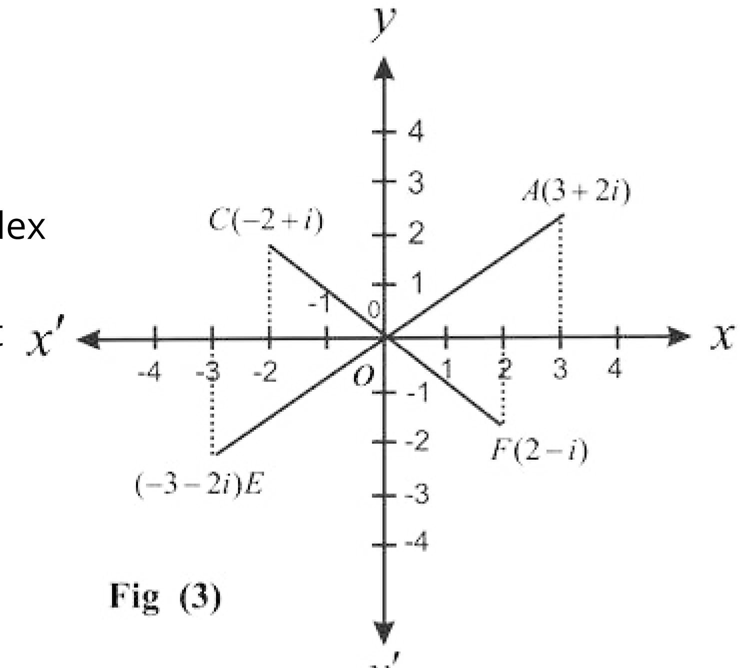

Fig (3)

In fig (4), *x*, *y* are the coordinates of a point.

It represents the complex number *x* + *iy*.

The *real number* $\sqrt{x^2+y^2}$ is called **the modulus of the complex number *a* + *ib*.**

In the figure $\overline{MA}\perp\overrightarrow{ox}$

$\therefore \overline{OM}=x,\ \overline{MA}=y$

In the right-angled triangle *OMA*, we have, by Pythagoras theorem,

$$\left|\overline{OA}\right|^2=\left|\overline{OM}\right|^2+\left|\overline{MA}\right|^2$$

$$\therefore \left|\overline{OA}\right|=\sqrt{x^2+y^2}$$

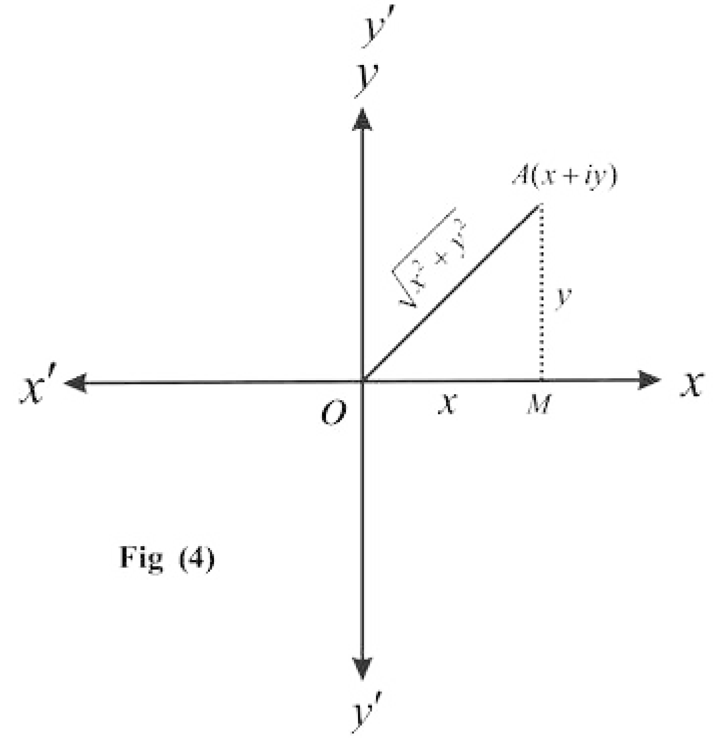

Fig (4)

Thus $\left|\overline{OA}\right|$ represents the modulus of *x* + *iy*. In other words: **The modulus of a complex number is the distance from the origin of the point representing the number.**

The modulus of a complex number is generally denoted as: $|x+iy|$ or $|(x, y)|$. For convenience, a complex number is denoted by z.

If $z = x + iy = (x, y)$, then

$$|z| = \sqrt{x^2 + y^2}$$

Example 1: Find moduli of the following complex numbers :

(i) $1 - i\sqrt{3}$ (ii) 3 (iii) $-5i$ (iv) $3 + 4i$

Solution:

i) Let $z = 1 - i\sqrt{3}$

or $z = 1 + i\left(-\sqrt{3}\right)$

$$\therefore |z| = \sqrt{(1)^2 + (-\sqrt{3})^2}$$

$$= \sqrt{1+3} = 2$$

ii) Let $z = 3$

or $z = 3 + 0.i$

$$\therefore |z| = \sqrt{(3)^2 + (0)^2} = 3$$

iii) Let $z = -5i$

or $z = 0 + (-5)i$

$$\therefore |z| = \sqrt{0^2 + (-5)^2} = 5$$

iv) Let $z = 3 + 4i$

$$\therefore |z| = \sqrt{(3)^2 + (4)^2}$$

Theorems: $\forall z, z_1, z_2 \in C$,

i) $|-z| = |z| = |\bar{z}| = |-\bar{z}|$

ii) $\bar{\bar{z}} = z$

iii) $z\bar{z} = |z|^2$

iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

v) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

vi) $|z_1.z_2| = |z_1|.|z_2|$

Proof :(i): Let $z = a + ib$,

So, $-z = -a - ib, \bar{z} = a - ib$ and $-\bar{z} = -a + ib$

$$\therefore |-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} \quad (1)$$

$$|z| = \sqrt{a^2 + b^2} \quad (2)$$

$$|\bar{z}| = \sqrt{(a)^2 + (b)^2} = \sqrt{a^2 + b^2} \quad (3)$$

$$|-\bar{z}| = \sqrt{(-a)^2 + (b)^2} = \sqrt{a^2 + b^2} \quad (4)$$

By equations (1), (2), (3) and (4) we conclude that

$$|-z| = |z| = |\bar{z}| = |-\bar{z}|$$

(ii) Let $z = a + ib$

So that $\bar{z} = a - ib$

Taking conjugate again of both sides, we have

$$\bar{\bar{z}} = a + ib = z$$

(iii) Let $z = a + ib$ so that $\bar{z} = a - ib$

$$\begin{aligned}\therefore z.\bar{z} &= (a+ib)(a-ib)\\ &= a^2 - iab + iab - i^2b^2\\ &= a^2 - (-1)b^2\\ &= a^2 + b^2 = |z|^2\end{aligned}$$

(iv) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$z_1 + z_2 = (a + ib) + (c + id)$

$= (a + c) + i(b + d)$

so, $\overline{z_1 + z_2} = \overline{(a+c) + i(b+d)}$ (Taking conjugate on both sides)

$= (a + c) - i(b + d)$

$= (a - ib) + (c - id) = \bar{z}_1 + \bar{z}_2$

(v) Let $z_1 = a + ib$ and $z_2 = c + id$, where $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$$

$$= \frac{a+ib}{c+id} \times \frac{c-id}{c-id} \qquad \text{(Note this step)}$$

$$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

$$\therefore \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}}$$

$$= \frac{ac+bd}{c^2+d^2} - i\frac{bc-ad}{c^2+d^2} \qquad (1)$$

Now $\frac{\overline{z_1}}{\overline{z_2}} = \frac{\overline{a+ib}}{\overline{c+id}} = \frac{a-ib}{c-id}$

$$= \frac{a-ib}{c-id} \times \frac{c+id}{c+id}$$

$$= \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} - i\frac{bc-ad}{c^2+d^2} \qquad (2)$$

From (1) and (2), we have

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

(vi) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$|z_1.z_2| = |(a+ib)(c+id)|$$

$$= |(ac-bd)+(ad+bc)i|$$

$$= \sqrt{(ac-bd)^2+(ad+bc)^2}$$

$$= \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$$

$$= \sqrt{(a^2+b^2)(c^2+d^2)}$$

$$= |z_1|.|z_2|$$

This result may be stated thus: -

The modulus of the product of two complex numbers is equal to the product of their moduli.

(vii) Algebraic proof of this part is tedius. Therefore, we prove it geometrically.

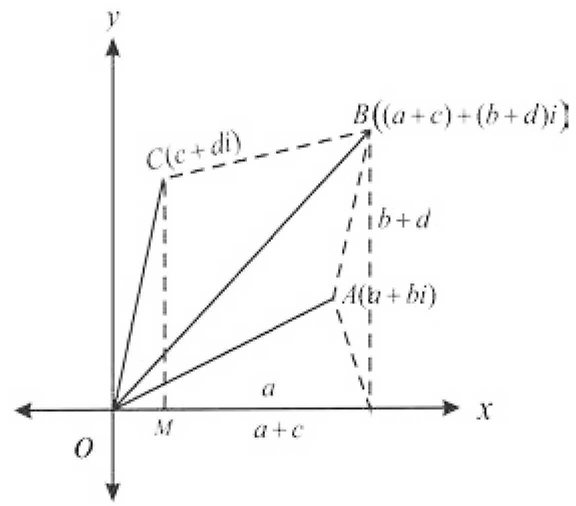

Fig (5)

In the figure point *A* represents $z_1 = a + ib$ and point *C* represents $z_2 = c + id$. We complete the parallelogram *OABC*. From the figure, it is evident that coordinates of *B* are $(a + c, b + d)$, therefore, *B* represents

$z_1 + z_2 = (a + c) + (b + d)i$ and $|\overline{OB}| = |z_1 + z_2|$.

Also $|\overline{OA}| = |z_1|$, $\qquad |\overline{AB}| = |\overline{OC}| = |z_2|$.

In the $\triangle OAB$; $OA + AB > OB$ $\qquad$ ($OA = m\overline{OA}$ etc.)

$\therefore |z_1| + |z_2| > |z_1 + z_2|$ $\qquad$ (1)

Also in the same triangle, $OA - AB < OB$

$\therefore |z_1| - |z_2| < |z_1 + z_2|$ $\qquad$ (2)

Combining (1) and (2), we have

$|z_1| - |z_2| < |z_1 + z_2| < |z_1| + |z_2|$ $\qquad$ (3)

which gives the required results with inequality signs.

Results with equality signs will hold when the points *A* and *C* representing z_1 and z_2 become collinear with *B*. This will be so when $\frac{a}{b}=\frac{c}{d}$ (see fig (6)).

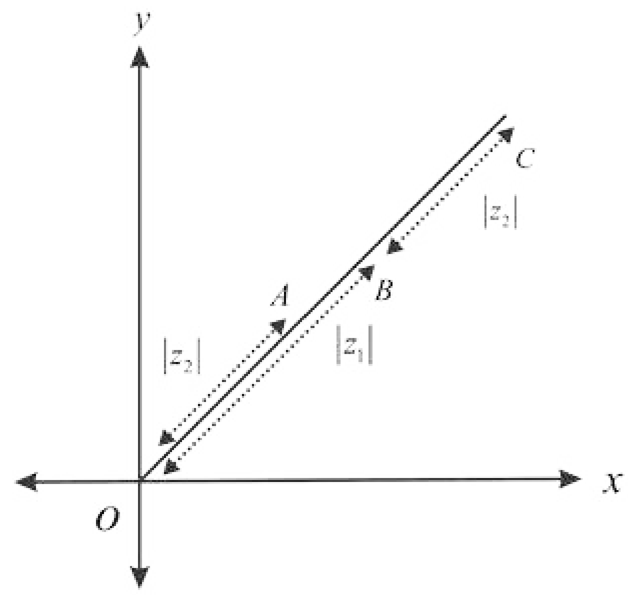

Fig (6)

In such a case $|z_1|+|z_2| = |\overline{OB}|+|\overline{OA}|$

$$= |\overline{OB}|+|\overline{BC}|$$
$$= |\overline{OC}|$$
$$= |z_1+z_2|$$

Thus $|z_1+z_2| = |z_1|+|z_2|$

The second part of result (vii) namely

$$|z_1+z_2| \le |z_1|+|z_2|$$

is analogue of the triangular inequality*. In words, it may be stated thus: - The modulus of the sum of two complex numbers is less than or equal to the sum of the moduli of the numbers.

Example 2: If $z_1 = 2+i$, $z_2 = 3-2i$, $z_3 = 1+3i$ then express $\frac{\bar{z}_1\,\bar{z}_3}{z_2}$ in the form $a+ib$

(Conjugate of a complex number z is denoted as $\bar{z}$)

Solution:

$$\frac{\bar{z}_1\,\bar{z}_3}{z_2} = \frac{\overline{(2+i)}\,\overline{(1+3i)}}{3-2i} = \frac{(2-i)(1-3i)}{3-2i}$$

$$= \frac{(2-3)+(-6-1)i}{3-2i} = \frac{-1-7i}{3-2i}$$
$$= \frac{(-1-7i)(3+2i)}{(3-2i)(3+2i)}$$
$$= \frac{(-3+14)+(-2-21)i}{3^2+2^2} = \frac{11}{13}\ \ \frac{23}{13}i$$

Example 3: Show that, $\forall\ z_1, z_2 \in C,\ \overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$

Solution: Let $z_1 = a + bi,\ z_2 = c + di$

$$\overline{z_1 z_2} = \overline{(a+bi)(c+di)} = \overline{(ac-bd)(ad+bc)i}$$
$$= (ac-bd)-(ad+bc)i \qquad (1)$$

$$\bar{z}_1.\bar{z}_2 = \overline{(a+bi)} = \overline{(c+di)}$$
$$= (a-bi)(c-di)$$
$$= (ac-bd)+(-ad-bc)i$$

$$\bar{z}_1.\bar{z}_2 = \overline{(a\ \ bi)}\,\overline{(c\ \ di)}$$
$$= (a\ \ bi)(c\ \ di)$$
$$= (ac-bd)+(-ad-bc)i \qquad (2)$$

Thus from (1) and (2) we have, $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$

Polar form of a Complex number: Consider adjoining diagram representing the complex number $z = x + iy$. From the diagram, we see that $x = r\cos\theta$ and $y = r\sin\theta$ where $r = |z|$ and θ is called argument of z.

Hence $x + iy = r\cos\theta + r\sin\theta$ ....(i)

where $r=\sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

Equation (i) is called the polar form of the complex number z.

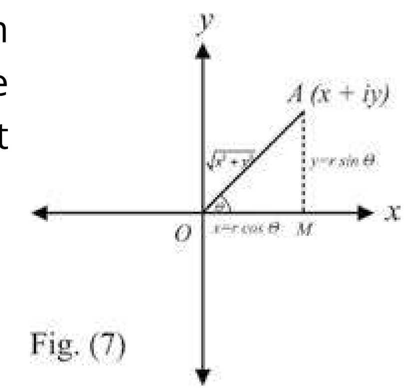

Fig. (7)

*In any triangle the sum of the lengths of any two sides is greater than the length of the third side and difference of the lengths of any two sides is less than the length of the third side.

Example 4: Express the complex number $1+i\sqrt{3}$ in polar form.

Solution:

Step-I: Put $r\cos\theta = 1$ and $r\sin\theta = \sqrt{\ }$

Step-II: $r^2 = (1)^2 + (\sqrt{3})^2$

$\Rightarrow r^2 = 1+3 = 4 \quad \Rightarrow r = 2$

Step-III: $\theta = \tan^{-1}\dfrac{\sqrt{3}}{1} = \tan^{-1}\sqrt{3} = 60^\circ$

Thus $1+i\sqrt{3} = 2\cos 60^\circ + i2\sin 60^\circ$

De Moivre's Theorem : -

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \ \forall n \in Z$

Proof of this theorem is beyond the scope of this book.

1.7 To find real and imaginary parts of

i) $(x + iy)^n$ **ii)** $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n, \quad x_2+iy_2 \neq 0$

for $n = \pm1, \pm2, \pm3, \ldots$

i) Let $x = r\cos\theta$ and $y = r\sin\theta$, then

$(x + iy)^n = (r\cos\theta + ir\sin\theta)^n$

$= (r\cos\theta + ir\sin\theta)^n$

$= [r(\cos\theta + i\sin\theta)]^n$

$= r^n(\cos\theta + i\sin\theta)^n$

$= r^n(\cos n\theta + i\sin n\theta)$ (By De Moivre's Theorem)

$= r^n \cos n\theta + ir^n \sin n\theta$

Thus $r^n \cos n\theta$ and $r^n \sin n\theta$ are respectively the real and imaginary parts of $(x + iy)^n$.

Where $r = \sqrt{x^2 \quad y^2}$ and $\theta \ \tan^{-1}\dfrac{x}{y}$.

ii) Let $x_1 + iy_1 = r_1\cos\theta_1 + r_1\sin n\theta_1$ and $x_2 + iy_2 = r_2\cos\theta_2 + r_2\sin n\theta_2$ then,

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{r_1\cos\theta_1 + r_1 i\sin\theta_1}{r_2\cos\theta_2 + r_2 i\sin\theta_2}\right)^n \quad \frac{r_1^n(\cos\theta_1 + i\sin\theta_1)^n}{r_2^n(\cos\theta_2 + i\sin\theta_2)^n}$$

$$= \frac{r_1^n}{r_2^n}(\cos\theta_1 \quad i\sin\theta_1)^n(\cos\theta_2 \quad i\sin\theta_2)^{-n}$$

$$= \frac{r_1^n}{r_2^n}(\cos n\theta_1 + i\sin n\theta_1)\big(\cos(-n\theta)_2 + i\sin(-n\theta_2)\big),$$

(By De Moivre's Theorem)

$$= \frac{r_1^n}{r_2^n}(\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 - i\sin n\theta_2), \ (\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta)$$

$$= \frac{r_1^n}{r_2^n}.[(\cos n\theta_1\cos n\theta_2 \quad \sin n\theta_1\sin n\theta_2)$$

$$+ i(\sin n\theta_1\cos n\theta_2 - \cos n\theta_1 \ \sin n\theta_2)]$$

$$= \frac{r_1^n}{r_2^n}[\cos(n\theta_1 - n\theta_2) + i\sin(n\theta_1 - n\theta_2)] \because \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\text{and } \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \frac{r_1^n}{r_2^n}[\cos n(\theta_1 - \theta_2) + i\sin n(\theta_1 - \theta_2)]$$

$$= \frac{r_1^n}{r_2^n}[\cos n(\theta_1 \quad \theta_2) + i\sin n\,(\theta_1 - \theta_2)]$$

Thus $\dfrac{r_1^n}{r_2^n}\cos n(\theta_1-\theta_2)$ and $\dfrac{r_1^n}{r_2^n}\sin n(\theta_1-\theta_2)$ are respectively the real and imaginary parts of $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n, \ x_2+iy_2 \neq 0$

where $r_1 = \sqrt{x_1^2 + y_1^2}\,;\ \theta_1 = \tan^{-1}\frac{y_1}{x_1}$ and $r_2 = \sqrt{x_2^2 \quad y_2^2}\,;\ \theta_2 = \tan^{-1}\frac{y_2}{x_2}$

Example 5: Find out real and imaginary parts of each of the following complex numbers.

i) $\left(\sqrt{3}+\mathrm{i}\right)^3$ ii) $\left(\dfrac{1-\sqrt{3}\,\mathrm{i}}{1+\sqrt{3}\,\mathrm{i}}\right)^5$

Solution:

i) Let $r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1$ where

$$r^2 = (\sqrt{3})^2 + 1^2 \text{ or } r = \sqrt{3+1} = 2 \text{ and } \theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30^o$$

So, $\left(\sqrt{3}+i\right)^3 = (r\cos\theta + ir\sin\theta)^3$

$= r^3(\cos3\theta + i\sin3\theta)$ (By De Moivre's Theorem)

$= 2^3(\cos90^\circ + i\sin90^\circ)$

$= 8(0 + i.1)$

$= 8i$

Thus 0 and 8 are respectively real and imaginary Parts of $\left(\sqrt{3}+\mathrm{i}\right)^3$.

ii) Let $r_1\cos\theta_1 = 1$ and $r_1\sin\theta_1 = -\sqrt{3}$

$$\Rightarrow r_1 = \sqrt{(1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2 \text{ and } \theta_1 = \tan^{-1} -\frac{\sqrt{3}}{1} - 60^o$$

Also Let $r_2\cos\theta_2 = 1$ and $r_2\sin\theta_2 = \sqrt{3}$

$$\Rightarrow r_2 = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2 \text{ and } \theta_2 \tan^{-1}\frac{\sqrt{3}}{1} = 60^o$$

So, $\left(\dfrac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^5 = \left[\dfrac{2\left(\cos(-60^o)+i\sin(-60^O)\right)}{2\left(\cos(60^o)+i\sin(60^O)\right)}\right]^5$

$$= \frac{\left(\cos(-60^\circ)+i\sin(-60^\circ)\right)^5}{\left(\cos(60^\circ)+i\sin(60^\circ)\right)^5}$$

$$= \left(\cos(-60^o)+i\sin(-60^O)\right)^5\left(\cos(60^o)+i\sin(60^O)\right)^{-5}$$

$$= \left(\cos(-300^\circ)+i\sin(-300^\circ)\right)\left(\cos(-300^\circ)+i\sin(-300^\circ)\right)$$

$$= \left(\cos(300^\circ) \quad i\sin(300^\circ)\right)\left(\cos(300^\circ) \quad i\sin(300^\circ)\right) - \because \cos(\quad\theta) = \cos\theta$$
$$\text{and } \sin(-\theta) = -\sin\theta$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$$

Thus $\dfrac{-1}{2}$, $\dfrac{\sqrt{3}}{2}$ are respectively real and imaginary parts of $\left(\dfrac{1-\sqrt{3}\,\mathrm{i}}{1+\sqrt{3}\,\mathrm{i}}\right)^5$

Exercise 1.3

1. Graph the following numbers on the complex plane: -

i) $2+3i$ ii) $2-3i$ iii) $-2-3i$ iv) $-2+3i$

v) -6 vi) i vii) $\frac{3}{5}-\frac{4}{5}i$ viii) $-5-6i$

2. Find the multiplicative inverse of each of the following numbers: -

i) $-3i$ ii) $1-2i$ iii) $-3-5i$ iv) (1, 2)

3. Simplify

i) i^{101} ii) $(-ai)^4$, $a \in \Re$ iii) i^{-3} iv) i^{-10}

4. Prove that $\bar{z} = z$ iff z is real.

5. Simplify by expressing in the form $a + bi$

i) $5+2\sqrt{-4}$ ii) $(2+\sqrt{-3})(3+\sqrt{-3})$

iii) $\dfrac{2}{\sqrt{5}+\sqrt{-8}}$ iv) $\dfrac{3}{\sqrt{6}-\sqrt{-12}}$

6. Show that $\forall z \in C$

i) $z^2 - \bar{z}^2$ is a real number. ii) $(z-\bar{z})^2$ is a real number.

7. Simplify the following

i) $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$ ii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3$

iiii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$ iv) $(a+bi)^2$

v) $(a+bi)^{-2}$ vi) $(a+bi)^3$

vii) $(a-bi)^3$ viii) $(3-\sqrt{-4})^{-3}$