CHAPTER

9

Fundamentals of Trigonometry

9.1 Introduction

Trigonometry is an important branch of Mathematics. The word **Trigonometry** has been derived from three Greek words: ***Trei*** (three), ***Goni*** (angles) and ***Metron*** (measurement). Literally it means **measurement of triangle.**

For study of calculus it is essential to have a sound knowledge of trigonometry.

It is extensively used in Business, Engineering, Surveying, Navigation, Astronomy, Physical and Social Sciences.

9.2 Units of Measures of Angles

Concept of an Angle

Two rays with a common starting point form an angle. One of the rays of angle is called initial side and the other as terminal side. The angle is identified by showing the direction of rotation from the initial side to the terminal side.

An angle is said to be positive/negative if the rotation is anti-clockwise/clockwise. Angles are usually denoted by Greek letters such as α (alpha), β (beta), γ (gamma), θ (theta) etc.

In figure 9.1 $\angle AOB$ is positive and $\angle COD$ is negative.

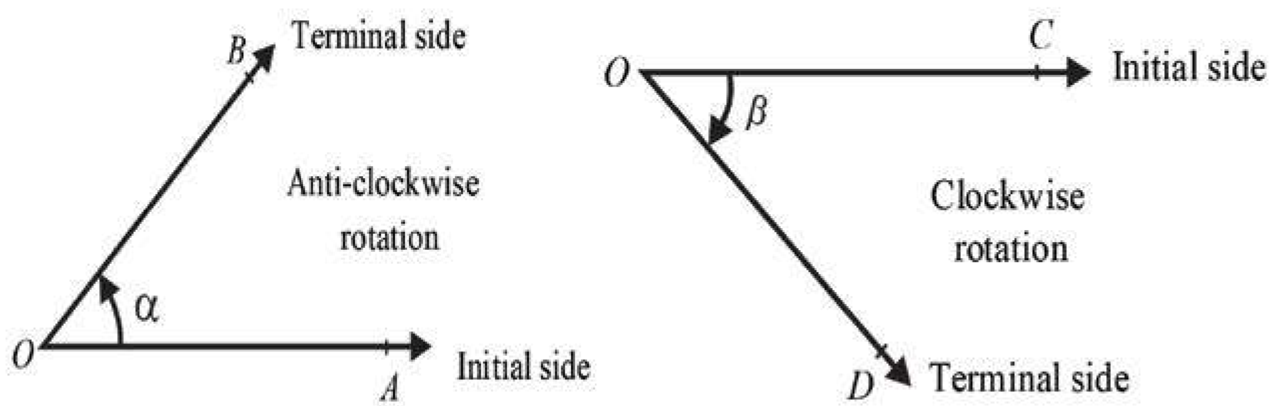


figure 9.1

There are two commonly used measurements for angles: **Degrees and Radians.** which are explained as below:

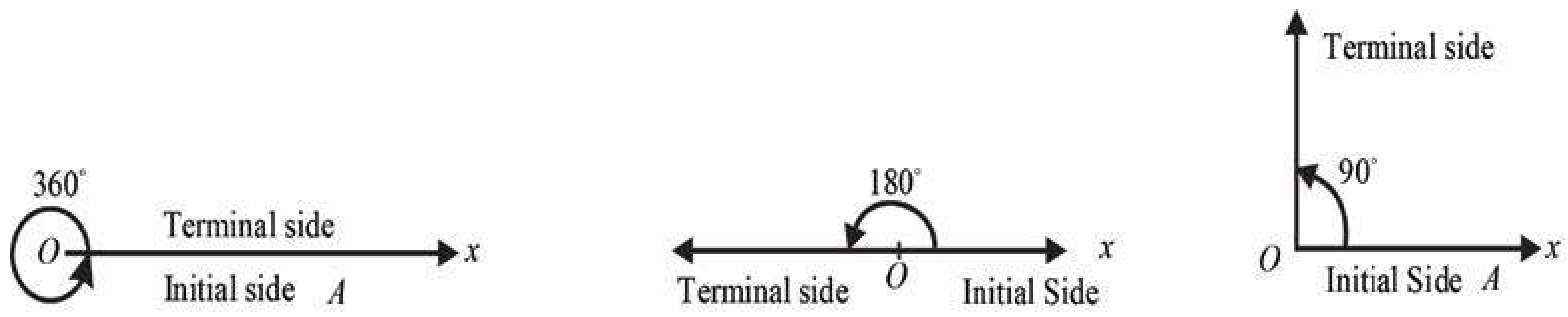
9.2.1. Sexagesimal System: (Degree, Minute and Second).

If the initial ray $\overrightarrow{OA}$ rotates in anti-clockwise direction in such a way that it coincides with itself, the angle then formed is said to be of 360 degrees (360°).

One rotation (anti-clockwise) = 360°

$\frac{1}{2}$ rotation (anti-clockwise) = 180° is called a straight angle

$\frac{1}{4}$ rotation (anti-clockwise) = 90° is called a right angle.



1 rotation =360° $\frac{1}{2}$ rotation = 180° $\frac{1}{4}$ rotation = 90°

1 degree (1°) is divided into 60 minutes ($60'$) and 1 minute ($1'$) is divided into 60 seconds ($60''$). As this system of measurement of angle owes its origin to the English and because 90, 60 are multiples of 6 and 10, so it is known as English system or Sexagesimal system.

Thus 1 rotation (anti-clockwise) = 360°.
One degree (1°) = 60′
One minute ($1'$) = 60″

9.2.2. Conversion from $D°M'S''$ to a decimal form and vice versa.

(i) $16°30' = 16.5°$ $\left(\text{As } 30' = \frac{1°}{2} = 0.5°\right)$

(ii) $45.25^\circ = 45^\circ 15' \quad \left(0.25^\circ = \frac{25^\circ}{100} = \frac{1^\circ}{4} = \frac{60'}{4} = 15'\right)$

Example 1: Convert $18^\circ\ 6'\ 21''$ to decimal form.

Solution: $1' = \left(\frac{1}{60}\right)^\circ$ and $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{60\times 60}\right)^\circ$

$$\therefore\ 18^\circ\ 6'\ 21'' = \left[18 + 6\left(\frac{1}{60}\right) + 21\left(\frac{1}{60\times 60}\right)\right]^\circ$$

= (18 + 0.1 + 0.005833)° =18.105833°

Example 2: Convert 21.256° to the $D^\circ\ M'\ S''$ form

Solution: $0.256^\circ = (0.256)(1^\circ)$

$= (0.256)(60') = 15.36'$

and $0.36' = (0.36)(1')$

$= (0.36)(60'') = 21.6''$

Therefore,

21.256°= 21°+ 0.256°

$= 21^\circ + 15.36'$

$= 21^\circ + 15' + 0.36'$

$= 21^\circ + 15' + 21.6''$

= 21° 15′ 22″ rounded off to nearest second

9.2.3. Circular System (Radians)

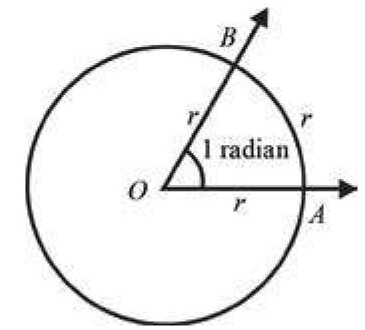
There is another system of angular measurement, called the **Circular System.** It is

most useful for the study of higher mathematics. Specially in Calculus, angles are measured in radians.

Definition: Radian is the measure of the angle subtended at the center of the circle by an arc, whose length is equal to the radius of the circle.

Consider a circle of radius r. Construct an angle $\angle AOB$ at the centre of the circle whose rays cut off an arc $\overset{\frown}{AB}$ on the circle whose length is equal to the radius r.

Thus $m\angle AOB = 1$ radian.

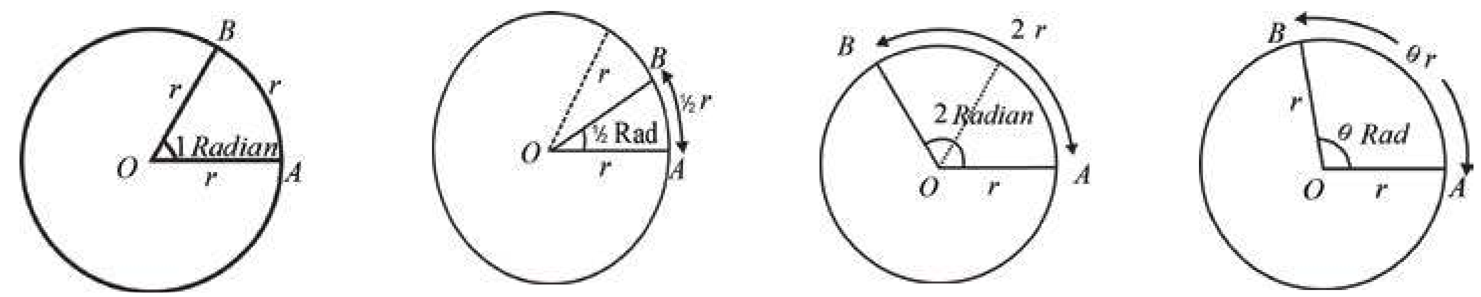


9.3 Relation between the length of an arc of a circle and the circular measure of its central angle.

Prove that $\theta = \frac{l}{r}$

where r is the radius of the circle l, is the length of the arc and θ is the circular measure of the central angle.

Proof:



By definition of radian;

An angle of 1 radian subtends an arc $\overset{\frown}{AB}$ on the circle of length = $1.r$

An angle of $\frac{1}{2}$ radian subtends an arc $\overset{\frown}{AB}$ on the circle of length= $\frac{1}{2}.r$

An angle of 2 radians subtends an arc $\overset{\frown}{AB}$ on the circle of length = $2.r$

∴ An angle of θ radian subtends an arc $\overset{\frown}{AB}$ on the circle of length= $\theta.r$

$$\Rightarrow \quad \overset{\frown}{AB} = \theta.r$$

$$\Rightarrow \quad l = \theta . r$$

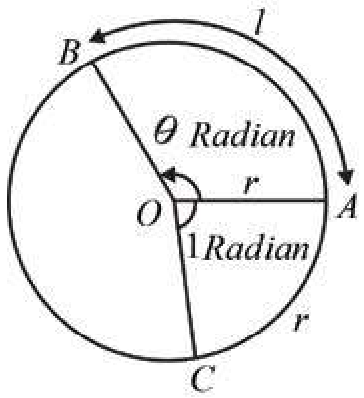
$$\therefore \quad \theta = \frac{l}{r}$$

Alternate Proof

Let there be a circle with centre *O* and radius *r*. Suppose that length of arc $\overset{\frown}{AB} = l$ and the central angle $m\angle AOB = \theta$ radian. Take an arc $\overset{\frown}{AC}$ of length = r.

By definition $m\angle AOC = 1$ radian.

We know from elementary geometry that measures of central angles of the arcs of a circle are proportional to the lengths of their arcs.



$$\Rightarrow \quad \frac{m\angle AOB}{m\angle AOC} = \frac{m\overset{\frown}{AB}}{m\overset{\frown}{AC}}$$

$$\Rightarrow \quad \frac{\theta \text{ radian}}{1 \text{ radian}} = \frac{l}{r}$$

$$\Rightarrow \quad \theta = \frac{l}{r}$$

Thus the central angle θ (in radian) subtended by a circular arc of length l is given by

$\theta = \frac{l}{r}$, where r is the radius of the circle.

Remember that r and l are measured in terms of the same unit and the radian measure is unit-less, i.e., it is a real number.

For example, if r = 3 cm and l = 6 cm

then $\quad \theta = \frac{l}{r} = \frac{6}{3} = 2$

9.3.1 Conversion of Radian into Degree and Vice Versa

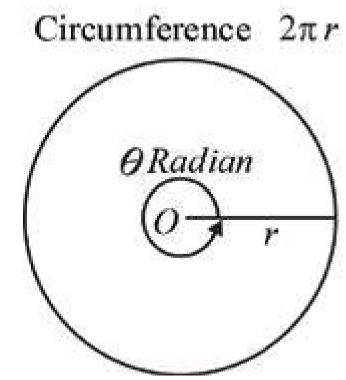
We know that circumference of a circle of radius r is $2\pi r = (l)$, and angle formed by one complete revolution is θ radian, therefore,

$$\theta = \frac{l}{r}$$

$$\Rightarrow \quad \theta = \frac{2\pi r}{r}$$

$$\Rightarrow \quad \theta = 2\pi \text{ radian}$$

Thus we have the relationship



$$2\pi \text{ radian} = 360^\circ$$

$$\Rightarrow \quad \pi \text{ radian} = 180^\circ$$

$$\Rightarrow \quad 1 \text{ radian} = \frac{180^\circ}{\pi} \approx \frac{180^\circ}{3.1416} \approx 57.296^\circ$$

Further $\quad 1^\circ = \frac{\pi}{180}$ radian.

$$\approx \frac{3.1416}{180} \approx 0.0175 \text{ radain}$$

Example 3: Convert the following angles in degree:

(i) $\frac{2\pi}{3}$ radain (ii) 3 radians.

Solution: (i) $\frac{2\pi}{3}$ radains $= \frac{2}{3}(\pi \text{ radain}) = \frac{2}{3}(180^\circ) \quad 120^\circ$

(ii) 3 radains $= 3(1 \text{ radain}) \approx 3(57.296^\circ) \approx 171.888^\circ$

Example 4: Convert 54° 45 ' into radians.

Solution: $54^\circ 45' = \left(54\frac{45}{60}\right)^\circ = \left(54\frac{3}{4}\right)^\circ \quad \frac{219^\circ}{4}$

$$= \frac{219}{4}(1^\circ)$$

$$\approx \frac{219}{4}(0.0175) \text{ radinas}$$

$$\approx 0.958 \text{ radains.}$$

Most calculators automatically would convert degrees into radians and radians into degrees.

Example 5: An arc subtends an angle of 70° at the center of a circle and its length is 132 m.m. Find the radius of the circle.

Solution:

$$70^\circ = 70 \quad \frac{3.1416}{180} \text{radains} = \frac{70}{180}(3.1416)\text{radain} \quad \frac{11}{9} \text{radains.} \ (\pi \quad 3.1416)$$

$$\therefore \quad \theta = \frac{11}{9} \text{ radain} \qquad \text{and} \qquad l \quad 132\text{m.m.}$$

$$\because \quad \theta = \frac{l}{r} \qquad \Rightarrow \qquad r = \frac{l}{\theta} = 132 \times \frac{9}{11} = 108 \text{ m.m.}$$

Example 6: Find the length of the equatorial arc subtending an angle of 1° at the centre of the earth, taking the radius of the earth as 6400 km.

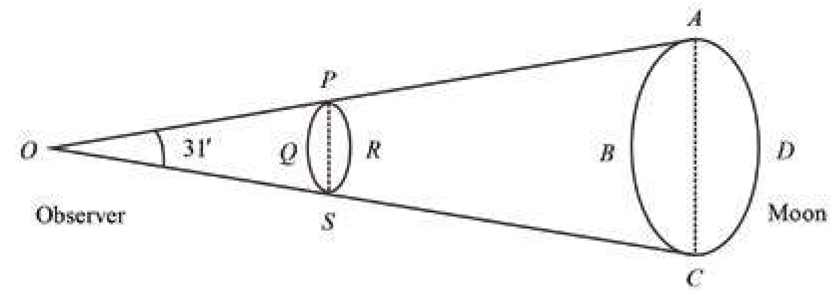
Solution: $1^\circ = \approx \frac{\pi}{180} \text{ radains} \qquad \frac{3.1416}{180} \text{ radain}$

$$\therefore \quad \theta \approx \frac{3.1416}{180} \qquad \text{and} \qquad r = 6400 \text{ km.}$$

$$\text{Now} \quad \theta = \frac{l}{r}$$

$$\Rightarrow \quad l = r\theta \approx 6400 \times \frac{31416}{1800000} \approx 111.7 \text{ km}$$

Example 7: Find correct to the nearest centimeter, the distance at which a coin of diameter '1' cm should be held so as to conceal the full moon whose diameter subtends an angle of 31' at the eye of the observer on the earth.



Solution: Let *O* be the eye of the observer. *ABCD* be the moon and *PQSR* be the coin, so that *APO* and *CSO* are straight line segments.

We know that $m\overline{PS} = 1 \ cm, \quad m\angle AOC = 31'$

Now since $m\angle ACO (\leq m \quad POC)$ is very very small.

$\therefore \quad \overline{PS}$ can be taken as the arc .of the circle with centre *O* and radius *OP*.

$$\text{Now} \quad OP = r \ , \ l = 1 \text{ cm} \ , \quad \theta = 31' \quad = \frac{31 \times \pi}{60 \times 180} \text{ radains}$$

$$\because \quad \theta = \frac{l}{r}$$

$$\therefore \quad r = \frac{l}{\theta} = \frac{1 \times 60 \times 180}{31 \times \pi} \approx \frac{60 \times 180}{31 \times 3.1416} \approx 110.89 \ cm.$$

Hence the coin should be held at an approximate distance of 111 *cm*. from the observer's eye.

Note: If the value of π is not given, we shall take $\pi \approx 3.1416$.

Exercise 9.1

1. Express the following sexagesimal measures of angles in radians:

i) 30° ii) 45° iii) 60° iv) 75°
v) 90° vi) 105° vii) 120° viii) 135°
ix) 150° x) $10^{\circ}\ 15'$ xi) $35^{\circ}\ 20'$ xii) $75^{\circ}\ 6'\ 30''$
xiii) $120'\ 40''$ xiv) $154^{\circ}\ 20''$ xv) 0° xvi) $3''$

2. Convert the following radian measures of angles into the measures of sexagesimal system:

i) $\frac{\pi}{8}$ ii) $\frac{\pi}{6}$ iii) $\frac{\pi}{4}$ iv) $\frac{\pi}{3}$ v) $\frac{\pi}{2}$
vi) $\frac{2\pi}{3}$ vii) $\frac{3\pi}{4}$ viii) $\frac{5\pi}{6}$ ix) $\frac{7\pi}{12}$ x) $\frac{9\pi}{5}$
xi) $\frac{11\pi}{27}$ xii) $\frac{13\pi}{16}$ xiii) $\frac{17\pi}{24}$ xiv) $\frac{25\pi}{36}$ xv) $\frac{19\pi}{32}$

3. What is the circular measure of the angle between the hands of a watch at 4 *O*'clock?

4. Find θ, when:

i) l = 1.5 cm, r = 2.5cm
ii) l = 3.2m, r = 2m

5. Find l, when:

i) $\theta = \pi$ randains, r = 6cm
ii) $\theta = 65^{\circ}\ 20'$ r = 18mm

6. Find r, when:

i) l = 5 cm, $\theta = \frac{1}{2}$ radian
ii) l = 56 cm, $\theta = 45^{\circ}$

7. What is the length of the arc intercepted on a circle of radius 14 cms by the arms of a central angle of 45°?

8. Find the radius of the circle, in which the arms of a central angle of measure 1 radian cut off an arc of length 35 cm.

9. A railway train is running on a circular track of radius 500 meters at the rate of 30 km per hour. Through what angle will it turn in 10 sec.?

10. A horse is tethered to a peg by a rope of 9 meters length and it can move in a circle with the peg as centre. If the horse moves along the circumference of the circle, keeping the rope tight, how far will it have gone when the rope has turned through an angle of 70°?

11. The pendulum of a clock is 20 cm long and it swings through an angle of 20° each second. How far does the tip of the pendulum move in 1 second?

12. Assuming the average distance of the earth from the sun to be 148 x 10^6 km and the angle subtended by the sun at the eye of a person on the earth of measure 9.3 x 10^{-3} radian. Find the diameter of the sun.

13. A circular wire of radius 6 cm is cut straightened and then bent so as to lie along the circumference of a hoop of radius 24 cm. Find the measure of the angle which it subtends at the centre of the hoop.

14. Show that the area of a sector of a circular region of radius r is $\frac{1}{2}r^2\theta$, where θ is the circular measure of the central angle of the sector.

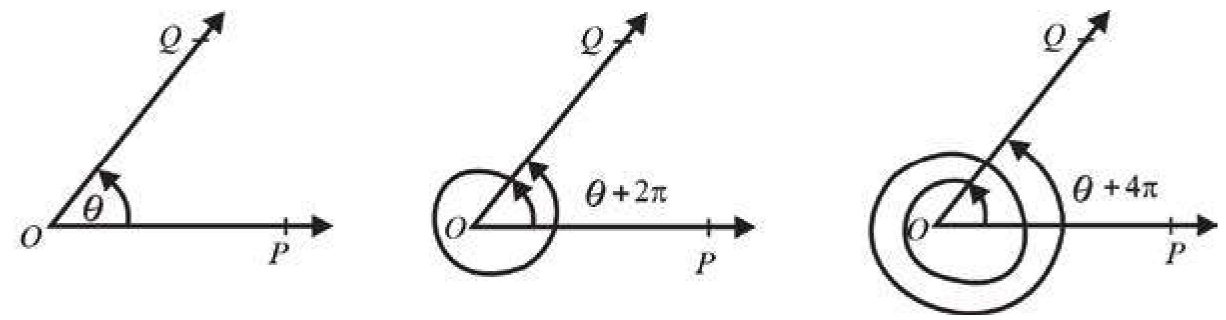
15. Two cities *A* and *B* lie on the equator such that their longitudes are 45°E and 25°W respectively. Find the distance between the two cities, taking radius of the earth as 6400 kms.

16. The moon subtends an angle of 0.5° at the eye of an observer on earth. The distance of the moon from the earth is 3.844 x 10^5 km approx. What is the length of the diameter of the moon?

17. The angle subtended by the earth at the eye of a spaceman, landed on the moon, is 1° 54′. The radius of the earth is 6400 km. Find the approximate distance between the moon and the earth.

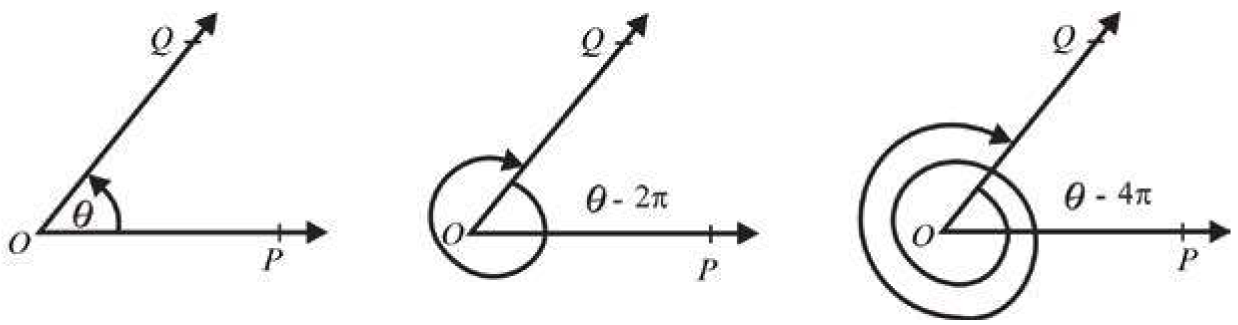
9.4 General Angle (Coterminal Angles)

There can be many angles with the same initial and terminal sides. These are called coterminal angles. Consider an angle $\angle POQ$ with initial side $\overrightarrow{OP}$ and terminal side $\overrightarrow{OQ}$ with vertex *O*. Let $m\angle POQ=\theta$ radian, where $0 \le \theta \le 2\pi$



Now, if the side $\overrightarrow{OQ}$ comes to its present position after one or more complete rotations in the anti-clockwise direction, then $m\angle POQ$
will be

i) $\theta+2\pi$, after one revolution ii) $\theta+4\pi$, after two revolutions,



However, if the rotations are made in the clock-wise direction as shown in the figure, $m\angle POQ$ will be:

i) $\theta-2\pi$, after one revolution,

ii) $\theta-4\pi$, after two revolution,

It means that $\overrightarrow{OQ}$ comes to its original position after every revolution of 2π radians in the postive or negative directions.

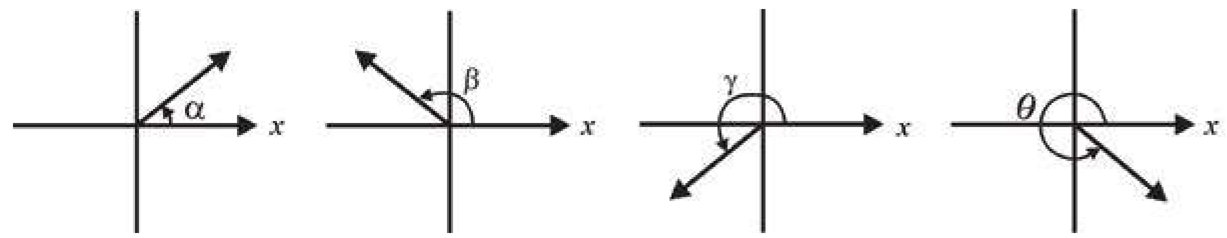
In general, if angle θ is in degrees, then $\theta + 360k$ *where* $k\in Z$, is an angle coterminal with θ. If angle θ is in radians, then $\theta + 2k\pi$ *where* $k\in Z$, is an angle coterminal with θ.

$\Rightarrow$ **General angle is** $\theta+2k\pi,\ \ k\in Z,$

9.5 Angle In The Standard Position

An angle is said to be in **standard position** if its vertex lies at the origin of a rectangular coordinate system and its initial side along the positive x-axis.

The following figures show four angles in standard position:



An angle in standard position is said to lie in a quadrant if its terminal side lies in that quadrant. In the above figure:

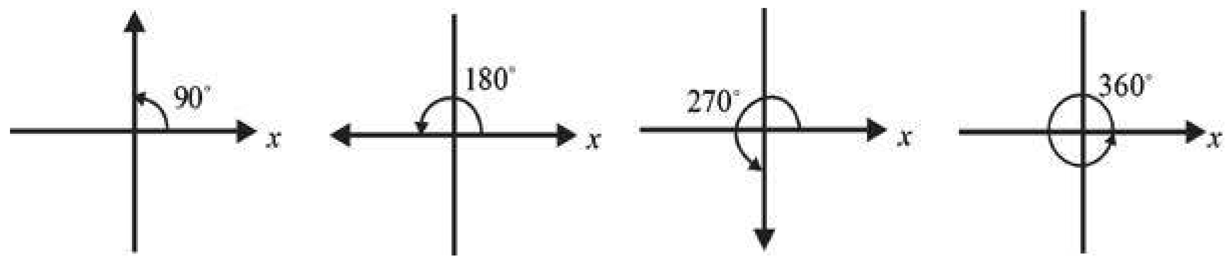
Angle α lies in I Quadrant as its terminal side lies is I Quadrant

Angle β lies in II Quadrant as its terminal side lies is II Quadrant

Angle γ lies in III Quadrant as its terminal side lies is III Quadrant

and Angle θ lies in IV Quadrant as its terminal side lies is IV Quadrant

If the terminal side of an angle falls on x-axis or y-axis, it is called a



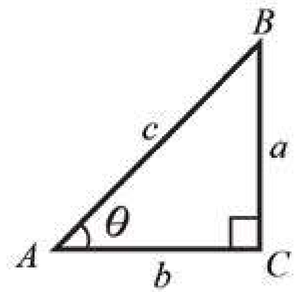
quadrantal angle.

i.e., 90°, 180°, 270° and 360° are quadrantal angles.

9.6 Trigonometric Functions

Consider a right triangle *ABC* with $\angle C=90^\circ$ and sides *a*, *b*, *c*, as shown in the figure. Let $m\angle A=\theta$ radian.

The side AB opposite to 90° is called the hypotenuse (hyp),
The side BC opposite to θ is called the opposite (opp) and
the side AC related to angle θ is called the adjacent (adj)



We can form six ratios as follows:

$$\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{c}{a}, \frac{c}{b} \text{ and } \frac{b}{a}$$

In fact these ratios depend only on the size of the angle and not on the triangle formed. Therefore, these ratios are called **trigonometric functions** of angle θ and are defined as below:

$$\text{Sin }\theta \quad : \text{Sin }\theta = \frac{a}{c} = \frac{\text{opp}}{\text{hyp}}; \quad \text{Cosecant }\theta : \csc\theta = \frac{c}{a} = \frac{\text{hyp}}{\text{opp}};$$

$$\text{Cosine }\theta : \text{Cos }\theta = \frac{b}{c} = \frac{\text{adj}}{\text{hyp}}; \quad \text{Secant }\theta : \sec\theta = \frac{c}{b} = \frac{\text{hyp}}{\text{adj}};$$

$$\text{Tangent }\theta : \tan\theta = \frac{a}{b} = \frac{\text{opp}}{\text{adj}}; \quad \text{Cotangent }\theta : \cot\theta = \frac{b}{a} = \frac{\text{adj}}{\text{opp}}.$$

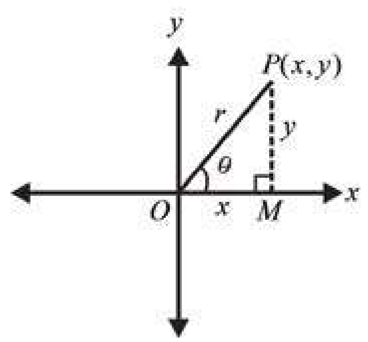
We observe useful relationships between these **six trigonometric functions** as follows:

$$\csc\theta = \frac{1}{\sin\theta}; \quad \sec\theta = \frac{1}{\cos\theta}; \quad \tan\theta = \frac{\sin\theta}{\cos\theta};$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}; \quad \cot\theta = \frac{1}{\tan\theta};$$

9.7 Trigonometric Functions of any angle

Now we shall define the trigonometric functions of any angle.
Consider an angle $\angle XOP = \theta$ radian in standard position.
Let coordinates of P (other than origin) on the terminal side of



the angle be (x, y).

If $r = \sqrt{x^2 + y^2}$ denote the distance from O (0, 0) to P (x, y), then six trigonometric functions of θ are defined as the ratios

$$\sin\theta = \frac{y}{r}; \quad \csc\theta = \frac{r}{y} \ (y \neq 0); \quad \tan\theta = \frac{y}{x} \ (x \neq 0)$$

$$\cos\theta = \frac{x}{r}; \quad \sec\theta = \frac{r}{x} \ (x \neq 0); \quad \cot\theta = \frac{x}{y} \ (y \neq 0)$$

Note: These definitions are independent of the position of the point P on the terminal side i.e., θ is taken as any angle.

9.8 Fundamental Identities

For any real number θ, we shall derive the following three fundamental identities:

i) $\sin^2\theta + \cos^2\theta = 1$

ii) $1 + \tan^2\theta = \sec^2\theta$

iii) $1 + \cot^2\theta = \csc^2\theta.$

Proof:

(i) Refer to right triangle ABC in fig. (I) by Pythagoras theorem, we have, Dividing $a^2 + b^2 = c^2$ both sides by c^2, we get

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{a^2}$$

$$\Rightarrow \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$\Rightarrow (\sin\theta)^2 + (\cos\theta)^2 = 1$$

$$\therefore \boxed{\sin^2\theta + \cos^2\theta = 1} \quad (1)$$

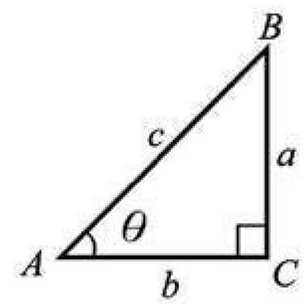


Fig.(1)

ii) Again as $a^2 + b^2 = c^2$
Dividing both sides by b^2, we get

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

$$\Rightarrow \left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

$$\Rightarrow (\tan\theta)^2 + 1 = (\sec\theta)^2$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta} \qquad (2)$$

(iii) Again as $a^2 + b^2 = c^2$
Dividing both sides by a^2, we get

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$

$$\Rightarrow 1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$$

$$\Rightarrow 1 + (\cot\theta)^2 = (\csc\theta)^2$$

$$\therefore \boxed{1 + \cot^2\theta = \csc^2\theta} \qquad (3)$$

Note: $(\sin\theta)^2 = \sin^2\theta$, $(\cos\theta)^2 = \cos^2\theta$ and $(\tan\theta)^2 = \tan^2\theta$ etc.

9.9 Signs of the Trigonometric functions

If θ is not a quadrantal angle, then it will lie in a particular quadrant. Because $r = \sqrt{x^2 + y^2}$ is always positive, it follows that the signs of the trigonometric functions can be found if the quadrant of 0 is known. For example,

(i) If θ lies in Quadrant I, then a point $P(x, y)$ on its terminal side has both x, y co-ordinates +ve

$\Rightarrow$ All trigonometric functions are +ve in Quadrant I.

(ii) If θ lies in Quadrant II, then a point $P(x, y)$ on its terminal side has negative x-coordinate. and positive y-coordinate.

$$\therefore \sin\theta = \frac{y}{r} = +ve > 0,\ \cos\theta = \frac{x}{r} = -ve < 0,\ \tan\theta = \frac{y}{x} = -ve < 0$$

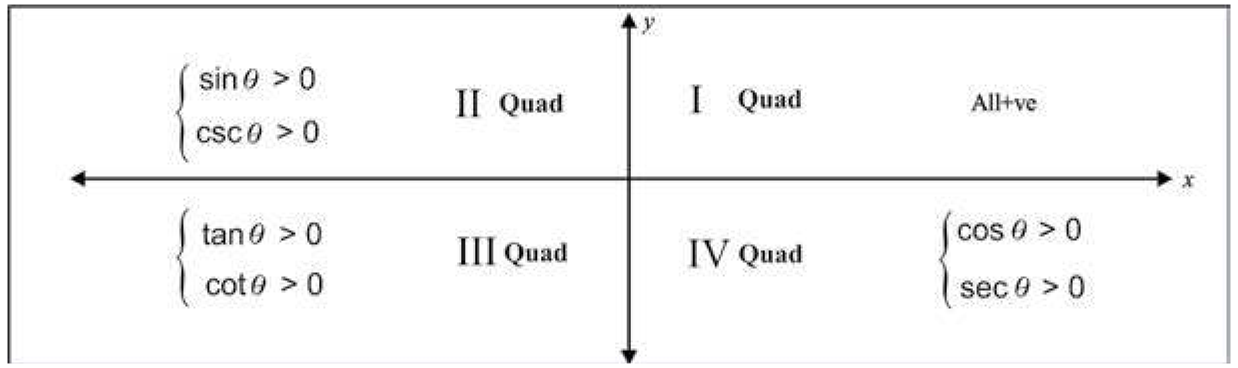
(iii) If θ lies in Quadrant III, then a point $P(x, y)$ on its terminal side has negative x-coordinate. and negative y-coordinate.

$$\therefore \sin\theta = \frac{y}{r} = -ve < 0,\ \cos\theta = \frac{x}{r} = -ve < 0,\ \tan\theta = \frac{y}{x} = +ve > 0$$

(iv) If θ lies in Quadrant IV, then a point $P(x, y)$ on its terminal side has positive x-coordinate. and negative y-coordinate.

$$\therefore \sin\theta = \frac{y}{r} = -ve < 0,\ \cos\theta = \frac{x}{r} = +ve > 0,\ \tan\theta = \frac{y}{x} = -ve < 0$$

These results are summarized in the following figure. Trigonometric functions mentioned are positive in these quardrants.



It is clear from the above figure that

$\sin(-\theta) = -\sin\theta$; $\quad \csc(-\theta) = -\csc\theta$

$\cos(-\theta) = \cos\theta$; $\quad \sec(-\theta) = \sec\theta$

$\tan(-\theta) = -\tan\theta$; $\quad \cot(-\theta) = -\cot\theta$

Example 1: If $\tan\theta = \frac{8}{15}$ and the terminal arm of the angle is in the III quadrant, find the values of the other trigonometric functions of θ.

Solution:

$$\tan\theta = \frac{8}{15} \qquad \therefore \cot\theta = \frac{1}{\tan\theta} = \frac{15}{8}$$

$$\sec^2\theta = 1+\tan^2\theta = 1+\left(\frac{8}{15}\right)^2 = 1+\frac{64}{225} = \frac{289}{225}$$

$$\therefore \sec\theta = \pm\sqrt{\frac{289}{225}} = \pm\frac{17}{15}$$

$\because$ The terminal arm of the angle is in the III quadrant where sec θ is negative

$$\therefore \sec\theta = -\frac{17}{15}$$

Now $$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{-\frac{17}{15}} = -\frac{15}{17}$$

$$\sin\theta = \tan\theta\,.\cos\theta = \frac{8}{15}\left(-\frac{15}{17}\right)$$

$$\therefore \sin\theta = -\frac{8}{17}$$

and $$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{8}{17}} = -\frac{17}{8}$$

Example 2: Find the value of other five trigonometric functions of θ, if $\cos\theta = \frac{12}{13}$ and the terminal side of the angle is not in the I quadrant.

Solution: The terminal side of the angle is not in the I quadrant but cos θ is positive,

$\therefore$ The terminal side of the angle is in the IV quadrant

Now $$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{12}{13}} = \frac{13}{12}$$

$$\sin^2\theta = 1-\cos^2\theta = 1-\left(\frac{12}{13}\right)^2 = 1-\frac{144}{169} = \frac{25}{169}$$

$$\therefore \sin\theta = \pm\frac{5}{13}$$

As the terminal side of the angle is in the IV quadrant where sin θ is negative.

$$\therefore \sin\theta = -\frac{5}{13}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5}$$

Exercise 9.2

1. Find the signs of the following:

i) $\sin 160^\circ$ ii) $\cos 190^\circ$ iii) $\tan 115^\circ$

iv) $\sec 245^\circ$ v) $\cot 80^\circ$ vi) $\text{cosec}\, 297^\circ$

2. Fill in the blanks:

i) $\sin(-310^\circ) = \ldots.\ \sin 310^\circ$ ii) $\cos(-75^\circ) = \ldots.\cos 75^\circ$

iii) $\tan(-182^\circ) = \ldots.\ \tan 182^\circ$ iv) $\cot(-173^\circ) = \ldots.\cot 137^\circ$

v) $\sec(-216^\circ) = \ldots \sec 216^\circ$ vi) $\text{cosec}(-15^\circ) = \ldots \text{cosec}\,15^\circ$

3. In which quadrant are the terminal arms of the angle lie when

i) $\sin\theta < 0$ and $\cos\theta > 0$, ii) $\cot\theta > 0$ and $\text{cosec}\,\theta > 0$,

iii) $\tan\theta < 0$ and $\cos\theta > 0$, iv) $\sec\theta < 0$ and $\sin\theta < 0$,

v) $\cot\theta > 0$ and $\sin\theta < 0$, vi) $\cos\theta < 0$ and $\tan\theta < 0$?

4. Find the values of the remaining trigonometric functions:

i) $\sin\theta = \frac{12}{13}$ and the terminal arm of the angle is in quad. I.

ii) $\cos\theta = \frac{9}{41}$ and the terminal arm of the angle is in quad. IV.

iii) $\cos\theta = -\frac{\sqrt{3}}{2}$ and the terminal arm of the angle is in quad. III.

iv) $\tan \quad = --$ and the terminal arm of the angle is in quad. II.

v) $\sin\theta = -\frac{1}{\sqrt{2}}$ and the terminal arm of the angle is not in quad. III.

5. Find $\cot\theta = \frac{15}{8}$ and the terminal arm of the angle is not is quad. I, find the values of $\cos\theta$ and $\text{cosec}\,\theta$.

6. If $\text{cosec}\ \theta = \frac{m^2+1}{2m}$ and $m \rangle 0 \left(0 < \theta < \frac{\pi}{2}\right)$, find the values of the remaining trigonometric

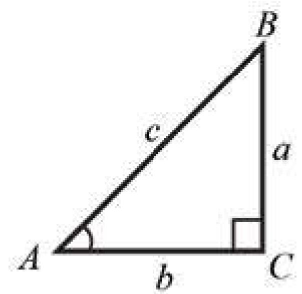
ratios.

7. If $\tan\theta = \frac{1}{\sqrt{7}}$ and the terminal arm of the angle is not in the III quad., find the values of $\frac{\csc^2\theta - \sec^2\theta}{\csc^2\theta + \sec^2\theta}$.

8. If $\cot\theta = \frac{5}{2}$ and the terminal arm of the angle is in the I quad., find the value of $\frac{3\sin\theta + 4\cos\theta}{\cos\theta - \sin\theta}$.

9.10 The values of Trigonometric Functions of acute angles 45°, 30° and 60°

Consider a right triangle *ABC* with $m\angle C = 90^\circ$ and sides *a*, *b*, *c* as shown in the figure on right hand side.



(a) **Case 1** when $m\angle A = 45^\circ = \frac{\pi}{4}$ randian

then $m\angle B = 45^\circ$

$\Rightarrow \Delta ABC$ is right isosceles.

As values of trigonometric functions depend only on the angle and not on the size of the triangle, we can take *a* = *b* = 1

By Pythagoras theorem,

$$c^2 = a^2 + b^2$$

$$\Rightarrow \quad c^2 = 1 + 1 = 2$$

$$\Rightarrow \quad c = \sqrt{2}$$

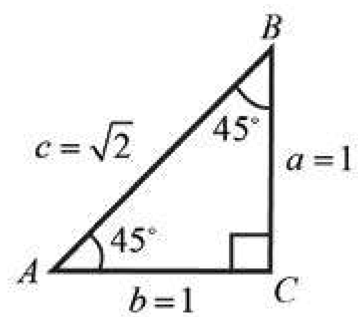


Fig (1)

∴ Using triangle of fig. 1, with *a*=*b*= 1 and *c* = $\sqrt{2}$

$$\sin 45^\circ = \frac{a}{c} = \frac{1}{\sqrt{2}}; \qquad \csc 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2};$$
$$\cos 45^\circ = \frac{b}{c} = \frac{1}{\sqrt{2}}; \qquad \sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2};$$
$$\tan 45^\circ = \frac{a}{b} = 1 \;; \qquad \cot 45^\circ = \frac{1}{\tan 45^\circ} = 1.$$

(b) **Case 2:** when $m\angle A = 30^\circ = \frac{\pi}{6}$ randian

then $m\angle B = 60^\circ$

By elementary geometry, in a right triangle the measure of the side opposite to 30° is half of the hypotenuse.

Let *c* = 2 then *a* = 1

∴ By Pythagoras theorem , $a^2+b^2 = c^4$

$$\Rightarrow b^2 = c^2 - a^2$$
$$= 4 - 1$$
$$= 3$$
$$\Rightarrow b = \sqrt{3}$$

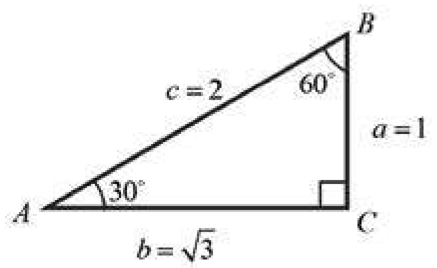


Fig (2)

∴ Using triangle of fig.2, with *a* = 1, $b = \sqrt{3}$ and *c* = 2

$$\sin 30^\circ = \frac{a}{c} = \frac{1}{2}; \qquad \csc 30^\circ = \frac{1}{\sin 30^\circ} = 2;$$
$$\cos 30^\circ = \frac{b}{c} = \frac{\sqrt{3}}{2}; \qquad \sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}};$$
$$\tan 30^\circ = \frac{a}{b} = \frac{1}{\sqrt{3}}; \qquad \cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}.$$

(c) **Case 3:** when $m\angle A = 60^\circ = \frac{\pi}{3}$ radian, then $m\angle B = 30^\circ$

By elementary geometry, in a right triangle the measure of the side opposite to 30° is

half the hypotenuse.

Let *c* = 2 then *b* = 1

∴ By Pythagoras theorem ,

$$\therefore \quad a^2 + b^2 = c^2$$
$$\Rightarrow a^2 = c^2 - b^2$$
$$= 4 - 1 = 3$$
$$\Rightarrow a = \sqrt{3}$$

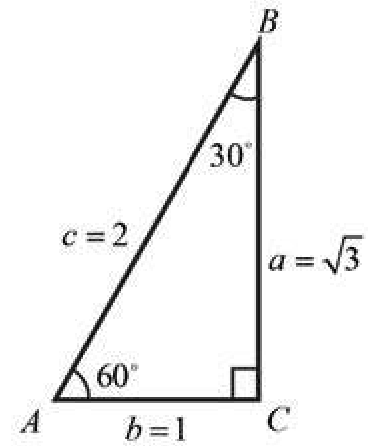


Fig (3)

∴ Using triangle of fig.3, with $a = \sqrt{3}$, , *b* = 1 and *c* = 2

$$\sin 60^\circ = \frac{a}{c} = \frac{\sqrt{3}}{2}; \qquad \csc 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}};$$
$$\cos 60^\circ = \frac{b}{c} = \frac{1}{2}; \qquad \sec 60^\circ = \frac{1}{\cos 60^\circ} = 2;$$
$$\tan 60^\circ = \frac{a}{b} = \sqrt{3}; \qquad \cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}.$$

Example 3: Find the values of all the trigonometric functions of

(i) 420° (ii) $\frac{-7\pi}{4}$ (iii) $\frac{19\pi}{3}$

Solution: We know that $\theta + 2k\pi = \theta$, where $k \in Z$

(i) 420° = 60°+ 1(360°) (*k*= 1)

= 60°

$$\therefore \sin 420^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}; \qquad \csc 420^\circ = \frac{2}{\sqrt{3}}$$
$$\cos 420^\circ = \cos 60^\circ = \frac{1}{2}\;; \quad = \qquad \sec 420^\circ \quad 2$$
$$\tan 420^\circ = \tan 60^\circ = \sqrt{3}\;; \quad = \qquad \cot 420^\circ \quad \frac{1}{\sqrt{3}}$$

(ii) $\frac{-7\pi}{4} = \frac{\pi}{4} + (-1)2\pi \qquad (k = -1)$

$$= \frac{\pi}{4}$$

$$\therefore \sin\left(\frac{-7\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}; \qquad \csc\left(\frac{-7\pi}{4}\right) = \csc\left(\frac{\pi}{4}\right) = \sqrt{2};$$

$$\cos\left(\frac{-7\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}; \qquad \sec\left(\frac{-7\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right) = \sqrt{2};$$

$$\tan\left(\frac{-7\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1; \qquad \cot\left(\frac{-7\pi}{4}\right) = \cot\left(\frac{\pi}{4}\right) = 1;$$

(iii) $\frac{19\pi}{3} = \frac{\pi}{3} + 3(2\pi) \qquad (k = 3)$

$$= \frac{\pi}{3}$$

$$\therefore \sin\left(\frac{19\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}; \qquad \csc\left(\frac{19\pi}{3}\right) = \csc\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}};$$

$$\cos\left(\frac{19\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}; \qquad \sec\left(\frac{19\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right) = 2;$$

$$\tan\left(\frac{19\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}; \qquad \cot\left(\frac{19\pi}{3}\right) = \cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}.$$

9.11 The values of the Trigonometric Functions of angles 0°, 90°, 180°, 270°, 360°.

When terminal line lies on the x– *axis* or the y– *axis*, the angle θ is called a quadrantal angle.

Now we shall find the values of trigonometric functions of quadrantal angles 0°, 90°, 180°, 270°, 360° and so on.

(a) **When θ = 0°**

The point (1,0) lies on the terminal side of angle 0°

$\Rightarrow$ $x = 1$ and $y = 0$

so $r = \sqrt{x^2 + y^2} = 1$

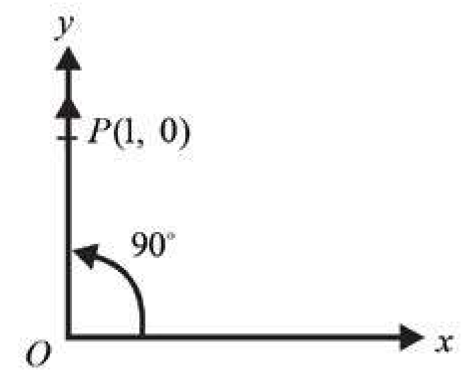
$$\therefore \sin 0^\circ = \frac{y}{r} = \frac{0}{1} = 0 \qquad \csc 0^\circ = \frac{1}{\sin 0^\circ} = \frac{1}{0} \text{ (undefined)}$$

$$\cos 0^\circ = \frac{x}{r} = \frac{1}{1} = 1 \qquad \sec 0^\circ = \frac{1}{\cos 0^\circ} = 1$$

$$\tan 0^\circ = \frac{y}{x} = \frac{0}{1} = 0 \qquad \cot 0^\circ = \frac{1}{\tan 0^\circ} = \frac{1}{0} \text{ (undefined)}$$

(b) **When $\theta = 90°$**

The point (0, 1) lies on the terminal side of angle 90°.



$\Rightarrow$ $x = 0$ and $y = 1$

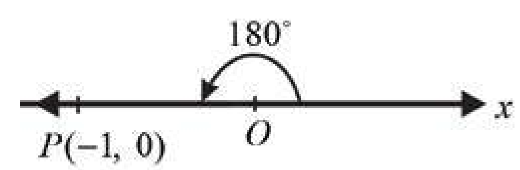
so $r = \sqrt{x^2 + y^2} = 1$

$$\therefore \sin 90^\circ = \frac{y}{r} = \frac{1}{1} = 1; \qquad \csc 90^\circ = \frac{1}{\sin 90^\circ} = 1;$$

$$\cos 90^\circ = \frac{x}{r} = \frac{0}{1} = 0; \qquad \sec 90^\circ = \frac{1}{\cos 90^\circ} = \frac{1}{0} \text{ (undefined)};$$

$$\tan 90^\circ = \frac{y}{x} = \frac{1}{0} \text{ (undefined)}; \qquad \cot 90^\circ = \frac{x}{y} = \frac{0}{1} = 0.$$

(c) **When $\theta = 180°$**



The point (-1, 0) lies on the terminal side of angle 180°.

$\Rightarrow$ $x = -1$ and $y = 0$

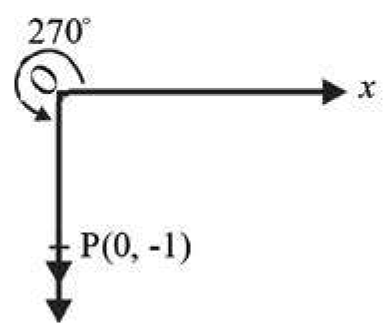
so $r=\sqrt{x^2+y^2}=1$

$\therefore \sin 180^\circ=\frac{y}{r}=\frac{0}{1}=0;$ $\csc 180^\circ=\frac{r}{y}=\frac{1}{0}$ (undefined);

$\cos 180^\circ=\frac{x}{r}=\frac{-1}{1}=-1;$ $\sec 180^\circ=\frac{r}{x}=\frac{1}{-1}=-1;$

$\tan 180^\circ=\frac{y}{x}=\frac{0}{-1}=0;$ $\cot 180^\circ=\frac{x}{y}=\frac{-1}{0}$ (undefined).

(d) **When $\theta = 270^\circ$**



The point (0, -1) lies on the terminal side of angle 270°.

$\Rightarrow$ $x = 0$ and $y = -1$

so $r=\sqrt{x^2+y^2}=1$

$\therefore \sin 270^\circ=\frac{y}{r}=\frac{-1}{1}=-1;$ $\csc 270^\circ=\frac{r}{y}=\frac{1}{-1}=-1;$

$\cos 270^\circ=\frac{x}{r}=\frac{0}{1}=0;$ $\sec 270^\circ=\frac{r}{x}=\frac{1}{0}$ (undefined);

$\tan 270^\circ=\frac{y}{x}=\frac{-1}{0}$ (undefined); $\cot 270^\circ=\frac{x}{y}=\frac{0}{-1}=0.$

Example 4: Find the values of all trigonometric functions of

(i) 360° (ii) $\frac{-\pi}{2}$ (iii) 5π

Solution: We know that $\theta+2k\pi=\theta$, where $k\in Z$

(i) Now 360° = 0° + 1(360°), (k = 1)
= 0°

$\sin 360^\circ=\sin 0^\circ=0;$ $\csc 360^\circ$ is undefined;

$\cos 360^\circ=\cos 0^\circ=1;$ $\sec 360^\circ=\frac{1}{\cos 0^\circ}=1$

$\tan 360^\circ=\tan 0^\circ=0;$ $\cot 360^\circ$ is undefined.

(ii) We know that $\theta+2k\pi=\theta$, where $k\in Z$

Now $-\frac{\pi}{2}=\frac{3\pi}{2}+(-1)2\pi$ $(k=-1)$

$=\frac{3\pi}{2}$

$\therefore \sin\left(-\frac{\pi}{2}\right)=\sin\left(\frac{3\pi}{2}\right)=-1;$ $\csc\left(-\frac{\pi}{2}\right)=-1;$

$\cos\left(-\frac{\pi}{2}\right)=\cos\left(\frac{3\pi}{2}\right)=0;$ $\sec\left(-\frac{\pi}{2}\right)$ is undefined;

$\tan\left(-\frac{\pi}{2}\right)=\tan\left(\frac{3\pi}{2}\right)$ is undefined; $\cot\left(-\frac{\pi}{2}\right)=0$

(iii) Now $5\pi=\pi+2(2\pi)$ $(k=2)$

$=\pi$

$\therefore \sin 5\pi=\sin\pi=0;$ $\csc 5\pi$ is undefined;

$\cos 5\pi=\cos\pi=-1;$ $\sec 5\pi=-1;$

$\tan 5\pi=\tan\pi=0;$ $\cot 5\pi$ is undefined;

Exercise 9.3

1. Verify the following:

(i) sin60° cos 30° – cos 60° sin 30° = sin 30°

(ii) $\sin^2\frac{\pi}{6}+\sin^2\frac{\pi}{3}+\tan^2\frac{\pi}{4}=2$

(iii) $2\sin 45^\circ + \frac{1}{2}\operatorname{cosec} 45^\circ = \frac{3}{\sqrt{2}}$

(iv) $\sin^2\frac{\pi}{6} : \sin^2\frac{\pi}{4} : \sin^2\frac{\pi}{3} : \sin^2\frac{\pi}{2} = 1:2:3:4.$

2. Evaluate the following:

i) $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{6}}$ ii) $\dfrac{1 - \tan^2\frac{\pi}{3}}{1 + \tan^2\frac{\pi}{3}}$

3. Verify the following when θ = 30°, 45°

i) $\sin 2\theta = 2\sin\theta\cos\theta$ ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

iii) $\cos 2\theta = 2\cos^2\theta - 1$ iv) $\cos 2\theta = 1 - 2\sin^2\theta$

v) $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

4. Find x, if $\tan^2 45° - \cos^2 60° = x \sin 45° \cos 45° \tan 60°$.

5. Find the values of the trigonometric functions of the following quadrantal angles:

i) $-\pi$ ii) -3π iii) $\frac{5}{2}\pi$

iv) $-\frac{9}{2}\pi$ v) -15π vi) 1530°

vii) – 2430° viii) $\frac{235}{2}\pi$ ix) $\frac{407}{2}\pi$

6. Find the values of the trigonometric functions of the following angles:

i) 390° ii) – 330° iii) 765°

iv) –675° v) $\frac{-17}{3}\pi$ vi) $\frac{13}{3}\pi$

vii) $\frac{25}{6}\pi$ viii) $\frac{-71}{6}\pi$ ix) –1035°

9.12 Domains of Trigonometric functions and of Fundamental Identities

We list the trigonometric functions and fundamental identities, learnt so far mentioning their domains as follows:

(i) $\sin\theta$, *for all* $\theta \in R$

(ii) $\cos\theta$, *for all* $\theta \in R$

(iii) $\csc\theta = \dfrac{1}{\sin\theta}$ ∈, *fo≠ all* θ R but ∈θ $n\pi$, n Z

(iv) $\sec\theta = \dfrac{1}{\cos\theta}$ ∈, *fo≠all* θ R but ∈θ $\left(\dfrac{2n+1}{2}\right)\pi$, n Z

(v) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ ∈ , ≠ *for all* θ R∈ but θ $(2n\ \ 1)\frac{\pi}{2}$, n Z

(vi) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ ∈, *fo≠ all* θ R but ∈θ $n\pi$, n Z

(vii) $\sin^2\theta + \cos^2\theta = 1$, ∈ *for all* θ R

(viii) $1 + \tan^2\theta = \sec^2\theta$∈ , ≠*for all* θ R but θ $(2n\ \ 1)\frac{\pi}{2}$, n Z

(ix) $1 + \cot^2\theta = \csc^2\theta$ ∈, *fo≠r all* θ R but∈ θ $n\pi$, n Z

Now we shall prove quite a few more identities with the help of the above mentioned identities.

Example 1: Prove that $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$, for all $\theta \in R$

Solution: L.H.S $= \cos^4\theta - \sin^4\theta$

$= \left(\cos^2\theta\right)^2 - \left(\sin^2\theta\right)^2$

$= \left(\cos^2\theta \quad \sin^2\theta\right)\left(\cos^2\theta \quad \sin^2\theta\right)$

$= (1)\left(\cos^2\theta - \sin^2\theta\right) \qquad (\because \ \sin^2\theta \quad \cos^2\theta \quad 1)$

$= \cos^2\theta - \sin^2\theta = \text{R.H.S.}$

Hence $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

Example 2: Prove that $\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \operatorname{cosec}^2 A \left(\textbf{Where } A \neq \frac{n\pi}{2}, n \in Z\right)$

Solution: L.H.S $= \sec^2 A + \operatorname{cosec}^2 A$

$= \dfrac{1}{\cos^2 A} + \dfrac{1}{\sin^2 A} = \dfrac{\sin^2 A + \cos^2 A}{\cos^2 A \sin^2 A}$

$= \dfrac{1}{\cos^2 A \sin^2 A} = \qquad [\because \ \sin^2 A \quad \cos^2 A \quad 1]$

$= \dfrac{1}{\cos^2 A} \cdot \dfrac{1}{\sin^2 A}$

$== \sec^2 A . \operatorname{cosec}^2 A \qquad \text{R.H.S}$

Hence $\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A . \operatorname{cosec}^2 A.$

Example 3: Prove that $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$, where θ is not an odd multiple of $\dfrac{\pi}{2}$.

Solution:

L.H.S. $= \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

$= \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}\sqrt{\dfrac{1-\sin\theta}{1-\sin\theta}}$ (rationalizing.)

$= \sqrt{\dfrac{(1-\sin\theta)^2}{1-\sin^2\theta}}$

$== \sqrt{\dfrac{(1-\sin\theta)^2}{\cos^2\theta}} \qquad \dfrac{1-\sin\theta}{\cos\theta}$

$= \dfrac{1}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{R.H.S}$

Hence $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta.$

Example 4: Show that $\cot^4\theta + \cot^2\theta = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta$, where θ is not an integral multiple of $\dfrac{\pi}{2}$.

Solution:

L.H.S. $= \cot^4\theta + \cot^2\theta$

$=+ \cot^2\theta(\cot^2\theta \quad 1)$

$= (\operatorname{cosec}^2\theta - 1)\operatorname{cosec}^2\theta$

$= \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta$

$=$ R.H.S.

Hence $\cot^4\theta + \cot^2\theta = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta.$

Exercise 9.4

Prove the following identities, state the domain of θ in each case:

1. $\tan\theta + \cot\theta = \operatorname{cosec}\theta \sec\theta$ **2.** $\sec\theta \operatorname{cosec}\theta \sin\theta \cos\theta = 1$

3. $\cos\theta + \tan\theta\sin\theta = \sec\theta$

4. $\text{cosec}\,\theta + \tan\theta\sec\theta = \text{cosec}\,\theta\sec^2\theta$

5. $\sec^2\theta - \text{cosec}^2\theta = \tan^2\theta - \cot^2\theta$

6. $\cot^2\theta - \cos^2\theta = \cot^2\theta\cos^2\theta$

7. $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

8. $2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

9. $\cos^2\theta - \sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

10. $\dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \dfrac{\cot\theta - 1}{\cot\theta + 1}$

11. $\dfrac{\sin\theta}{1+\cos\theta} + \cot\theta = \text{cosec}\,\theta$

12. $\dfrac{\cot^2\theta - 1}{1+\cot^2\theta} = 2\cos^2\theta - 1$

13. $\dfrac{1+\cos\theta}{1-\cos\theta} = (\text{cosec}\,\theta + \cot\theta)^2$

14. $(\sec\theta - \tan\theta)^2 = \dfrac{1-\sin\theta}{1+\sin\theta}$

15. $\dfrac{2\tan\theta}{1+\tan^2\theta} = 2\sin\theta\cos\theta$

16. $\dfrac{1-\text{sion}\,\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

17. $(\tan\theta + \cot\theta)^2 = \sec^2\theta\,\text{cosec}^2\theta$

18. $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$

19. $\dfrac{1}{\text{cosec}\,\theta - \cot\theta} - \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{1}{\text{cosec}\,\theta + \cot\theta}$

20. $\sin^3\theta - \cos^3\theta = (\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta)$

21. $\sin^6\theta - \cos^6\theta = (\sin^2\theta - \cos^2\theta)(1 - \sin^2\theta\cos^2\theta)$

22. $\sin^6\theta + \cos^6\theta = 1 \quad 3\sin^2\theta\cos^2\theta$

23. $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} = 2\sec^2\theta$

24. $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} + \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \dfrac{2}{1 - 2\sin^2\theta}$