CHAPTER

13

Invnerse Trignometric Functions

13.1 Introduction

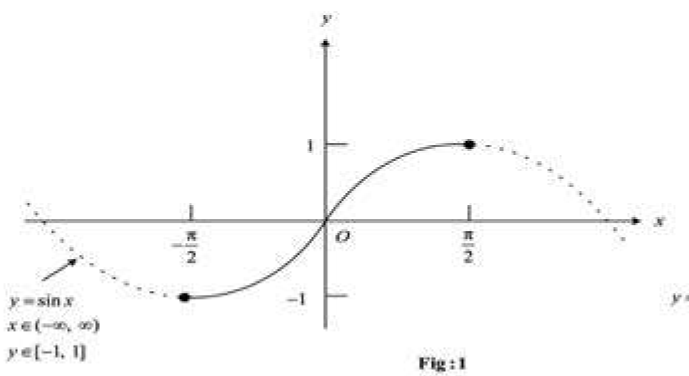
We have been finding the values of trigonometric functions for given measures of the angles. But in the application of trigonometry, the problem has also been the other way round and we are required to find the measure of the angle when the value of its trigonometric function is given. For this purpose, we need to have the knowledge of **inverse trigonometric functions.**

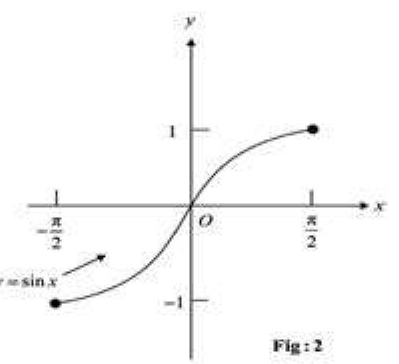
In chapter 2, we have discussed inverse functions. We learned that only a one-to-one function will have an inverse. If a function is not one-to-one, it may be possible to restrict its domain to make it one-to-one so that its inverse can be found.

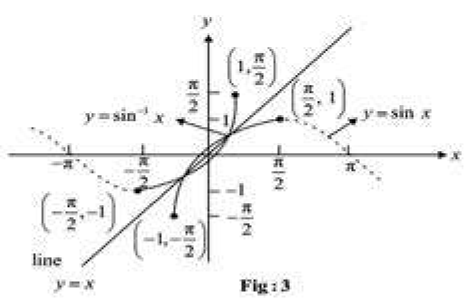
In this section we shall define the **inverse trigonometric functions.**

13.2 The Inverse sine Function:

The graph of $y = \sin x$, $-\infty < x < +\infty$, is shown in the figure 1.







We observe that every horizontal line between the lines y = 1 and y= -1 intersects the graph infinitly many times. It follows that the sine function is not one-to-one.However, if we

restrict the domain of y = sin$x$ to the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then the restricted function y = sin$x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ is called the **principal sine function;** which is now one-to-one and hence will have an inverse as shown in figure 2.

This inverse function is called the **inverse sin function** and is written as $\sin^{-1}x$ or arc sin$x$.

The **Inverse sine Function** is defined by:

y = $\sin^{-1}x$, if and only if x = sin y.

where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $-1 \le x \le 1$

Here y is the angle whose sine is x. The domain of the function

y = $\sin^{-1}x$ is $-1 \le x \le 1$, its range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

The graph of $y = \sin^{-1}x$ is obtained by reflecting the restricted portion of the graph of y = sin$x$ about the line y = x as shown in figure 3.

We notice that the graph of $y = \sin x$ is along the x - axis whereas the graph of $y = \sin^{-1}x$ is along the y - axis.

Note: It must be remembered that $\sin^{-1}x \neq (\sin x)^{-1}$.

Example 1: Find the value of (i) $\sin^{-1}\frac{\sqrt{3}}{2}$ (ii) $\sin^{-1}(-\frac{1}{2})$

Solution: (i) We want to find the angle y, whose sine is $\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \sin y = \frac{\sqrt{3}}{2}, \le \le \quad \frac{\pi}{2} \; y \; \frac{\pi}{2}$

$\Rightarrow \quad y = \frac{\pi}{3}$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

(ii) We want to find the angle y whose sine is $-\frac{1}{2}$

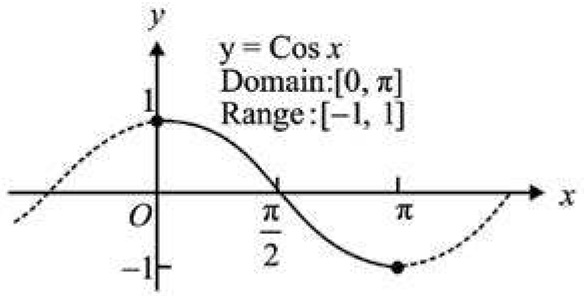
$\Rightarrow \quad \sin y = -\frac{1}{2}, \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\therefore \quad -y = \frac{\pi}{6}$

$\therefore \quad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

13.3 The Inverse Cosine Function:

The graph of $y = \cos x$, $-\infty < x < +\infty$, is shown in the figure 4.



(a) Fig : 4

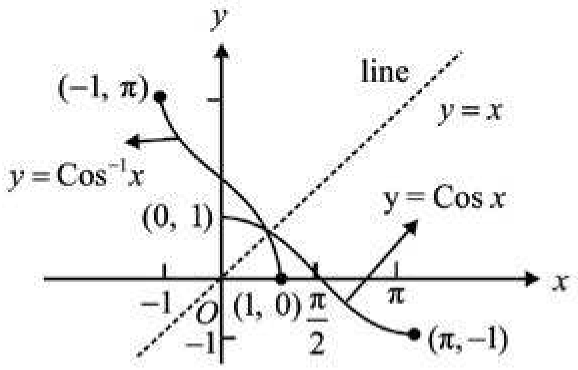


Fig : 6

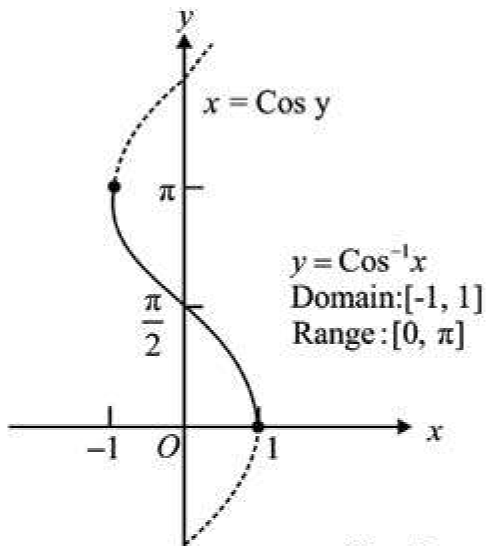


Fig : 5

We observe that every horizontal line between the lines y = 1 and y = -1 intersects the graph infinitly many times. It follows that the cosine function is not one-to-one. However, if we restrict the domain of y = cos$x$ to the interval [0, π], then the restricted function y = cos$x$, $0 \le x \le \pi$ is called the **principal cosine function;** which is now one-to-one and hence will have an inverse as shown in figure 5.

This inverse function is called the inverse cosine function and is written as $\cos^{-1}x$ or arc cos$x$.

The **Inverse Cosine Function** is defined by:

y = $\cos^{-1}x$, if and only if x= cos y.

where $0 \le y \le \pi$ and $-1 \le x \le 1$.

Here y is the angle whose cosine is x . The domain of the function y = $\cos^{-1}x$ is $-1 \le x \le 1$ and its range is $0 \le y \le \pi$.

The graph of y = $\cos^{-1}x$ is obtained by reflecting the restricted portion of the graph of y = cos x about the line y = x as shown in figure 6.

We notice that the graph of y = cos x is along the x - *axis* whereas the graph of y = $\cos^{-1}x$ is along the y - *axis* .

Note: It must be remembered that $\cos^{-1}x \neq (\cos x)^{-1}$

Example 2: Find the value of (i) $\cos^{-1}1$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution: (i) We want to find the angle y whose cosine is 1

$\Rightarrow \quad \cos y = 1, \qquad 0 \le y \le \pi$

$\Rightarrow \quad y = 0$

$\therefore \quad \cos^{-1}1 = 0$

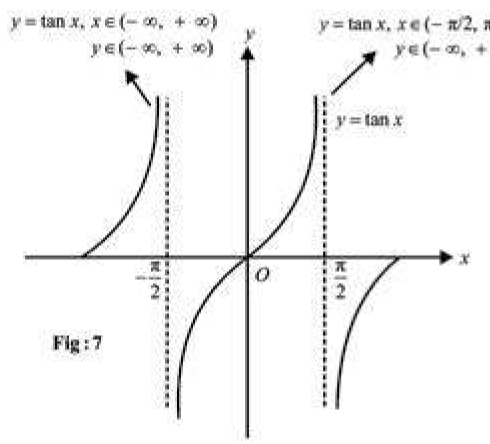
(ii) We want to find the angle y whose cosine is $-\frac{1}{2}$

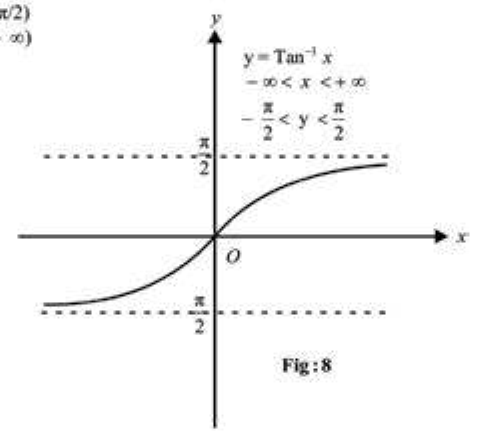
$\Rightarrow \quad \cos y = -\frac{1}{2}, \quad 0 \le y \le \pi$

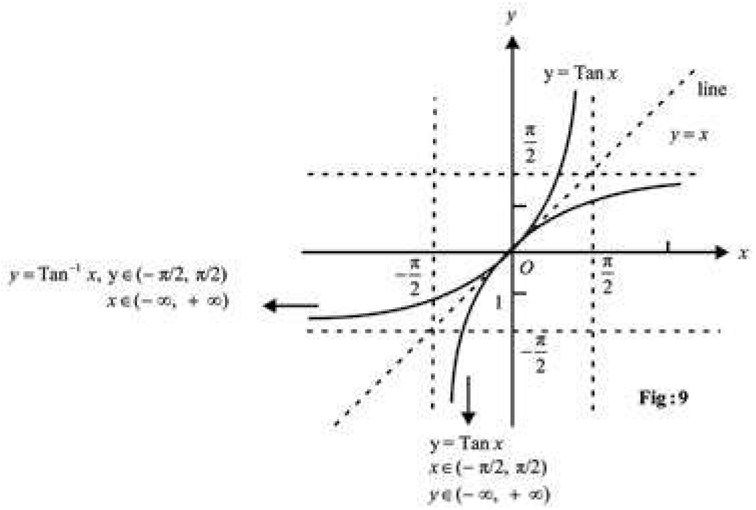
$$\therefore \quad y = \frac{2\pi}{3}$$

$$\therefore \quad \cos^{-1}(-\frac{1}{2}) = -\frac{2\pi}{3}$$

13.4 Inverse Tangent Function:







The graph of y = tan$x$, $-\infty < x < +\infty$, is shown in the figure 7.
We observe that every horizontal line between the lines y = 1 and $y = -1$ intersect the graph infinitly many times. It follows that the tangent function is not one-to-one.

However, if we restrict the domain of y = *Tan*x to the interval $\frac{-\pi}{2} < x < \frac{\pi}{2}$, then the restricted

function y = tan$x$ $\frac{-\pi}{2} < x < \frac{\pi}{2}$, is called the **Principal tangent function;** which is now one-to-one and hence will have an inverse as shown in figure 8.

This inverse function is called the **inverse tangent function** and is written as $\tan^{-1}x$ or arc tan$x$.

The **Inverse Tangent Function** is defined by:

y = $\tan^{-1}x$, if and only if x = tan y.

where $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $-\infty < x < +\infty$.

Here y is the angle whose tangent is x. The domain of the function y = $\tan^{-1}x$ is $-\infty < x < +\infty$ and its range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$

The graph of y = $\tan^{-1}x$ is obtained by reflecting the restricted portion of the graph of y = tan$x$ about the line y = x as shown in figure 9.

We notice that the graph of y = tan$x$ is along the x - axis whereas the graph of y = tan$x$ is along the y- axis.

Note: It must be remembered that $\tan^{-1}x \neq (\tan x)^{-1}$.

Example 3: Find the value of (i) $\tan^{-1}1$ (ii) $\tan^{-1}(-\sqrt{3})$

Solution: (i) We want to find the angle y, whose tangent is 1

$$\Rightarrow \quad \tan y = 1, \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\Rightarrow \quad y = \frac{\pi}{4}$$

$$\therefore \quad \tan^{-1}1 = \frac{\pi}{4}$$

(ii) We want to find the angle y whose tangent is $-\sqrt{3}$

$$\Rightarrow \quad \tan y = -\sqrt{3} \qquad \frac{\pi}{2} \not< y \not< \frac{\pi}{2}$$

$\therefore \quad y = \frac{2\pi}{3}$

$\therefore \quad \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$

13.5 Inverse Cotangent, Secant and Cosecant Functions

These inverse functions are not used frequently and most of the calculators do not even have keys for evaluating them. However, we list their definitions as below:

i) Inverse Cotangent function:

y = cot$x$, where $0 \le x \le \pi$ is called the **Principal Cotangent Function,** which is one-to-one and has an inverse.

The inverse cotangent function **is defined by:**

y = cot^{-1}x , if and only if x = coty

Where $0 < y < \pi$ and $-\infty < x < +\infty$

The students should draw the graph of y = cot^{-1} x by taking the reflection of y = cot$x$ in the line y = x. This is left as an exercise for them.

ii) Inverse Secant function

y = sec x, where $0 \le x \le \pi$ and $x \ne \frac{\pi}{2}$ is called the **Principal Secant Function,** which is one-to-one and has an inverse.

The Inverse Secant Function **is defined by:**

y = sec^{-1}x. if and only if x = secy

where $0 \le y \le \pi, \quad y \ne \frac{\pi}{2}$ and $|x| \ge 1$

The students should draw the graph of y = sec^{-1}x by taking the reflection of y = sec$x$ in the line y = x. This is left an exercise for them,

iii) Inverse Cosecant Function

y = csc x, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $x \ne 0$ is called the **Principal Cosecant Function,**

which is one-to-one and has an inverse.

The **Inverse Cosecant Function** is defined by:

$y = \csc^{-1} x$, if and only if $x = \csc y$

where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$ and $|x| \ge 1$

The students should draw the graph of $y = \csc^{-1}$ by taking the reflection of y = csc$x$ in the line y = x. This is left an exercise for them.

Note. While discusspjpihe Inverse Trigonometric Functions, we have seen that there are in general, no inverses of Trigonometric Functions, but restricting their domain to principal Functions, we have made them as functions.

13.6 Domains and Ranges of Principal Trigonometric Function and Inverse Trigonometric Functions.

From the discussion on the previous pages we get the following table showing domains and ranges of the Principal Trigonometric and Inverse Trigonometric Functions.

Functions	Domain	Range
$y = \sin x$	$\frac{-\pi}{2} \le x \le \frac{\pi}{2}$	$-1 \le x \le 1$
$y = \sin^{-1} x$	$-1 \le x \le 1$	$\frac{-\pi}{2} \le x \le \frac{\pi}{2}$
$y = \cos x$	$0 \le x \le \pi$	$-1 \le x \le 1$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le x \le \pi$
$y = \tan x$	$\frac{-\pi}{2} < x < \frac{\pi}{2}$	$(-\infty, \infty)$ or $\Re$

$y=\tan^{-1}x$	$(-\infty,\infty)$ or $\Re$	$\frac{-\pi}{2}<x<\frac{\pi}{2}$
$y=\cot x$	$0<x<\pi$	$(-\infty,\infty)$ or $\Re$
$y=\cot^{-1}x$	$(-\infty,\infty)$ or $\Re$	$0<x<\pi$
$y=\sec x$	$[0,\pi], x\neq\frac{\pi}{2}$	$y\leq-1$ or $y\geq1$
$y=\sec^{-1}x$	$x\geq-1$ or $x\leq1$	$[0,\pi], y\neq\frac{\pi}{2}$
$y=\csc x$	$[-\frac{\pi}{2},\frac{\pi}{2}], x\neq0$	$y\leq-1$ or $y\geq1$
$y=\csc^{-1}x$	$x\leq-1$ or $x\geq1$	$[-\frac{\pi}{2},\frac{\pi}{2}],\ y\neq0$

Example 4: Show that $\cos^{-1}\frac{12}{13}\quad\sin^{-1}\frac{5}{13}$

Solution: Let $\cos^{-1}\frac{12}{13}=\alpha \Rightarrow \cos\alpha \quad \frac{12}{13}$

$$\therefore \quad \sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \pm\sqrt{1-\left(\frac{12}{13}\right)^2}$$

$$= \pm\sqrt{1-\frac{144}{169}}$$

$$= \sqrt{\frac{169-144}{169}} \pm = \sqrt{\frac{25}{169}} \quad \frac{5}{13}$$

$\because$ cos α is +ve and domain of α is $[0, \pi]$, in which sine is +ve.

Thus $\quad \sin\alpha = \frac{5}{13} \Rightarrow \quad \alpha = \sin^{-1}\frac{5}{13}$

Hence $\quad \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{5}{13}$

Example 5: Find the value of

i) $\sin(\cos^{-1}\frac{\sqrt{3}}{2})$ ii) $\cos(\tan^{-1}0)$ iii) $\sec[\sin^{-1}(-\frac{1}{2})]$ **Solution:**

i) we first find the value of y, whose cosine is $\frac{\sqrt{3}}{2}$

$$\cos y = \frac{\sqrt{3}}{2}, \qquad 0\leq y\leq\pi$$

$$\Rightarrow \quad y=\frac{\pi}{6}$$

$$\Rightarrow \quad (\cos^{-1}\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$$

$$\therefore \quad \sin(\cos^{-1}\frac{\sqrt{3}}{2}) = \sin\frac{\pi}{6} \quad \frac{1}{2}$$

ii) we first find the value of y, whose tangent is 0

$$\tan y = 0, \qquad -\frac{\pi}{2}<y<\frac{\pi}{2}$$

$$\Rightarrow \quad y=0$$

$$\Rightarrow \quad (\tan^{-1}0) = 0$$

$$\therefore \quad \cos(\tan^{-1}0) = \cos 0 \quad 1$$

iii) we first find the value of y, whose sine is $-\frac{1}{2}$

$$\sin y = -\frac{1}{2}, \qquad -\frac{\pi}{2}\leq y\leq\frac{\pi}{2}$$

$$\Rightarrow \quad -y = \frac{\pi}{6}$$

$$\Rightarrow \quad \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$$

$\therefore \quad \sec[\sin^{-1}(-\frac{1}{2})]=\frac{2}{\sqrt{3}}$

Example: 6 Prove that the inverse trigonometric functions satisfy the following identities:

i) $\sin^{-1}x=\frac{\pi}{2}\ \cos^{-1}x$ and $\cos^{-1}x\ \frac{\pi}{2}\ \sin^{-1}x$

ii) $\tan^{-1}x=\frac{\pi}{2}\ \cot^{-1}x$ and $\cot^{-1}x\ \frac{\pi}{2}\ \tan^{-1}x$

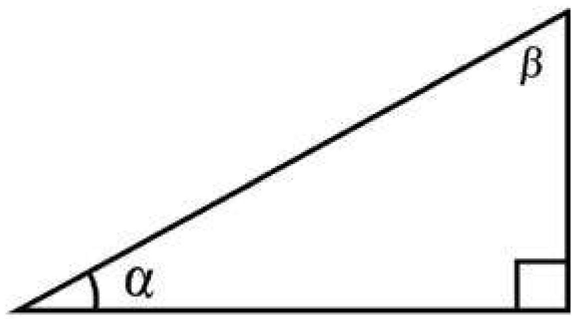
iii) $\sec^{-1}x=\frac{\pi}{2}\ \csc^{-1}x$ and $\csc^{-1}x\ \frac{\pi}{2}\ \sec^{-1}x$

Proof:

Consider the right triangle given in the figure Angles α and β are acute and complementary.

$\Rightarrow \quad \alpha+\beta=\frac{\pi}{2}$

$\Rightarrow \quad \alpha=\frac{\pi}{2}-\beta$ and $\beta=\frac{\pi}{2}-\alpha$ ...(i)



Now $\sin\alpha=\sin(\frac{\pi}{2}-\beta)=\cos\beta=x$ (say)

$\therefore \quad \alpha=\sin^{-1}x$ and $\beta=\cos^{-1}x$

Thus from (i) we have:

$\sin^{-1}x=\frac{\pi}{2}\ \cos^{-1}x$ and $\cos^{-1}x\ \frac{\pi}{2}\ \sin^{-1}x$

In a similar way, we can derive the identities (ii) and (iii).

Exercise 13.1

1. Evaluate without using tables / calculator:

i) $\sin^{-1}(1)$ ii) $\sin^{-1}(-1)$ iii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

iv) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ v) $\cos^{-1}\left(\frac{1}{2}\right)$ vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

vii) $\cot^{-1}(-1)$ viii) $\operatorname{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ ix) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

2. Without using table/ Calculator show that:

i) $\tan^{-1}\frac{5}{12}=\sin^{-1}\frac{5}{13}$ ii) $2\cos^{-1}\frac{4}{5}=\sin^{-1}\frac{24}{25}$

iii) $\cos^{-1}\frac{4}{5}=\cot^{-1}\frac{4}{3}$

3. Find the value of each expression:

i) $\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)$ ii) $\sec\left(\cos^{-1}\frac{1}{2}\right)$ iii) $\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$

iv) $\csc\left(\tan^{-1}(-1)\right)$ v) $\sec\left(\sin^{-1}(-\frac{1}{2})\right)$ vi) $\tan\left(\tan^{-1}(-1)\right)$

vii) $\sin\left(\sin^{-1}(\frac{1}{2})\right)$ viii) $\tan\left(\sin^{-1}(-\frac{1}{2})\right)$ ix) $\sin\left(\tan^{-1}(-1)\right)$

13.7 Addition and Subtraction Formulas

1) Prove that:

$\sin^{-1}A+\sin^{-1}B=\sin^{-1}(A\sqrt{1-B^2}+B\sqrt{1-A^2})$

Proof: Let $\sin^{-1}A=x \Rightarrow \sin x=A$

and $\sin^{-1}B=y \Rightarrow \sin y=B$

Now $\cos x=\pm\sqrt{1-\sin^2 x}=\pm\sqrt{1-A^2}$

In sin$x$ = A, domain = $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, in which Cosine is +ve,

$\therefore \quad \cos x=\sqrt{1\quad A^2}$

Similarly, $\cos y=\sqrt{1-B^2}$

Now $\sin(x+y)=\sin x\cos y+\cos x\sin y$

$$= A\sqrt{1-B^2} + B\sqrt{1-A^2}$$

$$\Rightarrow \quad x + y = \sin^{-1}(A\sqrt{1-B^2} - B\sqrt{1-A^2}) \quad \therefore$$

$$\boxed{\sin^{-1} A + \sin^{-1} B = \sin^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})}$$

In a similar way, we can prove that

2) $\boxed{\sin^{-1} A - \sin^{-1} B = \sin^{-1}(A\sqrt{1-B^2} - B\sqrt{1-A^2})}$

3) $\boxed{\cos^{-1} A + \cos^{-1} B = \cos^{-1}(AB - \sqrt{(1-A^2)(1-B^2)})}$

4) $\boxed{\cos^{-1} A - \cos^{-1} B = \cos^{-1}(AB + \sqrt{(1-A^2)(1-B^2)})}$

5) Prove that:

$$\tan^{-1} A + \tan^{-1} B = \tan^{-1}\frac{A+B}{1-AB}$$

Proof: Let $\tan^{-1} A = x \Rightarrow \tan x = A$

and $\tan^{-1} B = y \Rightarrow \tan y = B$

Now $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y} = \dfrac{A+B}{1-AB}$

$$\Rightarrow \quad x + y = \tan^{-1}\frac{A+B}{1-AB}$$

$$\therefore \quad \boxed{\tan^{-1} A + \tan B = \tan^{-1}\frac{A+B}{1-AB}}$$

In a similar way, we can prove that

6) $\boxed{\tan^{-1} A - \tan^{-1} B = \tan^{-1}\dfrac{A-B}{1+AB}}$

Cor. Putting *A - B* in

$\tan^{-1} A + \tan^{-1} B = \tan^{-1}\dfrac{A+B}{1-AB}$, we get

$$\tan^{-1} A + \tan^{-1} A = \tan^{-1}\frac{A+A}{1-A^2}$$

$$\Rightarrow 2\tan^{-1} A = \tan^{-1}\frac{2A}{1-A^2}$$

Exercise 13.2

Prove the following:

1. $\sin^{-1}\frac{5}{13} + \sin^{-1}\frac{7}{25} = \cos^{-1}\frac{253}{325}$

2. $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{9}{19}$

3. $2\tan^{-1}\frac{2}{3} = \sin^{-1}\frac{12}{13}$ $\left[\textbf{Hint: } \text{Let } \tan^{-1}\frac{2}{3} = x \text{ and shown } \sin 2x \ \frac{12}{13}\right]$

4. $\tan^{-1}\frac{120}{119} = 2\cos^{-1}\frac{12}{13}$

5. $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1} 3 = \frac{\pi}{4}$

6. $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{85}$

7. $\sin^{-1}\frac{77}{85} - \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{15}{17}$

8. $\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}$

9. $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$

$\left[\textbf{Hint: } \text{First add } \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} \text{ and then proceed}\right]$

10. $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$

11. $\tan^{-1}\frac{1}{11}+\tan^{-1}\frac{5}{6}=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}$

12. $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

13. Show that $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

14. Show that $\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}$

15. Show that $\cos(2\sin^{-1}x) = 1-2x^2$

16. Show that $\tan^{-1}(-x)- \quad = \quad \tan^{-1}x$

17. Show that $\sin^{-1}(-x)- \quad = \quad \sin^{-1}x$

18. Show that $\cos^{-1}(-x) \quad -=\pi \quad \cos^{-1}x$

19. Show that $\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$

20. Given that $x=\sin^{-1}\frac{1}{2}$, find the values of following trigonometric functions: sin$x$, cos$x$, tan$x$, cot$x$, sec$x$ and csc$x$.