CHAPTER

4

Introduction to Analytic Geometry

Animation 4.1: Coordinate System
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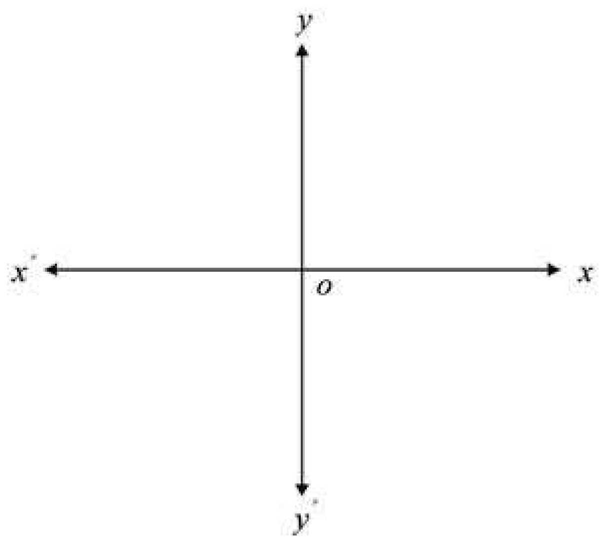
4.1 INTRODUCTION

Geometry is one of the most ancient branches of mathematics. The Greeks systematically studied it about four centuries B.C. Most of the geometry taught in schools is due to Euclid who expounded thirteen books on the subject (300 B.C.). A French philosopher and m athematician Rene Descartes (1596-1650 A.D.) introduced algebraic methods in geometry which gave birth to analytical geometry (or coordinate geometry). Our aim is to present fundamentals of the subject in this book.

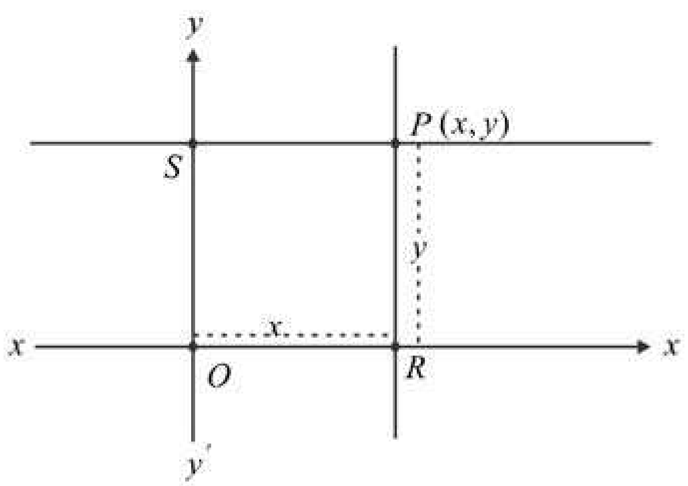
Coordinate System

Draw in a plane two mutually perpendicular number lines $x'x$ and $y'y$, one horizontal and the other vertical. Let their point of intersection be *O* , to which we call the **origin** and the real number 0 of both the lines is represented by *O*. The two lines are called the **coordinate axes**. The horizontal line $x'Ox$ is called the ***x*-axis** and the vertical line $y'Oy$ is called the ***y*-axis**.



As in the case of number line, we follow the convention that all points on the *y*-axis above $x'Ox$ are associated with positive real numbers, those below $x'Ox$ with negative real numbers. Similarly, all points on the *x*-axis and lying on the right of *O* will be positive and those on the left of *O* and lying on the *x*-axis will be negative.

Suppose *P* is any point in the plane. Then ***P*** can be **located** by using an ordered pair of real numbers. Through *P* draw lines parallel to the coordinates axes meeting x-axis at *R* and *y*-axis at *S*.



Let the directed distance $\overline{OR} = x$ and the directed distance $\overline{OS} = y$.

The ordered pair (*x*, *y*) gives us enough information to locate the point *P*. Thus, with every point *P* in the plane, we can associate an ordered pair of real numbers (*x*, *y*) and we say

that *P* has **coordinates** (*x*, *y*). It may be noted that *x* and *y* are the directed distances of *P* from the *y*-axis and the *x*-axis respectively. The reverse of this technique also provides method for associating exactly one point in the plane with any ordered pair (*x*, *y*) of real numbers. This method of pairing off in a one-to-one fashion the points in a plane with ordered pairs of real numbers is called the **two dimensional rectangular** (or **Cartesian**) **coordinate system**.

If (*x*, *y*) are the coordinates of a point *P*, then the first member (component) of the ordered pair is called the **x – coordinate** or **abscissa** of *P* and the second member of the ordered pair is called the **y – coordinate** or **ordinate** of *P*. Note that abscissa is always first element and the ordinate is second element in an ordered pair.

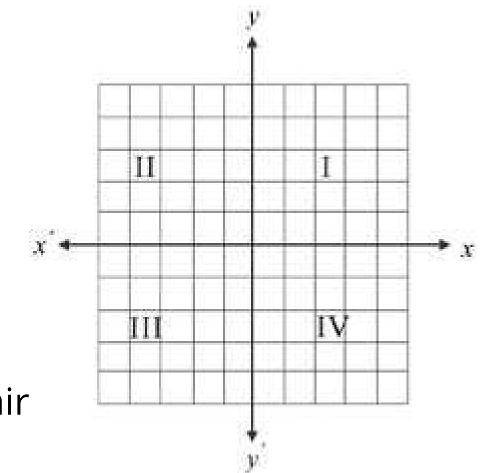
The coordinate axes divide the plane into four equal parts called **quadrants**. They are defined as follows:

Quadrant I: All points (*x*, *y*) with $x > 0$, $y > 0$

Quadrant II: All points (*x*, *y*) with $x < 0$, $y > 0$

Quadrant III: All points (*x*, *y*) with $x < 0$, $y < 0$

Quadrant IV: All points (*x*, *y*) with $x > 0$, $y < 0$



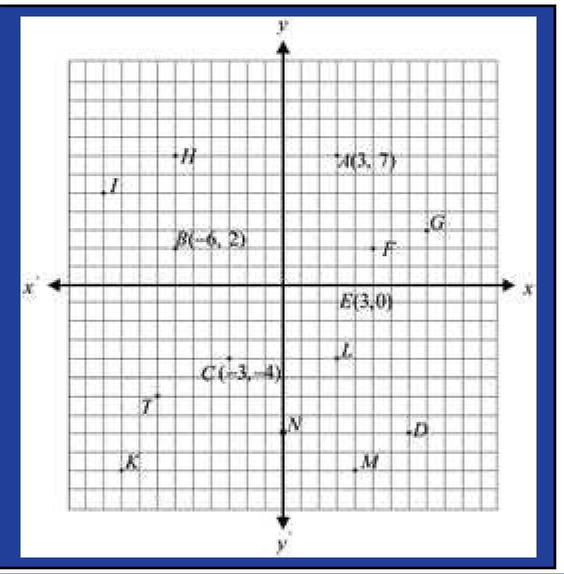
The point *P* in the plane that corresponds to an ordered pair (*x*, *y*) is called the graph of (*x*, *y*).

Thus given a set of ordered pairs of real numbers, the graph of the set is the aggregate of all points in the plane that correspond to ordered pairs of the set.

Challenge!

i- Write down the coordinates of the points if not mentioned.

ii- Locate (0, –1), (2, 2), (–4, 7) and (–3, –3).



4.1.1 The Distance Formula

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in the plane. We can find the distance $d = |\overline{AB}|$ from the right triangle AQB by using the Pythagorean theorem. We have

> **Note that :**
> AB stands for
> $m\overline{AB}$ or $|\overline{AB}|$

$$d = AB = AQ + QB^2 \qquad (1)$$

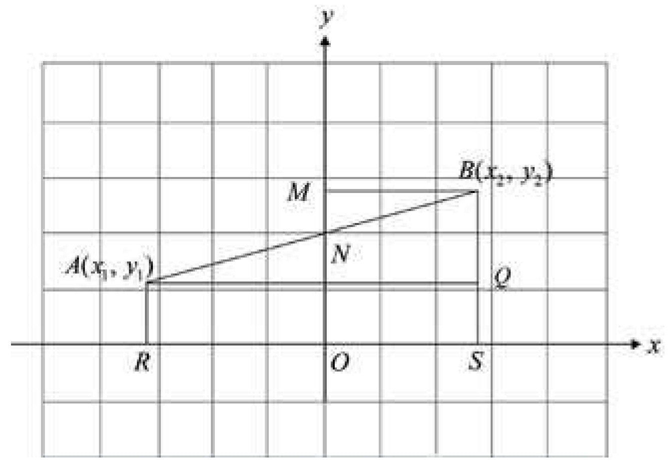
$$|AQ| = |RS| = |RO + OS|$$

$$= |-OR + OS|$$

$$= |x_2 - x_1|$$

$$|QB| = |SB - SQ| = |OM - ON|$$

$$= |y_2 - y_1|$$



Therefore, (1) takes the form

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

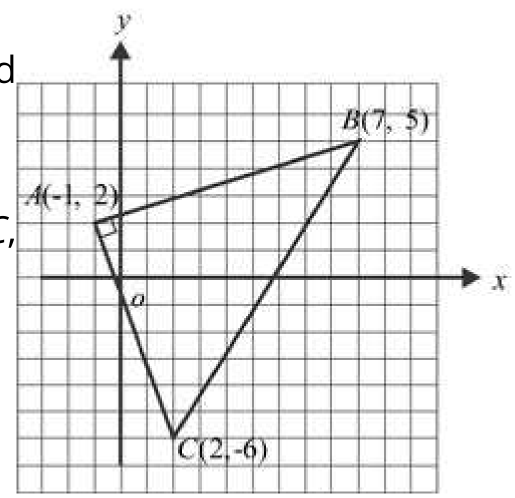
or $$d = |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad (2)$$

which is the formula for the distance d. The distance is always taken to be positive and it is not a directed distance from A to B when A and B do not lie on the same horizontal or vertical line.

If A and B lie on a line parallel to one of the coordinate axes, then by the formula (2), the distance AB is absolute value of the directed distance $\overline{AB}$.

The formula (2) shows that any of the two points can be taken as first point.

Example 1: Show that the points $A(-1, 2)$, $B(7, 5)$ and $C(2, -6)$ are vertices of a right triangle.



Solution: Let a, b and c denote the lengths of the sides BC, CA and AB respectively.

By the distance formula, we have

$$c = AB = \sqrt{(7-(-1))^2 + (5-2)^2} = \sqrt{73}$$

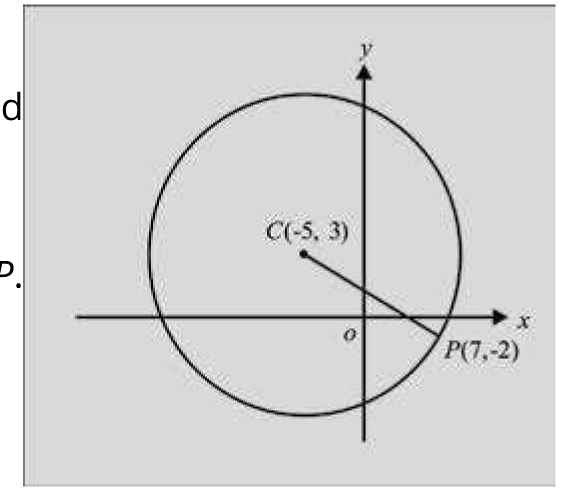
$$a = BC = \sqrt{(2-7)^2 + (-6-5)^2} = \sqrt{146}$$

$$b = CA = \sqrt{2-(-1)^2 + (-6-2)^2} = \sqrt{73}$$

Clearly: $a^2 = b^2 + c^2$.

Therefore, ABC is a right triangle with right angle at A.

Example 2: The point $C(-5, 3)$ is the centre of a circle and $P(7, -2)$ lies on the circle. What is the radius of the circle?



Solution: The radius of the circle is the distance from C to P. By the distance formula, we have

Radius $= CP = \sqrt{(7-(-5))^2 + (-2-3)^2}$

$= \sqrt{144 + 25} = 13$

4.1.2 Point Dividing the Join of Two Points in a given Ratio

Theorem: Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points in a plane. The coordinates of the point dividing the line segment AB in the ratio $k_1 : k_2$ are

$$\left(\frac{k_1x_2 + k_2x_1}{k_1 + k_2}, \frac{k_1y_2 + k_2y_1}{k_1 + k_2}\right)$$

Proof: Let $P(x, y)$ be the point that divides AB in the ratio $k_1 : k_2$

From A, B and P draw perpendiculars to the x-axis as shown in the figure. Also draw $BC \perp AQ$. Since LP is parallel to CA, in the triangle ACB, we have

$$\frac{k_1}{k_2} = \frac{AP}{PB} = \frac{CL}{LB} = \frac{QM}{MR} = \frac{x - x_1}{x_2 - x}$$

So, $$\frac{k_1}{k_2} = \frac{x - x_1}{x_2 - x}$$

or $$k_1x_2 - k_1x = k_2x - k_2x_1$$

or $$(k_1 + k_2)x = k_1x_2 + k_2x_1$$

or $\quad x = \dfrac{k_1x_2 + k_2x_1}{k_1 + k_2}$

Similarly, by drawing perpendiculars from *A* , *B* and *P* to the *y*-axis and

proceeding as before, we can show that $\quad y = \dfrac{k_1y_2 + k_2y_1}{k_1 + k_2}$

Note:

(i) If the directed distances *AP* and *PB* have the same sign, then their ratio is positive and *P* is said to divide *AB* internally.

(ii) If the directed distances *AP* and *PB* have opposite signs i.e, *P* is beyond *AB*. then their ratio is negative and P is said to divide *AB* **externally**.

$$\frac{AP}{BP} = \frac{k_1}{k_2} \text{ or } \frac{AP}{PB} = -\frac{k_1}{k_2}$$

Proceeding as before, we can show in this case that

$$x = \frac{k_1x_2 - k_2x_1}{k_1 - k_2} \quad y \quad \frac{k_1y_2 - k_2y_1}{k_1 - k_2}$$

Thus *P* is said to divide the line segment *AB* in ratio $k_1 : k_2$, **internally** or **externally** according as *P* lies between *AB* or beyond *AB*.

(iii) If $k_1 = k_2 = 1:1$, then *P* becomes **midpoint** of $\overline{AB}$ and coordinates of *P* are :

$$x = \frac{x_1 + x_2}{2}, \quad y \quad \frac{y_1 - y_2}{2}$$

(iv) The above theorem is valid in whichever quadrant *A* and *B* lie.

Example 1: Find the coordinates of the point that divides the join of *A* (–6, 3) and *B* (5, –2) in the ratio 2 : 3.

(i) internally (ii) externally

Solution: (i) Here $k_1 = 2, k_2 = 3, x_1 = \ 6, x_2 \ 5$.

By the formula, we have

$$x = \frac{2\times5+3\times(-6)}{2+3} \quad \frac{-8}{5} \quad \text{and} \quad y \quad \frac{2(-2)+3(3)}{2+3} \quad 1$$

Coordinates of the required point are $\left(\dfrac{-8}{5}, 1\right)$

(ii) In this case

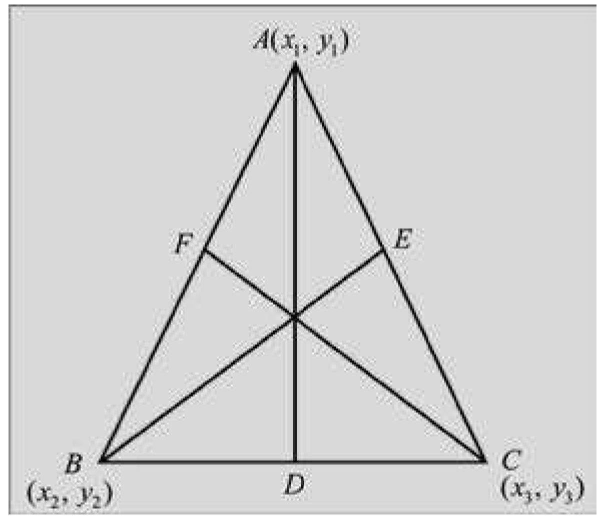
$$x = \frac{2\times5-3\times(-6)}{2-3} \quad 28 \quad \text{and} \quad y = \frac{2(-2)-3(3)}{2-3} \quad 13$$

Thus the required point has coordinates (–28, 13)

Theorem:

The centroid of a ΔABC is a point that divides each median in the ratio 2 : 1. Using this show that medians of a triangle are concurrent.

Proof: Let the vertices of a ΔABC have coordinates as shown in the figure.



Midpoint of *BC* is $D\left(\dfrac{x_2 + x_3}{2}, \dfrac{y_2 + y_3}{2}\right)$.

Let $G(\overline{x}, \overline{y})$ be the centroid of the Δ.

Then *G* divides *AD* in the ratio 2 : 1. Therefore

$$\overline{x} = \frac{2.\frac{x_2 + x_3}{2} + 1.x_1}{2+1} \quad \frac{x_1 + x_2 + x_3}{3}$$

Similarly, $\overline{y} = \dfrac{y_1 + y_2 + y_3}{3}$.

In the same way. we can show that coordinate of the point that divides *BE* and *CF* each in the ratio 2 : 1 are $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$.

Thus $(\overline{x}, \overline{y})$ lies on each median and so the medians of the ΔABC are concurrent.

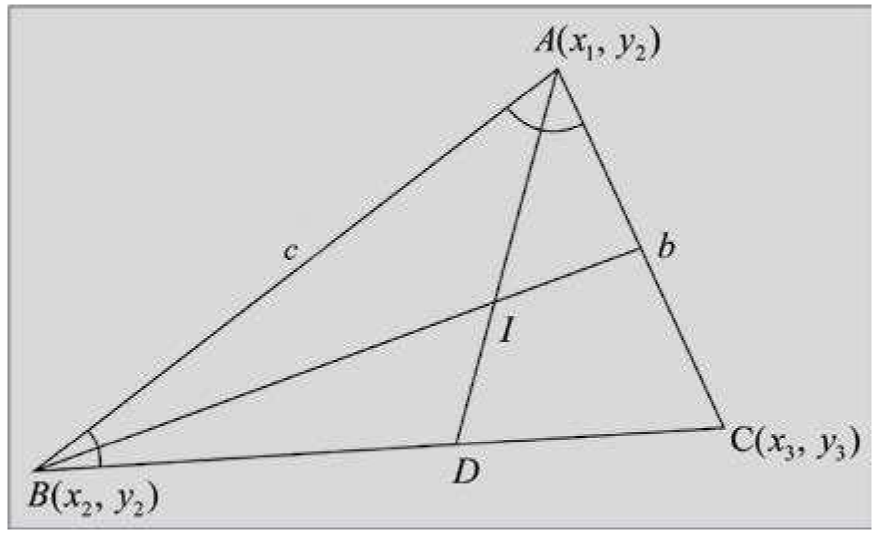
Theorem: Bisectors of angles of a triangle are concurrent.

Proof: Let the coordinates of the vertices of a triangle be as shown in the figure.

Suppose $|BC| = a, |CA| = b$ and $|AB| = c$

Let the bisector of $\angle A$ meet *BC* at *D*. Then *D* divides *BC* in the ratio *c* : *b*. Therefore coordinates of D are $\left(\frac{cx_3+bx_2}{b+c}, \frac{cy_3+by_2}{b+c}\right)$

The bisector of $\angle B$ meets AC at I and I divides AD in the ratio $c:|BD|$



Now $\frac{|BD|}{|DC|} = \frac{c}{b}$ or $\frac{|DC|}{|BD|} = \frac{b}{c}$

or $\frac{|DC|+|BD|}{|BD|} = \frac{b+c}{c}$

or $\frac{a}{|BD|} = \frac{b+c}{c}$ or $|BD| = \frac{ac}{b+c}$

Thus I divides AD in the ratio $c:\frac{ac}{b+c}$ or in the ratio $b+c:a$

Coordinates of I are

$$\left(\frac{(b+c)\frac{bx_2+cx_3}{b+c}+ax_1}{a+b+c}, \frac{(b+c)\frac{by_2+cy_3}{b+c}+ay_1}{a+b+c}\right)$$

i.e., $$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

The symmetry of these coordinates shows that the bisector of $\angle C$ will also pass through this point.

Thus the angle bisectors of a triangle are concurrent.

EXERCISE 4.1

1. Describe the location in the plane o f the point $P(x, y)$ for which

(i)	$x > 0$	(ii)	$x > 0$ and $y > 0$	(iii)	$x = 0$
(iv)	$y = 0$	(v)	$x < 0$ and $y \geq 0$	(vi)	$x = y$
(vii)	$\|x\| = \|y\|$	(viii)	$\|x\| \geq 3$	(ix)	$x > 2$ and $y = 2$

(x) x and y have opposite signs.

2. Find in each of the following:
   (i) the distance between the two given points
   (ii) midpoint of the line segment joining the two points
   (a) A (3 ,1); B (–2 ,–4)
   (b) A (–8 ,3); B (2, –1)
   (c) $A\left(-\sqrt{5}, -\frac{1}{3}\right)$; $B\left(-3\sqrt{5}, 5\right)$
3. Which of the following points are at a distance of 15 units from the origin?
   (a) $\left(\sqrt{176}, 7\right)$ (b) (10, –10) (c) (1, 15) (d) $\left(\frac{15}{2}, \frac{15}{2}\right)$
4. Show that
   (i) the points A (0, 2), $B\left(\sqrt{3}, 1\right)$ and C (0, –2) are vertices of a right triangle.
   (ii) the points A (3, 1), B (–2, –3) and C (2, 2) are vertices of an isosceles triangle.
   (iii) the points A (5, 2), B (–2, 3), C (–3, –4) and D (4, –5) are vertices of a parallelogram. Is the parallelogram a square?
5. The midpoints of the sides of a triangle are (1, –1), (–4, –3) and (–1, 1). Find coordinates of the vertices of the triangle.
6. Find h such that the points $A\left(\sqrt{3}, -1\right)$, B (0, 2) and C (h, –2) are vertices of a right triangle with right angle at the vertex A.
7. Find h such that A (–1, h), B (3, 2) and C (7, 3) are collinear.
8. The points A (–5, –2) and B (5, –4) are ends of a diameter of a circle. Find the centre and radius of the circle.
9. Find h such that the points A (h, 1), B (2, 7) and C (–6, –7) are vertices of a right triangle with right angle at the vertex A.
10. A quadrilateral has the points A (9, 3), B (-7, 7), C (–3, –7) and D(5, –5) as its vertices. Find the midpoints of its sides. Show that the figure formed by joining the midpoints consecutively is a parallelogram.

11. Find h such that the quadrilateral with vertices *A* (–3, 0), *B* (1, –2), *C* (5, 0) and *D* (1, h) is parallelogram. Is it a square?

12. If two vertices of an equilateral triangle are *A* (–3, 0) and *B* (3, 0), find the third vertex. How many of these triangles are possible?

13. Find the points trisecting the join of *A* (–1, 4) and *B* (6, 2).

14. Find the point three-fifth of the way along the line segment from *A* (–5, 8) to *B* (5, 3).

15. Find the point *P* on the join of *A* (1, 4) and *B* (5, 6) that is twice as far from *A* as *B* is from *A* and lies

(i) on the same side of *A* as *B* does.

(ii) on the opposite side of *A* as *B* does.

16. Find the point which is equidistant from the points *A* (5, 3), *B* (–2, 2) and *C* (4, 2). What is the radius of the circumcircle of the ΔABC?

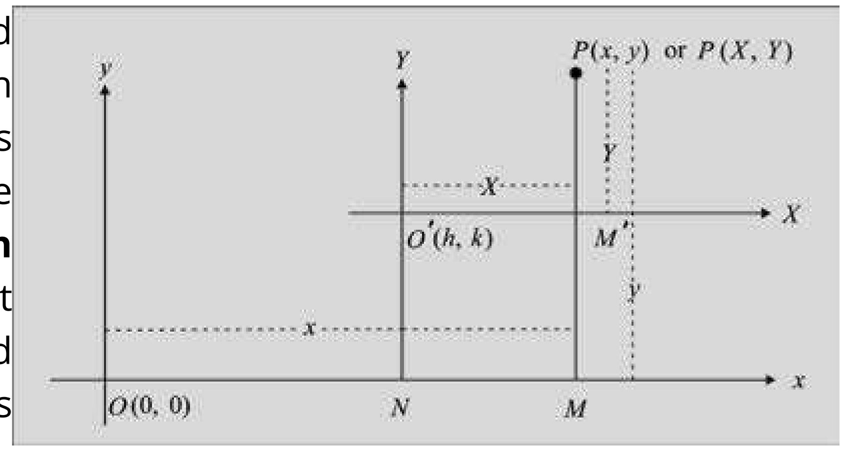
17. The points (4, –2), (–2, 4) and (5, 5) are the vertices of a triangle. Find in-centre of the triangle.

18. Find the points that divide the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ into four equal parts.

4.2 TRANSLATION AND ROTATION OF AXES

Translation of Axes

Let xy-coordinate system be given and $O'(h, k)$ be any point in the plane. Through *O′* draw two mutually perpendicular lines *O′X*, *O′Y* such that *O′X* is parallel to Ox. The new axes *O′X* and *O′Y* are called **translation** of the Ox – and Oy – axes through the point *O′*. In translation of axes, origin is shifted to another point in the plane but the axes remain parallel to the old axes.



Let *P* be a point with coordinates (x, y) referred to xy-coordinate system and the axes be translated through the point $O'(h, k)$ and *O′X, O′Y* be the new axes. If *P* has coordinates (*X, Y*) referred to the new axes, then we need to find *X, Y* in terms of x, y.

Draw *PM* and *O′ N* perpendiculars to Ox.

From the figure, we have

$OM = x, MP = y, \; ON = h, NO' = k = MM'$

Now $\quad X = O'M' = NM = OM - OM - ON = x - h$

Similarly, $\quad Y = M'P = MP - MM' = y - k$

Thus the coordinates of *P* referred to *XY*-system are $(x - h, y - k)$

i.e. $\quad X = x - h$

$\qquad Y = y - k$

Moreover, $x = X + h, \; y = Y + k$.

Example 1: The coordinates of a point *P* are (–6, 9). The axes are translated through the point *O′* (–3, 2). Find the coordinates of *P* referred to the new axes.

Solution. Here $h = -3, k = 2$

Coordinates of *P* referred to the new axes are (*X, Y*) given by

$X = -6 - (-3) = -3$ and $Y = 9 - 2 = 7$

Thus *P* (*X, Y*) = *P* (–3 ,7).

Example 2: The xy-coordinate axes are translated through the point *O′* (4, 6). The coordinates of the point *P* are (2, –3) referred to the new axes. Find the coordinates of *P* referred to the original axes.

Solution: Here $X = 2, Y = -3, h = 4, k = 6$.

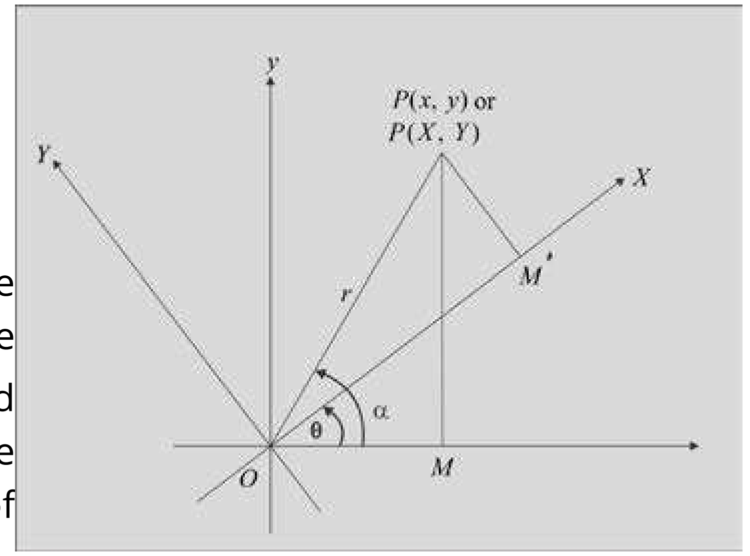
We have $\quad x = X + h = 4 + 2 = 6$

$\qquad y = Y + k = -3 + 6 = 3$

Thus required coordinates are *P* (6, 3).

Rotation of Axes

Let *xy*-system be given. We rotate Ox and Oy about the origin through an angle $\theta (0 < \theta < 90^\circ)$ so that the new axes are *OX* and *OY* as shown in the figure. Let a point *P* have coordinates (x, y) referred to the *xy*-system of



coordinates. Suppose *P* has coordinates (*X*, *Y*) referred to the *XY*-coordinate system. We have to find *X*, *Y* in terms of the given coordinates x, *y*. Let α be measure of the angle that *OP* makes with *O*.

From *P*, draw *PM* perpendicular to Ox and *PM′* perpendicular to *OX*. Let $|OP| = r$, From the right triangle OPM', we have

$$\left.\begin{array}{l} OM' = X = r\cos(\alpha-\theta) \\ M'P = Y = r\sin(\alpha-\theta) \end{array}\right] \qquad (1)$$

Also from the ΔOPM, we have

$$x = r\cos\alpha, \qquad y = r\sin\alpha$$

System of equations (1) may be re-written as:

$$\left.\begin{array}{l} X = r\cos\alpha\cos\theta + r\sin\alpha\sin\theta \\ Y = r\sin\alpha\cos\theta - r\cos\alpha\sin\theta \end{array}\right] \qquad (2)$$

Substituting from (2) into the above equations, we have

$$\left.\begin{array}{l} X = x\cos\theta + y\sin\theta \\ Y = y\cos\theta - x\sin\theta \end{array}\right] \qquad (3)$$

i.e., $(X, Y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$

are the coordinates of *P* referred to the new axes *OX* and *OY*.

Example 3: The *xy*-coordinate axes are rotated about the origin through an angle of 30^0. If the *xy*-coordinates of a point are (5, 7), find its *XY*-coordinates, where *OX* and *OY* are the axes obtained after rotation.

Solution. Let (*X*, *Y*) be the coordinates of *P* referred to the *XY*-axes. Here $\theta = 30^0$.

From equations (3) above, we have

$$X = 5\cos 30^\circ + 7\sin 30^\circ \text{ and } Y = -5\sin 30^\circ + 7\cos 30^\circ$$

or $$X = \frac{5\sqrt{3}}{2} + \frac{7}{2} \text{ and } Y = \frac{-5}{2} + \frac{7\sqrt{3}}{2}$$

i.e., $(X, Y) = \left(\frac{5\sqrt{3}+7}{2}, \frac{-5+7\sqrt{3}}{2}\right)$

are the required coordinates.

Example 4: The *xy*-axes are rotated about the origin through an angle of arctan $\frac{4}{3}$ lying in the first quadrant. The coordinates of a point *P* referred to the new axes *OX* and *OY* are *P* (–1, –7). Find the coordinates of *P* referred to the *xy*-coordinate system.

Solution. Let *P*(*x*, *y*) be the coordinates of *P* referred to the *xy*-coordinate system.

Angle of rotation is given by arctan $\theta = \frac{4}{3}$. Therefore, $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$.

From equations (3) above, we have

$$X = x\cos\theta + y\sin\theta \text{ and } Y = -x\sin\theta + y\cos\theta$$

or $$-1 = \frac{3}{5}x + \frac{4}{5}y \text{ and } -7 = -\frac{4}{5}x + \frac{3}{5}y$$

or $$3x + 4y + 5 = 0 \text{ and } -4x + 3y + 35 = 0$$

Solving these equations, we have

$$\frac{x}{125} = \frac{y}{-125} = \frac{1}{25} \qquad \Rightarrow x = 5,\ y = -5$$

Thus coordinates of *P* referred to the *xy*-system are (5, –5).

EXERCISE 4.2

1. The two points *P* and *O′* are given in *xy*-coordinate system. Find the *XY*-coordinates of *P* reffered to the translated axes *O′X* and *O′Y*.

(i) $P(3,2);\ O'(1,3)$ (ii) $P(-2,6);\ O'(-3,2)$

(iii) $P(-6,-8);\ O'(-4,-6)$ (iv) $P\left(\frac{3}{2},\frac{5}{2}\right);\ O'\left(-\frac{1}{2},\frac{7}{2}\right)$

2. The xy-coordinate axes are translated through the point whose coordinates are given in xy-coordinate system. The coordinates of P are given in the XY-coordinate system. Find the coordinates of P in xy-coordinate system.

(i) $P(8, 10)$; $O'(3, 4)$ (ii) $P(-5, -3)$; $O'(-2, -6)$

(iii) $P\left(-\frac{3}{4}, -\frac{7}{6}\right); O'\left(\frac{1}{4}, -\frac{1}{6}\right)$ (iv) $P(4, -3)$; $O'(-2, 3)$

3. The xy-coordinate axes are rotated about the origin through the indicated angle. The new axes are OX and OY. Find the XY-coordinates of the point P with the given xy-coordinates.

(i) $P(5, 3)$; $\theta = 45^0$ (ii) $P(3, -7)$; $\theta = 30^0$

(iii) $P(11, -15)$; $\theta = 60^0$ (iv) $P(15, 10)$: $\theta = \arctan\frac{1}{3}$

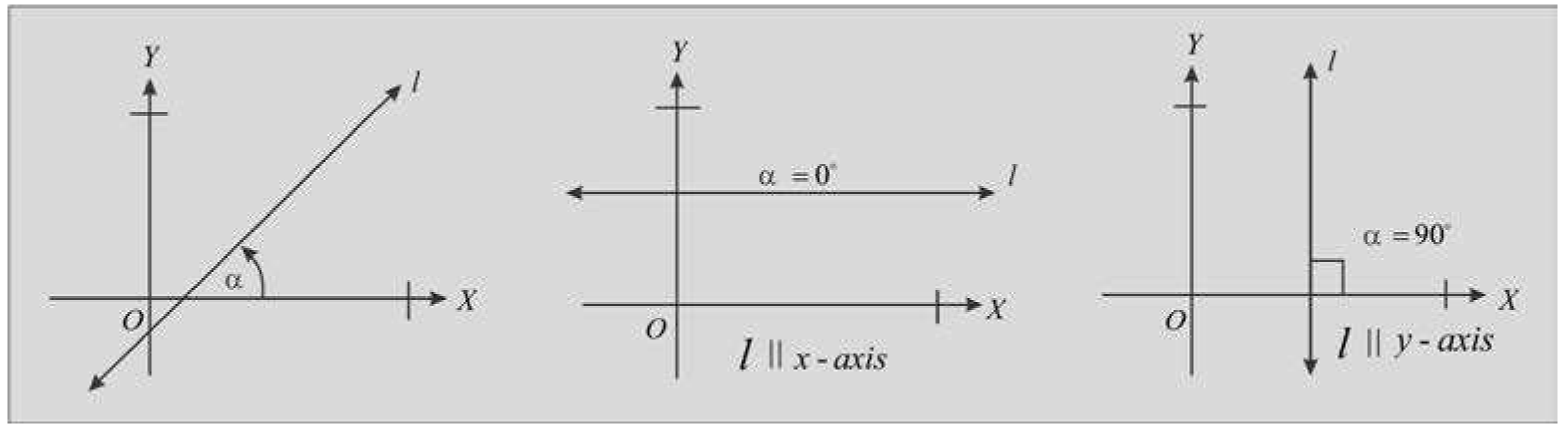
4. The xy-coordinate axes are rotated about the origin through the indicated angle and the new axes are OX and OY.

Find the xy-coordinates of P with the given XY-coordinates.

(i) $P(-5, 3)$; $\theta = 30^0$ (ii) $P\left(-7\sqrt{2}, 5\sqrt{2}\right)$; $\theta = 45^o$

4.3 EQUATIONS OF STRAIGHT LINES

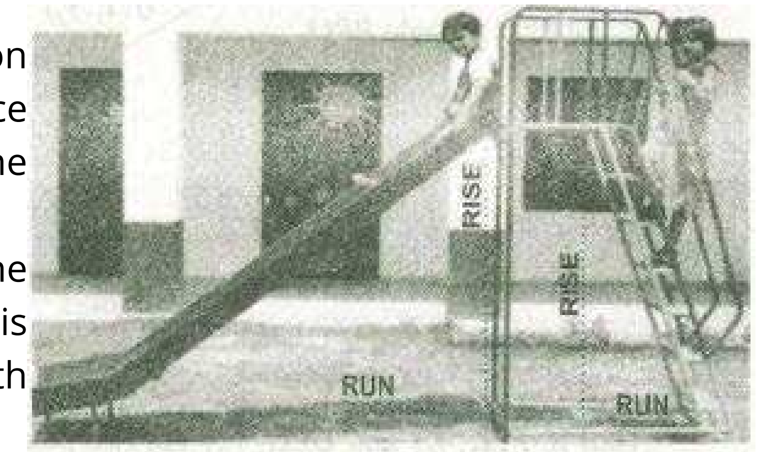
Inclination of a Line: The angle $\alpha\left(0^o < \alpha < 180^o\right)$ measured counterclockwise from positive x-axis to a non-horizontal straight line l is called the inclination of l.



Observe that the angle α in the different positions of the line l is α, 0^0 and 90^0 respectively.

Note: (i) If l is parallel to x-axis, then $\alpha = 0°$
(ii) If l is parallel to y-axis, then $\alpha = 90°$

Slope or gradient of a line: When we walk on an inclined plane, we cover horizontal distance (**run**) as well as vertical distance (**rise**) at the same time.



It is harder to climb a steeper inclined plane. The measure of steepness (ratio of rise to the run) is termed as slope or gradient of the inclined path and is denoted by m.

$$m = \frac{rise}{run} = \frac{y}{x} = \tan\alpha$$

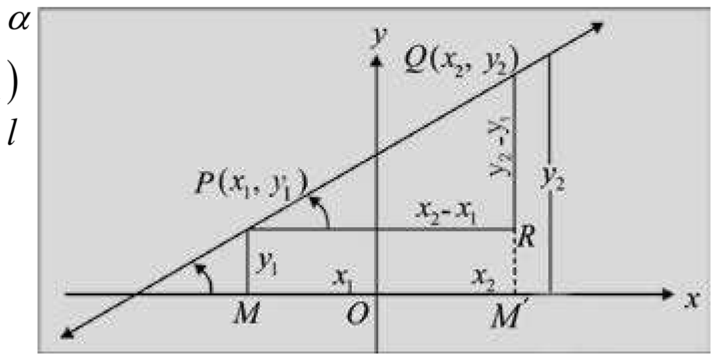
In analytical geometry, **slope or gradient *m*** of a non-vertical straight line with α as its inclination is defined by: $m : \tan\alpha$

If l is horizontal its slope is zero and if l is vertical then its slope is undefined.

If $0 < \alpha < 90^0$, m is positive and if $90^0 < \alpha < 180^0$, then m is negative

4.3.1 Slope or Gradient of a Straight Line Joining Two Points

If a non-vertical line l with inclination α passes through two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then the slope or gradient m of l is given by $m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\alpha$



Proof: Let m be the slope of the line l.

Draw perpendiculars PM and QM' on x-axis and a perpendicular PR on QM'

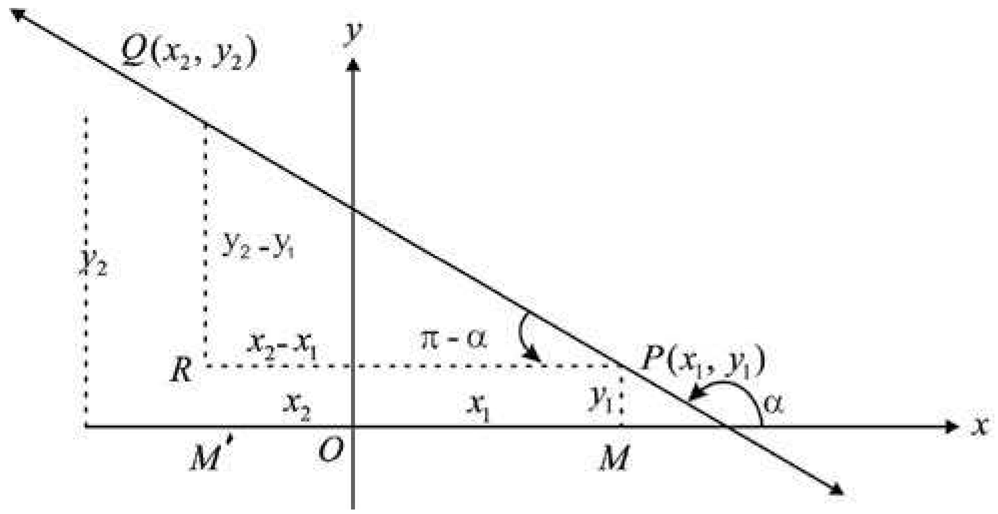
Then $\angle RPQ = \alpha$, $m\overline{PR} = x_2 - x_1$ and $m\overline{QR} = y_2 - y_1$

The slope or gradient of l is defined as: $m = \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$.

Case (i). When $0 < \alpha < \frac{\pi}{2}$

In the right triangle PRQ, we have

$$m = \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$$



Case (ii) When $\frac{\pi}{2} < \alpha < \pi$

In the right triangle PRQ

$$\tan(\pi - \alpha) = \frac{y_2 - y_1}{x_1 - x_2}$$

or $\quad -\tan\alpha = \frac{y_2 - y_1}{x_1 - x_2}$

or $\quad \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \frac{y_2 - y_1}{x_2 - x_1}$

Thus if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line, then slope of PQ is given by:

$$m == \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad m \quad \frac{y_1 - y_2}{x_1 - x_2}$$

Note:
(i) $m \neq \frac{y_2 - y_1}{x_1 - x_2}$ and $m \neq \frac{y_1 - y_2}{x_2 - x_1}$
(ii) *l* is horizontal, iff *m* = 0 ($\because$ $\alpha = 0^0$)
(iii) *l* is vertical, iff m is not defined ($\because$ $\alpha = 90^0$)
(iv) If slope of *AB* = slope of *BC*, then the points *A*, *B* and *C* are collinear.

Theorem: The two lines l_1 and l_2 with respective slopes m_1 and m_2 are

(i) parallel iff $m_1 = m_2$

(ii) perpendicular iff $m_1 = \frac{-1}{m_2}$

or $\quad m_1 m_2 + 1 = 0$

Remember that:
The symbol
(i) || stands for "parallel".
(ii) ∦ stands for "not parallel".
(iii) ⊥ stands for "perpendicular".

Example 1: Show that the points *A*(–3, 6), *B*(3, 2) and *C*(6, 0) are collinear.

Solution: We know that the points *A*, *B* and *C* are collinear if the line *AB* and *BC* have the same slopes. Here Slope of

Notice that:
Slope of *AB* = slope of *AC*

$$AB = \frac{2-6}{3-(-3)} = \frac{-4}{3+3} = \frac{-4}{6} = \frac{-2}{3} \quad \text{and slope of } BC = \frac{0-2}{6-3} = \frac{-2}{3}$$

$\because$ Slope of *AB* = Slope of *BC*

Thus *A*, *B* and *C* are collinear.

Example 2: Show that the triangle with vertices *A* (1, 1), *B* (4, 5) and *C* (12, –1) is a right triangle.

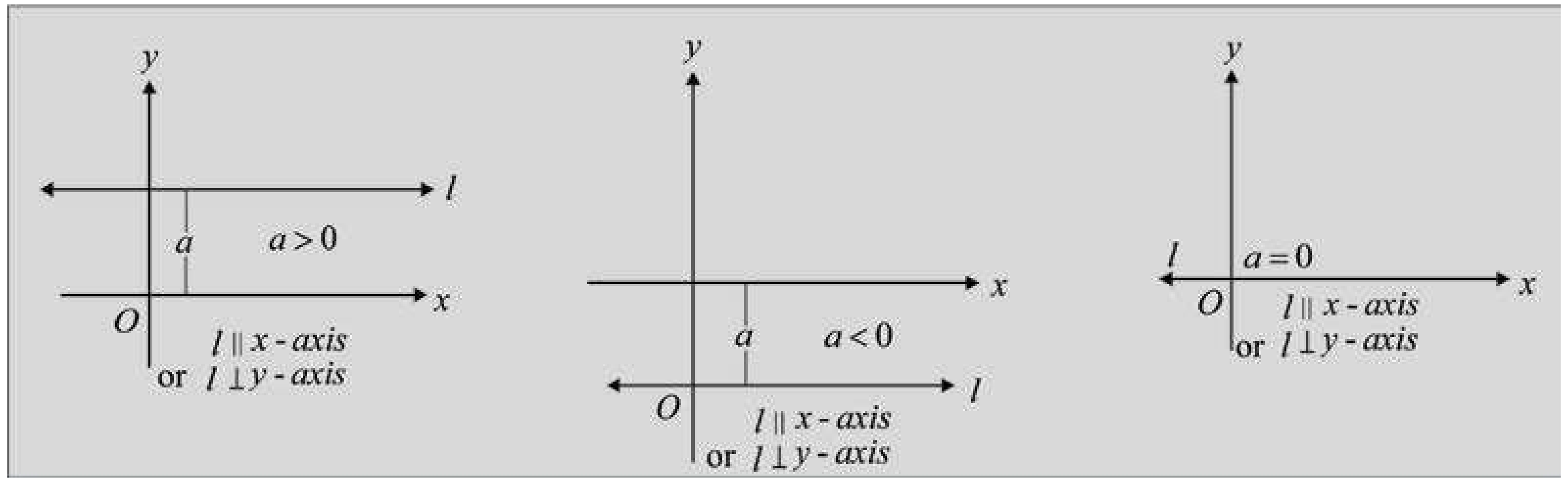
Solution: Slope of $AB = m_1 = \frac{5-1}{4-1} = \frac{4}{3}$

and Slope of $BC = m_2 = \frac{-1-5}{12-4} = \frac{-6}{8} = \frac{-3}{4}$

Since $m_1 m_2 = \left(\frac{4}{3}\right)\left(\frac{3}{4}\right) = \ 1$, therefore, $AB \perp BC$

So ΔABC is a right triangle.

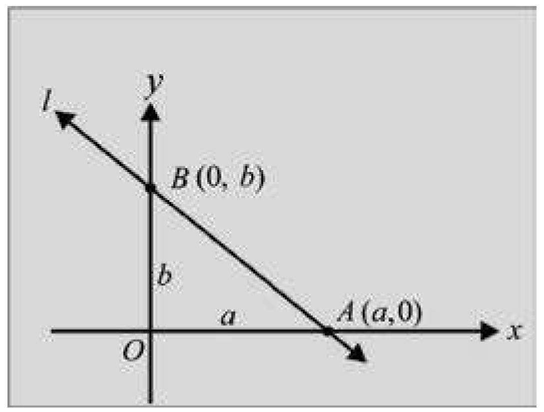
4.3.2 Equation of a Straight Line Parallel to the x-axis (or perpendicular to the y-axis)



All the points on the line l parallel to x-axis remain at a constant distance (say a) from x-axis. Therefore, each point on the line has its distance from x-axis equal to a, which is its y-coordinate (ordinate). So, all the points on this line satisfy the equation: $y = a$

Note: (i) If $a > 0$, then the line l is above the x-axis.
(ii) If $a < 0$, then the line l is below the x-axis.
(iii) If $a = 0$, then the line l becomes the x-axis.
Thus the equation of x-axis is $y = 0$

4.3.4 Derivation of Standard Forms of Equations of Straight Lines



Intercepts:

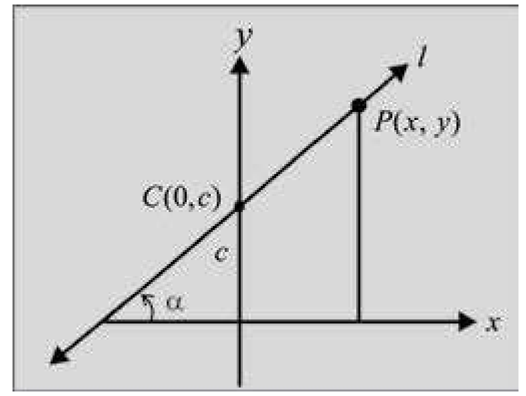
- If a line intersects x-axis at $(a, 0)$, then $\boldsymbol{a}$ is called ***x*-intercept** of the line.
- If a line intersects y-axis at $(0, b)$, then $\boldsymbol{b}$ is called ***y*-intercept** of the line.

1. Slope-Intercept form of Equation of a Straight Line:

Theorem: Equation of a non-vertical straight line with slope m and y-intercept c is given by:

$$y = mx + c$$

Proof: Let $P(x, y)$ be an arbitrary point of the straight line l with slope m and y-intercept c. As $C(0, c)$ and $P(x, y)$ lie on the line, so the slope of the line is:



$$m = \frac{y-c}{x-0} \text{ or } y - c = mx \text{ and } y = mx + c$$

is an equation of l.
The equation of the line for which
$c = 0$ is

$y = mc$

In this case the line passes through the origin.

Example 1: Find an equation of the straight line if

(a) its slope is 2 and y-intercept is 5

(b) it is perpendicular to a line with slope –6 and its y-intercept is $\frac{4}{3}$

Solution: (a) The slope and y-intercept of the line are respectively:

$m = 2$ and $c = 5$

Thus $y = 2x + 5$ (Slope-intercept form: $y = mx + c$)
is the required equation.

(b) The slope of the given line is

$$m_1 = -6$$

$\therefore$ The slope of the required line is: $m_2 = -\frac{1}{m_1} = \frac{1}{6}$

The slope and y-intercept of the required line are respectively:

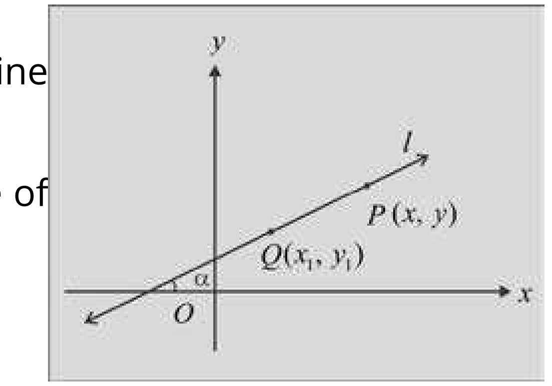
$$m = \frac{1}{6} \quad \text{(slope of } \perp \text{ line is –6)} \quad \text{and} \quad c = \frac{4}{3}$$

Thus $y = \frac{1}{6}(x) + \frac{4}{3}$ or $6y = x + 8$
is the required equation.

2. Point-slope Form of Equation of a Straight Line:

Theorem: Equation of a non-vertical straight line l with slope m and passing through a point $Q(x_1, y_1)$ is

$$y - y_1 = m(x - x_1)$$



Proof: Let $P(x, y)$ be an arbitrary point of the straight line with slope m and passing through $Q(x_1, y_1)$.

As $Q(x_1, y_1)$ and $P(x, y)$ both lie on the line, so the slope of the line is

$$m = \frac{y - y_1}{x - x_1} \text{ or } y - y_1 = m(x - x_1)$$

which is an equation of the straight line passing through x_1, y_1 with slope m.

3. Symmetric Form of Equation of a Straight Line:

We have $\frac{y - y_1}{x - x_1} = \tan\alpha$, where α is the inclination of the line.

$$\text{or } \frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\sin\alpha} = r(\text{say})$$

This is called **symmetric** form of equation of the line.

Example 2: Write down an equation of the straight line passing through (5, 1) and parallel to a line passing through the points (0,–1), (7, –15).

Solution: Let m be the slope of the required straight line, then

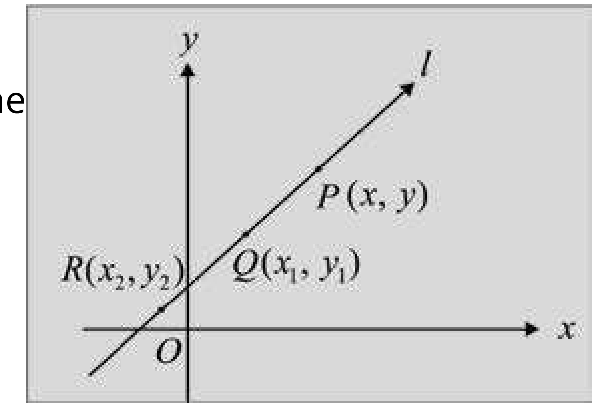
$$m = \frac{-15 - (-1)}{7 - 0} \qquad (\because \text{ Slopes of parallel lines are equal})$$
$$= -2$$

As the point (5, 1) lies on the required line having slope –2 so, by point-slope form of equation of the straight line, we have

$$y - (1) = -2(x - 5)$$
or $$y = -2x + 11$$
or $$2x + y - 11 = 0$$

is an equation of the required line.

4. Two-point Form of Equation of a Straight Line:



Theorem: Equation of a non-vertical straight line passing through two points $Q(x_1, y_1)$ and $R(x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

Proof: Let $P(x, y)$ be an arbitrary point of the line passing through $Q(x_1, y_1)$ and $R(x_2, y_2)$. So

$$\frac{y - y_1}{x - x_1} = \frac{y - y_2}{x - x_2} = \frac{y_2 - y_1}{x_2 - x_1} \qquad (P, Q \text{ and } R \text{ are collinear points})$$

We take

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

or $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ the required equation of the line PQ.

or $(y_2 - y_1)x - (x_2 - x_1)y + (x_1y_2 - x_2y_1) = 0$

We may write this equation in determinant form as: $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Note: **(i)** If $x_1 - x_2$, then the slope becomes undefined. So, the line is vertical.

(ii) $y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$ can be derived similarly.

Example 3: Find an equation of line through the points (–2, 1) and (6, –4).

Solution: Using two-points form of the equation of straight line, the required equation is

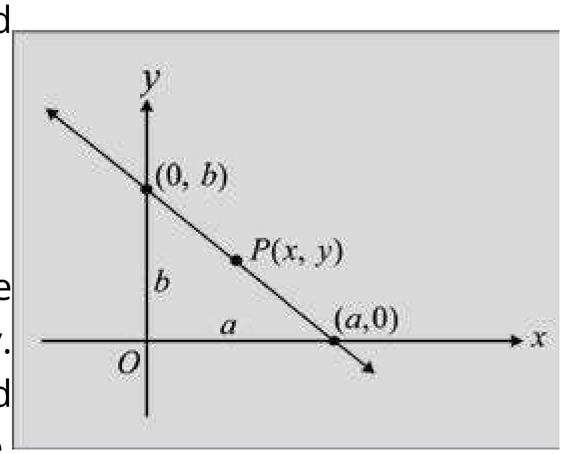
$$y-1=\frac{-4-1}{6-(-2)}\left[x-(-2)\right]$$

or $$y-1=\frac{-5}{8}(x+2) \text{ or } 5x+8y+2=0$$

5. Intercept Form of Equation of a Straight Line:

Theorem: Equation of a line whose non-zero *x* and *y*-intercepts are *a* and *b* respectively is

$$\frac{x}{a}+\frac{y}{b}=1$$



Proof: Let *P*(*x* , *y*) be an arbitrary point of the line whose non-zero *x* and *y*-intercepts are *a* and *b* respectively. Obviously, the points *A*(*a*, 0) and *B*(0, *b*) lie on the required line. So, by the two-point form of the equation of line, we have

$$y-0=\frac{b-0}{0-a}(x-a) \qquad (P, A \text{ and } B \text{ are collinear})$$

or $-ay=b(x-a)$

or $bx+ay=ab$

or $$\frac{x}{a}+\frac{y}{b}=1 \qquad \text{(dividing by } ab\text{)}$$

Hence the result.

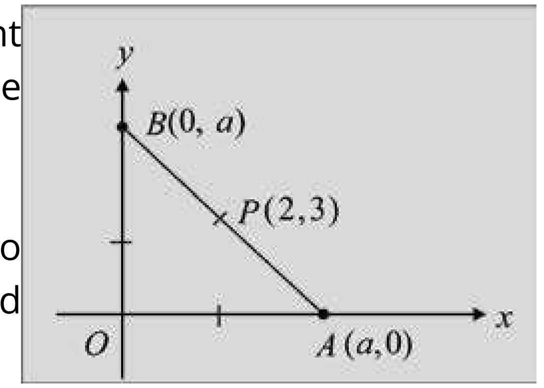
Example 4: Write down an equation of the line which cuts the *x*-axis at (2, 0) and *y*-axis at (0, –4).

Solution: As 2 and –4 are respectively *x* and *y*-intercepts of the required line, so by two-intercepts form of equation of a straight line, we have

$$\frac{x}{2}+\frac{y}{-4}=1 \; - \quad \text{or} \quad = 2x \quad y \quad 4 \quad 0$$

which is the required equation.

Example 5: Find an equation of the line through the point *P*(2, 3) which forms an isosceles triangle with the coordinate axes in the first quadrant.



Solution: Let *OAB* be an isosceles triangle so that the line *AB* passes through *A* = (*a*, 0) and *B*(0, *a*), where *a* is some positive real number.

Slope of $AB=\frac{a-0}{0-a}=-1$. But *AB* passes through *P* (2, 3).

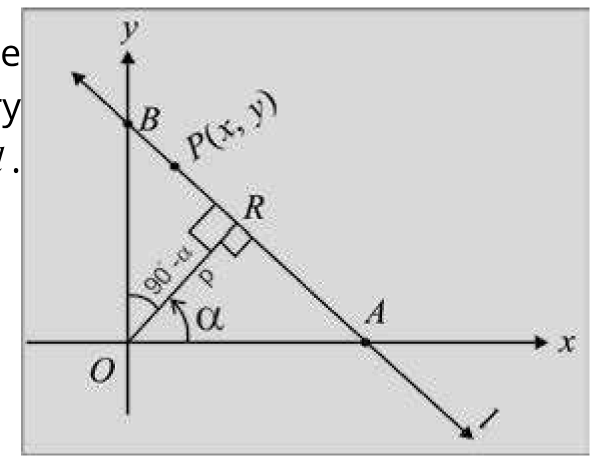
∵ Equation of the line through *P*(2, 3) with slope –1 is

$$y-3=-1(x-2) \quad \text{or} \quad x+y-5=0$$

6. Normal Form of Equation of a Straight Line:

Theorem: An equation of a non-vertical straight line l, such that length of the perpendicular from the origin to l is *p* and α is the inclination of this perpendicular, is

$$x\cos\alpha + y\sin\alpha = p$$



Proof: Let the line l meet the *x*-axis and *y*-axis at the points *A* and *B* respectively. Let *P* (*x*, *y*) be an arbitrary point of *AB* and let *OR* be perpendicular to the line l. Then $|OR|=p$.

From the right triangles *ORA* and *ORB*, we have,

$$\cos\alpha=\frac{p}{OA} \quad \text{or} \quad OA=\frac{p}{\cos\alpha}$$

and $$\cos(90^\circ-\alpha)=\frac{p}{OB} \quad \text{or} \quad OB=\frac{p}{\sin\alpha}$$

$[\because \cos(90^\circ - \alpha) = \sin\alpha)]$

As OA and OB are the x and y-intercepts of the line AB, so equation of AB is

$$\frac{x}{p/\cos\alpha} + \frac{y}{p/\sin\alpha} = 1 \qquad \text{(Two-intercept form)}$$

That is $x\cos\alpha + y\sin\alpha = p$ is the required equation.

Example 6: The length of perpendicular from the origin to a line is 5 units and the inclination of this perpendicular is 120^0. Find the slope and y-intercept of the line.

Solution. Here $p = 5$, $\alpha = 120^0$.

Equation of the line in normal form is

$$x\cos 120^\circ + y\sin 120^\circ = 5$$

$$\Rightarrow \quad -\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 5$$

$$\Rightarrow \quad x - \sqrt{3}y + 10 = 0 \qquad (1)$$

To find the slope of the line, we re-write (1) as: $y = \frac{x}{\sqrt{3}} + \frac{10}{\sqrt{3}}$

which is slope-intercept form of the equation.

Here $\quad m = \frac{1}{\sqrt{3}}$ and $c = \frac{10}{\sqrt{3}}$

4.3.5 A Linear Equation in two Variables Represents a Straight Line

Theorem: The linear equation $ax + by + c = 0$ in two variables x and y represents a straight line. A linear equation in two variables x and y is

$$ax + by + c = 0 \qquad (1)$$

where a, b and c are constants and a and b are not simultaneously zero.

Proof: Here a and b cannot be both zero. So the following cases arise:

Case I: $a \neq 0$, $b = 0$

In this case equation (1) takes the form:

$$ax + c = 0 \text{ or } x = -\frac{c}{a}$$

which is an equation of the straight line parallel to the y-axis at a directed distance $-\frac{c}{a}$ from the y-axis.

Remember that:
The equation (I) represents a straight line and is called the **general equation of a straight line.**

Case II: $a = 0$, $b \neq 0$

In this case equation (1) takes the form:

$$bx + c = 0 \text{ or } y = -\frac{c}{b}$$

which is an equation of the straight line parallel to x-axis at a directed distance $\frac{-c}{b}$ from the x-axis.

Case III: $a \neq 0$, $b \neq 0$

In this case equation (1) takes the form:

$$by = -ax - c \quad or \quad y = \frac{-a}{b}x - \frac{c}{b} = mx + c$$

which is the slope-intercept form of the straight line with slope $\frac{-a}{b}$ and y-intercept $\frac{-c}{b}$
.

Thus the equation $ax + by + c = 0$, always represents a straight line.

4.3.6 To Transform the General Linear Equation to Standard Forms

Theorem: To transform the equation $ax + by + c = 0$ in the standard form

1. **Slope-Intercept Form.**

We have

$by = -ax - c \quad or \quad y = \frac{-a}{b}x - \frac{c}{b} = mx + c,$ where $m = \frac{-a}{b},\ c = \frac{-c}{b}$

2. Point - Slope Form

We note from (1) above that slope o f the line $ax + by + c = 0$ is $\frac{-a}{b}$. A point on the line is $\left(\frac{-c}{a}, 0\right)$

Equation of the line becomes $y = \frac{-a}{b}\left(x + \frac{c}{a}\right)$

which is in the point-slope form.

3. Symmetric Form

$m = \tan\alpha = \frac{-a}{b}.\ \sin\alpha = \frac{a}{\pm\sqrt{a^2+b^2}},\ \cos\alpha \quad \frac{b}{\pm\sqrt{a^2+b^2}}$

A point on $ax + by + c = 0$ is $\left(\frac{-c}{a}, 0\right)$

Equation in the symmetric form becomes

$$\frac{x - \left(-\frac{c}{a}\right)}{b/\pm\sqrt{a^2+b^2}} = \frac{y-0}{a/\pm\sqrt{a^2+b^2}} \quad r$$

is the required transformed equation. Sign of the radical to be properly chosen.

4. Two -Point Form

We choose two arbitrary points on $ax + by + c = 0$. Two such points are $\left(\frac{-c}{a}, 0\right)$ and $\left(0, \frac{-c}{b}\right)$. Equation of the line through these points is

$$\frac{y-0}{0+\frac{c}{b}} = \frac{x+\frac{c}{a}}{-\frac{c}{a}-0} \qquad \text{i.e., } y \quad 0 \quad \frac{-a}{b}\left(x \quad \frac{c}{a}\right)$$

5. Intercept Form.

$$ax + by = -c \quad \text{or} \quad \frac{ax}{-c} + \frac{by}{-c} = 1 \quad \text{i.e} \quad \frac{x}{-c/a} + \frac{y}{-c/b} \quad 1 \quad =$$

which is an equation in two intercepts form.

6. Normal Form.

The equation: $ax + by + c = 0$ (1)

can be written in the normal form as:

$$\frac{ax+by}{\pm\sqrt{a^2+b^2}} = \frac{-c}{\pm\sqrt{a^2+b^2}} \qquad (2)$$

The sign of the radical to be such that the right hand side of (2) is positive.

Proof. We know that an equation of a line in normal form is

$$x\cos\alpha + y\sin\alpha = p \qquad (3)$$

If (1) and (3) are identical, we must have

$$\frac{a}{\cos\alpha} = \frac{b}{\sin\alpha} = \frac{-c}{p}$$

i.e., $\frac{p}{-c} = \frac{\cos\alpha}{a} = \frac{\sin\alpha}{b} = \frac{\sqrt{\cos^2\alpha + \sin^2\alpha}}{\pm\sqrt{a^2+b^2}} \quad \frac{1}{\pm\sqrt{a^2+b^2}}$

Hence, $\cos\alpha = \frac{a}{\pm\sqrt{a^2+b^2}}$ and $\sin\alpha \quad \frac{b}{\pm\sqrt{a^2+b^2}}$

Substituting for $\cos\alpha, \sin\alpha$ and p into (3), we have

$$\frac{ax+by}{\pm\sqrt{a^2+b^2}} = \frac{-c}{\pm\sqrt{a^2+b^2}}$$

Thus (1) can be reduced to the form (2) by dividing it by $\pm\sqrt{a^2+b^2}$. The sign of the radical to be chosen so that the right hand side of (2) is positive.

Example 1: Transform the equation $5x - 12y + 39 = 0$ into

(i) Slope intercept form. (ii) Two-intercept form.
(iii) Normal form. (iv) Point-slope form.
(v) Two-point form. (vi) Symmetric form.

Solution:

(i) We have $12y = 5x + 39$ or $y = \frac{5}{12}x + \frac{39}{12} \Rightarrow m = \frac{5}{12}$, y-intercept $c = \frac{39}{12}$

(ii) $5x - 12y = -39$ or $\frac{5x}{-39} + \frac{12y}{39} = 1$ or $\frac{x}{-39/5} + \frac{y}{39/12} = 1$ is the required equation.

(iii) $5x - 12y = -39$. Divide both sides by $\pm\sqrt{5^2 + 12^2} = \pm 13$. Since R.H.S is to be positive, we have to take negative sign.

Hence = $\frac{5x}{-13} + \frac{12y}{13} = 3$ is the normal form of the equation.

(iv) A point on the line is $\left(\frac{-39}{5}, 0\right)$ and its slope is $\frac{5}{12}$.

Equation can be written as: $y - 0 = \frac{5}{12}\left(x + \frac{39}{5}\right)$

(v) Another point on the line is $\left(0, \frac{39}{12}\right)$. Line through $\left(\frac{-39}{5}, 0\right)$ and $\left(0, \frac{39}{12}\right)$ is

$$\frac{y - 0}{0 - \frac{39}{12}} = \frac{x + \frac{39}{5}}{\frac{-39}{5} - 0}$$

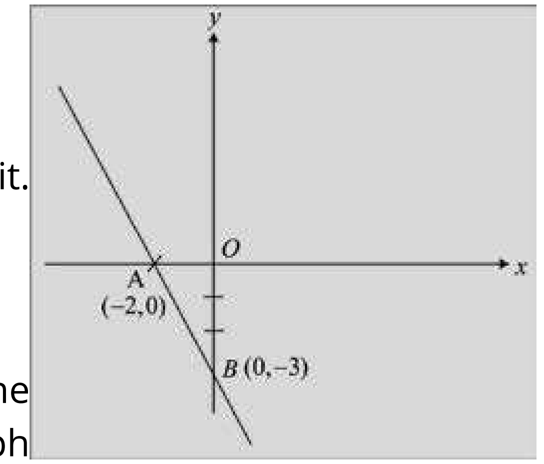
(vi) We have $\tan\alpha = \frac{5}{12} = m$, $\sin\alpha = \frac{5}{13}, \cos\alpha = \frac{12}{13}$. A point of the line is $\left(\frac{-39}{5}, 0\right)$.

Equation of the line in symmetric form is

$\frac{x + 39/5}{12/13} = \frac{y - 0}{5/13} = r$ (say)

Example 2: Sketch the line

$3x + 2y + 6 = 0$. (1)



Solution: To sketch the graph of (1), we find two points on it.
If $y = 0$, $x = -2$ and if $x = 0$, $y = -3$.

Thus x intercept = –2

y intercept = –3

The points $A(-2, 0)$, $B(0, -3)$ are on (1). Plot these points in the plane and draw the straight line through A and B. It is the graph of (1).

Example 3: Find the distance between the parallel lines

$2x + y + 2 = 0$ (1)

and $6x + 3y - 8 = 0$ (2)

Sketch the lines. Also find an equation of the line parallel to the given lines and lying midway between them.

Solution: We first convert both the lines into normal form. (1) can be written as

$2x + y = -2$

Dividing both sides by $-\sqrt{4+1}$, we have

$$\frac{-2}{\sqrt{5}}x + \frac{-y}{\sqrt{5}} = \frac{2}{\sqrt{5}} \quad (3)$$

which is normal form of (1). Normal form of (2) is

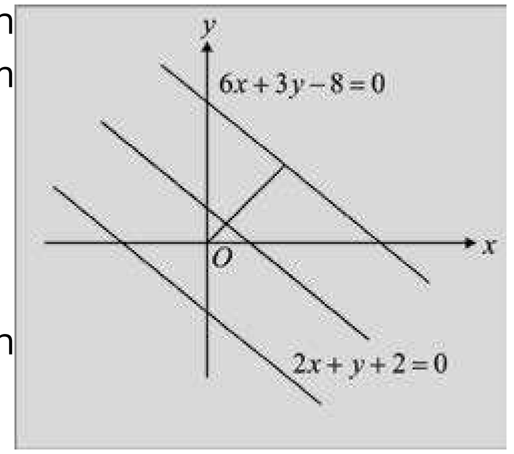
$$\frac{6x}{\sqrt{45}} + \frac{3y}{\sqrt{45}} = \frac{8}{\sqrt{45}}$$

i.e., $$\frac{2x}{\sqrt{5}} + \frac{y}{\sqrt{5}} = \frac{8}{3\sqrt{5}} \quad (4)$$

Length of the perpendicular from (0, 0) to the line (1) is $\frac{\ }{\sqrt{\ }}$ [From (3)]

Similarly, length of the perpendicular from (0, 0) to the line (2) is $\frac{8}{3\sqrt{5}}$ [From (4)]

From the graphs of the lines it is clear that the lines are on opposite sides of the origin, so the distance between them equals the sum of the two perpendicular lengths.



i.e., Required distance = $\frac{2}{\sqrt{5}} + \frac{8}{3\sqrt{5}} = \frac{14}{3\sqrt{5}}$

The line parallel to the given lines lying midway between them is such that length of the perpendicular

from O to the line = $\frac{8}{3\sqrt{5}} - \frac{7}{3\sqrt{5}} \left(\text{or } \frac{7}{3\sqrt{5}} - \frac{2}{\sqrt{5}}\right) = \frac{1}{3\sqrt{5}}$

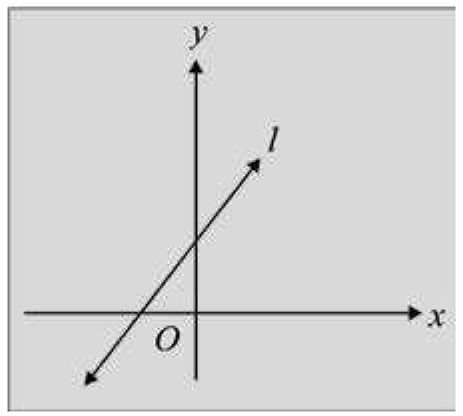
Required line is = $\frac{2x}{\sqrt{5}} + \frac{y}{\sqrt{5}} = \frac{1}{3\sqrt{5}}$ or $6x + 3y = 1$

4.3.7 Position of a point with respect to a line

Consider a non-vertical line l

$$l : ax + by + c = 0$$

in the *xy*-plane. Obviously, each point of the plane is either above the line or below the line or on the line.



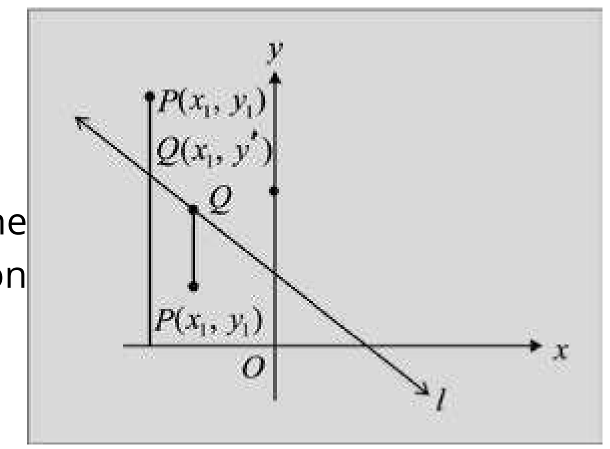
Theorem: Let $P(x_1, y_1)$ be a point in the plane not lying on l

$$l : ax + by + c = 0 \qquad (1)$$

then P lies

a) above the line (1) if $ax_1 + by_1 + c > 0$

b) below the line (1) if $ax_1 + by_1 + c < 0$



Proof: We can suppose that b > 0 (first multiply the equation by –1 if needed). Draw a perpendicular from P on x-axis meeting the line at $Q(x_1, y')$.

Thus $ax_1 + by' + c = 0$ so that

$$y' = \frac{a}{b}x_1 \quad \frac{c}{b}$$

The point $P(x_1, y_1)$ is above the line if $y_1 > y'$ that is

$$y_1 - y' > 0$$

i.e. $y_1 - \left(-\frac{a}{b}x_1 - \frac{c}{b}\right) > 0$

$\Rightarrow \quad ax_1 + by_1 + c > 0$

Similarly $P(x_1, y_1)$ is below the line if

$y_1 - y' < 0 \qquad$ i.e. $y_1 - \left(-\frac{a}{b}x_1 - \frac{c}{b}\right)$

or $\qquad ax_1 + by_1 + c < 0$

The point $P(x_1, y_1)$ is on the line if

$$ax_1 + by_1 + c =$$

Corollary 1. The point P is above or below l respectively if $ax_1 + by_1 + c$ and b have the same sign or have opposite signs.

Proof. If P is above l, then $y_1 - y' > 0$ i.e., $\frac{ax_1 + by_1 + c}{b} > 0$

Thus $ax_1 + by_1 + c$ and b have the same sign.
Similarly, P is below l if

$$y_1 - y' < 0 \quad \text{i.e.,} \quad \frac{ax_1 + by_1 + c}{b} < 0$$

Thus $ax_1 + by_1 + c$ and b have opposite signs.

Corollary 2. The point $P(x_1, y_1)$ and the origin are

(i) on the same side of l according as $ax_1 + by_1 + c$ and c have the same sign.

(ii) on the opposite sides of l according as $ax_1 + by_1 + c$ and c have opposite signs.

Proof. (i) The point $P(x_1, y_1)$ and O (0,0) are on the same side of l if $ax_1 + by_1 + c$ and a.0 + b.0 + c have the same sign.

(ii) Proof is left as an exercise

Example 1: Check whether the point (–2 , 4) lies above or below the line

$$4x+5y-3=0 \qquad (1)$$

Solution: Here b = 5 is positive. Also

4 (–2) + 5(4) – 3 = –8 + 20 – 3 = 9 > 0 (2)

The coefficient of y in (1) and the expression (2) have the same sign and so the point (–2, 4) lies above (1).

Example 2: Check whether the origin and the point P (5, –8) lie on the same side or on the opposite sides of the line:

$$3x+7y+15=0 \qquad (1)$$

Solution:

Here c = 15

For P (5, –8),

3(5) + 7(–8) + 15 = –26 < 0 (2)

But c = 15 >0

c and the expression (2) have opposite signs. Thus O (0, 0) and P (5, –8) are on the opposite sides of (1).

Note: To check whether a point $P(x_1, y_1)$ lies above or below the line

$$ax + by + c = 0$$

we make the co-efficient of y positive by multiplying the equation by (–1) if needed.

4.4 TWO AND THREE STRAIGHT LINES

For any two distinct lines l_1, l_2.

$l_1 : a_1x+b_1y+c=0$ and $l_2 : a_2x+b_2y+c=0$, one and only one of the following holds:

(i) $l_1 \parallel l_2$ (ii) $l_1 \perp l_2$ (iii) l_1 and l_2 are not related as (i) or (ii).

Recall that: Two non-parallel lines intersect each other at one and only one point.

The slopes of l_1 and l_2 are $m_1 = -\frac{a_1}{b_1}$, $m_2 = -\frac{a_2}{b_2}$

(i) $l_1 \parallel l_2 \Leftrightarrow$ slope of $l_1(m_1) =$ slope of $l_2(m_2)$.

$$\Leftrightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

$$\Leftrightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2} \qquad \Leftrightarrow \qquad a_1b_2 - b_1a_2 = 0$$

(ii) $l_1 \perp l_2 \Leftrightarrow m_1m_2 = -1$

$$\Leftrightarrow \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right) = -1 \quad \Leftrightarrow a_1a_2 + b_1b_2 = 0$$

(iii) If l_1 and l_2 are not related as in (i) and (ii), then there is no simple relation of the above forms.

4.4.1 The Point of Intersection of two Straight Lines

Let $l_1 : a_1x+b_1y+c_1=0$ (1)

and $l_2 : a_2x+b_2y+c_2=0$ (2)

be two non-parallel lines. Then $a_1b_2 - b_1a_2 \neq 0$

Let $P(x_1, y_1)$ be the point of intersection of l_1 and l_2. Then

$a_1x_1+b_1y_1+c_1=0$ (3)

$a_2x_1+b_2y_1+c_2=0$ (4)

Solving (3) and (4) simultaneously, we have

Recall that: Two non-parallel lines intersect each other at one and only one point.

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$x_1 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y_1 = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

is the required point of intersection.

Note: $a_1b_2 - a_2b_1 \neq 0$, otherwise $l_1 \parallel l_2$.

Examples 1: Find the point of intersection of the lines

$5x+7y=35$ (i)

$3x-7y=21$ (ii)

Solution: We note that the lines are not parallel and so they must intersect at a point. Adding (i) and (ii), we have

8*x* = 56 or *x* = 7

Setting this value of *x* into (1), we find, *y* = 0.

Thus (7, 0) is the point of intersection of the two lines.

Remember that:

* If the lines are parallel, then solution does not exist $(\because a_1b_2-a_2b_1=0)$

* Before solving equations one should ensure that lines are not parallel.

4.4.2 Condition of Concurrency of Three Straight Lines

Three non-parallel lines

$l_1: a_1x+b_1y+c_1=0$ (1)

$l_2: a_2x+b_2y+c_2=0$ (2)

$l_3: a_3x+b_3y+c_3=0$ (3)

are concurrent iff $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$

Proof: If the lines are concurrent then they have a common point of intersection $P(x_1,y_1)$ say. As $l_1 \nparallel l_2$, so their point of intersection (x,y) is

$$x == \frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1} \text{ and } y \ \frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}$$

This point also lies on (3), so

$$a_3\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}\right)+c_3=0$$

or $a_3(b_1c_2-b_2c_1)+b_3(a_2c_1-a_1c_2)+c_3(a_1b_2-a_2b_1)=0$

An easier way to write the above equation is in the following determinant form:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

This is a necessary and sufficient condition of concurrency of the given three lines.

Example 1: Check whether the following lines are concurrent or not. If concurrent, find the point of concurrency.

$3x-4y-3=0$ (1)

$5x+12y+1=0$ (2)

$32x+4y-17=0$ (3)

Solution. The determinant of the coefficients of the given equations is

$$\begin{vmatrix} 3 & -4 & -3 \\ 5 & 12 & 1 \\ 32 & 4 & -17 \end{vmatrix}=\begin{vmatrix} 18 & 32 & 0 \\ 5 & 12 & 1 \\ 117 & 208 & 0 \end{vmatrix}, \text{by } R_1+3R_2 \text{ and } R_3+17R_2$$

$$=-1\begin{vmatrix} 18 & 32 \\ 117 & 208 \end{vmatrix}=-(208\times 18-117\times 32)=0$$

Thus the lines are concurrent.

The point of intersection of any two lines is the required point of concurrency. From (1) and (2), we have

$$\frac{x}{-4+36}=\frac{y}{-15-3}=\frac{1}{36+20}$$

$$x=\frac{32}{56}=\frac{4}{7} \text{ and } y=\frac{-18}{56}=\frac{-9}{28} \text{ i.e. } \left(\frac{4}{7},\frac{-9}{28}\right)$$

is the point of intersection.

4.4.3 Equation of Lines through the point of intersection of two lines

We can find a family of lines through the point of intersection of two non parallel lines l_1 and l_2.

Do you remember?
An infinite number of lines can pass through a point

Let $l_1: a_1x+b_1y+c_1=0$ (1)
and $l_2: a_2x+b_2y+c_2=0$ (2)
For a non-zero real h, consider the equation

$$a_1x+b_1y+c_1+h(a_2x+b_2y+c_2)=0 \quad (3)$$

This, being a linear equation, represents a straight line. For different values of h, (3) represents different lines. Thus (3) is a family of lines.

If (x_1, y_1) is any point lying on both (1) and (2), then it is their point of intersection. Since (x_1, y_1) lies on both (1) and (2), we have

$$a_1x+b_1y+c_1=0 \text{ and } a_2x+b_2y+c_2=0$$

From the above two equations, we note that (x_1, y_1) also lies on (3).

Thus (3) is the required family of lines through the point of intersection of (1) and (2). Since h can assume an infinite number of values, (3) represents an infinite number of lines.

A particular line of the family (3) can be determined if one more condition is given.

Example 2: Find the family of lines through the point of intersection of the lines

$3x-4y-10=0$ (1)
$x+2y-10=0$ (2)

Find the member of the family which is

(i) parallel to a line with slope $\frac{-2}{3}$
(ii) perpendicular to the line $l: 3x-4y+1=0$.

Solution: (i) A family of lines through the point of intersection of equations (1) and (2) is

$3x-4y-10+k(x+2y-10)=0$
or $(3+k)x+(-4+2k)y+(-10-10k)=0$ (3)

Slope m of (3) is given by: $m = -\frac{3+k}{-4+2k}$

This is slope of any member of the family (3).

If (3) is parallel to the line with slope $-\frac{2}{3}$ then

$-\frac{3+k}{-4+2k} = \frac{-2}{3}$ or $9+3k = -8+4k$ i.e., $k = 17$

Substituting $k=17$ into (3), equation of the member of the family is
$20x+30y-180=0$ i.e., $2x+3y-18=0$

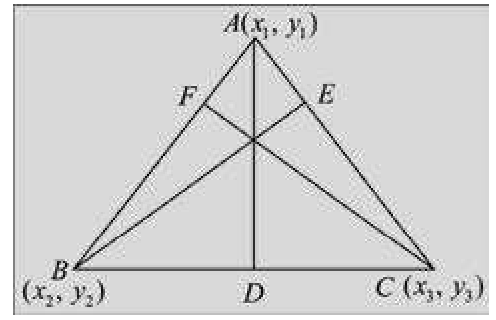
(ii) Slope of $3x-4y+1=0$ (4)

is $\frac{3}{4}$. Since (3) is to be perpendicular to (4), we have $-\frac{3+k}{-4+2k} \times \frac{3}{4} = -1$

or $9+3k = -16+8k$ or $k=5$

Inserting this value of k into (3), we get $4x+3y-30=0$ which is required equation of the line.

Theorem: Altitudes of a triangle are concurrent.



Proof. Let the coordinates of the vertices of ΔABC be as shown in the figure.

Then slope of $BC = \frac{y_2-y_3}{x_2-x_3}$

Therefore slope of the altitude $AD = -\frac{x_2-x_3}{y_2-y_3}$

Equation of the altitude AD is

$$y-y_1 = -\frac{x_2-x_3}{y_2-y_3}(x-x_1) \quad \text{(Point-slope form)}$$

or $x(x_2-x_3)+y(y_2-y_3)-x_1(x_2-x_3)-y_1(y_2-y_3)=0$ (1)

Equations of the altitudes BE and CF are respectively (by symmetry)

$x(x_3-x_1)+y(y_3-y_1)-x_2(x_3-x_1)-y_2(y_3-y_1)=0$ (2)
and $x(x_1-x_2)+y(y_1-y_2)-x_3(x_1-x_2)-y_3(y_1-y_2)=0$ (3)

The three lines (1), (2) and (3) are concurrent if and only if

$$D = \begin{vmatrix} x_2-x_3 & y_2-y_3 & -x_1(x_2-x_3)-y_1(y_2-y_3) \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-x_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix} \text{ is zero}$$

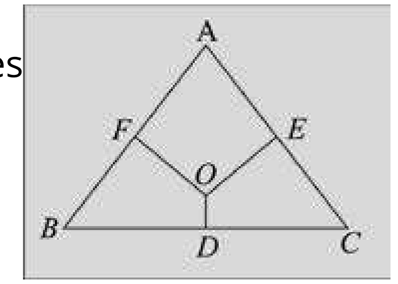
Adding 2nd and 3rd rows to the 1st row of the determihant, we have

$$\begin{vmatrix} 0 & 0 & 0 \\ x_3-x_1 & y_3-y_1 & -x_2\,(x_3-x_1)\ -y_2\,(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3\,(x_1-x_2)\ -y_3\,(y_1-y_2) \end{vmatrix} = 0$$

Thus the altitudes of a triangle are concurrent.

Theorem: Right bisectors of a triangle are concurrent.

Proof. Let $A\,(x_1, y_1)$, $B\,(x_2, y_2)$ and $C\,(x_3, y_3)$ be the vertices of ΔABC



The midpoint *D* of *BC* has coordinates

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$

Since the slope of *BC* is $\frac{y_2-y_3}{x_2-x_3}$, the slope of the right bisector *DO* of *BC* is $-\frac{x_2-x_3}{y_2-y_3}$

Equation of the right bisector DO of BC is

$$y - \frac{y_2+y_3}{2} = \frac{x_2-x_3}{y_2-y_3}\left(x \quad \frac{x_2+x_3}{2}\right) \quad \text{(Point-slope form)}$$

or $\quad x\,(x_2-x_3) + y\,(y_2-y_3) - \frac{1}{2}\,(y_2^2-y_3^2) - \frac{1}{2}\,(x_2^2-x_3^2) = 0 \quad (1)$

By symmetry, equations of the other two right bisectors EO and FO are respectively:

$$x\,(x_3-x_1) + y\,(y_3-y_1) - \frac{1}{2}\,(y_3^2-y_1^2) - \frac{1}{2}\,(x_3^2-x_1^2) = 0 \qquad (2)$$

and $\quad x\,(x_1-x_2) + y\,(y_1-y_2) - \frac{1}{2}\,(y_1^2-y_2^2) - \frac{1}{2}\,(x_1^2-x_2^2) = 0 \qquad (3)$

The lines (1), (2) and (3) will be concurrent if and only if

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -\frac{1}{2}\,(y_2^2-y_3^2)\ -\frac{1}{2}\,(x_2^2-x_3^2) \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}\,(y_3^2-y_1^2)\ -\frac{1}{2}\,(x_3^2-x_1^2) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}\,(y_1^2-y_2^2)\ -\frac{1}{2}\,(x_1^2-x_2^2) \end{vmatrix} = 0$$

Adding 2nd and 3rd rows to 1st row of the determinant, we have

$$\begin{vmatrix} 0 & 0 & 0 \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}\,(y_3^2-y_1^2)\ -\frac{1}{2}\,(x_3^2-x_1^2) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}\,(y_1^2-y_2^2)\ -\frac{1}{2}\,(x_1^2-x_2^2) \end{vmatrix} = 0$$

Thus the right bisectors of a triangle are concurrent.

Note: If equations of sides of the triangle are given, then intersection of any two lines gives a vertex of the triangle.

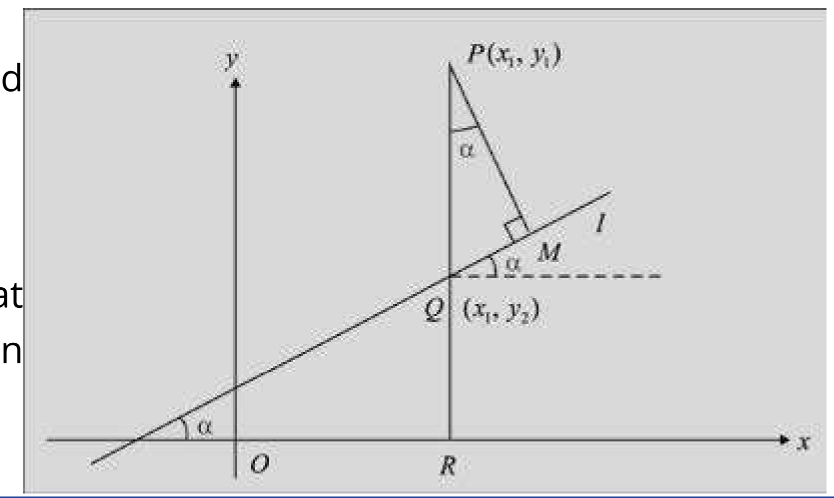
4.4.4 Distance of a point from a line

Theorem: The distance d from the point $P(x_1, y_1)$ to the line l

$l : ax + by + c = 0 \qquad (1)$

is given by $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

Proof: Let l be non-vertical and non-horizontal line.



From *P*, draw

$PQR \perp Ox$ and $PM \perp l$.

Let the ordinate of *Q* be y_2 so that coordinates of *Q* are (x_1, y_2). Since *Q* lies on l, we have $ax_1 + by_2 + c = 0$

or $\quad y_2 = \dfrac{-ax_1 - c}{b}$

From the figure it is clear that $\angle MPQ = \alpha$ = the inclination of l.

Now $\tan\alpha$ = slope of $l = \dfrac{-a}{b}$

Therefore, $|\cos\alpha| = \dfrac{|b|}{\sqrt{a^2+b^2}}$

Thus $\quad |PM| = d = |PQ|\,|\cos\alpha| = |y_1 - y_2|\,|\cos\alpha|$

$$= \left|y_1 - \frac{-ax_1 - c}{b}\right| \cdot \frac{|b|}{\sqrt{a^2+b^2}}$$

$$= \frac{|by_1 + ax_1 + c|}{|b|\sqrt{a^2+b^2}} \cdot |b| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$$

If l is horizontal, its equation is of the form $y = -\dfrac{c}{b}$ and the distance from $P(x_1, y_1)$ to l is simply the difference of the *y*-values.

$$\therefore \quad d = \left|y_1 - \left(-\frac{c}{b}\right)\right| = \left|\frac{by_1 + c}{b}\right|$$

Similarly, if the line is vertical and has equation: $x = \dfrac{-c}{a}$ then $d = \left|\dfrac{ax_1 + c}{a}\right|$

Note: If the point $P(x_1, y_1)$ lies on l, then the distance *d* is zero, since (x_1, y_1) satisfies the equation i.e., $ax_1 + by_1 + c = 0$

4.4.5 Distance Between two Parallel Lines

The distance between two parallel lines is the distance from any point on one of the lines to the other line.

Example: Find the distance between the parallel lines

$l: 2x - 5y + 13 = 0$ and

$l_2: 2x - 5y + 6 = 0$

Solution: First find any point on one of the lines, say l_1. If $x = 1$ lies on l_1, then y = 3 and the point (1,3) lies on it. The distance *d* from (1, 3) to l_2 is

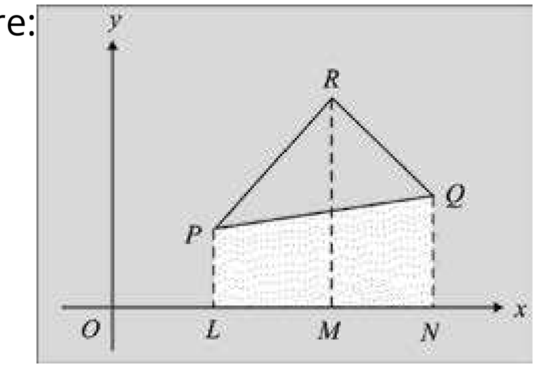
$$d = \frac{|2(1) - 5(3) + 6|}{\sqrt{(-2)^2 + 5^2}} = \frac{|2 - 15 + 6|}{\sqrt{4 + 25}} = \frac{7}{\sqrt{29}}$$

The distance between the parallel lines is $\dfrac{7}{\sqrt{29}}$.

Challenge!
Check the answer by taking
(i) any other point on l_1
(ii) any point of l_2 and finding its distance from l_1

4.4.6 Area of a Triangular Region Whose Vertices are Given

To find the area of a triangular region whose vertices are: $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$.



Draw perpendiculars $\overline{PL}$, $\overline{QN}$ and $\overline{RM}$ on x-axis.

Area of the triangular region PQR

= Area of the trapezoidal region $PLMR$

+ Area of the trapezoidal region $RMNQ$

– Area of the trapezoidal region $PLNQ$.

$$= \frac{1}{2}\left(\left|\overline{PL}\right| + \left|\overline{RM}\right|\right)\left(\left|\overline{LM}\right|\right) + \frac{1}{2}\left(\left|\overline{RM}\right| + \left|\overline{QN}\right|\right)\left(\left|\overline{MN}\right|\right) - \frac{1}{2}\left(\left|\overline{PL}\right| + \left|\overline{QN}\right|\right)\left(\left|\overline{LN}\right|\right)$$

$$= \frac{1}{2}[(y_1 + y_3)(x_3 - x_1) + (y_3 + y_2)(x_2 - x_3) - (y_1 + y_2)(x_2 - x_1)]$$

$$= \frac{1}{2}(x_3y_1 + x_3y_3 - x_1y_1 + x_1y_3 + x_2y_3 + x_2y_2 - x_3y_3 - x_2y_1 - x_2y_2 + x_1y_1 + x_1y_2)$$

$$= \frac{1}{2}(x_3y_1 - x_1y_3 + x_2y_3 - x_3y_2 - x_2y_1 + x_1y_2)$$

Thus required area A is given by:

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

Have you observed that:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Corollary: If the points P, Q and R are collinear, then

$$\Delta = 0$$

Note: In numerical problems, if sign of the area is negative, then it is to be omitted.

Example 1: Find the area of the region bounded by the triangle with vertices $(a, b+c)$, $(a, b-c)$ and $(-a, c)$.

Solution: Required area Δ is

$$\Delta = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ a & b-c & 1 \\ -a & c & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ 0 & 2c & 0 \\ -a & c & 1 \end{vmatrix}, \text{by } R_2 \quad R_1$$

$= \frac{1}{2}[-2c(a+a)]$, expanding by the second row

$= -2ca$

Thus $\Delta = 2ca$

Trapezium:
A quadrilateral having two parallel and two non-parallel sides.
Area of trapezoidal region:

$\frac{1}{2}$ (sum of || sides) (distance between || sides)

Example 2: By considering the area of the region bounded by the triangle with vertices *A* (1, 4), *B* (2, - 3) and *C* (3, - 10)
check whether the three points are collinear or not.

Solution: Area Δ of the region bounded by the triangle *ABC* is

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ 2 & -3 & 1 \\ 3 & -10 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ 1 & -7 & 0 \\ 3 & -14 & 0 \end{vmatrix} \text{ by } R_2 - R_1 \text{ and } R_3 - R_1$$

$= \frac{1}{2}[1(-14+14)]$, expanding by third column

= 0

Thus the points are collinear.

EXERCISE 4.3

1. Find the slope and inclination o f the line joining the points:
(i) (–2, 4) ; (5, 11) (ii) (3, –2) ; (2, 7) (iii) (4, 6) ; (4, 8)
Sketch each line in the plane.

2. In the triangle *A* (8, 6) *B* (–4, 2), *C* (–2 , –6) , find the slope of
(i) each side of the triangle
(ii) each median of the triangle
(iii) each altitude of the triangle.

3. By means of slopes, show that the following points lie on the same line:
(a) (–1, –3) ; (1, 5) ; (2, 9) (b) (4 ,–5) ; (7, 5) ; (10, 15)
(c) (–4, 6) ; (3, 8) ; (10, 10) (d) (a, 2*b*): (*c*, *a* + *b*); (2*c* – a, 2*a*)

4. Find k so that the line joining *A* (7, 3); *B* (*k*, –6) and the line joining *C* (–4, 5) ; *D* (–6, 4) are (i) parallel (ii) perpendicular.

5. Using slopes, show that the triangle with its vertices *A* (6, 1), *B* (2, 7) and *C* (–6, –7) is a right triangle.

6. The three points *A* (7, –1), *B* (–2, 2) and *C* (1, 4) are consecutive vertices of a parallelogram. Find the fourth vertex.

7. The points *A* (–1, 2), *B* (3, –1) and *C* (6, 3) are consecutive vertices of a rhombus. Find the fourth vertex and show that the diagonals of the rhombus are perpendicular to each other.

8. Two pairs of points are given. Find whether the two lines determined by these points are :
(i) parallel (ii) perpendicular (iii) none.
(a) (1, –2), (2, 4) and (4, 1), (–8, 2)
(b) (–3, 4), (6, 2) and (4, 5), (–2, –7)

9. Find an equation of
(a) the horizontal line through (7, –9)
(b) the vertical line through (–5, 3)

(c) the line bisecting the first and third quadrants.

(d) the line bisecting the second and fourth quadrants.

10. Find an equation of the line

(a) through *A* (–6, 5) having slope 7

(b) through (8, –3) having slope 0

(c) through (–8 , 5) having slope undefined

(d) through (–5, –3) and (9, –1)

(e) *y*-intercept: –7 and slope: –5

(f) *x*-intercept: –3 and *y*-intercept: 4

(g) *x*-intercept: –9 and slope: –4

11. Find an equation of the perpendicular bisector of the segment joining the points *A* (3 ,5) and *B* (9, 8).

12. Find equations of the sides, altitudes and medians of the triangle whose vertices are *A* (–3, 2), *B* (5, 4) and *C* (3, –8).

13. Find an equation of the line through (–4, –6) and perpendicular to a line having slope $\frac{-3}{2}$

14. Find an equation of the line through (11, –5) and parallel to a line with slope –24.

15. The points *A* (–1, 2), *B* (6, 3) and *C* (2, –4) are vertices of a triangle. Show that the line joining the midpoint *D* of *AB* and the midpoint *E* of *AC* is parallel to BC and $DE = \frac{1}{2}BC$.

16. A milkman can sell 560 litres of milk at Rs. 12.50 per litre and 700 litres of milk at Rs. 12.00 per litre. Assuming the graph of the sale price and the milk sold to be a straight line, find the number of litres of milk that the milkman can sell at Rs. 12.25 per litre.

17. The population of Pakistan to the nearest million was 60 million in 1961 and 95 million in 1981. Using *t* as the number of years after 1961, find an equation of the line that gives the population in terms of *t*. Use this equation to find the population in (a) 1947 (b) 1997.

18. A house was purchased for Rs.1 million in 1980. It is worth Rs. 4 million in 1996. Assuming that the value increased by the same amount each year, find an equation that gives the value of the house after *t* years of the date of purchase. What was its value in 1990?

19. Plot the Celsius (*C*) and Fahrenheit (*F*) temperature scales on the horizontal axis and the vertical axis respectively. Draw the line joining the freezing point and the boiling point of water. Find an equation giving *F* temperature in terms of *C*.

20. The average entry test score of engineering candidates was 592 in the year 1998 while the score was 564 in 2002. Assuming that the relationship between time and score is linear, find the average score for 2006.

21. Convert each of the following equation into

(i) Slope intercept form (ii) two intercept form (iii) normal form

(a) $2x - 4y + 11 = 0$ (b) $4x + 7y - 2 = 0$ (c) $15y - 8x + 13 = 0$

Also find the length of the perpendicular from (0, 0) to each line.

22. In each of the following check whether the two lines are

(i) parallel

(ii) perpendicular

(iii) neither parallel nor perpendicular

(a) $2x + y - 3 = 0$; $4x + 2y + 5 = 0$

(b) $3y = 2x + 5$; $3x + 2y - 8 = 0$

(c) $4y + 2x - 1 = 0$; $x - 2y - 7 = 0$

(d) $4x - y + 2 = 0 -$;+ $= 12x \quad 3y \quad 1 \quad 0$

(e) $12x + 35y - 7 = 0$; $105x - 36y + 11 = 0$

23. Find the distance between the given parallel lines. Sketch the lines. Also find an equation of the parallel line lying midway between them.

(a) $3x - 4y + 3 = 0$; $3x - 4y + 7 = 0$

(b) $12x + 5y - 6 = 0$; $12x + 5y + 13 = 0$

(c) $x + 2y - 5 = 0 +$;= $2x \quad 4y \quad 1$

24. Find an equation of the line through (–4, 7) and parallel to the line $2x-7y+4=0$.

25. Find an equation of the line through (5, –8) and perpendicular to the join of *A* (–15, –8), *B* (10, 7).

26. Find equations of two parallel lines perpendicular to $2x-y+3=0$ such that the product of the *x*-and *y*-intercepts of each is 3.

27. One vertex of a parallelogram is (1, 4); the diagonals intersect at (2, 1) and the sides have slopes 1 and $\frac{-1}{7}$. Find the other three vertices.

28. Find whether the given point lies above or below the given line
(a) (5, 8) ; $2x-3y+6=0$
(b) (–7, 6) ; $4x+3y-9=0$

29. Check whether the given points are on the same or opposite sides of the given line.
(a) (0, 0) and (–4, 7) ; $6x-7y+70=0$
(b) (2, 3) and (–2, 3) ; $3x-5y+8=0$

30. Find the distance from the point *P*(6, –1) to the line 6*x* – 4*y* + 9 = 0.

31. Find the area of the triangular region whose vertices are *A* (5, 3), *B* (–2, 2), *C* (4, 2).

32. The coordinates of three points are *A*(2, 3), *B*(–1, 1) and *C*(4, –5). By computing the area bounded by *ABC* check whether the points are collinear.

4.5. ANGLE BETWEEN TWO LINES

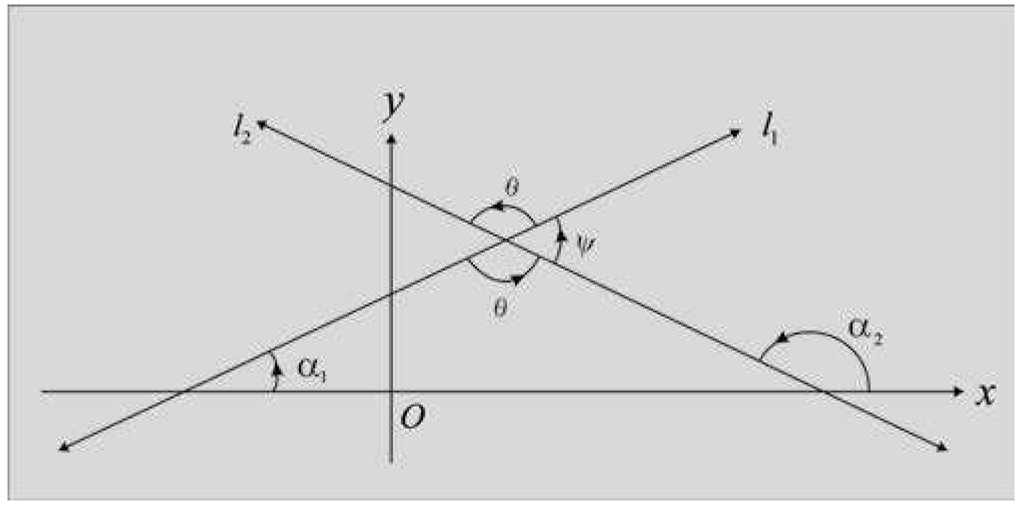
Let l_1 and l_2 be two intersecting lines, which meet at a point *P*. At the point *P* two supplementary angles are formed by the lines l_1 and l_2.

Unless $l_1 \perp l_2$ one of the two angles is acute. The angle from l_1 to l_2 is the angle θ through which l_1 is rotated anti-clockwise about the point *P* so that it coincides with l_2

In the figure below θ is angle of intersection of the two lines and it is measured from l_1 to l_2 in counterclockwise direction, ψ is also angle of intersection but it is measured from l_2 to l_1.

With this convention for angle of intersection, it is clear that the inclination of a line is the angle measured in the counterclockwise direction from the positive *x*-axis to the line, and it tallies with the earlier definition of the inclination of a line.

Theorem: Let l_1 and l_2 be two non-vertical lines such that they are not perpendicular to each other. If m_1 and m_2 are the slopes of l_1 and l_2 respectively: the angle θ from l_1 to l_2 is given by;



$$\tan\theta = \frac{m_2-m_1}{1+m_1m_2}$$

Proof: From the figure, we have

$$\alpha_2 = \alpha_1 + \theta$$

or $\quad \theta = \alpha_2 - \alpha_1$

$$\therefore \quad \tan\theta = \tan(\alpha_2-\alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_1\tan\alpha_2} = \frac{m_2-m_1}{1+m_1m_2}$$

Corollary 1. $l_1 \parallel l_2$ if and only if $m_1 = m_2$

$$\Leftrightarrow \quad \tan\theta = 0 \quad \frac{m_2-m_1}{1+m_1m_2}$$

$$\Leftrightarrow \quad m_2 = m_1$$

Corollary 2. $l_1 \perp l_2$ iff $1+m_1m_2=0$

$$\Leftrightarrow \quad \tan\theta = \frac{m_2-m_1}{1+m_1m_2} = \infty = \tan\frac{\pi}{2} \Leftrightarrow 1 + m_1m_2 = 0$$

These two results have already been stated in 4.3.1.

Example 1: Find the angle from the line with slope $\frac{-7}{3}$ to the line with slope $\frac{5}{2}$.

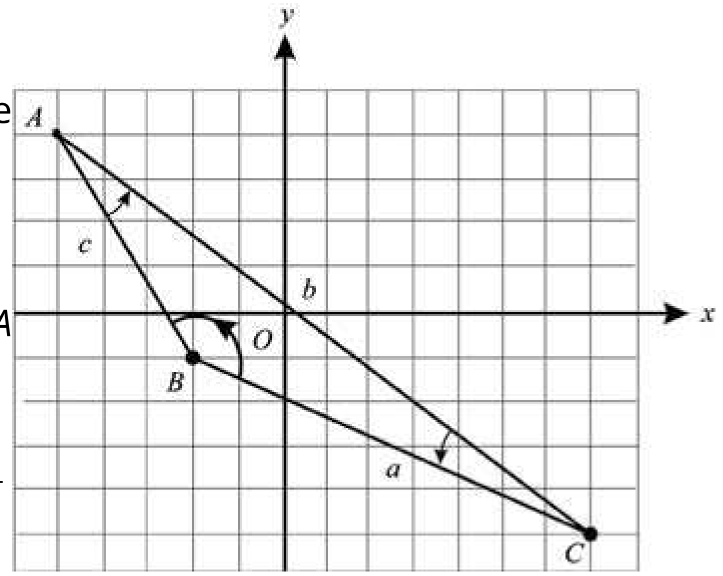
Solution: Here $m_2 = \frac{5}{2}, m_1 = \frac{-7}{3}$. If θ is measure of the required angle, then

$$\tan\theta=\frac{\frac{5}{2}-\left(\frac{-7}{3}\right)}{1+\frac{5}{2}\left(\frac{-7}{3}\right)}=\frac{29}{-29}=-1$$

Thus $\theta=135^\circ$

Example 2: Find the angles of the triangle whose vertices are

$A\,(-5, 4)$, $B\,(-2, -1)$, $C\,(7, -5)$



Solution: Let the slopes of the sides AB, BC and CA be denoted by m_c, m_a, m_b respectively. Then

$$m_c=\frac{4+1}{-5+2}=\frac{-5}{3},m_a=\frac{-5+1}{7+2}=\frac{-4}{9},m_b=\frac{-5-4}{7+5}=\frac{-3}{4}$$

Now angle A is measured from AB to AC.

$$\tan A=\frac{m_b-m_c}{1+m_bm_c}=\frac{\frac{-3}{4}+\frac{5}{3}}{1+\left(\frac{-3}{4}\right)\left(\frac{-5}{3}\right)}\quad\frac{11}{27}\quad\text{or}\quad m\angle A\quad 22.2^\circ$$

The angle B is measured from BC to BA

$$\therefore\quad\tan B=\frac{m_c-m_a}{1+m_cm_a}=\frac{\frac{-5}{3}+\frac{4}{9}}{1+\left(\frac{-5}{3}\right)\left(\frac{-4}{9}\right)}=\frac{-33}{47}\quad\text{or}\quad m\angle B=144.9^\circ$$

The angle C is measured from CA to CB.

$$\therefore\quad\tan A=\frac{m_a-m_b}{1+m_am_b}=\frac{\frac{-4}{9}+\frac{3}{4}}{1+\left(\frac{-4}{9}\right)\left(\frac{-3}{4}\right)}=\frac{11}{48}\quad\text{or}\quad m\angle C=12.9^\circ$$

4.5.1 Equation of a Straight Line in Matrix form

It is easy to solve two or three simultaneous linear equations by elementary methods. If the number of equations and variables become large, the solution of the equations by ordinary method becomes very difficult. In such a case, given equations are written in matrix form and solved.

One Linear Equation:

A linear equation

$$l: ax+by+c=0 \qquad (1)$$

in two variables x and y has its matrix form as:

$$[ax+by]=[-c]$$

or $\begin{bmatrix}a & b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=[-c]$

or $AX=C$

where $A=\begin{bmatrix}a & b\end{bmatrix}$, $X=\begin{bmatrix}x\\y\end{bmatrix}$ and $C=[-c]$

A System of Two Linear Equations:

A system of two linear equations

$$\left.\begin{array}{l}l_1: a_1x+b_1y+c=0\\ l: a_2x+b_2y+c=0\end{array}\right\} \qquad (2)$$

in two variables x and y can be written in matrix form as:

$$\begin{bmatrix}a_1x+b_1y\\a_2x+b_2y\end{bmatrix}=\begin{bmatrix}-c_1\\-c_2\end{bmatrix}$$

or $\begin{bmatrix}a_1 & b_1\\a_2 & b_2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-c_1\\-c_2\end{bmatrix} \qquad (3)$

or $AX=C$

where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, X $\begin{bmatrix} x \\ y \end{bmatrix}$ and $C = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$

Equations (2) have a solution iff det $A \neq 0$.

A System of Three Linear Equations:

A system of three linear equations

$$\left.\begin{array}{l} l_1 : a_1x + b_1y + c_1 = 0 \\ l_2 : a_2x + b_2y + c_2 = 0 \\ l_3 : a_3x + b_3y + c_3 = 0 \end{array}\right\} \qquad (5)$$

in two variables *y* and *y* takes the matrix form as

$$\begin{bmatrix} a_1x + b_1y + c_1 \\ a_2x + b_2y + c_2 \\ a_3x + b_3y + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the matrix

$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is singular, then the lines (5) are concurrent and so the system (5) has a unique solution.

Example 1: Express the system

$$\left.\begin{array}{l} 3x + 4y - 7 = 0 \\ 2x - 5y + 8 = 0 \\ x + y - 3 = 0 \end{array}\right\}$$

in matrix form and check whether the three lines are concurrent

Solution. The matrix form of the system is

$$\begin{bmatrix} 3 & 4 & -7 \\ 2 & -5 & 8 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Coefficient matrix of the system is

$$A = \begin{bmatrix} 3 & 4 & -7 \\ 2 & 5 & 8 \\ 1 & 1 & -3 \end{bmatrix} \text{ and det } A \begin{vmatrix} 0 & 1 & 2 \\ 0 & 7 & 14 \\ 1 & 1 & -3 \end{vmatrix} \quad \begin{array}{l} \text{by } R_1 - 3R_3 \\ \text{and } R_2 - 2R_3 \end{array}$$

and det A = 1(14+14) = 28 $\neq$ 0

As *A* is non-singular, so the lines are not concurrent.

Example 2: Find a system of linear equations corresponding to the matrix form

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 1 \\ 4 & 7 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1)$$

Are the lines represented by the system concurrent?

Solution: Multiplying the matrices on the L.H.S. of (1), we have

$$\begin{bmatrix} x + 2y + 5 \\ 3x + 5y + 1 \\ 4x + 7y + 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (2)$$

By using the definition of equality of two matrices, we have from (2),

$$\left.\begin{array}{l} x + 2y + 5 = 0 \\ 3x + 5y + 1 = 0 \\ 4x + 7y + 6 = 0 \end{array}\right]$$

as the required system of equations. The coefficient matrix *A* of the system is such that

$$\det \ A = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 5 & 1 \\ 4 & 7 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & -14 \\ 0 & -1 & -14 \end{vmatrix} = 0$$

Thus the lines of the system are concurrent.

EXERCISE 4.4

1. Find the point of intersection of the lines

(i) $x-2y+1=0-$ $+$and $2x \quad y \quad 2 \quad 0$

(ii) $3x+y+12=0+$ and$=$ $x \quad 2y \quad 1 \quad 0$

(iii) $x+4y-12=0-$ and $=$ $x \quad 3y \quad 3 \quad 0$

2. Find an equation of the line through

(i) the point (2, -9) and the intersection of the lines
$2x+5y-8=0-$ and $=$ $3x \quad 4y \quad 6 \quad 0$

(ii) the intersection of the lines
$x-y-4=0+$ $+$ and $7x \quad y \quad 20 \quad 0$ and

(a) parallel (b) perpendicular
to the line $6x+y-14=0$

(iii) through the intersection of the lines $x+2y+3=0$, $3x+4y+7=0$ and making equal intercepts on the axes.

3. Find an equation of the line through the intersection of
$16x - 10y - 33 = 0$; $12x - 14y - 29 = 0$ and the intersection of
$x - y + 4 = 0$; $x - 7y + 2 = 0$

4. Find the condition that the lines $y=m_1x+c_1; y=m_2x+c_2$ and $y=m_3x+c_3$ are concurrent.

5. Determine the value of p such that the lines $2x - 3y - 1 = 0$, $3x - y - 5 = 0$ and $3x + 4y + 8 = 0$ meet at a point.

6. Show that the lines $4x - 3y - 8 = 0$, $3x - 4y - 6 = 0$ and $x - y - 2 = 0$ are concurrent and the third-line bisects the angle formed by the first two lines.

7. The vertices of a triangle are A (–2, 3), B (–4, 1) and C (3, 5). Find coordinates of the

(i) centroid (ii) orthocentre

(iii) circumcentre of the triangle

Are these three points collinear?

8. Check whether the lines
$4x-3y-8=0$; $3x-4y-6=0$; $x-y-2=0$
are concurrent. If so, find the point where they meet

9. Find the coordinates of the vertices of the triangle formed by the lines
$x-2y-6=0$; $3x-y+3=0$; $2x+y-4=0$
Also find measures of the angles of the triangle.

10. Find the angle measured from the line l_1 to the line l_2 where

(a) l_1 : Joining $(2, 7)$ and $(7, 10)$
l_2 : Joining $(1, 1)$ and $(-5, 3)$

(b) l_1: Joining $(3,-1)$ and $(5,7)$
l_2: Joining $(2,4)$ and $(-8,2)$

Also find the acute angle in each case.

(c) l_1: Joining $(1,-7)$ and $(6,-4)$
l_2: Joining $(-1,2)$ and $(-6,-1)$

(d) l_1: Joining $(-9,-1)$ and $(3,-5)$
l_2:Joining $(2,7)$ and $(-6,-7)$

11. Find the interior angles of the triangle whose vertices are

(a) A (–2, 11), B (–6, –3), (4, –9)

(b) A (6, 1), B (2, 7), C(–6, –7)

(c) A (2, –5), B (–4, –3), (–1, 5)

(d) A (2, 8), B (–5, 4), C(4, –9)

12. Find the interior angles of the quadrilateral whose vertices are A (5, 2), B (–2, 3), C (–3, –4) and D (4, –5)

13. Show that the points
A (0, 0), B (2, 1), C (3, 3), D (1, 2) are the vertices of a rhombus.
Find its interior angles.

14. Find the area of the region bounded by the triangle whose sides are
$7x-y-10=0$; $10x+y-14=0$; $3x+2y+3=0$

15. The vertices of a triangle are A(–2, 3), B(–4, 1) and C(3, 5). Find the centre of the circumcircle of the triangle.

16. Express the given system of equations in matrix form. Find in each case whether the lines are concurrent.

(a) $x+3y-2=0;\quad 2x-y+4=0;\quad x-11y+14=0$

(b) $2x+3y+4=0;\quad x-2y-3=0;\quad 3x+y-8=0$

(c) $3x-4y-2=0;\quad x+2y-4=0;\quad 3x-2y+5=0.$

17. Find a system of linear equations corresponding to the given matrix form. Check whether the lines represented by the system are concurrent.

(a) $\begin{bmatrix}1 & 0 & -1\\ 2 & 0 & 1\\ 0 & -1 & 6\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$ (b) $\begin{bmatrix}1 & 1 & 2\\ 2 & 4 & -3\\ 3 & 6 & -5\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

4.6 HOMOGENEOUS EQUATION OF THE SECOND DEGREE IN TWO VARIABLES

We have already seen that if a graph is a straight line, then its equation is a linear equation in the variables *x* and *y*. Conversely, the graph of any linear equation in *x* and *y* is a straight line.

Suppose we have two straight lines represented by

$$a_1x+b_1y+c_1=0 \qquad (1)$$

and

$$a_2x+b_2y+c_2=0 \qquad (2)$$

Multiplying equations (1) and (2), we have

$$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)=0 \qquad (3)$$

It is a second degree equation in *x* and *y*.

Equation (3) is called joint equation of the pair of lines (1) and (2). On the other hand, given an equation of the second degree in *x* and *y*, say

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad (4)$$

where $a\neq 0$, represents equations of a pair of lines if (4) can be resolved into two linear factors. In this section, we shall study special joint equations of pairs of lines which pass through the origin.

Let $y=m_1x$ and $y=m_2x$ be two lines passing through the origin. Their joint equation is:

$$(y-m_1x)(y-m_2x)=0$$

or $$y^2-(m_1+m_2)xy+m_1m_2x^2=0 \qquad (5)$$

Equation (5) is a special type of a second degree homogeneous equation.

4.6.1 Homogeneous Equation

Let $$f(x,y)=0 \qquad (1)$$

be any equation in the variables *x* and *y*. Equation (1) is called a homogeneous equation of degree *n* (a positive integer) if

$$f(kx,ky)=k^n f(x,y)$$

for some real number *k*.

For example, in equation (5) above if we replace *x* and *y* by *kx* and *ky* respectively, we have

$$k^2y^2-k^2(m_1+m_2)xy+k^2m_1m_2x^2=0$$

or $k^2\left[y^2-(m_1+m_2)xy+m_1m_2x^2\right]=0$ i.e., $k^2 f(x,y)\quad 0$

Thus (5) is a homogeneous equation of degree 2.

$$ax^2+2hxy+by^2=0$$

A general second degree homogeneous equation can be written as:

$$ax^2+2hxy+by^2=0$$

provided *a*, *h* and *b* are not simultaneously zero.

Theorem: Every homogenous second degree equation

$$ax^2+2hxy+by^2=0 \qquad (1)$$

represents a pair of lines through the origin. The lines are

(i) real and distinct, if $h^2>ab$ (ii) real and coincident, if $h\quad ab$

(iii) imaginary, if $h^2<ab$

Proof: Multiplying (1) by b and re-arranging the terms, we have

$$b^2y^2 + 2bhxy + abx^2 = 0$$

or $$b^2y^2 + 2bhxy + h^2x^2 - h^2x^2 + abx^2 = 0$$

or $$(by + hx)^2 - x^2(h^2 - ab) = 0$$

or $$\left(by + hx + x\sqrt{h^2 - ab}\right)\left(by + hx - x\sqrt{h^2 - ab}\right) = 0$$

Thus (1) represents a pair of lines whose equations are:

$$by + x\left(h + \sqrt{h^2 - ab}\right) = 0 \qquad (2)$$

and $$by + x\left(h - \sqrt{h^2 - ab}\right) = 0 \qquad (3)$$

Clearly, the lines (2) and (3) are

(i) real and distinct if $h^2 > ab$. (ii) real and coincident, if $h^2 = ab$.

(iii) imaginary, if $h^2 < ab$.

It is interesting to note that even in case the lines are imaginary, they intersect in a real point viz (0, 0) since this point lies on their joint equation (1).

Example: Find an equation of each of the lines represented by

$$20x^2 + 17xy - 24y^2 = 0$$

Solution. The equation may be written as

$$24\left(\frac{y}{x}\right)^2 - 17\left(\frac{y}{x}\right) - 20 = 0$$

$$\Rightarrow \quad \frac{y}{x} = \frac{17 \pm \sqrt{289 + 1920}}{48} \quad \frac{17 \pm 47}{48} \quad \frac{4}{3}, \frac{-5}{8}$$

$$\Rightarrow \quad y = \frac{4}{3}x \quad \text{and} \quad y = \frac{-5}{8}x$$

$$\Rightarrow \quad 4x - 3y = 0 \quad \text{and} \quad 5x + 8y = 0$$

4.6.2 To find measure of the angle between the lines represented by

$$ax^2 + 2hxy + by^2 = 0 \qquad (1)$$

We have already seen that the lines represented by (1) are

$$by + x\left(h + \sqrt{h^2 - ab}\right) = 0 \qquad (2)$$

and $$by + x\left(h - \sqrt{h^2 - ab}\right) = 0 \qquad (3)$$

Now slopes of (2) and (3) are respectively given by:

$$m_1 = \frac{-\left(h + \sqrt{h^2 - ab}\right)}{b}, \quad \text{and} \quad m_2 \quad \frac{-\left(h - \sqrt{h^2 - ab}\right)}{b}$$

$$\text{Therefore,}\ m_1 + m_2 = \frac{-2h}{b} \qquad \text{and} \quad m_1m_2 \quad \frac{a}{b}$$

If θ is measure of the angle between the lines (2) and (3), then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2} \quad \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}} \quad \frac{2\sqrt{h^2 - ab}}{a + b}$$

The two lines are parallel, if $\theta = 0$, so that $\tan\theta = 0$ which implies $\boldsymbol{h^2 - ab = 0}$, which is the **condition for the lines to be coincident.**

If the lines are orthogonal, then $\theta = 90^\circ$, so that $\tan\theta$ is not defined. This implies ***a* + *b* = 0**. Hence the condition for (1) to represent a pair of orthogonal (perpendicular) lines is that **sum of the coefficients of x^2 and y^2 is 0.**

Example 1: Find measure of the angle between the lines represented by

$$x^2 - xy - 6y^2 = 0$$

Solution. Here $a = 1,\ h = \frac{1}{2},\ b \quad 6$

If θ is measure of the angle between the given lines, then

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b} = \frac{2\sqrt{\frac{1}{4}+6}}{-5} = -1 \quad \Rightarrow \theta = 135^\circ$$

Acute angle between the lines =180° - θ = 180° - 135° = 45°

Example2: Find a joint equation of the straight lines through the origin perpendicular to the lines represented by

$$x^2 + xy - 6y^2 = 0 \qquad (1)$$

Solution: (1) may be written as

$$(x-2y)(x+3y) = 0$$

Thus the lines represented by (1) are

$$x - 2y = 0 \qquad (2)$$

and $$x + 3y = 0 \qquad (3)$$

The line through (0, 0) and perpendicular to (2) is

$$y = \quad 2x \quad \text{or} \quad +y \quad 2x \quad 0 \qquad (4)$$

Similarly, the line through (0, 0) and perpendicular to (3) is

$$y = 3x - \quad \text{or}= \quad y \quad 3x \quad 0 \qquad (5)$$

Joint equation of the lines (4) and (5) is

$$(y+2x)(y-3x) = 0 \quad \text{or} \quad y^2 - xy - 6x^2 = 0$$

EXERCISE 4.5

Find the lines represented by each of the following and also find measure of the, angle between them **(Problems 1-6):**

1. $10x^2 - 23xy - 5y^2 = 0$

2. $3x^2 + 7xy + 2y^2 = 0$

3. $9x^2 + 24xy + 16y^2 = 0$

4. $2x^2 + 3xy - 5y^2 = 0$

5. $6x^2 - 19xy + 15y^2 = 0$

6. $x^2 + 2xy\sec\alpha + y^2 = 0$

7. Find a joint equation of the lines through the origin and perpendicular to the lines:

$x^2 - 2xy\tan\alpha - y^2 = 0$

8. Find a joint equation of the lines through the origin and perpendicular to the lines:

$ax^2 + 2hxy + by^2 = 0$

9. Find the area of the region bounded by:

$10x^2 - xy - 21y^2 = 0$ and $x + y + 1 = 0$