CHAPTER

1

MATRICES AND DETERMINANTS

Animation 1.1 : Matrix
Source & Credit : eLearn.punjab

Students Learning Outcomes

After studying this unit , the students will be able to:

1. Define
- a matrix with real entries and relate its rectangular layout (formation) with real life,
- rows and columns of a matrix,
- the order of a matrix,
- equality of two matrices.
2. Define and identify row matrix, column matrix, rectangular matrix, square matrix, zero/null matrix, diagonal matrix, scalar matrix, identity matrix, transpose of a matrix, symmetric and skew-symmetric matrices.
3. Know whether the given matrices are suitable for addition/ subtraction.
4. Add and subtract matrices.
5. Multiply a matrix by a real number.
6. Verify commutative and associative laws under addition.
7. Define additive identity of a matrix.
8. Find additive inverse of a matrix.
9. Know whether the given matrices are suitable for multiplication.
10. Multiply two (or three) matrices.
11. Verify associative law under multiplication.
12. Verify distributive laws.
13. Show with the help of an example that commutative law under multiplication does not hold in general (i.e., $AB \neq BA$).
14. Define multiplicative identity of a matrix.
15. Verify the result $(AB)^t = B^tA^t$.
16. Define the determinant of a square matrix.
17. Evaluate determinant of a matrix.
18. Define singular and non-singular matrices.
19. Define adjoint of a matrix.
20. Find multiplicative inverse of a non-singular matrix A and verify that $AA^{-1} = I = A^{-1}A$ where I is the identity matrix.
21. Use adjoint method to calculate inverse of a non-singular matrix.

22. Verify the result $(AB)^{-1} = B^{-1}A^{-1}$
23. Solve a system of two linear equations and related real life problems in two unknowns using
- Matrix inversion method,
- Cramer' s rule.

Introduction

The matrices and determinants are used in the field of Mathematics, Physics, Statistics, Electronics and other branches of science. The matrices have played a very important role in this age of Computer Science.

The idea of matrices was given by Arthur Cayley, an English mathematician of nineteenth century, who first developed, "Theory of Matrices" in 1858.

1.1 Matrix

A rectangular array or a formation of a collection of real numbers, say 0, 1, 2, 3, 4 and 7,such as, $\begin{matrix} 1 & 3 & 4 \\ 7 & 2 & 0 \end{matrix}$ and then enclosed by brackets `[ ]' is said to form a matrix $\begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 0 \end{bmatrix}$ Similarly $\begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}$ is another matrix.

We term the real numbers used in the formation of a matrix as entries or elements of the matrix. (Plural of matrix is matrices) The matrices are denoted conventionally by the capital letters A, B, C, M, N etc, of the English alphabets.

1.1.1 Rows and Columns of a Matrix

It is important to understand an entity of a matrix with the following formation

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 4 \\ 2 & 1 & -1 \end{bmatrix} \end{array} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array}$$

In matrix A, the entries presented in horizontal way are called rows.
In matrix A, there are three rows as shown by R_1, R_2 and R_3 of the matrix A.

$$\begin{array}{c} B = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix} \\ \quad\;\;\, \downarrow \;\; \downarrow \;\; \downarrow \\ \quad\;\;\, C_1 \; C_2 \; C_3 \end{array}$$

In matrix B, all the entries presented in vertical way are called columns of the matrix B.
In matrix B, there are three columns as shown by C_1, C_2 and C_3.

It is interesting to note that all rows have same number of elements and all columns have same number of elements but number of elements in rows and columns may not be same.

1.1.2 Order of a Matrix

The number of rows and columns in a matrix specifies its order. If a matrix M has m rows and n columns, then M is said to be of order m-by-n. For example,

$M = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ is of order 2-by-3, since it has two rows and three columns, whereas the matrix $N = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 2 & 3 & 7 \end{bmatrix}$ is a 3-by-3 matrix and $P = [\,3 \;\; 2 \;\; 5\,]$ is a matrix of order 1-by-3.

1.1.3 Equal Matrices

Let A and B be two matrices. Then A is said to be equal to B, and denoted by A = B, if and only if;
(i) the order of A = the order of B
(ii) their corresponding entries are equal.

Examples

(i) $A = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2+1 \\ -4 & 4-2 \end{bmatrix}$ are equal matrices.

We see that:
(a) the order of matrix A = the order of matrix B
(b) their corresponding elements are equal. Thus A = B

(ii) $L = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $M = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ are not equal matrices.

We see that order of L = order of M but entries in the second row and second column are not same, so L ≠ M.

(iii) $P = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \end{bmatrix}$ are not equal matrices. We see that order of P ≠ order of Q, so P ≠ Q.

EXERCISE 1.1

1. Find the order of the following matrices.

$A = \begin{bmatrix} 2 & 3 \\ -5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 3 & 5 \end{bmatrix}$, $C = [\,2 \;\; 4\,]$,

$D = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$, $E = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$, $F = [\,2\,]$

$G = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$, $H = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 6 \end{bmatrix}$

2. Which of the following matrices are equal?

$A = [\,3\,]$, $B = [\,3 \;\; 5\,]$, $C = [\,5-2\,]$,

$D = [\,5 \;\; 3\,]$, $E = \begin{bmatrix} 4 & 0 \\ 6 & 2 \end{bmatrix}$, $F = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$,

$G = \begin{bmatrix} 3-1 \\ 3+3 \end{bmatrix}$, $H = \begin{bmatrix} 4 & 0 \\ 6 & 2 \end{bmatrix}$, $I = [\,3 \;\; 3+2\,]$,

$J = \begin{bmatrix} 2+2 & 2-2 \\ 2+4 & 2+0 \end{bmatrix}$

3. Find the values of a, b, c and d which satisfy the matrix equation

$$\begin{bmatrix} a+c & a+2b \\ c-1 & 4d-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2d \end{bmatrix}$$

1.2 Types of Matrices

(i) Row Matrix

A matrix is called a row matrix, if it has only one row.
e.g., the matrix M = $[2 \ \ -1 \ \ 7]$ is a row matrix of order 1-by-3 and
M = $[1 \ \ -1]$ is a row matrix of order 1-by-2.

(ii) Column Matrix

A matrix is called a column matrix, if it has only one column.

e.g., $M = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $N = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ are column matrices of order 2-by-1 and 3-by-1 respectively.

(iii) Rectangular Matrix

A matrix M is called rectangular, if the number of rows of M is not equal to the number of M columns.

e.g., $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$; $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$; $C = [1 \ \ 2 \ \ 3]$ and $D = \begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix}$

are all rectangular matrices. The order of A is 3-by-2, the order of B is 2-by-3, the order of C is 1-by-3 and order of D is 3-by-1, which indicates that in each matrix the number of rows ≠ the number of columns.

(iv) Square Matrix

A matrix is called a square matrix, if its number of rows is equal to its number of columns.

e.g., $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ and $C = [3]$

are square matrices of orders, 2-by-2, 3-by-3 and 1-by-1 respectively.

(v) Null or Zero Matrix

A matrix is called a null or zero matrix, if each of its entries is 0.

e.g., $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $[0 \ \ 0]$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are null matrices of orders 2-by-2, 1-by-2, 2-by-1, 2-by-3 and 3-by-3 respectively. Note that null matrix is represented by O.

(vi) Transpose of a Matrix

A matrix obtained by interchanging the rows into columns or columns into rows of a matrix is called transpose of that matrix. If A is a matrix, then its transpose is denoted by A^t.

e.g., (i) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & 4 & -2 \end{bmatrix}$, then $A^t = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 3 & 0 & -2 \end{bmatrix}$

(ii) If $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$ then $B^t = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$

(iii) If $C = [0 \ \ 1]$, then $C^t = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

If a matrix A is of order 2-by-3, then order of its transpose A^t is 3-by-2.

(vii) Negative of a Matrix

Let A be a matrix. Then its negative, –A is obtained by changing the signs of all the entries of A, i.e.,

$$\text{If } A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}, \text{ then } -A = \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}.$$

(viii) Symmetric Matrix

A square matrix is symmetric if it is equal to its transpose i.e., matrix A is symmetric, if $A^t = A$.

e.g., (i) If $M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 0 \end{bmatrix}$ is a square matrix, then

$$M^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 0 \end{bmatrix} = M.$$ Thus M is a symmetric matrix.

(ii) If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$, then $A^t = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ 3 & 2 & 3 \end{bmatrix}$, $\neq A$

Hence A is not a symmetric matrix.

(ix) Skew-Symmetric Matrix

A square matrix A is said to be skew-symmetric, if $A^t = -A$.

e.g., if $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix}$

then $A^t = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -3 \\ -(-2) & 0 & -1 \\ -(-3) & -(-1) & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = -A$

Since $A^t = -A$, therefore A is a skew-symmetric matrix.

(x) Diagonal Matrix

A square matrix A is called a diagonal matrix if atleast any one of the entries of its diagonal is not zero and non-diagonal entries are zero.

e.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are all diagonal matrices of order 3-by-3.

$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ are diagonal matrices of order 2-by-2.

(xi) Scalar Matrix

A diagonal matrix is called a scalar matrix, if all the diagonal entries are same and non-zero.

For example $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ where k is a constant $\neq 0,1$.

Also $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $C = [5]$ are scalar matrices of order 3-by-3, 2-by-2 and 1-by-1 respectively.

(xii) Identity Matrix

A diagonal matrix is called identity (unit) matrix, if all diagonal entries are 1. It is denoted by I.

e.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3-by-3 identity matrix, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a 2-by-2 identity matrix, and $C = [1]$ is a 1-by-1 identity matrix.

> **Note:** (i) A scalar and identity matrix are diagonal matrices.
> (ii) A diagonal matrix is not a scalar or identity matrix.

EXERCISE 1.2

1. From the following matrices, identify unit matrices, row matrices, column matrices and null matrices.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = [2 \quad 3 \quad 4]$, $C = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$,

$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E = [\,0\,]$, $F = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

2. From the following matrices, identify
   (a) Square matrices (b) Rectangular matrices
   (c) Row matrices (d) Column matrices
   (e) Identity matrices (f) Null matrices

(i) $\begin{bmatrix} -8 & 2 & 7 \\ 12 & 0 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & 10 & -1 \end{bmatrix}$ (vii) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ix) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. From the following matrices, identify diagonal, scalar and unit (identity) matrices.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$D = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 5-3 & 0 \\ 0 & 1+1 \end{bmatrix}$$

4. Find negative of matrices A, B, C, D and E when:

$$A = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 6 \\ 3 & 2 \end{bmatrix},$$
$$D = \begin{bmatrix} -3 & 2 \\ -4 & 5 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$$

5. Find the transpose of each of the following matrices:

$$A = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \quad B = [5 \quad 1 \quad -6], \quad C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 0 \end{bmatrix},$$
$$D = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}, \quad E = \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

6. Verify that if $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, then

(i) $(A^t)^t = A$ (ii) $(B^t)^t = B$

1.3 Addition and Subtraction of Matrices

1.3.1 Addition of Matrices

Let A and B be any two matrices. The matrices A and B are conformable for addition, if they have the same order.

e.g., $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ are conformable for addition

Addition of A and B, written A + B is obtained by adding the entries of the matrix A to the corresponding entries of the matrix B.

$$\text{e.g., } A + B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 6 \end{bmatrix} + \begin{bmatrix} -2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2+(-2) & 3+3 & 0+4 \\ 1+1 & 0+2 & 6+3 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 4 \\ 2 & 2 & 9 \end{bmatrix}$$

1.3.2 Subtraction of Matrices

If A and B are two matrices of same order, then subtraction of matrix B from matrix A is obtained by subtracting the entries of matrix B from the corresponding entries of matrix A and it is denoted by A – B.

e.g., $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 4 & 3 \end{bmatrix}$ are conformable for subtraction.

$$\text{i.e., } A - B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 2 \\ -1 & 4 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2-0 & 3-2 & 4-2 \\ 1-(-1) & 5-4 & 0-3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 1 & -3 \end{bmatrix}$$

Some solved examples regarding addition and subtraction are given below.

(a) If $A = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & 3 \\ 2 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & 4 \\ 1 & -1 & 2 \\ 5 & -2 & 7 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & 3 \\ 2 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 4 \\ 1 & -1 & 2 \\ 5 & -2 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 1+0 & 2+3 & 7+4 \\ 0+1 & -1+(-1) & 3+2 \\ 2+5 & 5-2 & 1+7 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 11 \\ 1 & -2 & 5 \\ 7 & 3 & 8 \end{bmatrix}.$$

$$\text{and } A - B = A + (-B) = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & 3 \\ 2 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -3 & -4 \\ -1 & 1 & -2 \\ -5 & 2 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 2-3 & 7-4 \\ 0-1 & -1+1 & 3-2 \\ 2-5 & 5+2 & 1-7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 1 \\ -3 & 7 & -6 \end{bmatrix}.$$

(b) If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & -2 \\ 3 & 4 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+3 \\ -1+1 & 3-2 \\ 0+3 & 2+4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 3 & 6 \end{bmatrix}.$$

and $A - B = \begin{bmatrix} 1-2 & 2-3 \\ -1-1 & 3+2 \\ 0-3 & 2-4 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & 5 \\ -3 & -2 \end{bmatrix}.$

Note that the order of a matrix is unchanged under the operation of matrix addition and matrix subtraction.

1.3.3 Multiplication of a Matrix by a Real Number

Let A be any matrix and the real number k be a scalar. Then the scalar multiplication of matrix A with k is obtained by multiplying each entry of matrix A with k. It is denoted by kA.

Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$ be a matrix of order 3-by-3 and $k = -2$ be a real number.

Then,

$$KA = (-2)A = (-2)\begin{bmatrix} 1 & -1 & 4 \\ 2 & -1 & 0 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} (-2)(1) & (-2)(-1) & (-2)(4) \\ (-2)(2) & (-2)(-1) & (-2)(0) \\ (-2)(-1) & (-2)(3) & (-2)(2) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 & -8 \\ -4 & 2 & 0 \\ 2 & -6 & -4 \end{bmatrix}$$

Scalar multiplication of a matrix leaves the order of the matrix unchanged.

1.3.4 Commutative and Associative Laws of Addition of Matrices

(a) Commutative Law under Addition

If A and B are two matrices of the same order, then A + B = B + A is called commulative law under addition.

Let $A = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix}$

then $A + B = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix}$

$$= \begin{bmatrix} 2+3 & 3-2 & 0+5 \\ 5-1 & 6+4 & 1+1 \\ 2+4 & 1+2 & 3-4 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 5 \\ 4 & 10 & 2 \\ 6 & 3 & -1 \end{bmatrix}$$

Similarly

$$B + A = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 5 \\ 4 & 10 & 2 \\ 6 & 3 & -1 \end{bmatrix}$$

Thus the commutative law of addition of matrices is verified:

A + B = B + A

(b) Associative Law under Addition

If A, B and C are three matrices of same order, then (A + B) + C = A + (B + C) is called associative law under addition.

Let $A = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 1 & 2 & 0 \end{bmatrix}$

then $(A + B) + C = \left(\begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix} \right) + \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 1 & 2 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2+3 & 3-2 & 0+5 \\ 5-1 & 6+4 & 1+1 \\ 2+4 & 1+2 & 3-4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 & 5 \\ 4 & 10 & 2 \\ 6 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 8 \\ 2 & 10 & 6 \\ 7 & 5 & -1 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \left(\begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 1 & 2 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 3+1 & -2+2 & 5+3 \\ -1-2 & 4+0 & 1+4 \\ 4+1 & 2+2 & -4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 8 \\ -3 & 4 & 5 \\ 5 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 8 \\ 2 & 10 & 6 \\ 7 & 5 & -1 \end{bmatrix}$$

Thus the associative law of addition is verified: (A + B) + C = A + (B + C)

1.3.5 Additive Identity of a Matrix

If A and B are two matrices of same order and A + B = A = B + A, then matrix B is called additive identity of matrix A. For any matrix A and zero matrix O of same order, O is called additive identity of A as

$$A + O = A = O + A$$

e.g., let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

then $A + O = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = A$

$O + A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = A$

1.3.6 Additive Inverse of a Matrix

If A and B are two matrices of same order such that

$$A+B=O=B+A,$$

then A and B are called additive inverses of each other.

Additive inverse of any matrix A is obtained by changing to negative of the symbols (entries) of each non zero entry of A.

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & 0 \end{bmatrix}$

then $B = (-A) = -\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 1 & 2 \\ -3 & -1 & 0 \end{bmatrix}$

is additive inverse of A.

It can be verified as

$$A + B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -2 & -1 \\ 0 & 1 & 2 \\ -3 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1)+(-1) & (2)+(-2) & (1)+(-1) \\ 0+0 & (-1)+(1) & (-2)+(2) \\ (3)+(-3) & (1)+(-1) & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$$B + A = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 1 & 2 \\ -3 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)+(1) & (-2)+(2) & (-1)+(1) \\ 0+0 & (1)+(-1) & (2)+(-2) \\ (-3)+(3) & (-1)+(1) & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Since A + B = O = B + A .

Therefore, A and B are additive inverses of each other.

EXERCISE 1.3

1. Which of the following matrices are conformable for addition?

$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 2+1 \\ 3 \end{bmatrix}$,

$E = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, $F = \begin{bmatrix} 3 & 2 \\ 1+1 & -4 \\ 3+2 & 2+1 \end{bmatrix}$

2. Find additive inverse of the following matrices:

$A = \begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$,

$D = \begin{bmatrix} 1 & 0 \\ -3 & -2 \\ 2 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{2} \end{bmatrix}$

3. If $A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find,

(i) $A + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (ii) $B + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ (iii) c=+[-2 1 3]

(iv) $D + \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ (v) 2A (vi) (–1)B

(vii) (–2) C (viii) 3D (ix) 3C

4. Perform the indicated operations and simplify the following:

(i) $\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}\right) + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)$

(iii) $[2\ \ 3\ \ 1] + ([1\ \ 0\ \ 2] - [2\ \ 2\ \ 2])$ (iv) $= \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$ (vi) $\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\right) + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

5. For the matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

verify the following rules.

(i) A + C = C + A (ii) A + B = B + A
(iii) B + C = C + B (iv) A + (B + A) = 2A + B
(v) (C – B) + A = C + (A – B) (vi) 2A + B = A + (A + B)
(vii) (C–B) A = (C – A) – B (viii) (A + B) + C = A + (B + C)
(ix) A + (B – C) = (A – C) + B (x) 2A + 2B = 2(A + B)

6. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 7 \\ -3 & 8 \end{bmatrix}$, find (i) 3A – 2B

(ii) $2A^t - 3B^t$.

7. If $2\begin{bmatrix} 2 & 4 \\ -3 & a \end{bmatrix} + 3\begin{bmatrix} 1 & b \\ 8 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 18 & 1 \end{bmatrix}$, then find a and b.

8. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, then verify that

(i) $(A + B)^t = A^t + B^t$ (ii) $(A - B)^t = A^t - B^t$
(iii) $A + A^t$ is symmetric (iv) $A - A^t$ is skew symmetric
(v) $B + B^t$ is symmetric (vi) $B - B^t$ is skew symmetric

1.4 Multiplication of Matrices

Two matrices A and B are conformable for multiplication, giving product AB, if the number of columns of A is equal to the number of rows of B.

e.g., let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Here number of columns of A is equal to the number of rows of B. So A and B matrices are conformable for multiplication.

Multiplication of two matrices is explained by the following examples.

(i) If A = [1 2] and $B = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ then $AB = [1\ \ 2]\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$

$= [1 \times 2 + 2 \times 3 \quad 1 \times 0 + 2 \times 1] = [2 + 6 \quad 0 + 2] = [8 \quad 2]$, is a 1-by-2 matrix.

(ii) If $A = \begin{bmatrix} 1 & 3 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -3 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1\times(-1)+3\times3 & 1\times0+3\times2 \\ 2(-1)+(-3)(3) & 2\times0+(-3)(2) \end{bmatrix}$$

$$= \begin{bmatrix} -1+9 & 0+6 \\ -2-9 & 0-6 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ -11 & -6 \end{bmatrix}, \text{ is a 2-by-2 matrix.}$$

1.4.1 Associative Law under Multiplication

If A, B and C are three matrices conformable for multiplication then associative law under multiplication is given as (AB)C = A(BC)

e.g., $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$ then

L.H.S. = (AB)C $= \left(\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}\right)\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2\times0+3\times3 & 2\times1+3\times1 \\ -1\times0+0\times3 & -1\times1+0\times1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9 & 2+3 \\ 0+0 & -1+0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 5 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 9\times2+5\times(-1) & 9\times2+5\times0 \\ 0\times2+(-1)\times(-1) & 0\times2+(-1)\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 18-5 & 18+0 \\ 0+1 & 0+0 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ 1 & 0 \end{bmatrix}$$

R.H.S = A(BC) = $\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}\right)$

$$= \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0\times2+1\times(-1) & 0\times2+1\times0 \\ 3\times2+1\times(-1) & 3\times2+1\times0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)+3\times5 & 2\times0+3\times6 \\ (-1)(-1)+0\times5 & -1\times0+0\times6 \end{bmatrix} = \begin{bmatrix} -2+15 & 0+18 \\ 1+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 18 \\ 1 & 0 \end{bmatrix} = (AB)C$$

The associative law under multiplication of matrices is verified.

1.4.2 Distributive Laws of Multiplication over Addition and Subtraction

(a) Let A, B and C be three matrices. Then distributive laws of multiplication over addition are given below:

(i) A(B + C) = AB + AC (Left distributive law)

(ii) (A + B)C = AC + BC (Right distributive law)

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$, then in (i)

L.H.S = A (B+C)

$$= \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}+\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0+2 & 1+2 \\ 3-1 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2\times2+3\times2 & 2\times3+3\times1 \\ -1\times2+0\times2 & -1\times3+0\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+6 & 6+3 \\ -2+0 & -3+0 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ -2 & -3 \end{bmatrix}$$

R.H.S. = AB + AC

$$= \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}+\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times0+3\times3 & 2\times1+3\times1 \\ -1\times0+0\times3 & -1\times1+0\times1 \end{bmatrix}+\begin{bmatrix} 2\times2+3\times(-1) & 2\times2+3\times0 \\ -1\times2+0\times(-1) & -1\times2+0\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 5 \\ 0 & -1 \end{bmatrix}+\begin{bmatrix} 1 & 4 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 9+1 & 5+4 \\ 0-2 & -1-2 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ -2 & -3 \end{bmatrix} = \text{L.H.S.}$$

Which shows that

A(B + C) = AB + AC; Similarly we can verify (ii).

(b) Similarly the distributive laws of multiplication over subtraction are as follow.

(i) A(B - C) = AB -AC (ii) (A - B)C = AC - BC

Let $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then in (i)

L.H.S. = A(B − C)

$$= \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}-\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} -1-2 & 1-1 \\ 1-1 & 0-2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(-3)+(3)(0) & 2(0)+3(-2) \\ (0)(-3)+1\times0 & 0\times0+(1)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -6+0 & 0-6 \\ 0+0 & 0-2 \end{bmatrix} = \begin{bmatrix} -6 & -6 \\ 0 & -2 \end{bmatrix}$$

R.H.S. = AB – AC

$$= \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}-\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix}2(-1)+3(1) & 2(1)+3(0)\\ 0(-1)+1(1) & 0(1)+1(0)\end{bmatrix}-\begin{bmatrix}2\times2+3\times1 & 2\times1+3\times2\\ 0\times2+1\times1 & 0\times1+1\times2\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}-\begin{bmatrix}7 & 8\\ 1 & 2\end{bmatrix}=\begin{bmatrix}1-7 & 2-8\\ 1-1 & 0-2\end{bmatrix}=\begin{bmatrix}-6 & -6\\ 0 & -2\end{bmatrix}$$

which shows that

A(B – C) = AB – AC; Similarly (ii) can be verified.

1.4.3 Commutative Law of Multiplication of Matrices

Consider the matrices $A=\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$, then

$$AB=\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}=\begin{bmatrix}0\times1+1\times0 & 0\times0+1(-2)\\ 2\times1+3\times0 & 2\times0+3(-2)\end{bmatrix}=\begin{bmatrix}0 & -2\\ 2 & -6\end{bmatrix}$$

and $BA=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}=\begin{bmatrix}1\times0+0\times2 & 1\times1+0\times3\\ 0\times0+(-2)\times2 & 0\times1+3(-2)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -4 & -6\end{bmatrix}$

Which shows that, AB ≠ BA

Commutative law under multiplication in matrices does not hold in general i.e., if A and B are two matrices, then AB ≠ BA.

Commutative law under multiplication holds in particular case.

e.g., if $A=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-3 & 0\\ 0 & 4\end{bmatrix}$ then

$$AB=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-3 & 0\\ 0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}2\times(-3)+0\times0 & 2\times0+0\times4\\ 0\times(-3)+1\times0 & 0\times0+1\times4\end{bmatrix}=\begin{bmatrix}-6 & 0\\ 0 & 4\end{bmatrix}$$

and $BA=\begin{bmatrix}-3 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

$$=\begin{bmatrix}-3\times2+0\times0 & -3\times0+0\times1\\ 0\times2+4\times0 & 0\times0+4\times1\end{bmatrix}=\begin{bmatrix}-6 & 0\\ 0 & 4\end{bmatrix}$$

Which shows that AB = BA.

1.4.4 Multiplicative Identity of a Matrix

Let A be a matrix. Another matrix B is called the identity matrix of A under multiplication if

AB = A = BA

If $A=\begin{bmatrix}1 & 2\\ 0 & -3\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, then

$$AB=\begin{bmatrix}1 & 2\\ 0 & -3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1\times1+2\times0 & 1\times0+2\times1\\ 0\times1+(-3)\times0 & 0\times0+(-3)(1)\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\ 0 & -3\end{bmatrix}$$

$$BA=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & -3\end{bmatrix}=\begin{bmatrix}1\times1+0\times0 & 1\times2+0\times(-3)\\ 0\times1+1\times0 & 0\times2+1\times(-3)\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\ 0 & -3\end{bmatrix}$$

Which shows that AB = A = BA.

1.4.5 Verification of $(AB)^t = B^tA^t$

If A, B are two matrices and A^t, B^t are their respective transpose, then $(AB)^t = B^tA^t$.

e.g., $A=\begin{bmatrix}2 & 1\\ 0 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 3\\ -2 & 0\end{bmatrix}$

L.H.S. $=(AB)^t$

$$=\left(\begin{bmatrix}2 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ -2 & 0\end{bmatrix}\right)^t=\begin{bmatrix}2\times1+1\times(-2) & 2\times3+1\times0\\ 0\times1+(-1)\times(-2) & 0\times3+(-1)\times0\end{bmatrix}^t$$

$$=\begin{bmatrix}2-2 & 6+0\\ 0+2 & 0+0\end{bmatrix}^t=\begin{bmatrix}0 & 6\\ 2 & 0\end{bmatrix}^t=\begin{bmatrix}0 & 2\\ 6 & 0\end{bmatrix}$$

R.H.S. $=B^tA^t$,

$(A)^t=\begin{bmatrix}2 & 1\\ 0 & -1\end{bmatrix}^t=\begin{bmatrix}2 & 0\\ 1 & -1\end{bmatrix}$ and $(B)^t=\begin{bmatrix}1 & 3\\ -2 & 0\end{bmatrix}^t=\begin{bmatrix}1 & -2\\ 3 & 0\end{bmatrix}$

$$B^tA^t=\begin{bmatrix}1 & -2\\ 3 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\ 1 & -1\end{bmatrix}=\begin{bmatrix}1\times2+(-2)\times1 & 1\times0+(-2)(-1)\\ 3\times2+0\times1 & 3\times0+0\times(-1)\end{bmatrix}$$

$$=\begin{bmatrix}2-2 & 0+2\\ 6+0 & 0+0\end{bmatrix}=\begin{bmatrix}0 & 2\\ 6 & 0\end{bmatrix}=\text{L.H.S.}$$

Thus $(AB)^t = B^tA^t$

EXERCISE 1.4

1. Which of the following product of matrices is conformable for multiplication?

(i) $\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}\begin{bmatrix}-2\\ 3\end{bmatrix}$ (ii) $\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$

(iii) $\begin{bmatrix}1\\ -1\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}$ (iv) $\begin{bmatrix}1 & 2\\ 0 & -1\\ -1 & -2\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 2\end{bmatrix}$

(v) $\begin{bmatrix}3 & 2 & 1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 2\\ -2 & 3\end{bmatrix}$

2. If $A=\begin{bmatrix}3 & 0\\ -1 & 2\end{bmatrix}$, $B=\begin{bmatrix}6\\ 5\end{bmatrix}$, find (i) AB (ii) BA (if possible)

3. Find the following products.

(i) $[1\ \ 2]\begin{bmatrix}4\\ 0\end{bmatrix}$ (ii) $[1\ \ 2]\begin{bmatrix}5\\ -4\end{bmatrix}$ (iii) $[-3\ \ 0]\begin{bmatrix}4\\ 0\end{bmatrix}$

(iv) $[6\ \ -0]\begin{bmatrix}4\\ 0\end{bmatrix}$ (v) $\begin{bmatrix}1 & 2\\ -3 & 0\\ 6 & -1\end{bmatrix}\begin{bmatrix}4 & 5\\ 0 & -4\end{bmatrix}$

4. Multiply the following matrices.

(*a*) $\begin{bmatrix}2 & 3\\ 1 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}2 & -1\\ 3 & 0\end{bmatrix}$ (*b*) $\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\\ -1 & 1\end{bmatrix}$

(*c*) $\begin{bmatrix}1 & 2\\ 3 & 4\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$ (*d*) $\begin{bmatrix}8 & 5\\ 6 & 4\end{bmatrix}\begin{bmatrix}2 & -\frac{5}{2}\\ -4 & 4\end{bmatrix}$

(*e*) $\begin{bmatrix}-1 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

5. Let $A=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}$, $B=\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}$ and $C=\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$. verify whether

(i) AB = BA. (ii) A(BC) = (AB)C

(iii) A(B + C) = AB + AC (iv) A(B – C) = AB – AC

6. For the matrices

$$A=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix},\quad B=\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix},\quad C=\begin{bmatrix}-2 & 6\\ 3 & -9\end{bmatrix}$$

Verify that (i) $(AB)^t = B^t A^t$ (ii) $(BC)^t = C^t B^t$.

1.5 Multiplicative Inverse of a Matrix

1.5.1 Determinant of a 2-by-2 Matrix

Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ be a 2-by-2 square matrix. The determinant of A, denoted by det A or $|A|$ is defined as

$$|A| = \det A = \det\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{vmatrix}a & b\\ c & d\end{vmatrix}= ad - bc = \lambda \in \mathbb{R}$$

e.g., Let $B=\begin{bmatrix}1 & 1\\ -2 & 3\end{bmatrix}$.

Then $|B| = \det B = \begin{vmatrix}1 & 1\\ -2 & 3\end{vmatrix} = 1\times 3-(-2)(1) = 3+2 = 5$

If $M=\begin{bmatrix}2 & 6\\ 1 & 3\end{bmatrix}$, then $\det M = \begin{vmatrix}2 & 6\\ 1 & 3\end{vmatrix} = 2\times 3-1\times 6 = 0$

1.5.2 Singular and Non-Singular Matrix

A square matrix A is called singular, if the determinant of A is equal to zero. i.e., $|A| = 0$.

For example, $A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}$ is a singular matrix,

since det A = 1 × 0 – 0 × 2 = 0

A square matrix A is called non-singular, if the determinant of A is not equal to zero. i.e., $|A| \neq 0$. For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is non-singular, since det A = $1 \times 2 - 0 \times 1 = 2 \neq 0$.Note that, each square matrix with real entries is either singular or non-singular.

1.5.3 Adjoint of a Matrix

Adjoint of a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is obtained by interchanging the diagonal entries and changing the signs of other entries. Adjoint of matrix A is denoted as Adj A.

$$\text{i.e.,} \quad \text{Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{e.g., if } A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \text{ then Adj } A = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\text{If } B = \begin{bmatrix} 2 & -1 \\ 3 & -4 \end{bmatrix}, \text{ then Adj } B = \begin{bmatrix} -4 & 1 \\ -3 & 2 \end{bmatrix}$$

1.5.4 Multiplicative Inverse of a Non-singular Matrix

Let A and B be two non-singular square matrices of same order. Then A and B are said to be multiplicative inverse of each other if

$$AB = BA = I.$$

The inverse of A is denoted by A^{-1}, thus

$$AA^{-1} = A^{-1}A = I.$$

Inverse of Identity matrix is Identity matrix.

Inverse of a matrix is possible only if matrix is non-singular.

1.5.5 Inverse of a Matrix using Adjoint

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square matrix. To find the inverse of M, i.e., M^{-1}, first we find the determinant as inverse is possible only of a non-singular matrix.

$$|M| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

and $\text{Adj } M = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, then $M^{-1} = \dfrac{\text{Adj } M}{|M|}$

e.g., Let $A = \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}$, Then

$$|A| = \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} = -6 - (-1) = -6 + 1 = -5 \neq 0$$

$$\text{Thus } A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{\begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}}{-5} = \frac{-1}{5}\begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-1}{5} & \frac{-2}{5} \end{bmatrix}$$

$$\text{and } AA^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-1}{5} & \frac{-2}{5} \end{bmatrix} = \begin{bmatrix} \frac{6}{5} - \frac{1}{5} & \frac{2}{5} - \frac{2}{5} \\ -\frac{3}{5} + \frac{3}{5} & -\frac{1}{5} + \frac{6}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = AA^{-1}$$

1.5.6 Verification of $(AB)^{-1} = B^{-1}A^{-1}$

Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$

Then det A = 3 × 0 – (–1) × l = 1 ≠ 0

and det B = 0 × 2 – 3(–1) = 3 ≠ 0

Therefore, A and B are invertible i.e., their inverses exist.

Then, to verify the law of inverse of the product, take

$$AB = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3\times0+1\times3 & 3\times(-1)+1\times2 \\ -1\times0+0\times3 & -1\times(-1)+0\times2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \det(AB) = = \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = 3 \neq 0$$

$$\text{and L.H.S.} = (AB)^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & 1 \end{bmatrix}$$

R.H.S. = $B^{-1}A^{-1}$, where $B^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$, $A^{-1} = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$

$$=\frac{1}{3}\begin{bmatrix}2 & 1\\ -3 & 0\end{bmatrix}\cdot\frac{1}{1}\begin{bmatrix}0 & -1\\ 1 & 3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\times0+1\times1 & 2\times(-1)+1\times3\\ -3\times0+0\times1 & -3\times(-1)+0\times3\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}0+1 & -2+3\\ 0 & 3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & 1\\ 0 & 3\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ 0 & 1\end{bmatrix}$$

$= (AB)^{-1}$ Thus the law $(AB)^{-1} = B^{-1}A^{-1}$ is verified.

EXERCISE 1.5

1. Find the determinant of the following matrices.

(i) $A = \begin{bmatrix}-1 & 1\\ 2 & 0\end{bmatrix}$ (ii) $B = \begin{bmatrix}1 & 3\\ 2 & -2\end{bmatrix}$

(iii) $C = \begin{bmatrix}3 & 2\\ 3 & 2\end{bmatrix}$ (iv) $D = \begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}$

2. Find which of the following matrices are singular or non-singular?

(i) $A = \begin{bmatrix}3 & 6\\ 2 & 4\end{bmatrix}$ (ii) $B = \begin{bmatrix}4 & 1\\ 3 & 2\end{bmatrix}$

(iii) $C = \begin{bmatrix}7 & -9\\ 3 & 5\end{bmatrix}$ (iv) $D = \begin{bmatrix}5 & -10\\ -2 & 4\end{bmatrix}$

3. Find the multiplicative inverse (if it exists) of each.

(i) $A = \begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}$ (ii) $B = \begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}$

(iii) $C = \begin{bmatrix}-2 & 6\\ 3 & -9\end{bmatrix}$ (iv) $D = \begin{bmatrix}\frac{1}{2} & \frac{3}{4}\\ 1 & 2\end{bmatrix}$

4. If $A = \begin{bmatrix}1 & 2\\ 4 & 6\end{bmatrix}$ and $B = \begin{bmatrix}3 & -1\\ 2 & -2\end{bmatrix}$, then

(i) $A(\text{Adj } A) = (\text{Adj } A)\, A = (\det A)I$ (ii) $BB^{-1} = I = B^{-1}B$

5. Determine whether the given matrices are multiplicative inverses of each other.

(i) $\begin{bmatrix}3 & 5\\ 4 & 7\end{bmatrix}$ and $\begin{bmatrix}7 & -5\\ -4 & 3\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$ and $\begin{bmatrix}-3 & 2\\ 2 & -1\end{bmatrix}$

6. If $A = \begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}$, $B = \begin{bmatrix}-4 & -2\\ 1 & -1\end{bmatrix}$, $D = \begin{bmatrix}3 & 1\\ -2 & 2\end{bmatrix}$, then verify that

(i) $(AB)^{-1} = B^{-1}A^{-1}$ (ii) $(DA)^{-1} = A^{-1}D^{-1}$

1.6 Solution of Simultaneous Linear Equations

System of two linear equations in two variables in general form is given as

$$ax + by = m$$
$$cx + dy = n$$

where *a, b, c, d, m and n* are real numbers.

This system is also called simultaneous linear equations.

We discuss here the following methods of solution.

(i) **Matrix** inversion method

(ii) **Cramer's** rule

(i) Matrix Inversion Method

Consider the system of linear equations

$$ax + by = m$$
$$cx + dy = n$$

Then $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}m\\ n\end{bmatrix}$

or $AX = B$

where $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$, $X = \begin{bmatrix}x\\ y\end{bmatrix}$ and $B = \begin{bmatrix}m\\ n\end{bmatrix}$

or $X = A^{-1}B$ $|A| = ad - bc$

or $X = \dfrac{\text{Adj } A}{|A|}\times B$ $\because A^{-1} = \dfrac{\text{Adj } A}{|A|}$ and $|A| \neq 0$

$$\text{or}\quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}}{ad-bc}$$

$$= \begin{bmatrix} \dfrac{dm-bn}{ad-bc} \\ \dfrac{-cm+an}{ad-bc} \end{bmatrix}$$

$$\Rightarrow \quad x = \frac{dm-bn}{ad-bc} \quad \text{and} \quad y = \frac{an-cm}{ad-bc}$$

(ii) Cramer's Rule

Consider the following system of linear equations.

$ax + by = m$

$cx + dy = n$

We know that

$$AX = B, \text{ where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} m \\ n \end{bmatrix}$$

$$\text{or} \quad X = A^{-1}B \quad \text{or} \quad X = \frac{\text{Adj } A}{|A|} \times B$$

$$\text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}}{|A|} = \frac{\begin{bmatrix} dm-bn \\ -cm+an \end{bmatrix}}{|A|}$$

$$= \begin{bmatrix} \dfrac{dm-bn}{|A|} \\ \dfrac{-cm+an}{|A|} \end{bmatrix}$$

$$\text{or} \quad x = \frac{dm-bn}{|A|} = \frac{|A_x|}{|A|}$$

$$\text{and} \quad y = \frac{an-cm}{|A|} = \frac{|A_y|}{|A|}$$

$$\text{where } |A_x| = \begin{vmatrix} m & b \\ n & d \end{vmatrix} \quad \text{and} \quad |A_y| = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$$

Example 1

Solve the following system by using matrix inversion method.

$4x - 2y = 8$

$3x + y = -4$

Solution

Step 1 $\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$

Step 2 The coefficient matrix $M = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$ is non-singular,

since det M = $4 \times 1 - 3(-2) = 4 + 6 = 10 \neq 0$. So M^{-1} is possible.

Step 3 $\begin{bmatrix} x \\ y \end{bmatrix} = M^{-1}\begin{bmatrix} 8 \\ -4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 8 \\ -4 \end{bmatrix}$

$$= \frac{1}{10}\begin{bmatrix} 8-8 \\ -24-16 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 0 \\ -40 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$\Rightarrow \quad x = 0 \text{ and } y = -4$$

Example 2

Solve the following system of linear equations by using Cramer's rule.

$3x - 2y = 1$

$-2x + 3y = 2$

Solution

$3x - 2y = 1$

$-2x + 3y = 2$

We have

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}, A_x = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}, A_y = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$$

$|A| = \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = 9 - 4 = 5 \neq 0$ (A is non-singular

$$x=\frac{|A_x|}{|A|}=\frac{\begin{vmatrix}1 & -2\\ 2 & 3\end{vmatrix}}{5}=\frac{3+4}{5}=\frac{7}{5}$$

$$y=\frac{|A_y|}{|A|}=\frac{\begin{vmatrix}3 & 1\\ -2 & 2\end{vmatrix}}{5}=\frac{6+2}{5}=\frac{8}{5}$$

Example 3

The length of a rectangle is 6 cm less than three times its width. The perimeter of the rectangle is 140 cm. Find the dimensions of the rectangle. (by using matrix inversion method)

Solution

If width of the rectangle is x cm, then length of the rectangle is

$y = 3x - 6$,

from the condition of the question.

The perimeter = $2x + 2y = 140$ *(According to given condition)*

$\Rightarrow \quad x + y = 70$ ......(i)

and $3x - y = 6$ ......(ii)

In the matrix form

$$\begin{bmatrix}1 & 1\\ 3 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}70\\ 6\end{bmatrix}$$

$$\det\begin{bmatrix}1 & 1\\ 3 & -1\end{bmatrix}=\begin{vmatrix}1 & 1\\ 3 & -1\end{vmatrix}=1\times(-1)-3\times1=-1-3=-4\neq0$$

We know that

$$X=A^{-1}B \quad \text{and} \quad A^{-1}=\frac{\text{Adj}A}{|A|}$$

Hence $$\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{-4}\begin{vmatrix}-1 & -1\\ -3 & 1\end{vmatrix}\begin{bmatrix}70\\ 6\end{bmatrix}$$

$$=\frac{-1}{4}\begin{vmatrix}-70-6\\ -210+6\end{vmatrix}=\frac{-1}{4}\begin{bmatrix}-76\\ -204\end{bmatrix}=\begin{bmatrix}\frac{76}{4}\\ \frac{204}{4}\end{bmatrix}=\begin{bmatrix}19\\ 51\end{bmatrix}$$

Thus, by the equality of matrices, width of the rectangle x = 19 cm and the length y = 51 cm.

Verification of the solution to be correct, i.e.,

p = 2 × 19 + 2 × 51 = 38 + 102= 140 cm

Also y = 3(19) – 6 = 57 – 6 = 51 cm

EXERCISE 1.6

1 Use matrices, if possible, to solve the following systems of linear equations by:

(i) the matrix inversion method (ii) the Cramer's rule.

(i)	$2x - 2y = 4$ $3x + 2y = 6$	(ii)	$2x + y = 3$ $6x + 5y = 1$
(iii)	$4x + 2y = 8$ $3x - y = -1$	(iv)	$3x - 2y = -6$ $5x - 2y = -10$
(v)	$3x - 2y = 4$ $-6x + 4y = 7$	(vi)	$4x + y = 9$ $-3x - y = -5$
(vii)	$2x - 2y = 4$ $-5x - 2y = -10$	(viii)	$3x - 4y = 4$ $x + 2y = 8$

Solve the following word problems by using

(i) matrix inversion method (ii) Crammer's rule.

2 The length of a rectangle is 4 times its width. The perimeter of the rectangle is 150 cm. Find the dimensions of the rectangle.

3 Two sides of a rectangle differ by 3.5cm. Find the dimensions of the rectangle if its perimeter is 67cm.

4 The third angle of an isosceles triangle is 16° less than the sum of the two equal angles. Find three angles of the triangle.

5 One acute angle of a right triangle is 12° more than twice the other acute angle. Find the acute angles of the right triangle.

6 Two cars that are 600 km apart are moving towards each other. Their speeds differ by 6 km per hour and the cars are 123 km apart after $4\frac{1}{2}$ hours. Find the speed of each car.

REVIEW EXERCISE 1

2. Complete the following:
   (i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called ..... matrix.
   (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called ..... matrix.
   (iii) Additive inverse of $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$ is..........
   (iv) In matrix multiplication,in general, AB ...... BA.
   (v) Matrix A + B may be found if order of A and B is ......
   (vi) A matrix is called ..... matrix if number of rows and columns are equal.

3. If $\begin{bmatrix} a+3 & 4 \\ 6 & b-1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$, then find a and b.

4. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix}$, then find the following.

   (i) $2A + 3B$ (ii) $-3A + 2B$
   (iii) $-3(A + 2B)$ (iv) $\frac{2}{3}(2A - 3B)$

5. Find the value of X, if $\begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix} + X = \begin{bmatrix} 4 & -2 \\ -1 & -2 \end{bmatrix}$.

6. If $A = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 \\ 5 & -2 \end{bmatrix}$, then prove that

   (i) $AB \neq BA$ (ii) $A(BC) = (AB)C$

7. If $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix}$, then verify that

   (i) $(AB)^t = B^tA^t$ (ii) $(AB)^{-1} = B^{-1}A^{-1}$

SUMMARY

- A rectangular array of real numbers enclosed with brackets is said to form a matrix.
- A matrix A is called rectangular, if the number of rows and number of columns of A are not equal.
- A matrix A is called a square matrix, if the number of rows of A is equal to the number of columns.
- A matrix A is called a row matrix, if A has only one row.
- A matrix A is called a column matrix, if A has only one column.
- A matrix A is called a null or zero matrix, if each of its entry is 0.
- Let A be a matrix. The matrix A^t is a new matrix which is called transpose of matrix A and is obtained by interchanging rows of A into its respective columns (or columns into respective rows).
- A square matrix A is called symmetric, if $A^t = A$.
- Let A be a matrix. Then its negative, –A, is obtained by changing the signs of all the entries of A.
- A square matrix M is said to be skew symmetric, if $M^t = -M$,
- A square matrix M is called a diagonal matrix, if atleast any one of entry of its diagonal is not zero and remaining entries are zero.
- A diagonal matrix is called identity matrix, if all diagonal entries are

  1. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called a 3-by-3 identity matrix.

- Any two matrices A and B are called equal, if
  (i) order of A= order of B (ii) corresponding entries are same
- Any two matrices M and N are said to be conformable for addition, if order of M = order of N.
- Let A be a matrix of order 2-by-3. Then a matrix B of same order is said to be an additive identity of matrix A, if

  $B + A = A = A + B$

- Let A be a matrix. A matrix B is defined as an additive inverse of A, if B + A = O = A + B
- Let A be a matrix. Another matrix B is called the identity matrix of A under multiplication, if

$$B \times A = A = A \times B.$$

- Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2-by-2 matrix. A real number λ is called determinant of M, denoted by det M such that

$$\det M = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \lambda$$

- A square matrix M is called singular, if the determinant of M is equal to zero.
- A square matrix M is called non-singular, if the determinant of M is not equal to zero.
- For a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, adjoint of M is defined by

$$\text{Adj } M = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- Let M be a square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$M^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where det M = $ad - bc \neq 0$.

- The following laws of addition hold

M + N = N + M	(Commutative)
(M + N) + T = M + (N + T)	(Associative)

- The matrices M and N are conformable for multiplication to obtain MN if the number of columns of M = number of rows of N, where
  (i) (MN) $\neq$ (NM), in general
  (ii) (MN)T = M(NT) (Associative law)
  (iii) M(N + T) = MN + MT
  (iv) (N + T)M = NM + TM (Distributive laws)
- Law of transpose of product $(AB)^t = B^t A^t$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $AA^{-1} = I = A^{-1}A$

- The solution of a linear system of equations,

$$ax + by = m$$
$$cx + dy = n$$

by expressing in the matrix form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$

is given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} m \\ n \end{bmatrix}$

if the coefficient matrix is non-singular.

- By using the Cramer's rule the determinental form of solution of equations

$$ax + by = m$$
$$cx + dy = n$$

is

$$x = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & m \\ c & n \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad \text{where } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$