CHAPTER

5

FACTORIZATION

Animation 5.1: Factorization
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Students Learning Outcomes

After studying this unit, the students will be able to:

- Recall factorization of expressions of the following types.
  - $ka + kb + kc$
  - $ac + ad + bc + bd$
  - $a^2 \pm 2ab + b^2$
  - $a^2 - b^2$
  - $a^2 \pm 2ab + b^2 - c^2$

- Factorize the expressions of the following types.

  Type I:
  $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$

  Type II:
  $x^2 + px + q$

  Type III:
  $ax^2 + bx + c$

  Type IV:

  $$\begin{cases}(ax^2 + bx + c)(ax^2 + bx + d) + k \\ (x + a)(x + b)(x + c)(x + d) + k \\ (x + a)(x + b)(x + c)(x + d) + kx^2\end{cases}$$

  Type V:
  $a^3 + 3a^2b + 3ab^2 + b^3$
  $a^3 - 3a^2b + 3ab^2 - b^3$

  Type VI:
  $a^3 \pm b^3$

- State and prove remainder theorem and explain through examples.
- Find Remainder (without dividing) when a polynomial is divided by a linear polynomial.
- Define zeros of a polynomial.
- State and prove Factor theorem.
- Use Factor theorem to factorize a cubic polynomial.

Introduction

Factorization plays an important role in mathematics as it helps to reduce the study of a complicated expression to the study of simpler expressions. In this unit, we will deal with different types of factorization of polynomials.

5.1 Factorization

If a polynomial $p(x)$ can be expressed as $p(x) = g(x)h(x)$, then each of the polynomials $g(x)$ and $h(x)$ is called a factor of $p(x)$. For instance, in the distributive property

$$ab + ac = a(b + c),$$

a and $(b + c)$ are factors of $(ab + ac)$.

When a polynomial has been written as a product consisting only of prime factors, then it is said to be factored completely.

(a) Factorization of the Expression of the type $ka + kb + kc$

Example 1

Factorize $5a - 5b + 5c$

Solution

$5a - 5b + 5c = 5(a - b + c)$

Example 2

Factorize $5a - 5b - 15c$

Solution

$5a - 5b - 15c = 5(a - b - 3c)$

(b) Factorization of the Expression of the type $ac + ad + bc + bd$

We can write $ac + ad + bc + bd$ as

$(ac + ad) + (bc + db)$

$= a(c + d) + b(c + d)$

$= (a + b)(c + d)$

For explanation consider the following examples.

Example 1

Factorize $3x - 3a + xy - ay$

Solution

Regrouping the terms of given polynomial

$3x + xy - 3a - ay = x(3 + y) - a(3 + y)$ (monomial factors)

$= (3 + y)(x - a)$ $(3 + y)$ is common factor

Example 2

Factorize $pqr + qr^2 - pr^2 - r^3$

Solution

The given expression $= r(pq + qr - pr - r^2)$ (r is monomial common factor)

$= r[(pq + qr) - pr - r^2]$ (grouping of terms)

$= r[q(p + r) - r(p + r)]$ (monomial factors)

$= r(p + r)(q - r)$ $(p + r)$ is common factor

(c) Factorization of the Expression of the type $a^2 \pm 2ab + b^2$

We know that

(i) $a^2 + 2ab + b^2 = (a + b)^2 = (a + b)(a + b)$

(ii) $a^2 - 2ab + b^2 = (a - b)^2 = (a - b)(a - b)$

Now consider the following examples.

Example 1

Factorize $25x^2 + 16 + 40x$.

Solution

$25x^2 + 40x + 16 = (5x)^2 + 2(5x)(4) + (4)^2$

$= (5x + 4)^2$

$= (5x + 4)(5x + 4)$

Example 2

Factorize $12x^2 - 36x + 27$

Solution

$12x^2 - 36x + 27 = 3(4x^2 - 12x + 9)$

$= 3(2x - 3)^2$

$= 3(2x - 3)(2x - 3)$

(d) Factorization of the Expression of the type $a^2 - b^2$

For explanation consider the following examples.

Example 1

Factorize (i) $4x^2 - (2y - z)^2$ (ii) $6x^4 - 96$

Solution

(i) $4x^2 - (2y - z)^2 = (2x)^2 - (2y - z)^2$

$= [2x - (2y - z)][2x + (2y - z)]$

$= (2x - 2y + z)(2x + 2y - z)$

(ii) $6x^4 - 96 = 6(x^4 - 16)$

$= 6[(x^2)^2 - (4)^2]$

$= 6(x^2 - 4)(x^2 + 4)$

$= 6[(x)^2 - (2)^2](x^2 + 4)$

$= 6(x - 2)(x + 2)(x^2 + 4)$

(e) Factorization of the Expression of the type $a^2 \pm 2ab + b^2 - c^2$

We know that

$a^2 \pm 2ab + b^2 - c^2 = (a \pm b)^2 - (c)^2 = (a \pm b - c)(a \pm b + c)$

Example 1

Factorize (i) $x^2 + 6x + 9 - 4y^2$ (ii) $1 + 2ab - a^2 - b^2$

Solution

(i) $x^2 + 6x + 9 - 4y^2 = (x + 3)^2 - (2y)^2$

$= (x + 3 + 2y)(x + 3 - 2y)$

(ii) $1 + 2ab - a^2 - b^2 = 1 - (a^2 - 2ab + b^2)$
$= (1)^2 - (a - b)^2$
$= [1 - (a - b)]\,[1 + (a - b)]$
$= (1 - a + b)(1 + a - b)$

EXERCISE 5.1

Factorize

1. (i) $2abc - 4abx + 2abd$ (ii) $9xy - 12x^2y + 18y^2$
(iii) $-3x^2y - 3x + 9xy^2$ (iv) $5ab^2c^3 - 10a^2b^3c - 20a^3bc^2$
(v) $3x^3y(x - 3y) - 7x^2y^2(x - 3y)$ (vi) $2xy^3(x^2 + 5) + 8xy^2(x^2 + 5)$
2. (i) $5ax - 3ay - 5bx + 3by$ (ii) $3xy + 2y - 12x - 8$
(iii) $x^3 + 3xy^2 - 2x^2y - 6y^3$ (iv) $(x^2 - y^2)z + (y^2 - z^2)x$
3. (i) $144a^2 + 24a + 1$ (ii) $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}$
(iii) $(x + y)^2 - 14z(x + y) + 49z^2$ (iv) $12x^2 - 36x + 27$
4. (i) $3x^2 - 75y^2$ (ii) $x(x - 1) - y(y - 1)$
(iii) $128am^2 - 242an^2$ (iv) $3x - 243x^3$
5. (i) $x^2 - y^2 - 6y - 9$ (ii) $x^2 - a^2 + 2a - 1$
(iii) $4x^2 - y^2 - 2y - 1$ (iv) $x^2 - y^2 - 4x - 2y + 3$
(v) $25x^2 - 10x + 1 - 36z^2$ (vi) $x^2 - y^2 - 4xz + 4z^2$

(a) Factorization of the Expression of types $a^4 + a^2b^2 + b^4$ *or* $a^4 + 4b^4$

Factorization of such types of expression is explained in the following examples.

Example 1

Factorize $81x^4 + 36x^2y^2 + 16y^4$

Solution

$81x^4 + 36x^2y^2 + 16y^4$
$= (9x^2)^2 + 72x^2y^2 + (4y^2)^2 - 36x^2y^2$
$= (9x^2 + 4y^2)^2 - (6xy)^2$
$= (9x^2 + 4y^2 + 6xy)(9x^2 + 4y^2 - 6xy)$
$= (9x^2 + 6xy + 4y^2)\,(9x^2 - 6xy + 4y^2)$

Example 2

Factorize $9x^4 + 36y^4$

Solution

$9x^4 + 36y^4 = 9x^4 + 36y^4 + 36x^2y^2 - 36x^2y^2$
$= (3x^2)^2 + 2(3x^2)\,(6y^2) + (6y^2)^2 - (6xy)^2$
$= (3x^2 + 6y^2)^2 - (6xy)^2$
$= (3x^2 + 6y^2 + 6xy)(3x^2 + 6y^2 - 6xy)$
$= (3x^2 + 6xy + 6y^2)\,(3x^2 - 6xy + 6y^2)$

(b) Factorization of the Expression of the type $x^2 + px + q$

For explanation consider the following examples.

Example 1

Factorize (i) $x^2 - 7x + 12$ (ii) $x^2 + 5x - 36$

Solution

(i) $x^2 - 7x + 12$

From the factors of 12 the suitable pair of numbers is –3 and –4 since

$(-3) + (-4) = -7$ and $(-3)(-4) = 12$

Hence $x^2 - 7x + 12 = x^2 - 3x - 4x + 12$
$= x(x - 3) - 4(x - 3)$
$= (x - 3)\,(x - 4)$

(ii) $x^2 + 5x - 36$

From the possible factors of 36, the suitable pair is 9 and −4 because

$9 + (-4) = 5$ and $9 \times (-4) = -36$

Hence $x^2 + 5x - 36 = x^2 + 9x - 4x - 36$
$= x(x + 9) - 4(x + 9)$
$= (x + 9)\,(x - 4)$

(c) Factorization of the Expression of the type $ax^2 + bx + c,\ a \neq 0$

Let us explain the procedure of factorization by the following examples.

Example 1

Factorize (i) $9x^2 + 21x - 8$ (ii) $2x^2 - 8x - 42$ (iii) $10x^2 - 41xy + 21y^2$

Solution

(i) $9x^2 + 21x - 8$

In this case, on comparing with $ax^2 + bx + c$, $ac = (9)(-8) = -72$

From the possible factors of 72, the suitable pair of numbers (with proper sign) is 24 and –3 whose sum = 24 + (–3) = 21, (the coefficient of x)

and their product = (24) (–3) = –72 = ac

Hence $9x^2 + 21x - 8$

$= 9x^2 + 24x - 3x - 8$

$= 3x(3x + 8) - (3x + 8)$

$= (3x + 8)(3x - 1)$

(ii) $2x^2 - 8x - 42 = 2(x^2 - 4x - 21)$

Comparing $x^2 - 4x - 21$ with $ax^2 + bx + c$

we have $ac = (+1)(-21) = -21$

From the possible factors of 21, the suitable pair of numbers is –7 and +3 whose sum = –7 + 3 = – 4 and product = (–7) (3) = –21

Hence $x^2 - 4x - 21$

$= x^2 + 3x - 7x - 21$

$= x(x + 3) - 7(x + 3)$

$= (x + 3)(x - 7)$

Hence $2x^2 - 8x - 42 = 2(x^2 - 4x - 21) = 2(x + 3)(x - 7)$

(iii) $10x^2 - 41xy + 21y^2$

This type of question on factorization can also be done by the above procedures of splitting the middle term.

Here $ac = (10)(21) = 210$

Two suitable factors of 210 are –35 and –6

Their sum = –35 – 6 = –41

and product = (–35) (–6) = 210

Hence $10x^2 - 41xy + 21y^2$

$= 10x^2 - 35xy - 6xy + 21y^2$

$= 5x(2x - 7y) - 3y(2x - 7y)$

$= (2x - 7y)(5x - 3y)$

(d) Factorization of the following types of Expressions

$(ax^2 + bx + c)(ax^2 + bx + d) + k$

$(x + a)(x + b)(x + c)(x + d) + k$

$(x + a)(x + b)(x + c)(x + d) + kx^2$

We shall explain the method of factorizing these types of expressions with the help of following examples.

Example 1

Factorize $(x^2 - 4x - 5)(x^2 - 4x - 12) - 144$

Solution

$(x^2 - 4x - 5)(x^2 - 4x - 12) - 144$

Let $y = x^2 - 4x$. Then

$(y - 5)(y - 12) - 144 = y^2 - 17y - 84$

$= y^2 - 21y + 4y - 84$

$= y(y - 21) + 4(y - 21)$

$= (y - 21)(y + 4)$

$= (x^2 - 4x - 21)(x^2 - 4x + 4)$ (since $y = x^2 - 4x$)

$= (x^2 - 7x + 3x - 21)(x - 2)^2$

$= [x(x - 7) + 3(x - 7)](x - 2)^2$

$= (x - 7)(x + 3)(x - 2)(x - 2)$

Example 2

Factorize $(x + 1)(x + 2)(x + 3)(x + 4) - 120$

Solution

We observe that 1 + 4 = 2 + 3.

It suggests that we rewrite the given expression as

$[(x + 1)(x + 4)][(x + 2)(x + 3)] - 120$

$(x^2 + 5x + 4)(x^2 + 5x + 6) - 120$

Let $x^2 + 5x = y$, then

we get $(y + 4)(y + 6) - 120$

$= y^2 + 10y + 24 - 120$

$= y^2 + 10y - 96$

$= y^2 + 16y - 6y - 96$

$= y(y + 16) - 6(y + 16)$

$= (y + 16)(y - 6)$

$= (x^2 + 5x + 16)(x^2 + 5x - 6)$ since $y = x^2 + 5x$

$= (x^2 + 5x + 16)(x + 6)(x - 1)$

Example 3

Factorize $(x^2 - 5x + 6)(x^2 + 5x + 6) - 2x^2$

Solution

$(x^2 - 5x + 6)(x^2 + 5x + 6) - 2x^2$

$= [x^2 - 3x - 2x + 6][x^2 + 3x + 2x + 6] - 2x^2$

$= [x(x - 3) - 2(x - 3)][x(x + 3) + 2(x + 3)] - 2x^2$

$= [(x - 3)(x - 2)][(x + 3)(x + 2)] - 2x^2$

$= [(x - 2)(x + 2)][(x - 3)(x + 3)] - 2x^2$

$= (x^2 - 4)(x^2 - 9) - 2x^2$

$= x^4 - 13x^2 + 36 - 2x^2$

$= x^4 - 15x^2 + 36$

$= x^4 - 12x^2 - 3x^2 + 36$

$= x^2(x^2 - 12) - 3(x^2 - 12)$

$= (x^2 - 12)(x^2 - 3)$

$= [(x)^2 - (2\sqrt{3})^2][(x)^2 - (\sqrt{3})^2]$

$= (x - 2\sqrt{3})(x + 2\sqrt{3})(x - \sqrt{3})(x + \sqrt{3})$

(e) Factorization of Expressions of the following Types

$\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3}$

$\boldsymbol{a^3 - 3a^2b + 3ab^2 - b^3}$

For explanation consider the following examples.

Example 1

Factorize $x^3 - 8y^3 - 6x^2y + 12xy^2$

Solution

$x^3 - 8y^3 - 6x^2y + 12xy^2$.

$= (x)^3 - (2y)^3 - 3(x)^2(2y) + 3(x)(2y)^2$

$= (x)^3 - 3(x)^2(2y) + 3(x)(2y)^2 - (2y)^3$

$= (x - 2y)^3$

$= (x - 2y)(x - 2y)(x - 2y)$

(f) Factorization of Expressions of the following types $a^3 \pm b^3$

We recall the formulas,

$\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$

$\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$

For explanation consider the following examples.

Example 1

Factorize $27x^3 + 64y^3$

Solution

$27x^3 + 64y^3 = (3x)^3 + (4y)^3$

$= (3x + 4y)[(3x)^2 - (3x)(4y) + (4y)^2]$

$= (3x + 4y)(9x^2 - 12xy + 16y^2)$

Example 2

Factorize $1 - 125x^3$

Solution

$1 - 125x^3 = (1)^3 - (5x)^3$

$= (1 - 5x)[(1)^2 + (1)(5x) + (5x)^2]$

$= (1 - 5x)(1 + 5x + 25x^2)$

EXERCISE 5.2

Factorize

1. (i) $x^4+\frac{1}{x^4}-3$ (ii) $3x^4+12y^4$ (iii) $a^4+3a^2b^2+4b^4$
   (iv) $4x^4+81$ (v) x^4+x^2+25 (vi) x^4+4x^2+16
2. (i) $x^2+14x+48$ (ii) $x^2-21x+108$
   (iii) $x^2-11x-42$ (iv) $x^2+x-132$
3. (i) $4x^2+12x+5$ (ii) $30x^2+7x-15$
   (iii) $24x^2-65x+21$ (iv) $5x^2-16x-21$
   (v) $4x^2-17xy+4y^2$ (vi) $3x^2-38xy-13y^2$
   (vii) $5x^2+33xy-14y^2$ (viii) $\left(5x-\frac{1}{x}\right)^2+4\left(5x-\frac{1}{x}\right)+4,\ x\neq 0$
4. (i) $(x^2+5x+4)(x^2+5x+6)-3$
   (ii) $(x^2-4x)(x^2-4x-1)-20$
   (iii) $(x+2)(x+3)(x+4)(x+5)-15$
   (iv) $(x+4)(x-5)(x+6)(x-7)-504$
   (v) $(x+1)(x+2)(x+3)(x+6)-3x^2$
5. (i) $x^3+48x-12x^2-64$ (ii) $8x^3+60x^2+150x+125$
   (iii) $x^3-18x^2+108x-216$ (iv) $8x^3-125y^3-60x^2y+150xy^2$
6. (i) $27+8x^3$ (ii) $125x^3-216y^3$
   (iii) $64x^3+27y^3$ (iv) $8x^3+125y^3$

5.2 Remainder Theorem and Factor Theorem

5.2.1 Remainder Theorem

If a polynomial $p(x)$ is divided by a linear divisor $(x-a)$, then the remainder is $p(a)$.

Proof

Let $q(x)$ be the quotient obtained after dividing $p(x)$ by $(x-a)$. But the divisor $(x-a)$ is linear. So the remainder must be of degree zero i.e., a non-zero constant, say R. Consequently, by division Algorithm we may write

$$p(x) = (x-a)\,q(x) + R$$

This is an identity in x and so is true for all real numbers x. In particular, it is true for $x = a$. Therefore,

$$p(a) = (a-a)\,q(a) + R = 0 + R = R$$

i.e., $p(a)$= the remainder. Hence the theorem.

Note: Similarly, if the divisor is $(ax-b)$, we have

$$p(x) = (ax-b)\,q(x) + R$$

Substituting $x=\frac{a}{b}$ so that $ax-b=0$, we obtain

$$p\left(\frac{b}{a}\right)=0\,.\,q\left(\frac{b}{a}\right)+R=0-R=R$$

Thus if the divisor is linear, the above theorem provides an efficient way of finding the remainder without being involved in the process of long division.

5.2.2 To find Remainder (without dividing) when a polynomial is divided by a Linear Polynomial

Example 1

Find the remainder when $9x^2-6x+2$ is divided by

(i) $x-3$ (ii) $x+3$ (iii) $3x+1$ (iv) x

Solution

Let $p(x) = 9x^2-6x+2$

(i) When $p(x)$ is divided by $x-3$, by Remainder Theorem, the remainder is

$$R = p(3) = 9(3)^2 - 6(3) + 2 = 65$$

(ii) When $p(x)$ is divided by $x+3 = x-(-3)$, the remainder is

$$R = p(-3) = 9(-3)^2 - 6(-3) + 2 = 101$$

(iii) When $p(x)$ is divided by $3x+1$, the remainder is

$$R = p\left(-\frac{1}{3}\right) = 9\left(-\frac{1}{3}\right)^2 - 6\left(-\frac{1}{3}\right) + 2 = 5$$

(v) When $p(x)$ is divided by x, the remainder is

$$R = p(0) = 9(0)^2 - 6(0) + 2 = 2$$

Example 2

Find the value of k if the expression $x^3 + kx^2 + 3x - 4$ leaves a remainder of –2 when divided by $x + 2$.

Solution

Let $p(x) = x^3 + kx^2 + 3x - 4$

By the Remainder Theorem, when $p(x)$ is divided by $x + 2 = x - (-2)$, the remainder is

$$p(-2) = (-2)^3 + k(-2)^2 + 3(-2) - 4.$$
$$= -8 + 4k - 6 - 4$$
$$= 4k - 18$$

By the given condition, we have

$$p(-2) = -2 \quad \Rightarrow \quad 4k - 18 = -2 \quad \Rightarrow \quad k = 4$$

5.2.3 Zero of a Polynomial

Definition

If a specific number $x = a$ is substituted for the variable x in a polynomial $p(x)$ so that the value $p(a)$ is zero, then $x = a$ is called a zero of the polynomial $p(x)$.

A very useful consequence of the remainder theorem is what is known as the factor theorem.

5.2.4 Factor Theorem

The polynomial $(x - a)$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.

Proof

Let $q(x)$ be the quotient and R the remainder when a polynomial $p(x)$ is divided by $(x - a)$. Then by division Algorithm,

$$p(x) = (x - a)\,q(x) + R$$

By the Remainder Theorem, $R = p(a)$.

Hence $p(x) = (x - a)\,q(x) + p(a)$

(i) Now if $p(a) = 0$, then $p(x) = (x - a)\,q(x)$
i.e., $(x - a)$ is a factor of $p(x)$

(ii) Conversely, if $(x - a)$ is a factor of $p(x)$, then the remainder upon dividing $p(x)$ by $(x - a)$ must be zero i.e., $p(a) = 0$
This completes the proof.

Note: The Factor Theorem can also be stated as, "$(x - a)$ is a factor of $p(x)$ if and only if $x = a$ is a solution of the equation $p(x) = 0$".

The Factor Theorem helps us to find factors of polynomials because it determines whether a given linear polynomial $(x - a)$ is a factor of $p(x)$. All we need is to check whether $p(a) = 0$.

Example 1

Determine if $(x - 2)$ is a factor of $x^3 - 4x^2 + 3x + 2$.

Solution

For convenience, let

$$p(x) = x^3 - 4x^2 + 3x + 2$$

Then the remainder for $(x - 2)$ is

$$p(2) = (2)^3 - 4(2)^2 + 3(2) + 2$$
$$= 8 - 16 + 6 + 2 = 0$$

Hence by Factor Theorem, $(x - 2)$ is a factor of the polynomial $p(x)$.

Example 2

Find a polynomial $p(x)$ of degree 3 that has 2, –1, and 3 as zeros (i.e., roots).

Solution

Since $x = 2, -1, 3$ are roots of $p(x) = 0$

So by Factor Theorem $(x - 2)$, $(x + 1)$ and $(x - 3)$ are the factors of $p(x)$.

Thus $p(x) = a(x - 2)(x + 1)(x - 3)$

where any non-zero value can be assigned to a.

Taking $a = 1$, we get

$$p(x) = (x - 2)(x + 1)(x - 3)$$
$$= x^3 - 4x^2 + x + 6 \quad \text{as the required polynomial.}$$

EXERCISE 5.3

1. Use the remainder theorem to find the remainder when
   (i) $3x^3 - 10x^2 + 13x - 6$ is divided by $(x - 2)$
   (ii) $4x^3 - 4x + 3$ is divided by $(2x - 1)$
   (iii) $6x^4 + 2x^3 - x + 2$ is divided by $(x + 2)$
   (iv) $(2x - 1)^3 + 6(3 + 4x)^2 - 10$ is divided by $(2x + 1)$
   (v) $x^3 - 3x^2 + 4x - 14$ is divided by $(x + 2)$
2. (i) If $(x + 2)$ is a factor of $3x^2 - 4kx - 4k^2$, then find the value(s) of k.
   (ii) If $(x - 1)$ is a factor of $x^3 - kx^2 + 11x - 6$, then find the value of k.
3. Without actual long division determine whether
   (i) $(x - 2)$ and $(x - 3)$ are factors of $p(x) = x^3 - 12x^2 + 44x - 48$.
   (ii) $(x - 2)$, $(x + 3)$ and $(x - 4)$ are factors of $q(x) = x^3 + 2x^2 - 5x - 6$.
4. For what value of m is the polynomial $p(x) = 4x^3 - 7x^2 + 6x - 3m$ exactly divisible by $x + 2$?
5. Determine the value of k if $p(x) = kx^3 + 4x^2 + 3x - 4$ and $q(x) = x^3 - 4x + k$ leaves the same remainder when divided by $(x - 3)$.
6. The remainder after dividing the polynomial $p(x) = x^3+ax^2+7$ by $(x + 1)$ is $2b$. Calculate the value of a and b if this expression leaves a remainder of $(b + 5)$ on being divided by $(x - 2)$.
7. The polynomial $x^3 + lx^2 + mx + 24$ has a factor $(x + 4)$ and it leaves a remainder of 36 when divided by $(x - 2)$. Find the values of l and m.
8. The expression $lx^3 + mx^2 - 4$ leaves remainder of –3 and 12 when divided by $(x - 1)$ and $(x + 2)$ respectively. Calculate the values of l and m.
9. The expression $ax^3 - 9x^2 + bx + 3a$ is exactly divisible by $x^2 - 5x + 6$. Find the values of a and b.

5.3 Factorization of a Cubic Polynomial

We can use Factor Theorem to factorize a cubic polynomial

as explained below. This is a convenient method particularly for factorization of a cubic polynomial. We state (without proof) a very useful Theorem.

Rational Root Theorem

Let $a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0, \quad a_0 \neq 0$

be a polynomial equation of degree n with integral coefficients. If p/q is a rational root (expressed in lowest terms) of the equation, then p is a factor of the constant term a_n and q is a factor of the leading coefficient a_0.

Example 1

Factorize the polynomial $x^3 - 4x^2 + x + 6$, by using Factor Theorem.

Solution

We have $P(x) = x^3 - 4x^2 + x + 6$.

Possible factors of the constant term $p = 6$ are ±1, ±2, ±3 and ±6 and of leading coefficient $q = 1$ are ±1. Thus the expected zeros (or roots) of $P(x) = 0$ are $\frac{p}{q} = \pm1, \pm2, \pm3$ and ±6. If $x = a$ is a zero of $P(x)$, then $(x - a)$ will be a factor.

We use the hit and trial method to find zeros of $P(x)$. Let us try $x = 1$.

Now $P(1) = (1)^3 - 4(1)^2 + 1 + 6$
$= 1 - 4 + 1 + 6 = 4 \neq 0$

Hence $x = 1$ is not a zero of $P(x)$.

Again $P(-1) = (-1)^3 - 4(-1)^2 - 1 + 6$
$= -1 - 4 - 1 + 6 = 0$

Hence $x = -1$ is a zero of $P(x)$ and therefore,
$x - (-1) = (x + 1)$ is a factor of $P(x)$.

Now $P(2) = (2)^3 - 4(2)^2 + 2 + 6$
$= 8 - 16 + 2 + 6 = 0 \Rightarrow x = 2$ is a root.

Hence $(x - 2)$ is also a factor of $P(x)$.

Similarly $P(3) = (3)^3 - 4(3)^2 + 3 + 6$
$= 27 - 36 + 3 + 6 = 0 \Rightarrow x = 3$ is a zero of $P(x)$.

Hence $(x - 3)$ is the third factor of $P(x)$.

Thus the factorized form of

$P(x) = x^3 - 4x^2 + x + 6$

is $P(x) = (x + 1)(x - 2)(x - 3)$

EXERCISE 5.4

Factorize each of the following cubic polynomials by factor theorem.

1. $x^3 - 2x^2 - x + 2$
2. $x^3 - x^2 - 22x + 40$
3. $x^3 - 6x^2 + 3x + 10$
4. $x^3 + x^2 - 10x + 8$
5. $x^3 - 2x^2 - 5x + 6$
6. $x^3 + 5x^2 - 2x - 24$
7. $3x^3 - x^2 - 12x + 4$
8. $2x^3 + x^2 - 2x - 1$

REVIEW EXERCISE 5

1. Multiple Choice Questions. Choose the correct answer.

2. Completion Items. Fill in the blanks.
   (i) $x^2 + 5x + 6 =$ ....
   (ii) $4a^2 - 16 =$ ....
   (iii) $4a^2 + 4ab +$ (.........) is a complete square
   (iv) $\frac{x^2}{y^2} - 2 + \frac{y^2}{x^2} =$ ......
   (v) $(x + y)(x^2 - xy + y^2) =$ ....
   (vi) Factored form of $x^4 - 16$ is ....
   (vii) If $x - 2$ is factor of $p(x) = x^2 + 2kx + 8$, then $k =$ ....

3. Factorize the following.
   (i) $x^2 + 8x + 16 - 4y^2$
   (ii) $4x^2 - 16y^2$
   (iii) $9x^2 + 27x + 8$
   (iv) $1 - 64z^3$
   (v) $8x^3 - \frac{1}{27y^3}$
   (vi) $2y^2 + 5y - 3$
   (vii) $x^3 + x^2 - 4x - 4$
   (viii) $25m^2n^2 + 10mn + 1$
   (ix) $1 - 12pq + 36p^2q^2$

SUMMARY

- If a polynomial is expressed as a product of other polynomials, then each polynomial in the product is called a factor of the original polynomial.
- The process of expressing an algebraic expression in terms of its factors is called factorization. We learned to factorize expressions of the following types:
  - $ka + kb + kc$
  - $ac + ad + bc + bd$
  - $a^2 \pm 2ab + b^2$
  - $a^2 - b^2$
  - $(a^2 \pm 2ab + b^2) - c^2$
  - $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$
  - $x^2 + px + q$ • $ax^2 + bx + c$
  - $(ax^2 + bx + c)(ax^2 + bx + d) + k$
  - $(x + a)(x + b)(x + c)(x + d) + k$
  - $(x + a)(x + b)(x + c)(x + d) + kx^2$
  - $a^3 + 3a^2b + 3ab^2 + b^3$
  - $a^3 - 3a^2b + 3ab^2 - b^3$
  - $a^3 \pm b^3$
- If a polynomial $p(x)$ is divided by a linear divisor $(x - a)$, then the remainder is $p(a)$.
- If a specific number $x = a$ is substituted for the variable x in a polynomial $p(x)$ so that the value $p(a)$ is zero, then $x = a$ is called a zero of the polynomial $p(x)$.
- The polynomial $(x - a)$ is a factors of the polynomial $p(x)$ if and only if $p(a) = 0$. Factor theorem has been used to factorize cubic polynomials.
