version: 1.1

CHAPTER

10

CONGRUENT TRIANGLES

Animation 10.1: Algebraic Manipulation
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Students Learning Outcomes

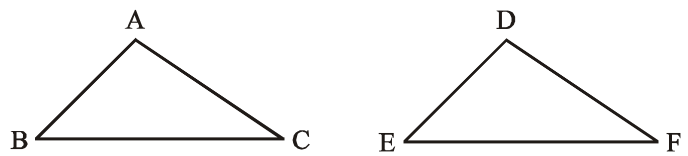
After studying this unit, the students will be able to:

- Prove that in any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, then the triangles are congruent.
- Prove that if two angles of a triangle are congruent, then the sides opposite to them are also congruent.
- Prove that in a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, the two triangles are congruent.
- Prove that if in the correspondence of two right-angled triangles, the hypotenuse and one side of one are congruent to the hypotenuses and the corresponding side of the other, then the triangles are congruent.

10.1. Congruent Triangles

Introduction

In this unit before proving the theorems, we will explain what is meant by 1 – 1 correspondence (the symbol used for 1 – 1 correspondence is $\longleftrightarrow$ and congruency of triangles. We shall also state S.A.S. postulate.



Let there be two triangles ABC and DEF. Out of the total six (1 – 1) correspondences that can be established between ΔABC and ΔDEF, one of the choices is explained below.

In the correspondence $\Delta ABC \longleftrightarrow \Delta DEF$ it means

$\angle A \longleftrightarrow \angle D$ ($\angle A$ corresponds to $\angle D$)
$\angle B \longleftrightarrow \angle E$ ($\angle B$ corresponds to $\angle E$)
$\angle C \longleftrightarrow \angle F$ ($\angle C$ corresponds to $\angle F$)

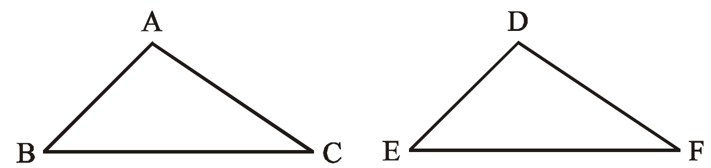
$\overline{AB} \longleftrightarrow \overline{DE}$ ($\overline{AB}$ corresponds to $\overline{DE}$)
$\overline{BC} \longleftrightarrow \overline{EF}$ ($\overline{BC}$ corresponds to $\overline{EF}$)
$\overline{CA} \longleftrightarrow \overline{FD}$ ($\overline{CA}$ corresponds to $\overline{FD}$)

Congruency of Triangles

Two triangles are said to be congruent written symbolically as, $\cong$, if there exists a correspondence between them such that all the corresponding sides and angles are congruent i.e.,

If $\begin{cases} \overline{AB} \cong \overline{DE} \\ \overline{BC} \cong \overline{EF} \\ \overline{CA} \cong \overline{FD} \end{cases}$ and $\begin{cases} \angle A \cong \angle D \\ \angle B \cong \angle E \\ \angle C \cong \angle F \end{cases}$

then $\Delta ABC \cong \Delta DEF$



Note:

(i) These triangles are congruent w.r.t. the above mentioned choice of the (1 – 1) correspondence.

(ii) $\Delta ABC \cong \Delta ABC$

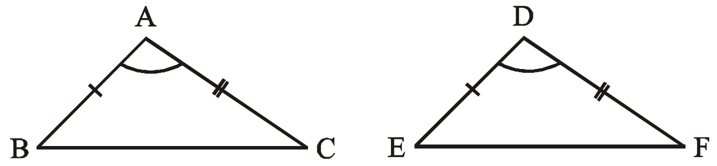
(iii) $\Delta ABC \cong \Delta DEF \Leftrightarrow \Delta DEF \cong \Delta ABC$

(iv) If $\Delta ABC \cong \Delta DEF$ and $\Delta ABC \cong \Delta PQR$, then $\Delta DEF \cong \Delta PQR$.

In any correspondence of two triangles, if two sides and their included angle of one triangle are congruent to the corresponding two sides and their included angle of the other, then the triangles are congruent.

In $\Delta ABC \longleftrightarrow \Delta DEF$, shown in the following figure,

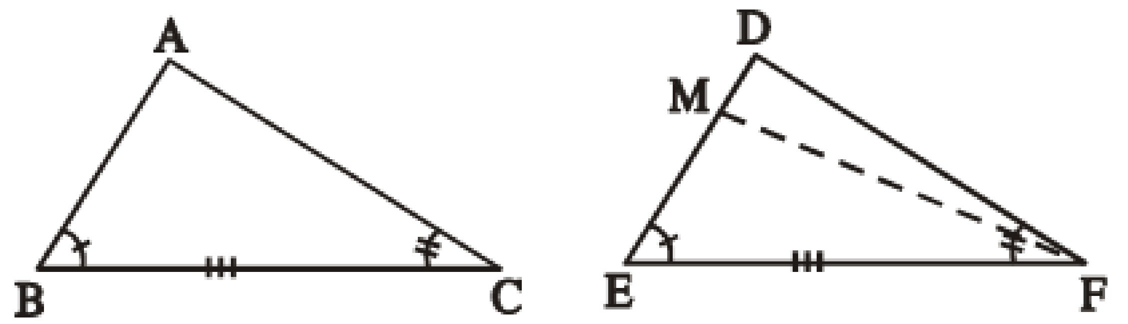
If $\begin{cases} \overline{AB} \cong \overline{DE} \\ \angle A \cong \angle D \\ \overline{AC} \cong \overline{DF} \end{cases}$



then $\Delta ABC \cong \Delta DEF$ (S. A. S. Postulate)

Theorem 10.1.1

In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, then the triangles are congruent. (A.S.A. ≅ A.S.A.)



Given

In $\Delta ABC \longleftrightarrow \Delta DEF$

$\angle B \cong \angle E$, $\overline{BC} \cong \overline{EF}$, $\angle C \cong \angle F$.

To Prove

$\Delta ABC \cong \Delta DEF$

Construction

Suppose $\overline{AB} \not\cong \overline{DE}$, take a point M on $\overline{DE}$ such that $\overline{AB} \cong \overline{ME}$. Join M to F

Proof

Statements	Reasons
In $\Delta ABC \longleftrightarrow \Delta MEF$	
$\overline{AB} \cong \overline{ME}$ ...... (i)	Construction
$\overline{BC} \cong \overline{EF}$ ...... (ii)	Given
$\angle B \cong \angle E$ ...... (iii)	Given
$\therefore \Delta ABC \cong \Delta MEF$	S.A.S. postulate
So, $\angle C \cong \angle MFE$	(Corresponding angles of congruent triangles)

But, $\angle C \cong \angle DFE$	Given
$\therefore \angle DFE \cong \angle MFE$	Both congruent to $\angle C$
This is possible only if D and M are the same points, and $\overline{ME} \cong \overline{DE}$	
So, $\overline{AB} \cong \overline{DE}$ .... (iv)	$\overline{AB} \cong \overline{ME}$ (construction) and $\overline{ME} \cong \overline{DE}$ (proved)
Thus from (ii), (iii) and (iv), we have $\Delta ABC \cong \Delta DEF$	S.A.S. postulate

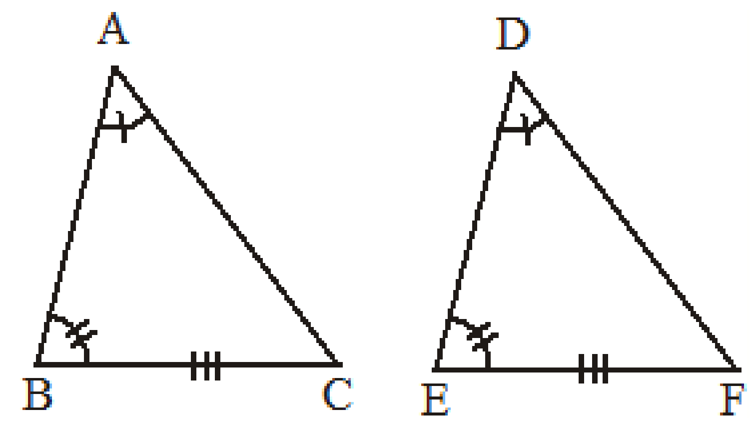
Corollary

In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, then the triangles are congruent. (S.A.A. ≅ S.A.A.)

Given

In $\Delta ABC \longleftrightarrow \Delta DEF$

$\overline{BC} \cong \overline{EF}$, $\angle A \cong \angle D$, $\angle B \cong \angle E$



To Prove

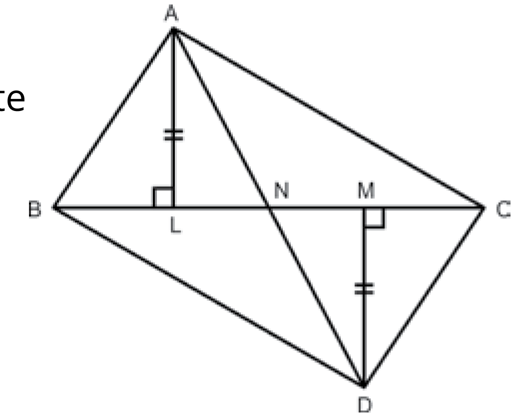
$\Delta ABC \cong \Delta DEF$

Proof

Statements	Reasons
In $\Delta ABC \longleftrightarrow \Delta DEF$	
$\angle B \cong \angle E$	Given
$\overline{BC} \cong \overline{EF}$	Given
$\angle C \cong \angle F$	$\angle A \cong \angle D$, $\angle B \cong \angle E$, (Given)
$\therefore \Delta ABC \cong \Delta DEF$	A.S.A. $\cong$ A.S.A.

Example

If ΔABC and ΔDCB are on the opposite sides of common base $\overline{BC}$ such that $\overline{AL} \perp \overline{BC}$, $\overline{DM} \perp \overline{BC}$ and $\overline{AL} \cong \overline{DM}$, then $\overline{BC}$ bisects $\overline{AD}$.



Given

ΔABC and ΔDCB are on the opposite sides of $\overline{BC}$ such that $\overline{AL} \perp \overline{BC}$, $\overline{DM} \perp \overline{BC}$, $\overline{AL} \cong \overline{DM}$, and $\overline{AD}$ is cut by $\overline{BC}$ at N.

To Prove

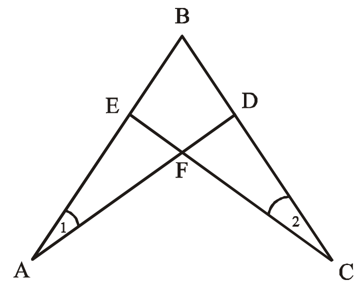
$\overline{AN} \cong \overline{DN}$

Proof

Statements	Reasons
In $\Delta ALN \longleftrightarrow \Delta DMN$	
$\overline{AL} \cong \overline{DM}$	Given
$\angle ALN \cong \angle DMN$	Each angle is right angle
$\Delta ANL \cong \angle DNM$	Vertical angels
$\therefore \Delta ALN \cong \Delta DMN$	S.A.A. $\cong$ S.A.A.
Hence $\overline{AN} \cong \overline{DN}$	Corresponding sides of $\cong$ Δs.

EXERCISE 10.1

1. In the given figure,
   $\overline{AB} \cong \overline{CB}$, $\angle 1 \cong \angle 2$.
   Prove that
   $\Delta ABD \cong \Delta CBE$.



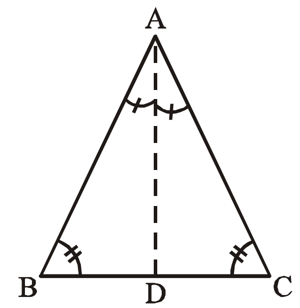
2. From a point on the bisector of an angle, perpendiculars are drawn to the arms of the angle. Prove that these perpendiculars are equal in measure.
3. In a triangle ABC, the bisectors of $\angle B$ and $\angle C$ meet in a point I. Prove that I is equidistant from the three sides of ΔABC.

Theorem 10.1.2

If two angles of a triangle are congruent, then the sides opposite to them are also congruent.

Given

In ΔABC, $\angle B \cong \angle C$



To Prove

$\overline{AB} \cong \overline{AC}$

Construction

Draw the bisector of $\angle A$, meeting BC at the point D.

Proof

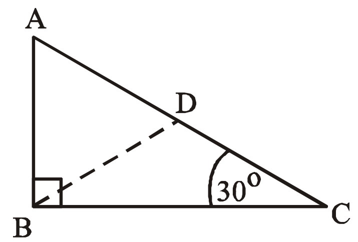
Statements	Reasons
In $\Delta ABD \longleftrightarrow \Delta ACD$	
$\overline{AD} \cong \overline{AD}$	Common
$\angle B \cong \angle C$	Given
$\angle BAD \cong \angle CAD$	Construction
$\therefore \Delta ABD \cong \Delta ACD$	S.A.A. $\cong$ S.A.A.
Hence $\overline{AB} \cong \overline{AC}$	(Corresponding sides of congruent triangles)

Example 1

If one angle of a right triangle *d* is of 30°, the hypotenuse is twice as long as the side opposite to the angle.

Given

In ΔABC, $m\angle B = 90^0$ and
$m\angle C = 30^0$



To Prove

$m\overline{AC} = 2m\overline{AB}$

Construction

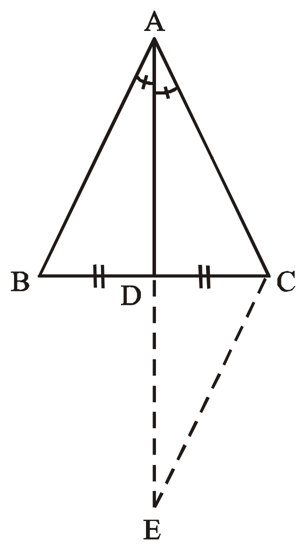
At B, construct $\angle CBD$ of 30°. Let $\overline{BD}$ cut $\overline{AC}$ at the point D.

Proof

Statements	Reasons
In ΔABD, $m\angle A = 60^0$	$m\angle ABC = 90°$, $m\angle C = 30°$
$m\angle ABD = m\angle ABC - m\angle CBD$	
$= 60^0$	$m\angle ABC = 90°$, $m\angle CBD = 30°$
$\therefore$ $m\angle ADB = 60^0$	Sum of measures of $\angle$s of a Δ is 180°
$\therefore$ ΔABD is equilateral	Each of its angles is equal to 60°
$\therefore$ $\overline{AB} \cong \overline{BD} \cong \overline{AD}$	Sides of equilateral Δ
In ΔBCD, $\overline{BD} \cong \overline{CD}$	$\angle C = \angle CBD$ (each of 30),
Thus $m\overline{AC} = m\overline{AD} + m\overline{CD}$	
$= m\overline{AB} + m\overline{AB}$	$\overline{AD} \cong \overline{AB}$ and $\overline{CD} \cong \overline{BD} \cong \overline{AB}$
$= 2(m\overline{AB})$	

Example 2

If the bisector of an angle of a triangle bisects the side opposite to it, the triangle is isosceles.



Given

In ΔABC, $\overline{AD}$ bisects $\angle A$ and $\overline{BD} \cong \overline{CD}$
$m\angle C = 30^0$

To Prove

$\overline{AB} \cong \overline{AC}$

Construction

Produce $\overline{AD}$ to E, and take $\overline{ED} \cong \overline{AD}$
Joint C to E.

Proof

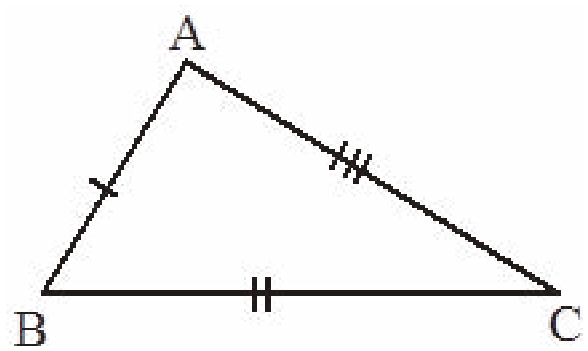
Statements	Reasons
In $\Delta ADB \longleftrightarrow \Delta EDC$	
$\overline{AD} \cong \overline{ED}$	Construction
$\angle ADB = \angle EDC$	Vertical angles
$\overline{BD} \cong \overline{CD}$	Given
$\therefore$ $\Delta ADB \cong \Delta EDC$	S.A.S. Postulate
$\therefore$ $\overline{AB} \cong \overline{EC}$ I	Corresponding sides of $\cong \Delta$s
and $\angle BAD \cong \angle E$	Corresponding angles of $\cong \Delta$s
But $\angle BAD \cong \angle CAD$	Given
$\therefore$ $\angle E \cong \angle CAD$	Each $\cong \angle BAD$
In ΔACE, $\overline{AC} \cong \overline{EC}$ II	$\angle E \cong \angle CAD$ (proved)
Hence $\overline{AB} \cong \overline{AC}$	From I and II

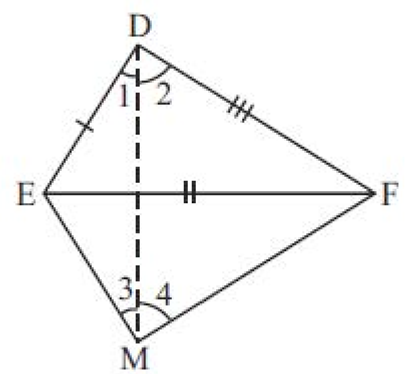
EXERCISE 10.2

1. Prove that any two medians of an equilateral triangle are equal in measure.
2. Prove that a point, which is equidistant from the end points of a line segment, is on the right bisector of the line segment.

Theorem 10.1.3

In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, then the two triangles are congruent (S.S.S $\cong$ S.S.S).





Given

In $\Delta ABC \leftrightarrow \Delta DEF$

$\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\overline{CA} \cong \overline{FD}$

To Prove

$\Delta ABC \cong \Delta DEF$

Construction

Suppose that in ΔDEF the side $\overline{EF}$ is not smaller than any of the remaining two sides. On $\overline{EF}$ costruct a ΔMEF in which, $\angle FEM \cong \angle B$ and $\overline{ME} \cong \overline{AB}$. Join D and M. As shown in the above figures we label some of the angles as 1, 2, 3 and 4.

Proof

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta MEF$	
$\overline{BC} \cong \overline{EF}$	Given
$\angle B = \angle FEM$	Construction
$\overline{AB} \cong \overline{ME}$	Construction
$\therefore \Delta ABC \cong \Delta MEF$	S.A.S postulate
and $\overline{CA} \cong \overline{FM}$ .... (i)	(corresponding sides of congruent triangles)
Also $\overline{CA} \cong \overline{FD}$ .... (ii)	Given
$\therefore \overline{FM} \cong \overline{FD}$	{From (i) and (ii)}
In ΔFDM	
$\angle 2 \cong \angle 4$ .... (iii)	$\overline{FM} \cong \overline{FD}$ (proved)
Similarly $\angle 1 \cong \angle 3$ .... (iv)	
$\therefore m\angle 2 + m\angle 1 = m\angle 4 + m\angle 3$	{from (iii) and (iv)}
$\therefore m\angle EDF = m\angle EMF$	
Now, in $\Delta ADB \leftrightarrow \Delta EDC$	
$\overline{FD} \cong \overline{FM}$	Proved
andm $\angle EDF \cong m\angle EMF$	Proved
$\overline{DE} \cong \overline{ME}$	Each one $\cong \overline{AB}$
$\therefore \Delta DEF \cong \Delta MEF$	S.A.S. postulate
Also $\Delta ABC \cong \Delta MEF$	Proved
Hence $\Delta ABC \cong \Delta DEF$	Each $\Delta \cong \Delta MEF$ (Proved)

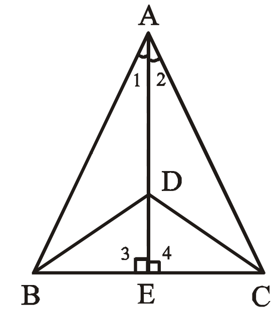
Corollary

If two isosceles triangles are formed on the same side of their common base, the line through their vertices would be the right bisector of their common base.

Given

ΔABC and ΔDBC are formed on the same side of $\overline{BC}$ such that

$\overline{AB} \cong \overline{AC}$, $\overline{DB} \cong \overline{DC}$, $\overline{AD}$ meets $\overline{BC}$ at E.



To Prove

$\overline{BE} \cong \overline{CE}$, $\overline{AE} \perp \overline{BC}$

Proof

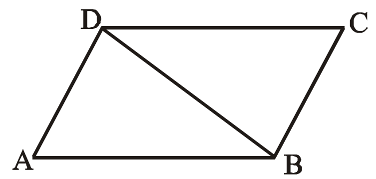
Statements	Reasons
In $\Delta ADB \longleftrightarrow \Delta ADC$	
$\overline{AB} \cong \overline{AC}$	Given
$\overline{DB} \cong \overline{DC}$	Given
$\overline{AD} \cong \overline{AD}$	Common
$\therefore \Delta ADB \cong \Delta ADC$	S.S.S $\cong$ S.S.S.
$\therefore \angle 1 \cong \angle 2$	Corresponding angles of $\cong \Delta$s
In $\Delta ABE \longleftrightarrow \Delta ACE$	
Also $\overline{AB} \cong \overline{AC}$	Given
$\therefore \angle 1 \cong \angle 2$	Proved
$\overline{AE} \cong \overline{AE}$	Common
$\therefore \Delta ABE \cong \Delta ACE$	S.A.S. postulate
$\therefore \overline{BE} \cong \overline{CE}$	Corresponding sides of $\cong \Delta$s
$\angle 3 \cong \angle 4$ I	Corresponding sides of $\cong \Delta$s
$m\angle 3 + m\angle 4 = 180^0$ II	Supplementary angles Postulate
$\therefore m\angle 3 = m\angle 4 = 90^0$	From I and II
Hence $\overline{AE} \perp \overline{BC}$	

Corollary:

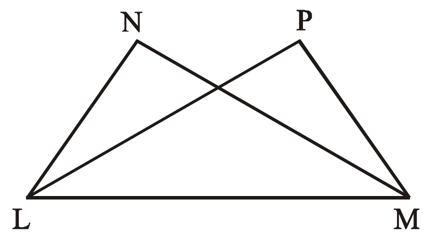
An equilateral triangle is an equiangular triangle.

EXERCISE 10.3

1. In the figure, $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$.
   Prove that $\angle A \cong \angle C$, $\angle ABC \cong \angle ADC$.



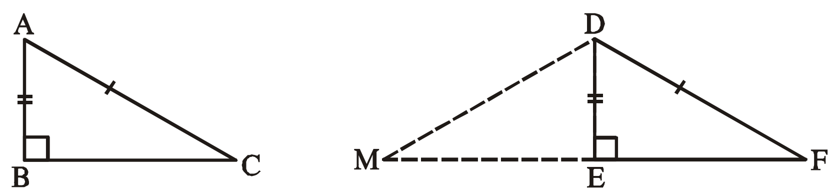
2. In the figure, $\overline{LN} \cong \overline{MP}$, $\overline{MN} \cong \overline{LP}$.
   Prove that $\angle N \cong \angle P$, $\angle NML \cong \angle PLM$.



3. Prove that the median bisecting the base of an isosceles triangle bisects the vertex angle and it is prependicular to the base.

Theorem 10.1.4

If in the correspondence of the two right-angled triangles, the hypotenuse and one side of one triangle are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent. (H.S ≅ H.S).



Given

In $\Delta ABC \longleftrightarrow \Delta DEF$

$\angle B \cong \angle E$ (right angles)

$\overline{CA} \cong \overline{FD}$, $\overline{AB} \cong \overline{DE}$

To Prove

$\Delta ABC \cong \Delta DEF$

Construction

Produce $\overline{FE}$ to a point M such that $\overline{EM} \cong \overline{BC}$ and join the points D and M.

Proof

Statements	Reasons
$m\angle DEF + m\angle DEM = 180°$ ......(i)	(Supplementary angles)
Now $m\angle DEF = 90°$ .....(ii)	Given
$\therefore$ $m\angle DEM = 90°$	{from (i) and (ii)}
In $\Delta ABC \leftrightarrow \Delta DEM$	
$\overline{BC} \cong \overline{EM}$	(construction)
$\angle ABC \cong \angle DEM$	(each $\angle$ equal to 90°)
$\overline{AB} \cong \overline{DE}$	(given)
$\therefore$ $\Delta ABC \cong \Delta DEM$	S.A.S. postulate
and $\angle C \cong \angle M$	(Corresponding angles of congruent triangles)
$\overline{CA} \cong \overline{MD}$	(Corresponding sides of congruent triangles)
But $\overline{CA} \cong \overline{FD}$	(given)
$\therefore$ $\overline{MD} \cong \overline{FD}$	each is congruent to $\overline{CA}$
In ΔDMF	
$\angle F \cong \angle M$	$\overline{MD} \cong \overline{FD}$ (proved)
But $\angle C \cong \angle M$	(proved)
$\angle C \cong \angle F$	(each is congruent to $\angle M$)
In $\Delta ABC \leftrightarrow \Delta DEF$	
$\overline{AB} \cong \overline{DE}$	(given)
$\angle ABC \cong \angle DEF$	(given)
$\angle C \cong \angle F$	(proved)
Hence $\Delta ABC \cong \Delta DEF$	(S.A.A. ≅ S.A.A)

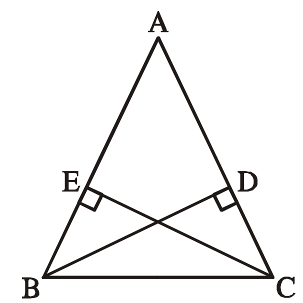
Example

If perpendiculars from two vertices of a triangle to the opposite sides are congruent, then the triangle is isosceles.

Given

In ΔABC, $\overline{BD} \perp \overline{AC}$, $\overline{CE} \perp \overline{AB}$

Such that $\overline{BD} \cong \overline{CE}$



To Prove

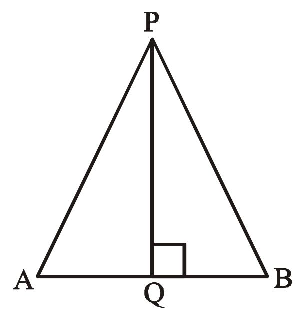
$\overline{AB} \cong \overline{AC}$

Proof

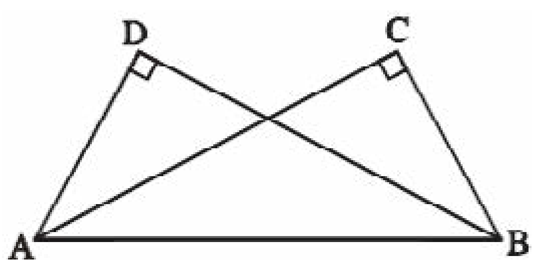
Statements	Reasons
In $\Delta BCD \longleftrightarrow \Delta CBE$	
$\angle BCD \cong \angle BEC$	$\overline{BD} \perp \overline{AC}$, $\overline{CE} \perp \overline{AB}$ (given) $\Rightarrow$ each angle = 90^0
$\overline{BC} \cong \overline{BC}$	Common hypotenuse
$\overline{BD} \cong \overline{CE}$	Given
$\therefore$ $\Delta BCD \cong \Delta CBE$	H.S. $\cong$ H.S.
$\therefore$ $\angle BCD \cong \angle CBE$	Corresponding angles Δs
Thus $\angle BCD \cong \angle CBE$	
Hence $\overline{AB} \cong \overline{AC}$	In ΔABC, $\angle BCA \cong \angle CBA$

EXERCISE 10.4

1. In ΔPAB of figure, $\overline{PQ} \perp \overline{AB}$ and $\overline{PA} \cong \overline{PB}$, proved that $\overline{AQ} \cong \overline{BQ}$ and $\angle APQ \cong \angle BPQ$.



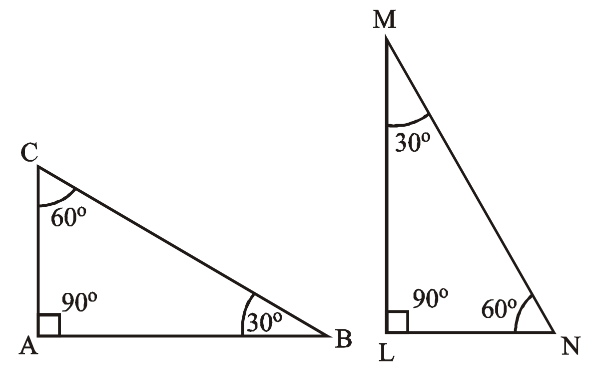
2. In the figure, $m\angle C = m\angle D = 90^0$ and $\overline{BC} \cong \overline{AD}$. Prove that $\overline{AC} \cong \overline{BD}$, and $\angle BAC \cong \angle ABD$.



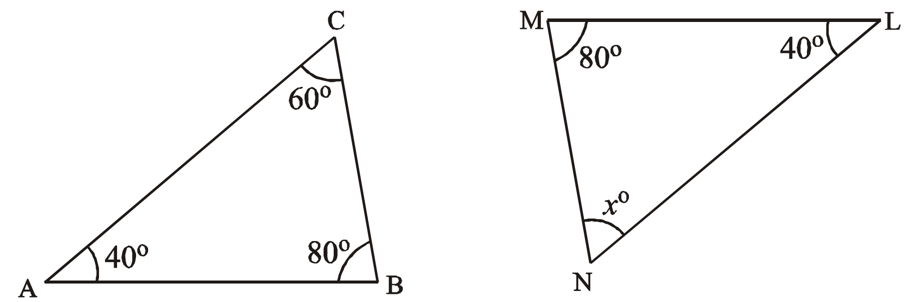
3. In the figure, $m\angle B = m\angle D = 90^0$ and $\overline{AD} \cong \overline{BC}$. Prove that ABCD is a rectangle.

REVIEW EXERCISE 10

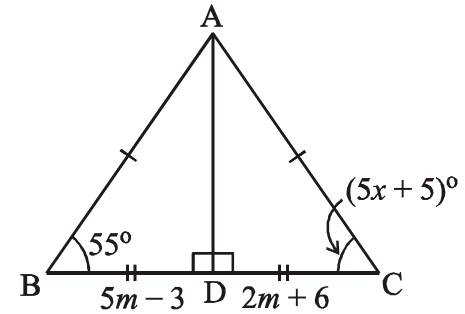
1. Which of the following are true and which are false?
   (i) A ray has two end points. ......
   (ii) In a triangle, there can be only one right angle. ......
   (iii) Three points are said to be collinear, if they lie on same line. ...
   (iv) Two parallel lines intersect at a point. ......
   (v) Two lines can intersect only at one point. ......
   (vi) A triangle of congruent sides has non-congruent angles. ......
2. If $\Delta ABC \cong \Delta LMN$, then
   (i) $m\angle M \cong$ ....
   (ii) $m\angle N \cong$ ....
   (iii) $m\angle A \cong$ ....



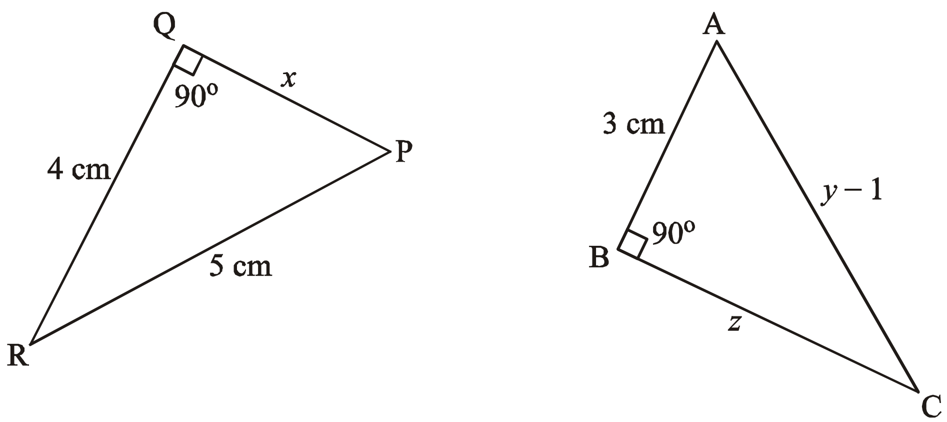
3. If $\Delta ABC \cong \Delta LMN$, then find the unknown x.



4. Find the value of unknowns for the given congruent triangles.



5. If PQR ≅ ABC, then find the unknowns.



SUMMARY

In this unit we stated and proved the following theorems:

- In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, the two triangles are congruent. (A.S.A ≅ A.S.A.)
- If two angles of a triangle are congruent, then the sides opposite to them are also congruent.
- In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, then the two triangles are congruent (S.S.S ≅ S.S.S).
- If in the correspondence of the two right-angled triangles, the hypotenuse and one side of one triangle are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent. (H.S ≅ H.S).
- Two triangles are said to be congruent, if there exists a correspondence between them such that all the corresponding sides and angles are congruent.