CHAPTER

12

LINE BISECTORS AND ANGLE BISECTORS

Animation 12.1: Angle- Bisectors
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Students Learning Outcomes

After studying this unit, the students will be able to:

- Prove that any point on the right bisector of a line segment is equidistant from its end points.
- Prove that any point equidistant from the end points of a line segment is on the right bisector of it.
- Prove that the right bisectors of the sides of a triangle are concurrent.
- Prove that any point on the bisector of an angle is equidistant from its arms.
- Prove that any point inside an angle, equidistant from its arms, is on the bisector of it.
- Prove that the bisectors of the angles of a triangle are concurrent.

Introduction

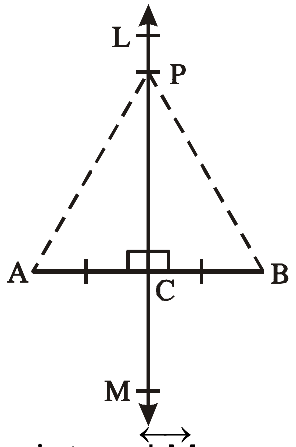
In this unit, we will prove theorems and their converses, if any, about right bisector of a line segment and bisector of an angle. But before that it will be useful to recall the following definitions:

Right Bisector of a Line Segment

A line is called a right bisector of a line segment if it is perpendicular to the line segment and passes through its midpoint.

Bisector of an Angle

A ray BP is called the bisector of $\angle ABC$, if P is a point in the interior of the angle and $m\angle ABP = m\angle PBC$.



Theorem 12.1.1

Any point on the right bisector of a line segment is equidistant from its end points.

Given

A line LM intersects the line segment AB at the point C. Such that $\overleftrightarrow{LM} \perp \overline{AB}$ and $\overline{AC} \cong \overline{BC}$. P is a point on $\overleftrightarrow{LM}$.

To Prove

$\overline{PA} \cong \overline{PB}$

Construction

Join P to the points A and B.

Proof

Statements	Reasons
In $\Delta ACP \longleftrightarrow \Delta BCP$	
$\overline{AC} \cong \overline{BC}$	given
$\angle ACP \cong \angle BCP$	given $\overline{PC} \perp \overline{AB}$, so that each $\angle$ at C = 90^0
$\overline{PC} \cong \overline{PC}$	Common
$\Delta ACP \cong \Delta BCP$	S.A.S. postulate
Hence $\overline{PA} \cong \overline{PB}$	(corresponding sides of congruent triangles)

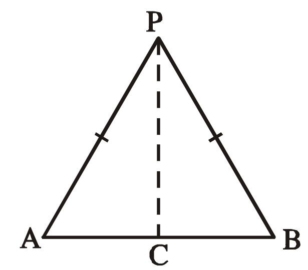
Theorem 12.1.2

{Converse of Theorem 12.1.1}

Any point equidistant from the end points of a line segment is on the right bisector of it.

Given

$\overline{AB}$ is a line segment. Point P is such that $\overline{PA} \cong \overline{PB}$.



To Prove

The point P is on the right bisector of $\overline{AB}$.

Construction

Joint P to C, the mid-point of $\overline{AB}$.

Proof

Statements	Reasons
In $\Delta ACP \longleftrightarrow \Delta BCP$	
$\overline{PA} \cong \overline{PB}$	given
$\overline{PC} \cong \overline{PC}$	Common

$\overline{AC} \cong \overline{BC}$	Construction
$\therefore$ $\Delta ACP \cong \Delta BCP$	S.S.S. ≅ S.S.S.
$\angle ACP \cong \angle BCP$ ......(i)	(corresponding angles of congruent triangles)
But $m\angle ACP + m\angle BCP = 180^\circ$ ..... (ii)	Supplementary angles
$\therefore$ $m\angle ACP = m\angle BCP = 90^\circ$	from (i) and (ii)
i.e., $\overline{PC} \perp \overline{AB}$ ...... (iii)	$mACP = 90^\circ$ (proved)
Also $\overline{CA} \cong \overline{CB}$ ...... (iv)	construction
$\therefore$ $\overline{PC}$ is a right bisector of $\overline{AB}$.	from (iii) and (iv)
i.e., the point P is on the right bisector of $\overline{AB}$.	

EXERCISE 12.1

1. Prove that the centre of a circle is on the right bisectors of each of its chords.
2. Where will be the centre of a circle passing through three non-collinear points? And why?
3. Three villages P, Q and R are not on the same line. The people of these villages want to make a Children Park at such a place which is equidistant from these three villages. After fixing the place of Children Park, prove that the Park is equidistant from the three villages.

Theorem 12.1.3

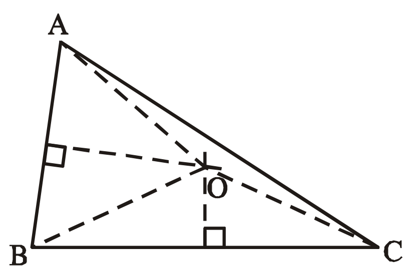
The right bisectors of the sides of a triangle are concurrent.

Given

ΔABC

To Prove

The right bisectors of $\overline{AB}$, BC and $\overline{CA}$ are concurrent.



Construction

Draw the right bisectors of $\overline{AB}$ and $\overline{BC}$ which meet each other at the point O. Join O to A, B and C.

Proof

Statements	Reasons
In $\overline{OA} \cong \overline{OB}$ ...... (i)	(Each point on right bisector of a segment is equidistant from its end points)
$\overline{OB} \cong \overline{OC}$ ...... (ii)	as in (i)
$\overline{OA} \cong \overline{OC}$ ...... (iii)	from (i) and (ii)
$\therefore$ Point O is on the right bisector of $\overline{CA}$. ...... (iv)	(O is equidistant from A and C)
But point O is on the right bisector of $\overline{AB}$ and of $\overline{BC}$ ...... (v)	construction
Hence the right bisectors of the three sides of a triangle are concurrent at O.	{from (iv) and (v)}

Observe that

(a) The right bisectors of the sides of an acute triangle intersect each other inside the triangle.
(b) The right bisectors of the sides of a right triangle intersect each other on the hypotenuse.
(c) The right bisectors of the sides of an obtuse triangle intersect each other outside the triangle.

Theorem 12.1.4

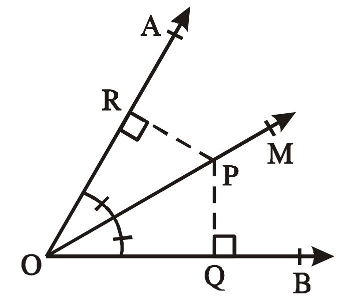
Any point on the bisector of an angle is equidistant from its arms.

Given

A point P is on $\overrightarrow{OM}$, the bisector of $\angle AOB$.

To Prove

$\overline{PQ} \cong \overline{PR}$ i.e., P is equidistant from $\overline{OA}$ and $\overrightarrow{OB}$.



Construction

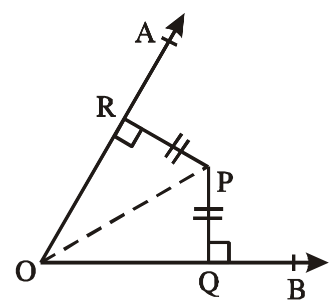
Draw $\overline{PR} \perp \overrightarrow{OA}$ and $\overline{PQ} \perp \overrightarrow{OB}$

Proof

Statements	Reasons
In $\Delta POQ \longleftrightarrow \Delta POR$	
$\overline{OP} \cong \overline{OP}$	common
$\angle PQO \cong \angle PRO$	construction
$\angle POQ \cong \angle POR$	given
$\therefore$ $\Delta POQ \cong \Delta POR$	S.A.A. $\cong$ S.A.A.
Hence $\overline{PQ} \cong \overline{PR}$	(corresponding sides of congruent triangles)

Theorem 12.1.5 (Converse of Theorem 12.1.4)

Any point inside an angle, equidistant from its arms, is on the bisector of it.



Given

Any point P lies inside $\angle AOB$ such that $\overline{PQ} \cong$ PR, where $\overline{PQ} \perp \overrightarrow{OB}$ and $\overline{PR} \perp \overrightarrow{OA}$.

To Prove

Point P is on the bisector of $\angle AOB$.

Construction

Join P to O.

Proof

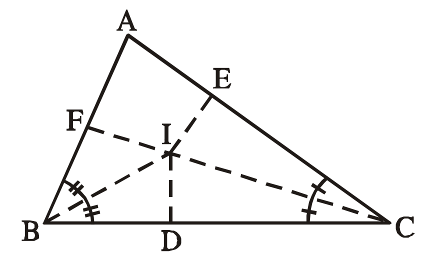
Statements	Reasons
In $\Delta POQ \longleftrightarrow \Delta POR$	
$\angle PQO \cong \angle PRO$	given (right angles)
$\overline{PO} \cong \overline{PO}$	common
$\overline{PQ} \cong \overline{PR}$	given
$\therefore$ $\Delta POQ \cong \Delta POR$	H.S. $\cong$ H.S.
Hence $\angle POQ \cong \angle POR$	(corresponding angels of congruent triangles)
i.e., P is on the bisector of $\angle AOB$.	

EXERCSISE 12.2

1. In a quadrilateral ABCD, $\overline{AB} \cong \overline{BC}$ and the right bisectors of $\overline{AD}$, $\overline{CD}$ meet each other at point N. Prove that $\overline{BN}$ is a bisector of $\angle ABC$.
2. The bisectors of $\angle A$, $\angle B$ and $\angle C$ of a quadrilateral ABCP meet each other at point O. Prove that the bisector of $\angle P$ will also pass through the point O.
3. Prove that the right bisectors of congruent sides of an isoscles triangle and its altitude are concurrent.
4. Prove that the altitudes of a triangle are concurrent.

Theorem 12.1.6

The bisectors of the angles of a triangle are concurrent.



Given

ΔABC

To Prove

The bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent.

Construction

Draw the bisectors of $\angle B$ and $\angle C$ which intersect at point I. From I, draw $\overline{IF} \perp \overline{AB}$, $\overline{ID} \perp \overline{BC}$ and $\overline{IE} \perp \overline{CA}$.

Proof

Statements	Reasons
$\overline{ID} \cong \overline{IF}$	(Any point on bisector of an angle is equidistant from its arms)
Similarly, $\overline{ID} \cong \overline{IE}$	
$\therefore$ $\overline{IE} \cong \overline{IF}$	Each ID, proved.
So, the point I is on the bisector of $\angle A$ ...... (i)	

Also the point *I* is on the bisectors of ∠ABC and ∠BCA ...... (ii)	Construction
Thus the bisectors of ∠A, ∠B and ∠C are concurrent at I.	{from (i) and (ii)}

Note. In practical geometry also, by constructing angle bisectors of a triangle, we shall verify that they are concurrent.

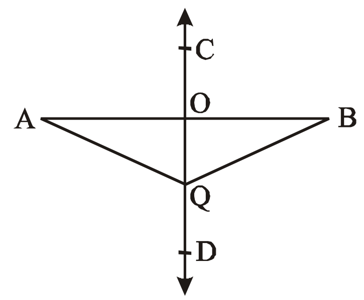
EXERCISE 12.3

1. Prove that the bisectors of the angles of base of an isoscles triangle intersect each other on its altitude.
2. Prove that the bisectors of two exterior and third interior angle of a triangle are concurrent.

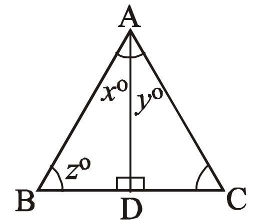
REVIEW EXERCISE 12

1. Which of the following are true and which are false?
   (i) Bisection means to divide into two equal parts. ......
   (ii) Right bisection of line segment means to draw perpendicular which passes through the mid point. ......
   (iii) Any point on the right bisector of a line segment is not equidistant from its end points. ......
   (iv) Any point equidistant from the end points of a line segment is on the right bisector of it. ......
   (v) The right bisectors of the sides of a triangle are not concurrent. ......
   (vi) The bisectors of the angles of a triangle are concurrent. ......
   (vii) Any point on the bisector of an angle is not equidistant from its arms. ......
   (viii) Any point inside an angle, equidistant from its arms, is on the bisector of it. ......

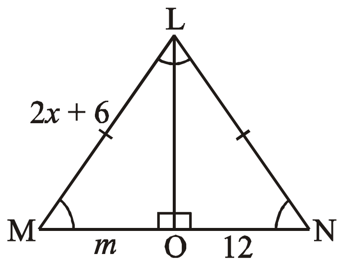
2. If $\overleftrightarrow{CD}$ is a right bisector of line segment $\overline{AB}$, then
   (i) m$\overline{OA}$ = ...
   (ii) m$\overline{AQ}$ = ...



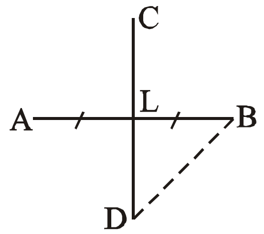
3. Define the following
   (i) Bisector of a line segment
   (ii) Bisector of an angle
4. The given triangle ABC is equilateral triangle and $\overline{AD}$ is bisector of angle A, then find the values of unknowns x^o, y^o and z^o.



5. In the given congruent triangles LMO and LNO, find the unknowns x and m.



6. $\overline{CD}$ is right bisector of the line segment AB.
   (i) If m$\overline{AB}$ = 6cm, then find the m$\overline{AL}$ and m$\overline{LB}$.
   (ii) If m$\overline{BD}$ = 4cm, then find m$\overline{AD}$.



SUMMARY

In this unit we stated and proved the following theorems:

- Any point on the right bisector of a line segment is equidistant from its end points.
- Any point equidistant from the end points of a line segment is on the right bisector of it.
- The right bisectors of the sides of a triangle are concurrent.
- Any point on the bisector of an angle is equidistant from its arms.
- Any point inside an angle, equidistant from its arms, is on the bisector of it.

- The bisectors of the angles of a triangle are concurrent.
- Right bisection of a line segment means to draw a perpendicular at the mid point of line segment.
- Bisection of an angle means to draw a ray to divide the given angle into two equal parts.