CHAPTER

14

RATIO AND PROPORTION

Animation 14.1: Ratio and Proportion
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Students Learning Outcomes

After studying this unit, the students will be able to:

- prove that a line parallel to one side of a triangle, intersecting the other two sides, divides them proportionally.
- prove that if a line segment intersects the two sides of a triangle in the same ratio, then it is parallel to the third side.
- prove that the internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angle.
- prove that if two triangles are similar, the measures of their corresponding sides are proportional

Introduction

In this unit we will prove some theorems and corollaries involving ratio and proportions of sides of triangle and similarity of triangles. A knowledge of ratio and proportion is necessary requirement of many occupations like food service occupation, medications in health, preparing maps for land survey and construction works, profit to cost ratios etc.

Recall that we defined ratio $a : b = \frac{a}{b}$ as the comparison of two alike quantities a and b, called the elements (terms) of a ratio. (Elements must be expressed in the same units). Equality of two ratios was defined as proportion.

That is, if $\boldsymbol{a : b = c : d}$, then $\boldsymbol{a, b, c}$ and $\boldsymbol{d}$ are said to be in proportion.

Similar Triangles

Equally important are the similar shapes. In particular the similar triangles that have many practical applications. For example, we know that a photographer can develop prints of different sizes from the same negative. In spite of the difference in sizes, these pictures look like each other. One photograph is simply an enlargement of another. They are said to be similar in shape. Geometrical figures can also be similar. e.g., If

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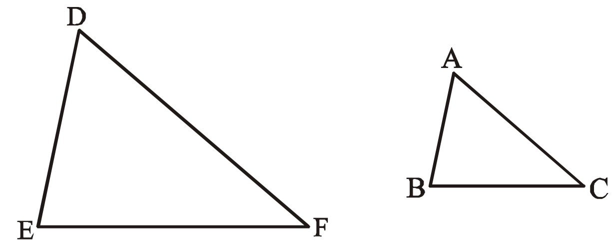
In $\quad \Delta ABC \longleftrightarrow \Delta DEF$

$\angle A \cong \angle D,\ \angle B \cong \angle E,\ \angle C \cong \angle F$, and $\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{BC}}{m\overline{EF}} = \frac{m\overline{CA}}{m\overline{FD}}$



then ΔABC and ΔDEF are called similar triangles which is symbolically written as

$\Delta ABC \sim \Delta DEF$

It means that corresponding angles of similar triangles are equal and measures of their corresponding sides are proportional.

$\Delta PQR \cong \Delta LMN$ means that in

$\Delta PQR \longleftrightarrow \Delta LMN$

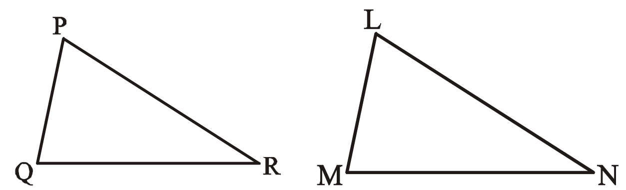
$\angle P \cong \angle L,\quad \angle Q \cong \angle M,$

$\angle R \cong \angle N,\quad \overline{PQ} \cong \overline{LM},$

$\overline{QR} \cong \overline{MN},\quad \overline{RP} \cong \overline{NL}$

Now as $\frac{m\overline{PQ}}{m\overline{LM}} = \frac{m\overline{QR}}{m\overline{MN}} = \frac{m\overline{RP}}{m\overline{NL}} = 1$

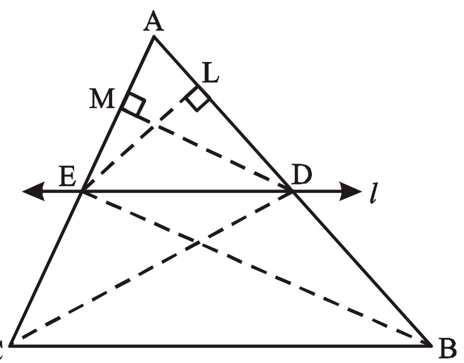
$\therefore \quad \Delta PQR \sim \Delta LMN$



In other words, two congruent triangles are similar also. But two similar triangles are not necessarily congruent, as congruence of their corresponding sides is not necessary.

Theorem 14.1.1

A line parallel to one side of a triangle and intersecting the other two sides divides them proportionally.



Given

In ΔABC, the line ℓ is intersecting the sides $\overline{AC}$ and $\overline{AB}$ at points E and D respectively such that $\overline{ED} \parallel \overline{CB}$.

To Prove

$m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$

Construction

Join B to E and C to D. From D draw $\overline{DM} \perp \overline{AC}$ and from E draw $\overline{EL} \perp \overline{AB}$.

Proof

Statements	Reasons
In triangles BED and AED, $\overline{EL}$ is the common perpendicular.	
$\therefore$ Area of $\Delta BED = \frac{1}{2} \times m\overline{BD} \times m\overline{EL}$ .... (i)	Area of a $\Delta = \frac{1}{2}$(base)(height)
and Area of $\Delta AED = \frac{1}{2} \times m\overline{AD} \times m\overline{EL}$ .... (ii)	
Thus $\frac{\text{Area of } \Delta BED}{\text{Area of } \Delta AED} = \frac{m\overline{BD}}{m\overline{AD}}$ .... (iii)	Dividing (i) by (ii)
Similarly	
$\frac{\text{Area of } \Delta CDE}{\text{Area of } \Delta ADE} = \frac{m\overline{EC}}{m\overline{AE}}$ .... (iv)	
But $\Delta BED \cong \Delta CDE$	(Areas of triangles with common base and same altitudes are equal). Given that $\overline{ED} \parallel m\overline{CB}$, so altitudes are equal.
$\therefore$ From (iii) and (iv), we have	
$\frac{m\overline{DB}}{m\overline{AD}} = \frac{m\overline{EC}}{m\overline{AE}}$ or $\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{AE}}{m\overline{EC}}$	Taking reciprocal of both sides.
Hence $m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$	

Observe that

From the above theorem we also have

$$\frac{m\overline{BD}}{m\overline{AB}} = \frac{m\overline{CE}}{m\overline{AC}} \text{ and } \frac{m\overline{AD}}{m\overline{AB}} = \frac{m\overline{AE}}{m\overline{AC}}$$

Corollaries

(a) If $\frac{m\overline{AD}}{m\overline{AB}} = \frac{m\overline{AE}}{m\overline{AC}}$, then $\overline{DE} \parallel \overline{BC}$ (b) If $\frac{m\overline{AB}}{m\overline{DB}} = \frac{m\overline{AC}}{m\overline{EC}}$, then $\overline{DE} \parallel \overline{BC}$

Points to be noted

(i) Two points determine a line and three non-collinear points determine a plane.

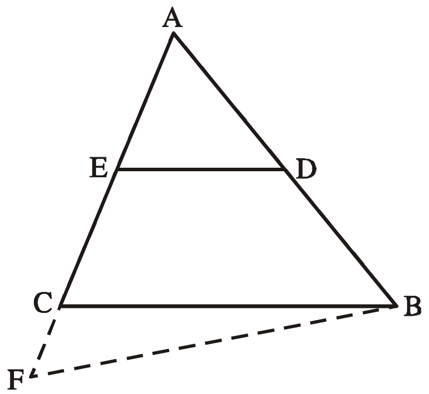
(ii) A line segment has exactly one midpoint.

(iii) If two intersecting lines form equal adjacent angles, the lines are perpendicular.

Theorem 14.1.2

(Converse of Theorem 14.1.1)

If a line segment intersects the two sides of a triangle in the same ratio, then it is parallel to the third side.



Given

In ΔABC, $\overline{ED}$ intersects $\overline{AB}$ and $\overline{AC}$ such that $m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$

To Prove

$\overline{ED} \parallel \overline{CB}$

Construction

If $\overline{ED} \nparallel \overline{CB}$, then draw $\overline{BF} \parallel \overline{DE}$ to meet $\overline{AC}$ produced at F.

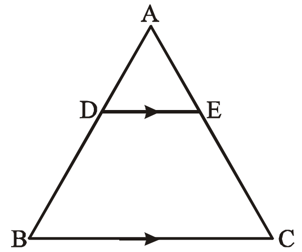
Proof

Statements	Reasons
In ΔABF	

$\overline{DE} \parallel \overline{BF}$	Construction
∴ $\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{AE}}{m\overline{EF}}$ ....(i)	(A line parallel to one side of a triangle divides the other two sides proportionally Theorem 14.1.1)
But $\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{AE}}{m\overline{EC}}$ ....(ii)	Given
∴ $\frac{m\overline{AE}}{m\overline{EF}} = \frac{m\overline{AE}}{m\overline{EC}}$	From (i) and (ii)
or $m\overline{EF} = m\overline{EC}$, which is possible only if point F is coincident with C.	(Property of real numbers.)
∴ Our supposition is wrong Hence $\overline{ED} \parallel \overline{CB}$	

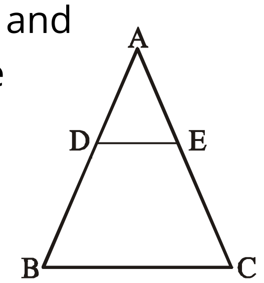
EXERCISE 14.1

1. In ΔABC, $\overline{DE} \parallel \overline{BC}$.

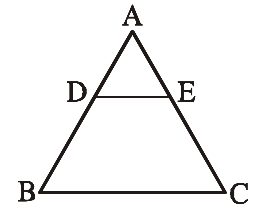


   (i) If $m\overline{AD} = 1.5cm$, $m\overline{BD} = 3cm$, $m\overline{AE} = 1.3cm$, then find $m\overline{CE}$.
   (ii) If $m\overline{AD} = 2.4cm$, $m\overline{AE} = 3.2cm$, $m\overline{EC} = 4.8cm$, find $m\overline{AB}$.
   (iii) If $\frac{m\overline{AD}}{m\overline{DB}} = \frac{3}{5}$ and $m\overline{AC} = 4.8cm$, find $m\overline{AE}$.
   (iv) If $m\overline{AD} = 2.4cm$, $m\overline{AE} = 3.2cm$, $m\overline{DE} = 2cm$, $m\overline{BC} = 5cm$, find $m\overline{AB}$, $m\overline{DB}$, $m\overline{AC}$, $m\overline{CE}$.
   (v) If $\overline{AD} = 4x - 3$, $\overline{AE} = 8x - 7$, $\overline{BD} = 3x - 1$, and $\overline{CE} = 5x - 3$, find the value of x.

2. If ΔABC is an isosceles triangle, $\angle A$ is vertex angle and $\overline{DE}$ intersects the sides $\overline{AB}$ and $\overline{AC}$ as shown in the figure so that
   $m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$.
   Prove that ΔADE is also an isosceles triangle.



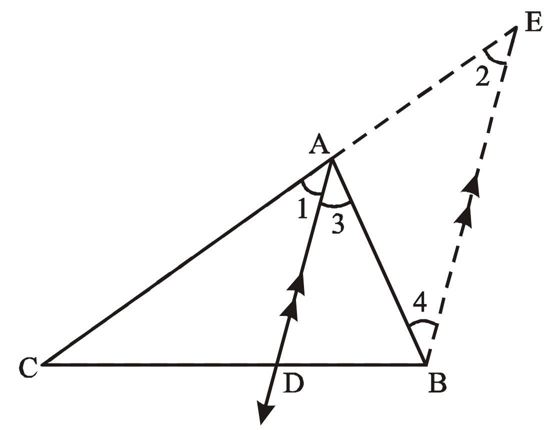
3. In an equilateral triangle ABC shown in the figure, $m\overline{AE} : m\overline{AC} = m\overline{AD} : m\overline{AB}$
   Find all the three angles of ΔADE and name it also.



4. Prove that the line segment drawn through the mid-point of one side of a triangle and parallel to another side bisects the third side.
5. Prove that the line segment joining the mid-points of any two sides of a triangle is parallel to the third side.

Theorem 14.1.3

The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angle.



Given

In ΔABC internal angle bisector of $\angle A$ meets $\overline{CB}$ at the point D.

To Prove

$m\overline{BD} : m\overline{DC} = m\overline{AB} : m\overline{AC}$

Construction

Draw a line segment $\overline{BE} \parallel \overline{DA}$ to meet $\overline{CA}$ produced at E.

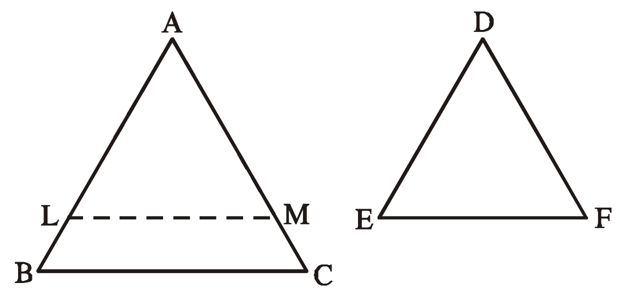
Proof

Statements	Reasons
∵ $\overline{AD} \parallel \overline{EB}$ and EC intersects them,	Construction
∴ $m\angle 1 = m\angle 2$ ......(i)	Corresponding angles
Again $\overline{AD} \parallel \overline{EB}$	

and $\overline{AB}$ intersects them.	
$\therefore$ $m\angle 3 = m\angle 4$ ....(ii)	Alternate angles
But $m\angle 1 = m\angle 3$	Given
$\therefore$ $m\angle 2 = m\angle 4$	From (i) and (ii)
and $\overline{AB} \cong \overline{AE}$ or $\overline{AE} \cong \overline{AB}$	In a Δ, the sides opposite to congruent angles are also congruent.
Now $\overline{AD} \parallel \overline{EB}$	Construction
$\therefore$ $\frac{m\overline{BD}}{m\overline{DC}} = \frac{m\overline{EA}}{m\overline{AC}}$	by Theorem 14.1.1
or $\frac{m\overline{BD}}{m\overline{DC}} = \frac{m\overline{AE}}{m\overline{AC}}$	$m\overline{EA} = m\overline{AB}$ (proved)
Thus $m\overline{BD} : m\overline{DC} = m\overline{AB} : \overline{AC}$	

Theorem 14.1.4

If two triangles are similar, then the measures of their corresponding sides are proportional.



Given

$\Delta ABC \sim \Delta DEF$

i.e., $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$

To Prove

$$\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{AC}}{m\overline{DF}} = \frac{m\overline{BC}}{m\overline{EF}}$$

Construction

(i) Suppose that $m\overline{AB} > m\overline{DE}$

(ii) $m\overline{AB} \leq m\overline{DE}$

On $\overline{AB}$ take a point L such that $m\overline{AL} = m\overline{DE}$.

On $\overline{AC}$ take a point M such that $m\overline{AM} = m\overline{DF}$. Join L and M by the line segment $\overline{LM}$.

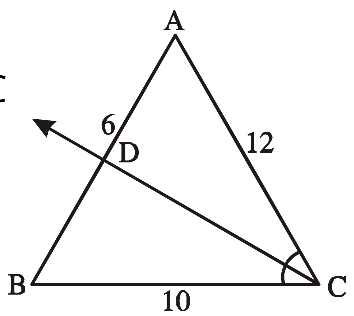
Proof

Statements	Reasons
(i) In $\Delta ALM \leftrightarrow \Delta DEF$	
$\angle A \cong \angle D$	Given
$\overline{AL} \cong \overline{DE}$	Construction
$\overline{AM} \cong \overline{DF}$	Construction
Thus $\Delta ALM \cong \Delta DEF$	S.A.S. Postulate
and $\angle L \cong \angle E$, $\angle M \cong \angle F$	(Corresponding angles of congruent triangles)
Now $\angle E \cong \angle B$ and $\angle F \cong \angle C$	Given
$\therefore$ $\angle L \cong \angle B$, $\angle M \cong \angle C$	Transitivity of congruence
Thus $\overline{LM} \parallel \overline{BC}$	Corresponding angles are equal.
Hence $\frac{m\overline{AL}}{m\overline{AB}} = \frac{m\overline{AM}}{m\overline{AC}}$	by Theorem 14.1.1
or $\frac{m\overline{DE}}{m\overline{AB}} = \frac{m\overline{DF}}{m\overline{AC}}$ ....(i)	$m\overline{AL} = m\overline{DE}$ and $m\overline{AM} = m\overline{DF}$ (construction)
Similarly by intercepting segments on BA and BC, we can prove that	
$\frac{m\overline{DE}}{m\overline{AB}} = \frac{m\overline{EF}}{m\overline{BC}}$ .....(ii)	
Thus $\frac{m\overline{DE}}{m\overline{AB}} = \frac{m\overline{DF}}{m\overline{AC}} = \frac{m\overline{EF}}{m\overline{BC}}$	by (i) and (ii)
or $\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{AC}}{m\overline{DF}} = \frac{m\overline{BC}}{m\overline{EF}}$	by taking reciprocals
(ii) If $m\overline{AB} < m\overline{DE}$, it can	

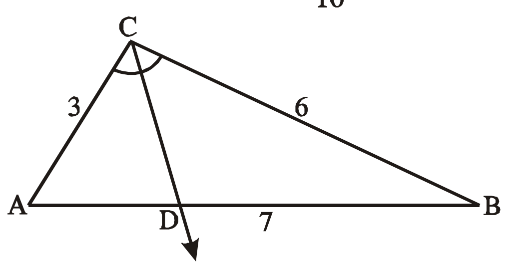
similarly be proved by taking intercepts on the sides of ΔDEF.	
If $m\overline{AB} = m\overline{DE}$,	
then $\Delta ABC \leftrightarrow \Delta DEF$	
$\angle A \cong \angle D$	Given
$\angle B \cong \angle E$	Given
and $\overline{AB} \cong \overline{DE}$	
so $\Delta ABC \cong \Delta DEF$	A.S.A. ≅ A.S.A.
Thus $\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{AC}}{m\overline{DF}} = \frac{m\overline{BC}}{m\overline{EF}} = 1$	$\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$
Hence the result is true for all cases.	

EXERCISE 14.2

1. In ΔABC as shown in the figure, $\overrightarrow{CD}$ bisects $\angle C$ and meets $\overline{AB}$ at D. $m\overline{BD}$ is equal to
   (a) 5 (b) 16 (c) 10 (d) 18



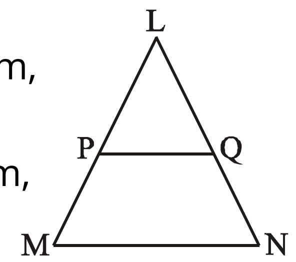
2. In ΔABC shown in the figure, $\overrightarrow{CD}$ bisects $\angle C$. If $m\overline{AC} = 3$, $m\overline{CB} = 6$ and $m\overline{AB} = 7$, then find $m\overline{AD}$ and $m\overline{DB}$.



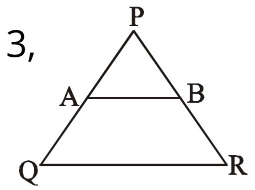
3. Show that in any correspondence of two triangles, if two angles of one triangle are congruent to the corresponding angles of the other, then the triangles are similar.
4. If line segments AB and CD are intersecting at point X and $\frac{m\overline{AX}}{m\overline{XB}} = \frac{m\overline{CX}}{m\overline{XD}}$ then show that ΔAXC and ΔBXD are similar.

REVIEW EXERCISE 14

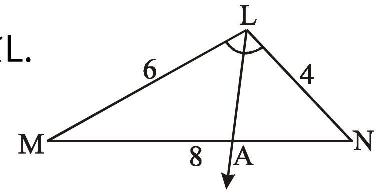
1. Which of the following are true and which are false?
   (i) Congruent triangles are of same size and shape. ......
   (ii) Similar triangles are of same shape but different sizes. .....
   (iii) Symbol used for congruent is '~'. ......
   (iv) Symbol used for similarity is '≅'. ......
   (v) Congruent triangles are similar. ......
   (vi) Similar triangles are congruent. ......
   (vii) A line segment has only one mid point. ......
   (viii) One and only one line can be drawn through two points. ......
   (ix) Proportion is non-equality of two ratios. ......
   (x) Ratio has no unit. ......
2. Define the following:
   (i) Ratio (ii) Proportion
   (iii) Congruent Triangles (iv) Similar Triangles
3. In ΔLMN shown in the figure, $\overline{MN} \parallel \overline{PQ}$
   (i) If $m\overline{LM} = 5cm$, $m\overline{LP} = 2.5cm$, $m\overline{LQ} = 2.3cm$, then find $m\overline{LN}$.
   (ii) If $m\overline{LM} = 6cm$, $m\overline{LQ} = 2.5cm$, $m\overline{QN} = 5cm$, then find $m\overline{LP}$.



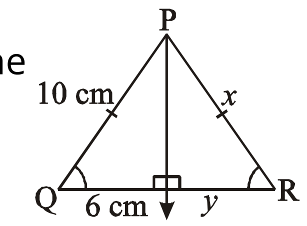
4. In the shown figure, let $m\overline{PA} = 8x - 7$, $m\overline{PB} = 4x - 3$, $m\overline{AQ} = 5x - 3$, $m\overline{BR} = 3x - 1$. Find the value of x if $\overline{AB} \parallel \overline{QR}$.



5. In ΔLMN shown in the figure, $\overrightarrow{LA}$ bisects $\angle L$. If $m\overline{LN} = 4$, $m\overline{LM} = 6$, $m\overline{MN} = 8$, then find $m\overline{MA}$ and $m\overline{AN}$.



6. In isosceles ΔPQR shown in the figure, find the value of x and y.



SUMMARY

In this unit we stated and proved the following theorems and gave some necessary definitions:

- A line parallel to one side of a triangle and intersecting the other two sides divides them proportionally.
- If a line segment intersects the two sides of a triangle in the same ratio, then it is parallel to the third side.
- The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angle.
- If two triangles are similar, then the measures of their corresponding sides are proportional.
- The ratio between two alike quantities is defined as $a : b = \frac{a}{b}$, where a and b are the elements of the ratio.
- Proportion is defined as the equality of two ratios i.e., $a : b = c : d$.
- Two triangles are said to be similar if they are equiangular and corresponding sides are proportional.