CHAPTER 15

PYTHAGORAS' THEOREM

Animation 15.1: Pythagoras-2a
Source & Credit: wikipedia

Students Learning Outcomes

After studying this unit, the students will be able to:

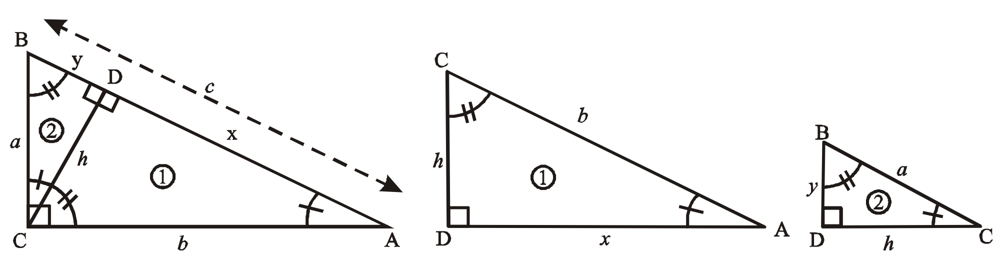
- prove that in a right-angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides. (Pythagoras' theorem).
- prove that if the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right angled triangle (converse to Pythagoras' theorem).

Introduction

Pythagoras, a Greek philosopher and mathematician discovered the simple but important relationship between the sides of a right-angled triangle. He formulated this relationship in the form of a theorem called Pythagoras' Theorem after his name. There are various methods of proving this theorem. We shall prove it by using similar triangles. We shall state and prove its converse also and then apply them to solve different problems.

Pythagoras Theorem 15.1.1

In a right angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides.



(i) (ii)-a (ii)-b

Given

ΔACB is a right angled triangle in which $mC = 90^0$ and $m\overline{BC} = a$, $m\overline{AC} = b$ and $m\overline{AB} = c$.

To Prove

$c^2 = a^2 + b^2$

Construction

Draw $\overline{CD}$ perpendicular from C on $\overline{AB}$.

Let $m\overline{CD} = h$, $m\overline{AD} = x$ and $m\overline{BD} = y$. Line segment CD splits ΔABC into two Δs ADC and BDC which are separately shown in the figures (ii) –a and (ii) –b respectively.

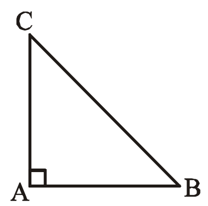
Proof (Using similar Δs)

Statements	Reasons
In $\Delta ADC \leftrightarrow \Delta ACB$	Refer to figure (ii) -a and (i)
$\angle A \cong \angle A$	
$\angle ADC \cong \angle ACB$	Construction – given, each angle = 90^0
$\angle C \cong \angle B$	$\angle C$ and $\angle B$, complements of $\angle A$
$\therefore$ $\Delta ADC \sim \Delta ACB$	Congruency of three angles
$\therefore$ $\frac{x}{b} = \frac{b}{c}$	(Measures of corresponding sides of similar triangles are proportional)
or $x = \frac{b^2}{c}$ .....(I)	
Again in $\Delta BDC \leftrightarrow \Delta BCA$	Refer to figure (ii)-b and (i)
$\angle B \cong \angle B$	Common - self congruent
$\angle BDC \cong \angle BCA$	Construction – given, each angle = 90^0
$\angle C \cong \angle A$	$\angle C$ and $\angle A$, complements of $\angle B$
$\therefore$ $\Delta BDC \sim \Delta BCA$	Congruency of three angles
$\therefore$ $\frac{y}{a} = \frac{a}{c}$	(Corresponding sides of similar triangles are proportional)
$\therefore$ $y = \frac{a^2}{c}$ .....(II)	
But $y + x = c$	Supposition.
$\therefore$ $\frac{a^2}{c} + \frac{b^2}{c} = c$	By (I) and (II)
or $a^2 + b^2 = c^2$ i.e., $c^2 = a^2 + b^2$	Multiplying both sides by c.

Corollary

In a right angled ΔABC, right angle at A,

(i) $\overline{AB}^2 = \overline{BC}^2 - \overline{CA}^2$
(ii) $\overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2$

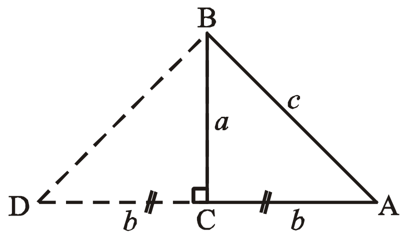


Remark

Pythagoras' Theorem has many proofs. The one we have given is based on the proportionality of the sides of two similar triangles. For convenience Δs ADC and CDB have been shown separately. Otherwise, the theorem is usually proved using figure (i) only.

Theorem 15.1.2 [Converse of Pythagoras' Theorem 15.1.1]

If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right angled triangle.



Given

In a ΔABC, $m\overline{AB} = c$, $m\overline{BC} = a$ and $m\overline{AC} = b$ such that $a^2 + b^2 = c^2$.

To Prove

ΔACB is a right angled triangle.

Construction

Draw $\overline{CD}$ perpendicular to $\overline{BC}$ such that $\overline{CD} \cong \overline{CA}$. Join the points B and D.

Proof

Statements	Reasons
ΔDCB is a right-angled triangle.	Construction
$\therefore$ $(m\overline{BD})^2 = a^2 + b^2$	Pythagoras theorem
But $a^2 + b^2 = c^2$	Given
$\therefore$ $(m\overline{BD})^2 = c^2$	
or $m\overline{BD} = c$	Taking square root of both sides.

Now in	
$\Delta DCB \leftrightarrow \Delta ACB$	
$\overline{CA} \cong \overline{CA}$	Construction
$\overline{BC} \cong \overline{BC}$	Common
$\overline{DB} \cong \overline{AB}$	Each side = c.
$\therefore$ $\Delta DCB \cong \Delta ACB$	S.S.S. $\cong$ S.S.S.
$\therefore$ $\angle DCB \cong \angle ACB$	(Corresponding angles of congruent triangles)
But $m\angle DCB = 90^0$	Construction
$\therefore$ $\angle ACB = 90^0$	
Hence the ΔACD is a right-angled triangle.	

Corollaries

Let c be the longest of the sides *a, b* and *c* of a triangle.

* If $a^2 + b^2 = c^2$, then the triangle is right.
* If $a^2 + b^2 > c^2$, then the triangle is acute.
* If $a^2 + b^2 < c^2$, then the triangle is obtuse.

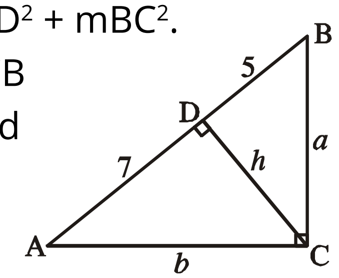
EXERCISE 15

1. Verify that the Δs having the following measures of sides are right - angled.
   (i) $a = 5$ cm, $b = 12$ cm, $c = 13$ cm
   (ii) $a = 1.5$ cm, $b = 2$ cm, $c = 2.5$ cm
   (iii) $a = 9$ cm, $b = 12$ cm, $c = 15$ cm
   (iv) $a = 16$ cm, $b = 30$ cm, $c = 34$ cm
2. Verify that $a^2 + b^2$, $a^2 - b^2$ and 2ab are the measures of the sides of a right angled triangle where *a* and *b* are any two real numbers ($a > b$).
3. The three sides of a triangle are of measure 8, *x* and 17 respectively. For what value of *x* will it become base of a right angled triangle?
4. In a isosceles Δ, the base $m\overline{BC} = 28$cm, and $m\overline{AB} = m\overline{AC} = 50$cm. If $m\overline{AD} \perp m\overline{BC}$, then find

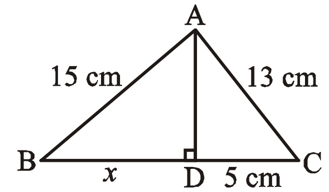
(i) length of $\overline{AD}$ (ii) area of ΔABC

5. In a quadrilateral ABCD, the diagonals AC and BD are perpendicular to each other. Prove that $m\overline{AB}^2 + m\overline{CD}^2 = m\overline{AD}^2 + m\overline{BC}^2$.

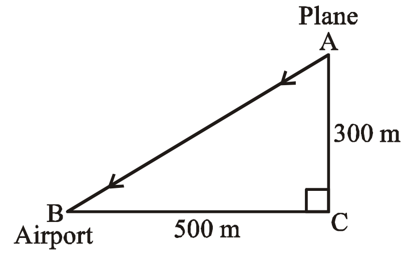
6. (i) In the ΔABC as shown in the figure, $m\angle ACB = 90^0$ and $\overline{CD} \perp \overline{AE}$ Find the lengths a, h and b if $m\overline{BD} = 5$ units and $m\overline{AD} = 7$ units.



(ii) Find the value of x in the shown figure.

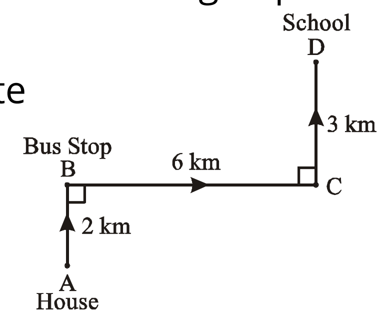


7. A plane is at a height of 300 m and is 500 m away from the airport as shown in the figure. How much distance will it travel to land at the airport?



8. A ladder 17 m long rests against a vertical wall. The foot of the ladder is 8 m away from the base of the wall. How high up the wall will the ladder reach?

9. A student travels to his school by the route as shown in the figure. Find $m\overline{AD}$, the direct distance from his house to school.



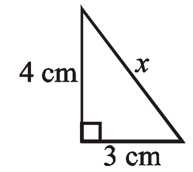
REVIEW EXERCISE 15

1. Which of the following are true and which are false?
   (i) In a right angled triangle greater angle is of 90^0. ......
   (ii) In a right angled triangle right angle is of 60^0. ......
   (iii) In a right triangle hypotenuse is a side opposite to right angle. ......
   (iv) If a, b, c are sides of right angled triangle with c as longer side, then $c^2 = a^2 + b^2$......
   (v) If 3 cm and 4 cm are two sides of a right angled triangle, then hypotenuse is 5 cm. ......

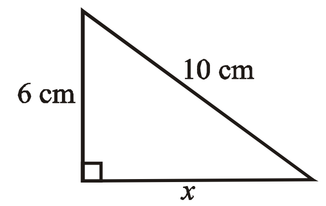
   (vi) If hypotenuse of an isosceles right triangle is $\sqrt{2}$ cm, then each of other side is of length 2 cm. ......

2. Find the unknown value in each of the following figures.

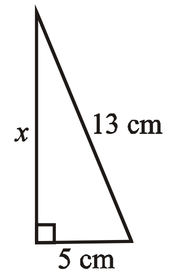
(i)



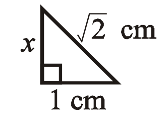
(ii)



(iii)



(iv)



SUMMARY

In this unit we learned to state and prove Pythagoras' Theorem and its converse with corollaries.

- In a right angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides.
- If the square of one side of a triangle is equal to the sum of the squares of the other two sides then the triangle is a right angled triangle.

Moreover, these theorems were applied to solve some questions of practical use.