CHAPTER

16

THEOREMS RELATED WITH AREA

Animation 16.1: mirandamolina
Source & Credit: The Math Kid

Students Learning Outcomes

After studying this unit, the students will be able to:

- Prove that parallelograms on the same base and lying between the same parallel lines (or of the same altitude) are equal in area.
- Prove that parallelograms on equal bases and having the same altitude are equal in area.
- Prove that triangles on the same base and of the same altitude are equal in area.
- Prove that triangles on equal bases and of the same altitude are equal in area.

Introduction

In this unit we will state and prove some important theorems related with area of parallelograms and triangles along with corollaries. We shall apply them to solve appropriate problems and to prove some useful results.

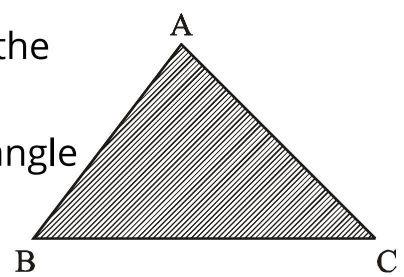
Some Preliminaries

Area of a Figure

The region enclosed by the bounding lines of a closed figure is called the area of the figure.

The area of a closed region is expressed in square units (say, sq. m or m^2) i.e. a positive real number.

Triangular Region



The interior of a triangle is the part of the plane enclosed by the triangle.

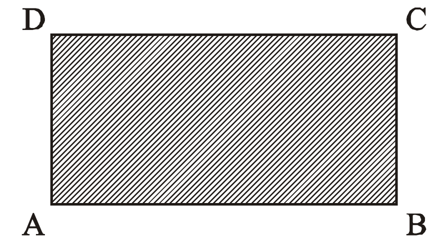
A triangular region is the union of a triangle and its interior i.e., the three line segments forming the triangle and its interior.

By area of a triangle, we mean the area of its triangular region.

Congruent Area Axiom

If $\Delta ABC \cong \Delta PQR$, then area of (region ΔABC) = area of (region ΔPQR)

Rectangular Region



The interior of a rectangle is the part of the plane enclosed by the rectangle.

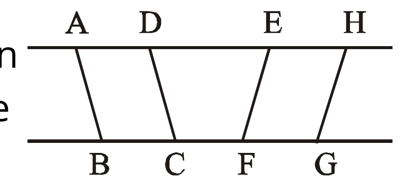
A rectangular region is the union of a rectangle and its interior.

A rectangular region can be divided into two or more than two triangular regions in many ways.

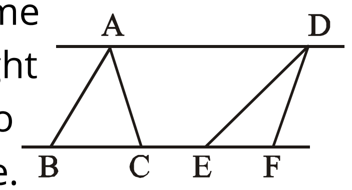
Recall that if the length and width of a rectangle are ***a*** units and ***b*** units respectively, then the area of the **rectangle** is equal to $a \times b$ square units.

If a is the side of a square, its area = a^2 square units.

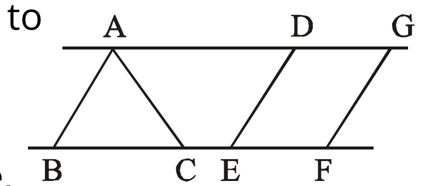
Between the same Parallels



Two parallelograms are said to be between the same parallels, when their bases are in the same straight line and their sides opposite to these bases are also in a straight line; as the parallelograms ABCD, EFGH in the given figure.



Two triangles are said to be between the same parallels, when their bases are in the same straight line and the line joining their vertices is parallel to their bases; as the Δs ABC, DEF in the given figure.



A triangle and a parallelogram are said to be between the same parallels, when their bases are in the same straight line, and the side of the parallelogram opposite the base, produced if necessary, passes through the vertex of the triangle as are the ΔABC and the parallelogram DEFG in the given figure.

Definition

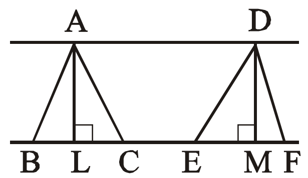
If one side of a parallelogram is taken as its base, the perpendicular distance between that side and the side parallel to it, is called the Altitude or Height of the parallelogram.

Definition

If one side of a triangle is taken as its base, the perpendicular to that side, from the opposite vertex is called the Altitude or Height of the triangle.

Useful Result

Triangles or parallelograms placed between the same or equal parallels will have the same or equal altitudes or heights.



Place the triangles ABC, DEF so that their bases $\overline{BC}$, $\overline{EF}$ are in the same straight line and the vertices on the same side of it, and suppose $\overline{AL}$, $\overline{DM}$ are the equal altitudes. We have to show that $\overline{AD}$ is parallel to BCEF.

Proof

$\overline{AL}$ and $\overline{DM}$ are parallel, for they are both perpendicular to $\overline{BF}$. Also $m\overline{AL} = m\overline{DM}$. (given)

$\therefore$ $\overline{AD}$ is parallel to $\overline{LM}$.

A similar proof may be given in the case of parallelograms.

Useful Result

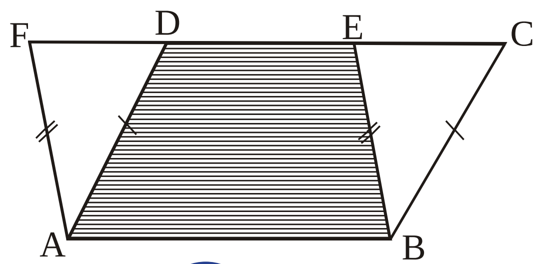
A diagonal of a parallelogram divides it into two congruent triangles (S.S.S.) and hence of equal area.

Theorem 16.1.1

Parallelograms on the same base and between the same parallel lines (or of the same altitude) are equal in area.

Given

Two parallelograms ABCD and ABEF having the same base $\overline{AB}$ and between the same parallel lines $\overline{AB}$ and $\overline{DE}$.



To Prove

area of parallelogram ABCD = area of parallelogram ABEF

Proof

Statements	Reasons
area of (parallelogram ABCD)	
= area of (quad. ABED) + area of (ΔCBE) ... (1)	[Area addition axiom]
area of (parallelogram ABEF)	
= area of (quad. ABED) + area of (ΔDAF) ... (2)	[Area addition axiom]
In Δs CBE and DAF	
$m\overline{CB} = m\overline{DA}$	[opposite sides of a parallelogram]
$m\overline{BE} = m\overline{AF}$	[opposite sides of a parallelogram]
$m\angle CBE = m\angle DAF$	[$\because$ $\overline{BC} \parallel \overline{AD}$, $\overline{BE} \parallel \overline{AF}$]
$\therefore$ $\Delta CBE \cong \Delta DAF$	[S.A.S. cong. axiom]
$\therefore$ area of (ΔCBE) = area of (ΔDAF) ...(3)	[cong. area axiom]
Hence area of (parallelogram ABCD)	
= area of (parallelogram ABEF)	from (1), (2) and (3)

Corollary

(i) The area of a parallelogram is equal to that of a rectangle on the same base and having the same altitude.

(ii) Hence area of parallelogram = base x altitude

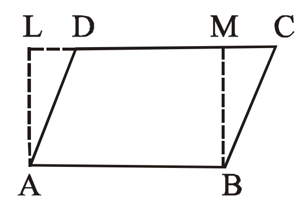
Proof

Let ABCD be a parallelogram. $\overline{AL}$ is an altitude corresponding to side $\overline{AB}$.

(i) Since parallelogram ABCD and rectangle ALMB are on the same base $\overline{AB}$ and between the same parallels,

$\therefore$ by above theorem it follows that

area of (parallelogram ABCD) = area of (rect. ALMB)

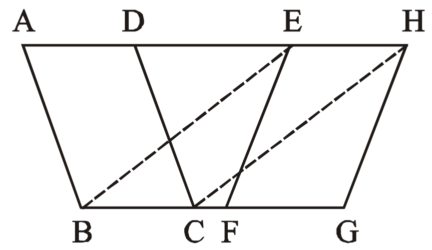


(ii) But area of (rect. ALMB) = $\overline{AB} \times \overline{AL}$

Hence area of (parallelogram ABCD) = $\overline{AB} \times \overline{AL}$.

Theorem 16.1.2

Parallelograms on equal bases and having the same (or equal) altitude are equal in area.



Given

Parallelograms ABCD, EFGH are on the equal bases $\overline{BC}$, $\overline{FG}$, having equal altitudes.

To Prove

area of (parallelogram ABCD) = area of (parallelogram EFGH)

Construction

Place the parallelograms ABCD and EFGH so that their equal bases $\overline{BC}$, $\overline{FG}$ are in the straight line BCFG. Join $\overline{BE}$ and $\overline{CH}$.

Proof

Statements	Reasons
The given $\|^{gms}$ ABCD and EFGH are between the same parallels	Their altitudes are equal (given)
Hence ADEH is a straight line $\parallel \overline{BC}$	
$\therefore$ $m\overline{BC} = m\overline{FG}$	Given
$= m\overline{EH}$	EFGH is a parallelogram
Now $m\overline{BC} = m\overline{EH}$ and they are $\parallel$	
$\therefore$ $\overline{BE}$ and $\overline{CH}$ are both equal and $\parallel$	
Hence EBCH is a parallelogram	A quadrilateral with two opposite sides congruent and parallel is a parallelogram
Now Area of $\|^{gm}$ ABCD = Area of $\|^{gm}$ EBCH ....(i)	Being on the same base $\overline{BC}$ and between the same parallels
But Area of $\|^{gm}$ EBCH = Area of $\|^{gm}$ EFGH .....(ii)	Being on the same base $\overline{EH}$ and between the

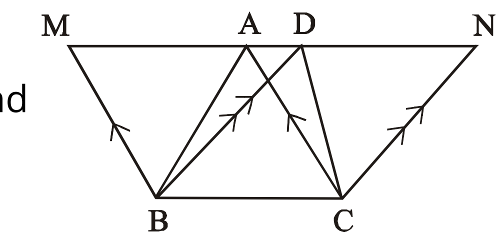
	same parallels
Hence area ($\|^{gm}$ ABCD) = area ($\|^{gm}$ EFGH)	From (i) and (ii)

EXERCISE 16.1

1. Show that the line segment joining the mid-points of opposite sides of a parallelogram, divides it into two equal parallelograms.
2. In a parallelogram ABCD, $m\overline{AB}$ = 10 cm. The altitudes corresponding to sides AB and AD are respectively 7 cm and 8 cm. Find $\overline{AD}$.
3. If two parallelograms of equal areas have the same or equal bases, their altitudes are equal.

Theorem 16.1.3

Triangles on the same base and of the same (i.e. equal) altitudes are equal in area.



Given

Δs ABC, DBC on the same base $\overline{BC}$, and having equal altitudes.

To Prove

area of (ΔABC) = area of (ΔDBC)

Construction

Draw $\overline{BM} \parallel$ to $\overline{CA}$, $\overline{CN} \parallel$ to $\overline{BD}$ meeting $\overline{AD}$ produced in M, N.

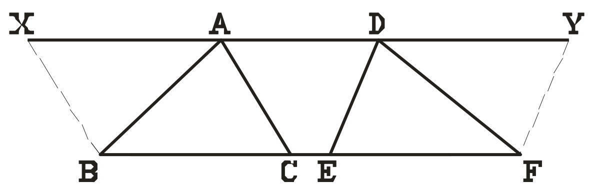
Proof

Statements	Reasons
Δ ABC and Δ DBC are between the same $\|^{s}$	Their altitudes are equal
Hence $\overline{MADN}$ is parallel to $\overline{BC}$	
$\therefore$ Area ($\|^{gm}$ BCAM) = Area ($\|^{gm}$ BCND) ....(i)	These $\|^{gm}$ are on the same base $\overline{BC}$ and between the same $\|^{s}$
But Area of $\Delta ABC = \frac{1}{2}$(Area of $\|^{gm}$ BCAM) ....(ii)	Each diagonal of a $\|^{gm}$ bisects it into two

and Area of $\Delta DBC = \frac{1}{2}$(Area of ($\|^{gm}$ BCND) ....(iii)	congruent triangles
Hence Area (Δ ABC) = Area (Δ DBC)	From (i), (ii) and (iii)

Theorem 16.1.4

Triangles on equal bases and of equal altitudes are equal in area.



Given

Δs ABC, DEF on equal base $\overline{BC}$, $\overline{EF}$ and having altitudes equal.

To Prove

Area of (Δ ABC) = Area of (Δ DEF)

Construction

Place the Δs ABC and DEF so that their equal bases $\overline{BC}$ and $\overline{EF}$ are in the same straight line BCEF and their vertices on the same side of it. Draw $\overline{BX} \parallel \overline{CA}$ and $\overline{FY} \parallel$ ED meeting $\overline{AD}$ produced in X, Y respectively.

Proof

Statements	Reasons
ΔABC, ΔDEF are between the same parallels	Their altitudes are equal (given)
$\therefore$ XADY is $\parallel$ to BCEF	
$\therefore$ area ($\|^{gm}$ BCAX) = area ($\|^{gm}$ EFYD) ......(i)	These $\|^{gm}$ are on equal bases and between the same parallels
But Area of $\Delta ABC = \frac{1}{2}$ Area of ($\|^{gm}$ BCAX) ......(ii)	Diagonal of a $\|^{gm}$ bisects it

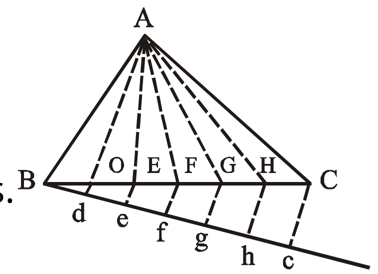
and area of $\Delta DFE = \frac{1}{2}$ area of ($\|^{gm}$ EFYD) ......(iii)	
$\therefore$ area (ΔABC) = area (ΔDEF)	From (i), (ii) and (iii)

Corollaries

1. Triangles on equal bases and between the same parallels are equal in area.
2. Triangles having a common vertex and equal bases in the same straight line, are equal in area.

EXERCISE 16.2

1. Show that a median of a triangle divides it into two triangles of equal area.
2. Prove that a parallelogram is divided by its diagonals into four triangles of equal area.
3. Divide a triangle into six equal triangular parts.

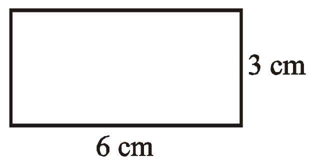


REVIEW EXERCISE 16

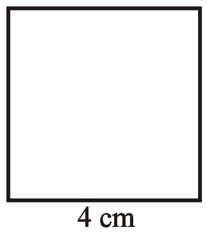
1. Which of the following are true and which are false?
   (i) Area of a figure means region enclosed by bounding lines of closed figure.
   (ii) Similar figures have same area.
   (iii) Congruent figures have same area.
   (iv) A diagonal of a parallelogram divides it into two non-congruent triangles.
   (v) Altitude of a triangle means perpendicular from vertex to the opposite side (base).
   (vi) Area of a parallelogram is equal to the product of base and height.

2. Find the area of the following.

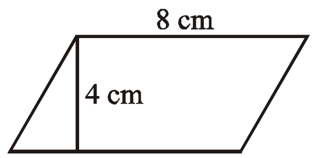
(i)



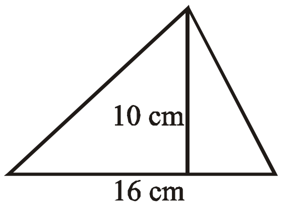
(ii)



(iii)



(iv)



3. Define the following
   - (i) Area of a figure
   - (ii) Triangular Region
   - (iii) Rectangular Region
   - (iv) Altitude or Height of a triangle

SUMMARY

In this unit we mentioned some necessary preliminaries, stated and proved the following theorems alongwith corollaries, if any.

- Area of a figure means region enclosed by the boundary lines of a closed figure.
- A triangular region means the union of triangle and its interior.
- By area of triangle means the area of its triangular region
- Altitude or height of a triangle means perpendicular distance to base from its opposite vertex.
- Parallelograms on the same base and between the same parallel lines (or of the same altitude) are equal in area.
- Parallelograms on equal bases and having the same (or equal) altitude are equal in area.
- Triangles on the same base and of the same (i.e. equal) altitudes are equal in area.
- Triangles on equal bases and of equal altitudes are equal in area.