Unit-2

THEORY OF QUADRATIC EQUATIONS

In this unit, students will learn how to

- *define discriminant ($b^2 - 4ac$) of the quadratic expression $ax^2 + bx + c$.*
- *find discriminant of a given quadratic equation.*
- *discuss the nature of roots of a quadratic equation through discriminant.*
- *determine the nature of roots of a given quadratic equation and verify the result by solving the equation.*
- *determine the value of an unknown involved in a given quadratic equation when the nature of its roots is given.*
- *find cube roots of unity.*
- *recognize complex cube roots of unity as ω and ω^2.*
- *prove the properties of cube roots of unity.*
- *use properties of cube roots of unity to solve appropriate problems.*
- *find the relation between the roots and the coefficients of a quadratic equation.*
- *find the sum and product of roots of a given quadratic equation without solving it.*
- *find the value(s) of unknown(s) involved in a given quadratic equation when*
    - *sum of roots is equal to a multiple of the product of roots,*
    - *sum of the squares of roots is equal to a given number,*
    - *roots differ by a given number,*
    - *roots satisfy a given relation (e.g., the relation $2\alpha + 5\beta = 7$ where α and β are the roots of given equation),*
    - *both sum and product of roots are equal to a given number.*
- *define symmetric functions of roots of a quadratic equation.*
- *evaluate a symmetric function of the roots of a quadratic equation in terms of its coefficients.*

- *establish the formula,*
  $x^2 - (\text{sum of roots})\, x + (\text{product of roots}) = 0,$
  to find a quadratic equation from the given roots.
- *form the quadratic equation whose roots, for example, are of the type:*
  - $2\alpha + 1,\ 2\beta + 1,$
  - $\alpha^2,\ \beta^2,$
  - $\frac{1}{\alpha},\ \frac{1}{\beta},$
  - $\frac{\alpha}{\beta},\ \frac{\beta}{\alpha},$
  - $\alpha + \beta,\ \frac{1}{\alpha} + \frac{1}{\beta},$

  where α, β are the roots of a given quadratic equation.
- *describe the method of synthetic division.*
- *use synthetic division to*
  - *find quotient and remainder when a given polynomial is divided by a linear polynomial,*
  - *find the value(s) of unknown(s) if the zeros of a polynomial are given,*
  - *find the value(s) of unknown(s) if the factors of a polynomial are given,*
  - *solve a cubic equation if one root of the equation is given,*
  - *solve a biquadratic (quartic) equation if two of the real roots of the equation are given.*
- *solve a system of two equations in two variables when*
  - *one equation is linear and the other is quadratic,*
  - *both the equations are quadratic.*
- *solve the real life problems leading to quadratic equations.*

2.1 Nature of the roots of a quadratic equation

On solving quadratic equations, we get different kinds of roots. Now we will discuss the nature or characteristics of the roots of the quadratic equation without actually solving it.

2.1.1 Discriminant ($b^2 - 4ac$) of the quadratic expression $ax^2 + bx + c$.

We know that two roots of the equation $ax^2 + bx + c = 0$, $a \neq 0$ (i)

are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

The nature of these roots depends on the value of the expression "$b^2 - 4ac$" which is called the **"discriminant"** of the quadratic equation (i) or the quadratic expression $ax^2 + bx + c$.

2.1.2 To find the discriminant of a given quadratic equation.

We explain the procedure to find the discriminant of a given quadratic equation through the following example:

Example 1: Find the discriminant of the following equations.

(a) $2x^2 - 7x + 1 = 0$ (b) $x^2 - 3x + 3 = 0$

Solution:

(a) $2x^2 - 7x + 1 = 0$

Here $a = 2$, $b = -7$, $c = 1$

Disc. $= b^2 - 4ac$

$= (-7)^2 - 4(2)(1)$

$= 49 - 8 = 41$

(b) $x^2 - 3x + 3 = 0$

Here $a = 1$, $b = -3$, $c = 3$

Disc. $= b^2 - 4ac$

$= (-3)^2 - 4(1)(3)$

$= 9 - 12 = -3$

2.1.3 Nature of the roots of a quadratic equation through discriminant.

The roots of the quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and its discriminant is $b^2 - 4ac$.

When a, b and c are rational numbers.

(i) If $b^2 - 4ac > 0$ and is a perfect square, then the roots are rational (real) and unequal.

(ii) If $b^2 - 4ac > 0$ and is not a perfect square, then the roots are irrational (real) and unequal.

(iii) If $b^2 - 4ac = 0$, then the roots are rational (real) and equal.

(iv) If $b^2 - 4ac < 0$, then the roots are imaginary (complex conjugates).

2.1.4 Determine the nature of the roots of a given quadratic equation and verify the result by solving the equation.

We illustrate the procedure through the following examples:

Example 2: Using discriminant, find the nature of the roots of the following equations and verify the results by solving the equations.

(a) $x^2 - 5x + 5 = 0$ (b) $2x^2 - x + 1 = 0$

(c) $x^2 + 8x + 16 = 0$ (d) $7x^2 + 8x + 1 = 0$

Solution: **(a)** $x^2 - 5x + 5 = 0$

Compare with the standard quadratic equation

$$ax^2 + bx + c = 0$$

Here $a = 1, b = -5$ and $c = 5$

$$\text{Disc.} = b^2 - 4ac$$

$$= (-5)^2 - 4(1)(5) = 25 - 20 = 5 > 0$$

As the Disc. is positive and is not a perfect square.

Therefore, the roots are irrational (real) and unequal.

Now solving the equation $x^2 - 5x + 5 = 0$ by quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)} = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2}$$

Evidently, the roots are irrational (real) and unequal.

(b) $2x^2 - x + 1 = 0$

Here $a = 2$, $b = -1$ and $c = 1$

$$\text{Disc.} = b^2 - 4ac$$

$$= (-1)^2 - 4(2)(1) = 1 - 8 = -7 < 0$$

As the Disc. is negative,

therefore, the roots of the equation are imaginary and unequal.

Verification by solving the equation.

$$2x^2 - x + 1 = 0$$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(1)}}{2(2)}$$

$$= \frac{1 \pm \sqrt{1 - 8}}{4} = \frac{1 \pm \sqrt{-7}}{4}$$

Evidently, the roots are imaginary and unequal.

(c) $x^2 + 8x + 16 = 0$

Here $a = 1, b = 8$ and $c = 16$

$$\text{Disc.} = b^2 - 4ac$$

$$= (8)^2 - 4(1)(16)$$

$$= 64 - 64 = 0$$

As the discriminant is zero, therefore
the roots are rational (real) and equal.

Verification by solving the equation.

$$x^2 + 8x + 16 = 0$$
$$(x + 4)^2 = 0$$
$$\Rightarrow \quad x = -4, -4$$

So the roots are rational (real) and equal.

(d) $7x^2 + 8x + 1 = 0$

Here $a = 7$, $b = 8$ and $c = 1$

$$\text{Disc.} = b^2 - 4ac$$
$$= (8)^2 - 4(7)(1)$$
$$= 64 - 28 = 36 = (6)^2$$

which is positive and perfect square.

$\therefore$ The roots are rational (real) and unequal.

Now solving the equation by factors, we get

$$7x^2 + 8x + 1 = 0$$
$$7x^2 + 7x + x + 1 = 0$$
$$7x(x + 1) + 1(x + 1) = 0$$
$$(x + 1)(7x + 1) = 0$$

Either $x + 1 = 0$ or $7x + 1 = 0$, that is

$x = -1$ or $7x = -1 \Rightarrow x = -\frac{1}{7}$

Thus, the roots are (real) rational and unequal.

2.1.5 To determine the value of an unknown involved in a given quadratic equation when nature of its roots is given.

We illustrate the procedure through the following example:

Examples 3: Find k, if the roots of the equation

$(k + 3)x^2 - 2(k + 1)x - (k + 1) = 0$ are equal, if $k \neq -3$.

Solution: $(k + 3)x^2 - 2(k + 1)x - (k + 1) = 0$

Here $a = k + 3$, $b = -2(k + 1)$ and $c = -(k + 1)$

$\because$ As roots are equal, so Disc. = 0, that is,

$$\therefore \quad b^2 - 4ac = 0$$
$$[-2(k + 1)]^2 - 4(k + 3)[-(k + 1)] = 0$$
$$4[k + 1]^2 + 4(k + 3)(k + 1) = 0 \quad \text{or} \quad 4(k + 1)(k + 1 + k + 3) = 0$$
$$\Rightarrow \quad 4(k + 1)(2k + 4) = 0 \quad \text{or} \quad 8(k + 1)(k + 2) = 0$$
$$\Rightarrow \quad k + 1 = 0 \quad \text{or} \quad k + 2 = 0$$
$$\Rightarrow \quad k = -1 \quad \text{or} \quad k = -2$$

Thus, roots will be equal if $k = -2, -1$.

EXERCISE 2.1

1. Find the discriminant of the following given quadratic equations:

   (i) $2x^2 + 3x - 1 = 0$ (ii) $6x^2 - 8x + 3 = 0$

   (iii) $9x^2 - 30x + 25 = 0$ (iv) $4x^2 - 7x - 2 = 0$

2. Find the nature of the roots of the following given quadratic equations and verify the result by solving the equations:

   (i) $x^2 - 23x + 120 = 0$ (ii) $2x^2 + 3x + 7 = 0$

   (iii) $16x^2 - 24x + 9 = 0$ (iv) $3x^2 + 7x - 13 = 0$

3. For what value of k, the expression

   $k^2x^2 + 2(k + 1)x + 4$ is perfect square.

4. Find the value of k, if the roots of the following equations are equal.

   (i) $(2k - 1)x^2 + 3kx + 3 = 0$

   (ii) $x^2 + 2(k + 2)x + (3k + 4) = 0$

   (iii) $(3k + 2)x^2 - 5(k + 1)x + (2k + 3) = 0$

5. Show that the equation $x^2 + (mx + c)^2 = a^2$ has equal roots,

   if $c^2 = a^2(1 + m^2)$

6. Find the condition that the roots of the equation $(mx + c)^2 - 4ax = 0$ are equal.

7. If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal, then $a = 0$ or $a^3 + b^3 + c^3 = 3abc$.

8. Show that the roots of the following equations are rational.

   (i) $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$

   (ii) $(a + 2b)x^2 + 2(a + b + c)x + (a + 2c) = 0$

9. For all values of k, prove that the roots of the equation

   $x^2 - 2\left(k + \frac{1}{k}\right)x + 3 = 0$, $k \neq 0$ are real.

10. Show that the roots of the equation

    $(b - c)x^2 + (c - a)x + (a - b) = 0$ are real.

2.2 Cube Roots of Unity and Their Properties.

2.2.1 The cube roots of unity.

Let a number x be the cube root of unity,

i.e., $x = (1)^{1/3}$

or $x^3 = 1$

$\Rightarrow$ $x^3 - 1 = 0$

$(x^3) - (1)^3 = 0$

$(x - 1)(x^2 + x + 1) = 0$ [using $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$]

Either $x - 1 = 0$ or $x^2 + x + 1 = 0$

$\Rightarrow \quad x = 1 \quad$ or $\quad x = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$

$$= \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$\therefore$ Three **cube roots** of unity are

$1, \dfrac{-1 + i\sqrt{3}}{2}$ and $\dfrac{-1 - i\sqrt{3}}{2}$, where $i = \sqrt{-1}$.

2.2.2 Recognise complex cube roots of unity as ω and ω².

The two complex cube roots of unity are $\dfrac{-1+\sqrt{-3}}{2}$ and $\dfrac{-1-\sqrt{-3}}{2}$.

If we name anyone of these as ω (pronounced as omega), then the other is ω^2. We shall prove this statement in the next article.

2.2.3 Properties of cube roots of unity.

(a) Prove that each of the complex cube roots of unity is the square of the other.

Proof: The complex cube roots of unity are $\dfrac{-1+\sqrt{-3}}{2}$ and $\dfrac{-1-\sqrt{-3}}{2}$.

We prove that

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2 = \frac{-1-\sqrt{-3}}{2} \quad \text{and} \quad \left(\frac{-1-\sqrt{-3}}{2}\right)^2 = \frac{-1+\sqrt{-3}}{2}$$

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2 = \frac{1+(-3)-2\sqrt{-3}}{4} = \frac{-2-2\sqrt{-3}}{4} = \frac{2(-1-\sqrt{-3})}{4} = \frac{-1-\sqrt{-3}}{2}$$

$$\left(\frac{-1-\sqrt{-3}}{2}\right)^2 = \frac{1+(-3)+2\sqrt{-3}}{4} = \frac{-2+2\sqrt{-3}}{4} = \frac{2(-1+\sqrt{-3})}{4} = \frac{-1+\sqrt{-3}}{2}$$

Thus, each of the complex cube root of unity is the square of the other, that is,

if $\omega = \dfrac{-1+\sqrt{-3}}{2}$, then $\omega^2 = \dfrac{-1-\sqrt{-3}}{2}$ and if $\omega = \dfrac{-1-\sqrt{-3}}{2}$, then $\omega^2 = \dfrac{-1+\sqrt{-3}}{2}$.

(b) Prove that the product of three cube roots of unity is one.

Proof: Three cube roots of unity are

$1, \dfrac{-1+\sqrt{-3}}{2}$ and $\dfrac{-1-\sqrt{-3}}{2}$

The **product** of cube roots of unity $= (1)\left(\dfrac{-1+\sqrt{-3}}{2}\right)\left(\dfrac{-1-\sqrt{-3}}{2}\right)$

$$= \frac{(-1)^2 - (\sqrt{-3})^2}{4} = \frac{1-(-3)}{4} = \frac{1+3}{4} = \frac{4}{4} = 1$$

i.e., $(1)(\omega)(\omega^2) = 1$ or $\omega^3 = 1$

Remember that:
$\omega^4 = \omega^3 . \omega = 1 . \omega = \omega$

(c) Prove that each complex cube root of unity is reciprocal of the other.

Proof: We know that $\omega^3 = 1 \Rightarrow \omega . \omega^2 = 1$, so

$$\omega = \frac{1}{\omega^2} \quad \text{or} \quad \omega^2 = \frac{1}{\omega}$$

Thus, each complex cube root of unity is **reciprocal** of the other.

(d) Prove that the sum of all the cube roots of unity is zero.

i.e., $\mathbf{1 + \omega + \omega^2 = 0}$

Proof: The cube roots of unity are

$$1, \frac{-1+\sqrt{-3}}{2} \text{ and } \frac{-1-\sqrt{-3}}{2}.$$

If $\omega = \dfrac{-1+\sqrt{-3}}{2}$, then $\omega^2 = \dfrac{-1-\sqrt{-3}}{2}$

The **sum** of all the roots $= 1 + \omega + \omega^2$

$$= 1 + \frac{-1+\sqrt{-3}}{2} + \frac{-1-\sqrt{-3}}{2}$$

$$= \frac{2-1+\sqrt{-3}-1-\sqrt{-3}}{2} = \frac{0}{2} = 0$$

Thus, $1 + \omega + \omega^2 = 0$

We can easily deduce the following results, that is,

(i) $1 + \omega^2 = -\omega$ (ii) $1 + \omega = -\omega^2$ (iii) $\omega + \omega^2 = -1$

2.2.4 Use of properties of cube roots of unity to solve appropriate problems.

We can reduce the higher powers of ω into 1, ω and ω^2.

e.g., $\omega^7 = (\omega^3)^2 . \omega = (1)^2 . \omega = \omega$

$\omega^{23} = (\omega^3)^7 . \omega^2 = (1)^7 . \omega^2 = \omega^2$

$\omega^{63} = (\omega^3)^{21} = (1)^{21} = 1$

$$\omega^{-5} = \frac{1}{\omega^5} = \frac{1}{\omega^3 . \omega^2} = \frac{1}{1 . \omega^2} = \frac{\omega^3}{\omega^2} = \omega$$

$$\omega^{-16} = \frac{1}{\omega^{16}} = \frac{1}{(\omega^3)^5 . \omega}$$

$$= \frac{1}{(1)^5 . \omega} = \frac{\omega^3}{\omega} = \omega^2$$

$\omega^{-27} = \frac{1}{\omega^{27}} = \frac{1}{(\omega^3)^9} = \frac{1}{(1)^9} = 1$

Example 1: Evaluate $(-1+\sqrt{-3})^8 + (-1-\sqrt{-3})^8$.

Solution: $(-1+\sqrt{-3})^8 + (-1-\sqrt{-3})^8$

$= \left[2\left(\frac{-1+\sqrt{-3}}{2}\right)\right]^8 + \left[2\left(\frac{-1-\sqrt{-3}}{2}\right)\right]^8$

$= (2\omega)^8 + (2\omega^2)^8$

$= 256\,\omega^8 + 256\,\omega^{16}$

$= 256\,[\omega^8 + \omega^{16}]$

$= 256\,[(\omega^3)^2 \,.\, \omega^2 + (\omega^3)^5 \,.\, \omega]$ $\quad (\because \omega^3 = 1)$

$= 256\,[\omega^2 + \omega]$ $\quad (\omega + \omega^2 = -1)$

$= 256\,(-1) = -256$

Example 2: Prove that $x^3 - y^3 = (x-y)(x-\omega y)(x-\omega^2 y)$.

Solution: $x^3 - y^3 = (x-y)(x-\omega y)(x-\omega^2 y)$

R.H.S $= (x-y)(x-\omega y)(x-\omega^2 y)$

$= (x-y)[x^2 - \omega^2 xy - \omega xy + \omega^3 y^2]$

$= (x-y)[x^2 - xy(\omega^2+\omega) + (1)y^2]$

$= (x-y)[x^2 - xy(-1) + y^2]$

$= (x-y)[x^2 + xy + y^2]$

$= x^3 - y^3 =$ L.H.S

EXERCISE 2.2

1. Find the cube roots of –1, 8, –27, 64.
2. Evaluate
   - (i) $(1-\omega-\omega^2)^7$ (ii) $(1-3\omega-3\omega^2)^5$
   - (iii) $(9+4\omega+4\omega^2)^3$ (iv) $(2+2\omega-2\omega^2)(3-3\omega+3\omega^2)$
   - (v) $(-1+\sqrt{-3})^6 + (-1-\sqrt{-3})^6$ (vi) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9 + \left(\frac{-1-\sqrt{-3}}{2}\right)^9$
   - (vii) $\omega^{37} + \omega^{38} - 5$ (viii) $\omega^{-13} + \omega^{-17}$
3. Prove that $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$.
4. Prove that $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$ ..... $2n$ factors = 1.

2.3 Roots and co-efficients of a quadratic equation.

We know that $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$ are roots of the equation $ax^2+bx+c=0$ where a, b are coefficients of x^2 and x respectively. While c is the constant term.

2.3.1 Relation between roots and co-efficients of a quadratic equation.

If $\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$,

then we can find the **sum** and the **product** of the roots as follows.

Sum of the roots $=\alpha+\beta$

$$=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=\frac{-2b}{2a}=-\frac{b}{a}$$

Product of the roots $=\alpha\beta$

$$=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}=\frac{b^2-(b^2-4ac)}{4a^2}$$

$$=\frac{b^2-b^2+4ac}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

If we denote the sum of roots and product of roots by S and P respectively, then

$$S=-\frac{b}{a}=-\frac{\text{Co-efficient of } x}{\text{Co-efficient of } x^2}$$

and $$P=\frac{c}{a}=\frac{\text{Constant term}}{\text{Co-efficient of } x^2}.$$

2.3.2 The sum and the product of the roots of a given quadratic equation without solving it.

We illustrate the method through the following example.

Example 1: Without solving, find the sum and product of the roots of the equations.

(a) $3x^2-5x+7=0$ (b) $x^2+4x-9=0$

Solution: **(a)** Let α and β be the roots of the equation

$3x^2-5x+7=0$

Then sum of roots $=\alpha+\beta=-\frac{b}{a}=-\left(\frac{-5}{3}\right)=\frac{5}{3}$

and product of roots $=\alpha\beta=\frac{c}{a}=\frac{7}{3}$

(b) Let α and β be the roots of the equation $x^2+4x-9=0$

Then $\alpha+\beta=-\frac{b}{a}=-\frac{4}{1}=-4$

and $\alpha\beta=\frac{c}{a}=\frac{-9}{1}=-9$

2.3.3 To find unknown values involved in a given quadratic equation.

The procedure is illustrated through the following examples.

(a) Sum of the roots is equal to a multiple of the product of the roots.

Example 1: Find the value of h, if the sum of the roots is equal to 3-times the product of the roots of the equation $3x^2+(9-6h)x+5h=0$.

Solution: Let α, β be the roots of the equation

$$3x^2+(9-6h)x+5h=0$$

Then $\alpha+\beta=-\frac{b}{a}=-\left(\frac{9-6h}{3}\right)=\frac{6h-9}{3}$

$\alpha\beta=\frac{c}{a}=\frac{5h}{3}$

Since $\alpha+\beta=3(\alpha\beta)$

$$\frac{6h-9}{3}=3\left(\frac{5h}{3}\right) \quad \text{or} \quad \frac{3(2h-3)}{3}=5h$$

$$2h-3=5h \Rightarrow 2h-5h=3$$

$$-3h=3 \Rightarrow h=\frac{3}{-3}=-1$$

(b) Sum of the squares of the roots is equal to a given number.

Example 2: Find p, if the sum of the squares of the roots of the equation $4x^2+3px+p^2=0$ is unity.

Solution: If α, β are the roots of $4x^2+3px+p^2=0$,

then $\alpha+\beta=-\frac{b}{a}=-\frac{3p}{4}$

and $\alpha\beta=\frac{c}{a}=\frac{p^2}{4}$

Since $\alpha^2+\beta^2=1$ (Given)

$$(\alpha+\beta)^2-2\alpha\beta=1$$

$$\Rightarrow \left(\frac{-3p}{4}\right)^2-2\left(\frac{p^2}{4}\right)=1 \quad \text{or} \quad \frac{9p^2}{16}-\frac{p^2}{2}=1$$

$$\Rightarrow 9p^2-8p^2=16 \Rightarrow p^2=16 \Rightarrow p=\pm 4$$

(c) Two roots differ by a given number.

Example 3: Find h, if the roots of the equation $x^2 - hx + 10 = 0$ differ by 3.

Solution: Let α and $\alpha - 3$ be the roots of $x^2 - hx + 10 = 0$.

Then $\alpha + \alpha - 3 = -\frac{b}{a} = -\left(\frac{-h}{1}\right) = h$

$2\alpha - 3 = h \Rightarrow 2\alpha = h + 3 \Rightarrow \alpha = \frac{h+3}{2}$ (i)

and $\alpha(\alpha - 3) = \frac{c}{a} = \frac{10}{1} = 10$ or $\alpha(\alpha - 3) = 10$ (ii)

Putting value of α from equation (i) in equation (ii), we get

$\left(\frac{h+3}{2}\right)\left(\frac{h+3}{2} - 3\right) = 10 \Rightarrow \left(\frac{h+3}{2}\right)\left(\frac{h+3-6}{2}\right) = 10$

$\left(\frac{h+3}{2}\right)\left(\frac{h-3}{2}\right) = 10 \Rightarrow h^2 - 9 = 40$, that is,

$h^2 = 49 \Rightarrow h = \pm 7$

(d) Roots satisfy a given relation

(e.g. $2\alpha + 5\beta = 7$, where α, β are the roots of a given equation).

Example 4: Find p, if the roots α, β of the equation $x^2 - 5x + p = 0$, satisfy the relation $2\alpha + 5\beta = 7$.

Solution: If α, β are the roots of the equation $x^2 - 5x + p = 0$.

Then $\alpha + \beta = -\frac{b}{a} = -\left(\frac{-5}{1}\right) = 5$

$\alpha + \beta = 5 \Rightarrow \beta = 5 - \alpha$ (i)

and $\alpha\beta = \frac{c}{a} = \frac{p}{1} = p \Rightarrow \alpha\beta = p$ (ii)

Since $2\alpha + 5\beta = 7$ (Given) (iii)

Put the value of β from equation (i) in equation (iii)

$2\alpha + 5(5 - \alpha) = 7$

$2\alpha + 25 - 5\alpha = 7$ or $-3\alpha = 7 - 25$, that is

$-3\alpha = -18 \Rightarrow \alpha = 6$ (iv)

$\beta = 5 - 6 = -1$ Use (i) and (iv)

Put the values of α and β in eq. (ii)

$6(-1) = p \Rightarrow p = -6$

(e) Both sum and product of the roots are equal to a given number.

Example 5: Find m, if sum and product of the roots of the equation $5x^2 + (7 - 2m)x + 3 = 0$ is equal to a given number, say λ.

Solution: Let α, β be the roots of the equation

$$5x^2 + (7 - 2m)x + 3 = 0$$

Then $\alpha + \beta = -\frac{b}{a} = -\frac{7-2m}{5} = \frac{2m-7}{5}$

and $\alpha\beta = \frac{c}{a} = \frac{3}{5}$

Let $\alpha + \beta = \lambda$ (i) and $\alpha\beta = \lambda$ (ii)

Then from (i) and (ii) $\alpha + \beta = \alpha\beta$, that is,

$\therefore \quad \frac{2m-7}{5} = \frac{3}{5} \Rightarrow 2m - 7 = 3 \Rightarrow 2m = 10 \Rightarrow m = 5$

EXERCISE 2.3

1. Without solving, find the sum and the product of the roots of the following quadratic equations.
   (i) $x^2 - 5x + 3 = 0$ (ii) $3x^2 + 7x - 11 = 0$
   (iii) $px^2 - qx + r = 0$ (iv) $(a + b)x^2 - ax + b = 0$
   (v) $(l + m)x^2 + (m + n)x + n - l = 0$ (vi) $7x^2 - 5mx + 9n = 0$
2. Find the value of k, if
   (i) sum of the roots of the equation $2kx^2 - 3x + 4k = 0$ is twice the product of the roots.
   (ii) sum of the roots of the equation $x^2 + (3k - 7)x + 5k = 0$ is $\frac{3}{2}$ times the product of the roots.
3. Find k, if
   (i) sum of the squares of the roots of the equation $4kx^2 + 3kx - 8 = 0$ is 2.
   (ii) sum of the squares of the roots of the equation $x^2 - 2kx + (2k + 1) = 0$ is 6.
4. Find p, if
   (i) the roots of the equation $x^2 - x + p^2 = 0$ differ by unity.
   (ii) the roots of the equation $x^2 + 3x + p - 2 = 0$ differ by 2.
5. Find m, if
   (i) the roots of the equation $x^2 - 7x + 3m - 5 = 0$ satisfy the relation $3\alpha + 2\beta = 4$
   (ii) the roots of the equation $x^2 + 7x + 3m - 5 = 0$ satisfy the relation $3\alpha - 2\beta = 4$
   (iii) the roots of the equation $3x^2 - 2x + 7m + 2 = 0$ satisfy the relation $7\alpha - 3\beta = 18$
6. Find m, if sum and product of the roots of the following equations is equal to a given number λ.
   (i) $(2m + 3)x^2 + (7m - 5)x + (3m - 10) = 0$
   (ii) $4x^2 - (3 + 5m)x - (9m - 17) = 0$

2.4 Symmetric functions of the roots of a quadratic equation.

2.4.1 Define symmetric functions of the roots of a quadratic equation

Definition:

Symmetric functions are those functions in which the roots involved are such that the value of the expressions involving them remain unaltered, when roots are interchanged. For example, if

$f(\alpha, \beta) = \alpha^2 + \beta^2$, then

$f(\beta, \alpha) = \beta^2 + \alpha^2 = \alpha^2 + \beta^2$ $\quad (\because\ \beta^2 + \alpha^2 = \alpha^2 + \beta^2)$

$= f(\alpha, \beta)$

Example: Find the value of $\alpha^3 + \beta^3 + 3\alpha\beta$, if $\alpha = 2$, $\beta = 1$. Also find the value of $\alpha^3 + \beta^3 + 3\alpha\beta$ if $\alpha = 1$, $\beta = 2$.

Solution: When $\alpha = 2$ and $\beta = 1$,

$\alpha^3 + \beta^3 + 3\alpha\beta = (2)^3 + (1)^3 + 3(2)(1)$

$= 8 + 1 + 6 = 15$

When $\alpha = 1$ and $\beta = 2$,

$\alpha^3 + \beta^3 + 3\alpha\beta = (1)^3 + (2)^3 + 3(1)(2)$

$= 1 + 8 + 6 = 15$

The expression $\alpha^3 + \beta^3 + 3\alpha\beta$ represents a symmetric function of α and β.

2.4.2. Evaluate a symmetric function of roots of a quadratic equation in terms of its co-efficients

If α, β are the roots of the quadratic equation

$ax^2 + bx + c = 0, \ (a \neq 0)$ (i)

Then $\alpha + \beta = -\dfrac{b}{a}$ (ii)

and $\alpha\beta = \dfrac{c}{a}$ (iii)

The functions given in equations (ii) and (iii) are the symmetric functions for the quadratic equation (i).

Some more symmetric functions of two variables α, β are given below:

(i) $\alpha^2 + \beta^2$ (ii) $\alpha^3 + \beta^3$

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ (iv) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

Example 1: If α, β are the roots of the quadratic equation

$px^2 + qx + r = 0 \quad , \quad (p \neq 0)$

then evaluate $\alpha^2\beta + \alpha\beta^2$

Solution: Since α, β are the roots of $px^2 + qx + r = 0$, therefore,

$$\alpha+\beta=-\frac{q}{p} \quad \text{and} \quad \alpha\beta=\frac{r}{p}$$

$$\alpha^2\beta+\alpha\beta^2=\alpha\beta(\alpha+\beta)$$

$$=\frac{r}{p}\left(-\frac{q}{p}\right)=\frac{-qr}{p^2}$$

Example 2: If α, β are the roots of the equation $2x^2+3x+4=0$, then find the value of (i) $\alpha^2+\beta^2$ (ii) $\frac{1}{\alpha}+\frac{1}{\beta}$

Solution: Since α, β are the roots of the equation $2x^2+3x+4=0$, therefore,

$$\alpha+\beta=-\frac{3}{2} \quad \text{and} \quad \alpha\beta=\frac{4}{2}=2$$

(i) $$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(\frac{-3}{2}\right)^2-2\,(2)$$

$$=\frac{9}{4}-4=\frac{9-16}{4}=-\frac{7}{4}$$

(ii) $$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta}=\frac{\alpha+\beta}{\alpha\beta}=(\alpha+\beta)\frac{1}{\alpha\beta}$$

$$=\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right)=\frac{-3}{4}$$

EXERCISE 2.4

1. If α, β are the roots of the equation $x^2+px+q=0$, then evaluate
   (i) $\alpha^2+\beta^2$ (ii) $\alpha^3\beta+\alpha\beta^3$
   (iii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$
2. If α, β are the roots of the equation $4x^2-5x+6=0$, then find the values of
   (i) $\frac{1}{\alpha}+\frac{1}{\beta}$ (ii) $\alpha^2\beta^2$
   (iii) $\frac{1}{\alpha^2\beta}+\frac{1}{\alpha\beta^2}$ (iv) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$
3. If α, β are the roots of the equation $lx^2+mx+n=0$ $(l\neq 0)$, then find the values of
   (i) $\alpha^3\beta^2+\alpha^2\beta^3$ (ii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

2.5 Formation of a quadratic equation.

If α and β are the roots of the required quadratic equation.

Let $x = \alpha$ and $x = \beta$

i.e., $x - \alpha = 0$, $x - \beta = 0$

and $(x - \alpha)(x - \beta) = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

which is the required quadratic equation in the standard form.

2.5.1 Find a quadratic equation from given roots and establish the formula x^2 – (sum of the roots) x + product of the roots = 0.

Let α, β be the roots of the quadratic equation

$$ax^2 + bx + c = 0 \quad , \quad (a \neq 0) \qquad \text{(i)}$$

Then $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

Rewrite eq. (i) as $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$

or $x^2 - \left(-\dfrac{b}{a}\right)x + \dfrac{c}{a} = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

or x^2 – (sum of roots) x + product of roots = 0 , that is,

$x^2 - Sx + P = 0$ where $S = \alpha + \beta$ and $P = \alpha\beta$

Example 1: Form a quadratic equation with roots 3 and 4.

Solution: Since 3 and 4 are the roots of the required quadratic equation, therefore,

S = Sum of the roots = 3 + 4 = 7

P = Product of the roots = (3) (4) = 12

As $x^2 - Sx + P = 0$, so the required quadratic equation is $x^2 - 7x + 12 = 0$

2.5.2 Form quadratic equations whose roots are of the type

(i) $2\alpha + 1 , 2\beta + 1$ (ii) α^2 , β^2 (iii) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ (iv) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

(v) $\alpha + \beta, \dfrac{1}{\alpha} + \dfrac{1}{\beta}$ where α, β are the roots of a given quadratic equation.

Example 2: If α, β are the roots of the equation $2x^2 - 3x - 5 = 0$, form quadratic equations having roots

(i) $2\alpha + 1 , 2\beta + 1$ (ii) α^2, β^2 (iii) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$

(iv) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$ (v) $\alpha + \beta, \dfrac{1}{\alpha} + \dfrac{1}{\beta}$

Solution: As α, β are the roots of the equation

$$2x^2 - 3x - 5 = 0,$$

therefore, $\alpha + \beta = -\left(\frac{-3}{2}\right) = \frac{3}{2}$ and $\alpha\beta = \frac{-5}{2} = -\frac{5}{2}$

(i) S = Sum of the roots $= 2\alpha + 1 + 2\beta + 1$

$$= 2(\alpha + \beta) + 2 = 2\left(\frac{3}{2}\right) + 2 = 5$$

P = Product of the roots $= (2\alpha + 1)(2\beta + 1)$

$$= 4\alpha\beta + 2(\alpha + \beta) + 1$$

$$= 4\left(-\frac{5}{2}\right) + 2\left(\frac{3}{2}\right) + 1$$

$$= -10 + 3 + 1 = -6$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - 5x - 6 = 0$$

(ii) S = Sum of the roots $= \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right) = \frac{9}{4} + 5 = \frac{29}{4}$$

P = Product of the roots $= \alpha^2 . \beta^2 = (\alpha\beta)^2 = \left(-\frac{5}{2}\right)^2 = \frac{25}{4}$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \frac{29}{4}x + \frac{25}{4} = 0 \quad \Rightarrow \quad 4x^2 - 29x + 25 = 0$$

(iii) S = Sum of the roots $= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = (\alpha + \beta) \cdot \frac{1}{\alpha\beta}$

$$= \frac{3}{2} \cdot \left(-\frac{2}{5}\right) \qquad (\because \ \alpha\beta = -\frac{5}{2})$$

$$= -\frac{3}{5}$$

P = Product of the roots $= \frac{1}{\alpha} \cdot \frac{1}{\beta} = \frac{1}{\alpha\beta} = -\frac{2}{5}$ $\qquad (\because \ \alpha\beta = -\frac{5}{2})$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \left(-\frac{3}{5}\right)x + \left(-\frac{2}{5}\right) = 0 \quad \Rightarrow \quad 5x^2 + 3x - 2 = 0$$

(iv) S = Sum of the roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = [(\alpha + \beta)^2 - 2\alpha\beta] \cdot \frac{1}{\alpha\beta}$

$$= \left[\left(\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right)\right] \times \left(-\frac{2}{5}\right) \qquad (\because \ \alpha\beta = -\frac{5}{2})$$

$$= \left(\frac{9}{4} + 5\right) \times \left(-\frac{2}{5}\right) = \frac{29}{4} \times \left(-\frac{2}{5}\right) = -\frac{29}{10}$$

P = Product of the roots $= \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \left(-\frac{29}{10}\right)x + 1 = 0 \quad \Rightarrow \quad 10x^2 + 29x + 10 = 0$$

(v) S = Sum of the roots $= \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta} = \alpha + \beta + \frac{\beta + \alpha}{\alpha\beta}$

$$= (\alpha + \beta)\left(1 + \frac{1}{\alpha\beta}\right) = \frac{3}{2}\left(1 - \frac{2}{5}\right) = \frac{3}{2} \times \frac{3}{5}$$

$$= \frac{9}{10}$$

P = Product of the roots $= (\alpha + \beta) \cdot \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = (\alpha + \beta)\left(\frac{\beta + \alpha}{\alpha\beta}\right)$

$$= (\alpha + \beta)^2 \times \frac{1}{\alpha\beta} = \left(\frac{3}{2}\right)^2 \times \left(-\frac{2}{5}\right)$$

$$= \frac{9}{4} \times \left(-\frac{2}{5}\right) = -\frac{9}{10}$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \frac{9}{10}x + \left(-\frac{9}{10}\right) = 0 \quad \Rightarrow \quad 10x^2 - 9x - 9 = 0$$

Example 3: If α, β are the roots of the equation $x^2 - 7x + 9 = 0$, then form an equation whose roots are 2α and 2β.

Solution: Since α, β are the roots of the equation $x^2 - 7x + 9 = 0$, therefore,

$$\alpha + \beta = -\frac{b}{a} = -\left(\frac{-7}{1}\right) = 7$$

and $$\alpha\beta = \frac{c}{a} = \frac{9}{1} = 9$$

The roots of the required equation are $2\alpha, 2\beta$

S = Sum of roots $= 2\alpha + 2\beta = 2(\alpha + \beta) = 2(7) = 14$

P = Product of roots $= (2\alpha)(2\beta) = 4\alpha\beta = 4(9) = 36$

Thus the required quadratic equation will be

$x^2 - Sx + P = 0$, that is,

$x^2 - 14x + 36 = 0$

EXERCISE 2.5

1. Write the quadratic equations having following roots.

(a)	1, 5	(b)	4, 9	(c)	−2, 3
(d)	0, −3	(e)	2, −6	(f)	−1, −7
(g)	$1+i, 1-i$	(h)	$3+\sqrt{2}, 3-\sqrt{2}$		

2. If α, β are the roots of the equation $x^2 - 3x + 6 = 0$.
Form equations whose roots are

(a) $2\alpha + 1,\ 2\beta + 1$ (b) $\alpha^2,\ \beta^2$ (c) $\frac{1}{\alpha}, \frac{1}{\beta}$

(d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (e) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

3. If α, β are the roots of the equation $x^2 + px + q = 0$.
Form equations whose roots are

(a) α^2, β^2 (b) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

2.6 Synthetic Division

Synthetic division is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial. In fact synthetic division is simply a shortcut of long division method.

2.6.1 Describe the synthetic division method.

The method of synthetic division is described through the following example.

Example 1: Using synthetic division, divide the polynomial $P(x) = 5x^4 + x^3 - 3x$ by $x - 2$.

Solution: $(5x^4 + x^3 - 3x) \div (x - 2)$

From divisor, $x - a$, here $a = 2$

Now write the co-efficients of the dividend in a row with zero as the co-efficient of the missing powers of x in the descending order as shown below.

Dividend $P(x) = 5x^4 + 1 \times x^3 + 0 \times x^2 - 3x + 0 \times x^0$

Now write the co-efficients of x from dividend in a row and $a = 2$ on the left side.

	5	1	0	−3	0
2	↓	10	22	44	82
	5	11	22	41	82

(i) Write 5 the first co-efficient as it is in the row under the horizontal line.

(ii) Multiply 5 with 2 and write the result 10 under 1. write the sum of 1 + 10 = 11 under the line.

(iii) Multiply 11 with 2 and place the result 22 under 0. Add 0 and 22 and write the result 22 under the line.

(iv) Multiply 22 with 2, place the result 44 under –3. Write 41 as the sum of 44 and –3 under the line.

(v) Multiply 41 with 2 and put the result 82 under 0. The sum of 0 and 82 is 82.

In the resulting row 82 separated by the vertical line segment is the remainder and 5, 11, 22, 41 are the co-efficients of the quotient.

As the highest power of x in dividend is 4, therefore the highest power of x in quotient will be $4 - 1 = 3$.

Thus **Quotient** $= Q(x) = 5x^3 + 11x^2 + 22x + 41$ and the **Remainder** $= \boldsymbol{R} = 82$

2.6.2 Use synthetic division to

(a) find quotient and remainder, when a given polynomial is divided by a linear polynomial.

Example 2: Using synthetic division, divide $P(x) = x^4 - x^2 + 15$ by $x + 1$

Solution: $(x^4 - x^2 + 15) \div (x + 1)$

As $x + 1 = x - (-1)$, so $a = -1$

Now write the co-efficients of dividend in a row and $a = -1$ on the left side.

	1	0	–1	0	15
–1	↓	–1	1	0	0
	1	–1	0	0	15

$\therefore$ Quotient $= Q(x) = x^3 - x^2 + 0.\,x + 0 = x^3 - x^2$

and Remainder = 15

(b) find the value (s) of unknown (s), if the zeros of a polynomial are given.

Example 3: Using synthetic division, find the value of h. If the zero of polynomial

$P(x) = 3x^2 + 4x - 7h$ is 1.

Solution: $P(x) = 3x^2 + 4x - 7h$ and its zero is 1.

Then by synthetic division.

	3	4	$-7h$
1	↓	3	7
	3	7	$7 - 7h$

Remainder $= 7 - 7h$

Since 1 is the zero of the polynomial, therefore,

Remainder = 0, that is,

$7 - 7h = 0$

$7 = 7h \Rightarrow h = 1$

c) find the value (s) of unknown (s), if the factors of a polynomial are given.

Example 4: Using synthetic division, find the values of l and m, if $x - 1$ and $x + 1$ are the factors of the polynomial $P(x) = x^3 + 3lx^2 + mx - 1$

Solution: Since $x - 1$ and $x + 1$ are the factors of $P(x) = x^3 + 3lx^2 + mx - 1$.

therefore, 1 and –1 are zeros of polynomial $P(x)$.

Now by synthetic division

	1	$3l$	m	-1
1	↓	1	$3l+1$	$3l+m+1$
	1	$3l+1$	$3l+m+1$	$3l+m$

Since 1 is the zero of polynomial, therefore remainder is zero, that is,

$3l + m = 0$ (i)

and

	1	$3l$	m	-1
-1	↓	-1	$-3l+1$	$3l-m-1$
	1	$3l-1$	$-3l+m+1$	$3l-m-2$

Since -1 is the zero of polynomial, therefore, remainder is zero, that is,

$3l - m - 2 = 0$ (ii)

Adding eqs. (i) and (ii), we get

$6l - 2 = 0$

$6l = 2 \Rightarrow l = \frac{2}{6} = \frac{1}{3}$

Put the value of l in eq. (*i*) $3\left(\frac{1}{3}\right) + m = 0$ or $1 + m = 0 \Rightarrow m = -1$

Thus $l = \frac{1}{3}$ and $m = -1$

(d) solve a cubic equation, if one root of the equation is given.

Example 5: Using synthetic division, solve the equation $3x^3 - 11x^2 + 5x + 3 = 0$ when 3 is the root of the equation.

Solution: Since 3 is the root of the equation $3x^3 - 11x^2 + 5x + 3 = 0$.

Then by synthetic division, we get

	3	-11	5	3
3	↓	9	-6	-3
	3	-2	-1	0

The depressed equation is $3x^2 - 2x - 1 = 0$

$3x^2 - 3x + x - 1 = 0$

$3x(x - 1) + 1(x - 1) = 0$

$(x - 1)(3x + 1) = 0$

Either $x - 1 = 0$ or $3x + 1 = 0$, that is,

$$x = 1 \quad \text{or} \quad 3x = -1 \Rightarrow x = -\frac{1}{3}$$

Hence 3, 1 and $-\frac{1}{3}$ are the roots of the given equation.

(e) solve a biquadratic (quartic) equation, if two of the real roots of the equation are given.

Example 6: By synthetic division, solve the equation

$x^4 - 49x^2 + 36x + 252 = 0$ having roots -2 and 6.

Solution: Since -2 and 6 are the roots of the given equation $x^4 - 49x^2 + 36x + 252 = 0$.

Then by synthetic division, we get

	1	0	−49	36	252
−2	↓	−2	4	90	−252
	1	−2	−45	126	0
6		6	24	−126	
	1	4	−21	0	

∴ The depressed equation is

$$x^2 + 4x - 21 = 0$$
$$x^2 + 7x - 3x - 21 = 0$$
$$x(x + 7) - 3(x + 7) = 0$$
$$(x + 7)(x - 3) = 0$$

Either $x + 7 = 0$ or $x - 3 = 0$

$x = -7$ or $x = 3$

Thus $-2, 6, -7$ and 3 are the roots of the given equation.

EXERCISE 2.6

1. Use synthetic division to find the quotient and the remainder, when
   (i) $(x^2 + 7x - 1) \div (x + 1)$ (ii) $(4x^3 - 5x + 15) \div (x + 3)$
   (iii) $(x^3 + x^2 - 3x + 2) \div (x - 2)$
2. Find the value of h using synthetic division, if
   (i) 3 is the zero of the polynomial $2x^3 - 3hx^2 + 9$
   (ii) 1 is the zero of the polynomial $x^3 - 2hx^2 + 11$
   (iii) -1 is the zero of the polynomial $2x^3 + 5hx - 23$
3. Use synthetic division to find the values of l and m, if
   (i) $(x + 3)$ and $(x - 2)$ are the factors of the polynomial $x^3 + 4x^2 + 2lx + m$
   (ii) $(x - 1)$ and $(x + 1)$ are the factors of the polynomial $x^3 - 3lx^2 + 2mx + 6$

4. Solve by using synthetic division, if

(i) 2 is the root of the equation $x^3 - 28x + 48 = 0$

(ii) 3 is the root of the equation $2x^3 - 3x^2 - 11x + 6 = 0$

(iii) -1 is the root of the equation $4x^3 - x^2 - 11x - 6 = 0$

5. Solve by using synthetic division, if

(i) 1 and 3 are the roots of the equation $x^4 - 10x^2 + 9 = 0$

(ii) 3 and -4 are the roots of the equation $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$

2.7 Simultaneous equations

A system of equations having a common solution is called a system of **simultaneous equations**.

The set of all the ordered pairs (x, y), which satisfies the system of equations is called the **solution set** of the system.

2.7(i) Solve a system of two equations in two variables,

(*a*) when one equation is linear and the other is quadratic.

To solve a system of equations in two variables x and y. Find 'y' in terms of x from the given linear equation. Substitute the value of y in the quadratic equation, we get an other quadratic equation in one variable x. Solve this equation for x and then find the values of y.

The values of x and y provide the solution set of the system of equations.

> In an ordered pair (x, y), x always occupies first place and y second place.

Example 1: Solve the system of equations

$3x + y = 4$ and $3x^2 + y^2 = 52.$

Solution: The given equations are

$3x + y = 4$ (i)

and $3x^2 + y^2 = 52$ (ii)

From eq. (i) $y = 4 - 3x$ (iii)

Put value of y in eq. (ii)

$3x^2 + (4 - 3x)^2 = 52$

$3x^2 + 16 - 24x + 9x^2 - 52 = 0$

$12x^2 - 24x - 36 = 0$ or $x^2 - 2x - 3 = 0$

By factorization

$x^2 - 3x + x - 3 = 0$

$x(x - 3) + 1(x - 3) = 0$

$\Rightarrow$ $(x - 3)(x + 1) = 0$

Either $x - 3 = 0$ or $x + 1 = 0$, that is,

$x = 3$ or $x = -1$

Put the values of x in eq. (iii)

When $x = 3$ When $x = -1$

$y = 4 - 3x$ $\quad\quad$ $y = 4 - 3x$

$y = 4 - 3\,(3) = 4 - 9$ $\quad\quad$ $y = 4 - 3(-1) = 4 + 3$

$y = -5$ $\quad\quad$ $y = 7$

$\therefore$ The ordered pairs are (3, −5) and (−1,7)

Thus, the solution set is $\{(3, -5), (-1, 7)\}$

(*b*) **when both the equations are quadratic.**

The method to solve the equations is illustrated through the following examples.

Example 2: Solve the equations

$$x^2 + y^2 + 2x = 8 \quad \text{and} \quad (x-1)^2 + (y+1)^2 = 8$$

Solution: The given equations are

$$x^2 + y^2 + 2x = 8 \qquad \text{(i)}$$

$$(x-1)^2 + (y+1)^2 = 8 \qquad \text{(ii)}$$

From equation (ii), we get

$$x^2 - 2x + 1 + y^2 + 2y + 1 = 8$$

or $\quad x^2 + y^2 - 2x + 2y = 6 \qquad \text{(iii)}$

Subtracting eq. (iii) from eq. (i), we have

$$4x - 2y = 2 \quad \text{or} \quad 2x - y = 1$$

$\Rightarrow \quad y = 2x - 1 \qquad \text{(iv)}$

Put the value of y in eq. (ii)

$$(x-1)^2 + (2x - 1 + 1)^2 = 8$$

$$x^2 - 2x + 1 + 4x^2 - 8 = 0$$

$$5x^2 - 2x - 7 = 0$$

$$5x^2 - 7x + 5x - 7 = 0 \quad \text{or} \quad x(5x-7) + 1\,(5x-7) = 0$$

$\Rightarrow \quad (5x-7)(x+1) = 0$

Either $\quad 5x - 7 = 0 \quad$ or $\quad x + 1 = 0$, that is,

$$5x = 7 \;\Rightarrow\; x = \frac{7}{5} \quad \text{or} \quad x = -1$$

Now putting the values of x in eq. (iv), we have

When $x = \frac{7}{5}$ $\quad\quad$ When $x = -1$

$$y = 2\left(\frac{7}{5}\right) - 1 \qquad\qquad y = 2(-1) - 1$$

$$y = \frac{14}{5} - 1 = \frac{14-5}{5} = \frac{9}{5} \qquad\qquad y = -3$$

Thus, the solution set is $\left\{(-1, -3), \left(\frac{7}{5}, \frac{9}{5}\right)\right\}$.

Example 3: Solve the equations

$x^2 + y^2 = 7$ and $2x^2 + 3y^2 = 18$.

Solution: Given equations are

$$x^2 + y^2 = 7 \qquad \text{(i)}$$

$$2x^2 + 3y^2 = 18 \qquad \text{(ii)}$$

Multiply equation (i) with 3

$$3x^2 + 3y^2 = 21 \qquad \text{(iii)}$$

Subtracting equations (ii) from (iii)

$$x^2 = 3 \quad \Rightarrow \quad x = \pm\sqrt{3}$$

When $x = \sqrt{3}$, then from equation (i)

$$x^2 + y^2 = 7 \quad \text{or} \quad 3 + y^2 = 7 \quad \Rightarrow \ y^2 = 4 \quad \Rightarrow \ y = \pm 2$$

When $x = -\sqrt{3}$, then $y = \pm 2$

Thus, the required solution set is $\{(\pm\sqrt{3}, \pm 2)\}$.

Example 4: Solve the equations

$x^2 + y^2 = 20$ and $6x^2 + xy - y^2 = 0$

Solution: Given equations are

$$x^2 + y^2 = 20 \qquad \text{(i)}$$

$$6x^2 + xy - y^2 = 0 \qquad \text{(ii)}$$

The equation (ii) can be written as

$$y^2 - xy - 6x^2 = 0$$

$$\Rightarrow \qquad y = \frac{-(-x) \pm \sqrt{(-x)^2 - 4 \times 1 \times (-6x^2)}}{2 \times 1}$$

$$= \frac{x \pm \sqrt{x^2 + 24x^2}}{2} = \frac{x \pm \sqrt{25x^2}}{2}$$

$$= \frac{x \pm 5x}{2}$$

We have $y = \frac{x + 5x}{2} = \frac{6x}{2} = 3x$ or $y = \frac{x - 5x}{2} = \frac{-4x}{2} = -2x$

Substituting $y = 3x$ in the equation (i), we get

$$x^2 + (3x)^2 = 20 \quad \text{or} \quad x^2 + 9x^2 = 20$$

$$\Rightarrow \qquad 10x^2 = 20 \quad \Rightarrow \ x^2 = 2 \quad \Rightarrow \ x = \pm\sqrt{2}$$

When $x = \sqrt{2}$, $y = 3(\sqrt{2}) = 3\sqrt{2}$ and when $x = -\sqrt{2}$, $y = 3(-\sqrt{2}) = -3\sqrt{2}$

Substituting $y = -2x$ in the equation (i), we have

$$x^2 + (-2x)^2 = 20 \quad \text{or} \quad x^2 + 4x^2 = 20$$

$$\Rightarrow \qquad 5x^2 = 20 \quad \Rightarrow \ x^2 = 4 \quad \Rightarrow \ x = \pm 2$$

When $x = 2$, $y = -2(2) = -4$ and when $x = -2$, $y = -2(-2) = 4$

Thus, the solution is $\{(\sqrt{2}, 3\sqrt{2}), (-\sqrt{2}, -3\sqrt{2}), (2, -4), (-2, 4)\}$.

Example 5: Solve the equations

$x^2 + y^2 = 40$ and $3x^2 - 2xy - y^2 = 80.$

Solution: Given equations are

$x^2 + y^2 = 40$ (i)

$3x^2 - 2xy - y^2 = 80$ (ii)

Multiplying equation (i) by 2, we have

$2x^2 + 2y^2 = 80$ (iii)

Subtracting the equation (iii) from equation (ii), we get

$x^2 - 2xy - 3y^2 = 0$ (iv)

The equation (iv) can be written as

$x^2 - 3xy + xy - 3y^2 = 0$

or $x(x - 3y) + y(x - 3y) = 0$

$\Rightarrow$ $(x - 3y)(x + y) = 0$

Either $x - 3y = 0$ or $x + y = 0$

$x = 3y$ or $x = -y$

Put in eq. (i),

$x = 3y$	$x = -y$
$(3y)^2 + y^2 = 40$	$(-y)^2 + y^2 = 40$
$10y^2 = 40$	$2y^2 = 40$
$y^2 = 4$	$y^2 = 20$
$y = \pm 2$	$y = \pm 2\sqrt{5}$

$y = 2$	$y = -2$	$y = 2\sqrt{5}$	$y = -2\sqrt{5}$
$x = 3y$	$x = 3y$	$x = -y$	$x = -y$
$x = 3(2)$	$x = 3(-2)$	$x = -\left(2\sqrt{5}\right)$	$x = -\left(-2\sqrt{5}\right)$
$x = 6$	$x = -6$	$x = -2\sqrt{5}$	$= 2\sqrt{5}$

$\therefore$ The solution set is $\left\{(6, 2), (-6, -2), \left(2\sqrt{5}, -2\sqrt{5}\right), \left(-2\sqrt{5}, 2\sqrt{5}\right)\right\}$

EXERCISE 2.7

Solve the following simultaneous equations.

1.	$x + y = 5$	;	$x^2 - 2y - 14 = 0$
2.	$3x - 2y = 1$	;	$x^2 + xy - y^2 = 1$
3.	$x - y = 7$	;	$\frac{2}{x} - \frac{5}{y} = 2$
4.	$x + y = a - b$	;	$\frac{a}{x} - \frac{b}{y} = 2$
5.	$x^2 + (y - 1)^2 = 10$	;	$x^2 + y^2 + 4x = 1$
6.	$(x + 1)^2 + (y + 1)^2 = 5$	;	$(x + 2)^2 + y^2 = 5$
7.	$x^2 + 2y^2 = 22$	;	$5x^2 + y^2 = 29$
8.	$4x^2 - 5y^2 = 6$	;	$3x^2 + y^2 = 14$
9.	$7x^2 - 3y^2 = 4$	;	$2x^2 + 5y^2 = 7$
10.	$x^2 + 2y^2 = 3$	;	$x^2 + 4xy - 5y^2 = 0$
11.	$3x^2 - y^2 = 26$	;	$3x^2 - 5xy - 12y^2 = 0$
12.	$x^2 + xy = 5$	;	$y^2 + xy = 3$
13.	$x^2 - 2xy = 7$	;	$xy + 3y^2 = 2$

2.7(ii) Solving Real Life Problems with Quadratic Equations

There are many problems which lead to quadratic equations. To form an equation, we use symbols for unknown quantities in the problems. Then roots of the equation may provide the answer to these problems.

The procedure to solve these problems is explained in the following examples.

Example 1: Three less than a certain number multiplied by 9 less than twice the number is 104. Find the number.

Solution: Let the required number be x. Then

three less than the number = $x - 3$

and 9 less than twice the number = $2x - 9$

According to the given condition, we have

$$(x - 3)(2x - 9) = 104$$

$$2x^2 - 15x + 27 = 104$$

$$2x^2 - 15x - 77 = 0$$

Factorizing, we get

$$(2x + 7)(x - 11) = 0 \quad \Rightarrow \quad x = -\frac{7}{2}, \quad x = 11$$

i.e., $x = -\frac{7}{2}$ and 11 are the required numbers.

Example 2: The length of a rectangle is 4cm more than its breadth. If the area of the rectangle is $45cm^2$. Find its sides.

Solution: Let the breadth in cm be x.

Then the length in cm will be $x + 4$.

By the given condition rectangular area = $45\,cm^2$, that is,

$$x(x + 4) = 45$$
$$x^2 + 4x - 45 = 0$$
$$(x + 9)(x - 5) = 0$$
$$x + 9 = 0 \quad \text{or} \quad x - 5 = 0$$
$$x = -9 \quad \text{or} \quad x = 5$$

If $x = 5$, then $x + 4 = 5 + 4 = 9$ (neglecting –ve value)

Thus the breadth is 5cm and length is 9cm.

Example 3: The sum of the co-ordinates of a point is 6 and the sum of their squares is 20. Find the co-ordinates of the point.

Solution: Let (x, y) be the co-ordinates of the point. Then by the given conditions, we have

$$x + y = 6 \quad \text{(i)}$$
$$x^2 + y^2 = 20 \quad \text{(ii)}$$

From eq. (i) $y = 6 - x$ (iii)

Putting $y = 6 - x$ in eq. (ii), we get

$$x^2 + (6 - x)^2 = 20$$
$$x^2 + 36 + x^2 - 12x - 20 = 0$$
$$2x^2 - 12x + 16 = 0 \quad \text{or} \quad x^2 - 6x + 8 = 0$$

Factorizing, we get

$$(x - 4)(x - 2) = 0 \quad \Rightarrow \quad x = 4 \quad \text{or} \quad x = 2$$

using eq. (iii), $y = 6 - 4 = 2$ or $y = 6 - 2 = 4$

$\therefore$ the co-ordinates of the point are (4, 2) or (2, 4)

EXERCISE 2.8

1. The product of two positive consecutive numbers is 182. Find the numbers.
2. The sum of the squares of three positive consecutive numbers is 77. Find them.
3. The sum of five times a number and the square of the number is 204. Find the number.
4. The product of five less than three times a certain number and one less than four times the number is 7. Find the number.
5. The difference of a number and its reciprocal is $\frac{15}{4}$. Find the number.
6. The sum of the squares of two digits of a positive integral number is 65 and the number is 9 times the sum of its digits. Find the number.

7. The sum of the co-ordinates of a point is 9 and sum of their squares is 45. Find the co-ordinates of the point.

8. Find two integers whose sum is 9 and the difference of their squares is also 9.

9. Find two integers whose difference is 4 and whose squares differ by 72.

10. Find the dimensions of a rectangle, whose perimeter is 80cm and its area is $375cm^2$.

MISCELLANEOUS EXERCISE - 2

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) If α, β are the roots of $3x^2 + 5x - 2 = 0$, then $\alpha + \beta$ is

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $\frac{-5}{3}$ (d) $\frac{-2}{3}$

(ii) If α, β are the roots of $7x^2 - x + 4 = 0$, then $\alpha\beta$ is

(a) $\frac{-1}{7}$ (b) $\frac{4}{7}$ (c) $\frac{7}{4}$ (d) $\frac{-4}{7}$

(iii) Roots of the equation $4x^2 - 5x + 2 = 0$ are

(a) irrational (b) imaginary (c) rational (d) none of these

(iv) Cube roots of -1 are

(a) $-1, -\omega, -\omega^2$ (b) $-1, \omega, -\omega^2$ (c) $-1, -\omega, \omega^2$ (d) $1, -\omega, -\omega^2$

(v) Sum of the cube roots of unity is

(a) 0 (b) 1 (c) -1 (d) 3

(vi) Product of cube roots of unity is

(a) 0 (b) 1 (c) -1 (d) 3

(vii) If $b^2 - 4ac < 0$, then the roots of $ax^2 + bx + c = 0$ are

(a) irrational (b) rational (c) imaginary (d) none of these

(viii) If $b^2 - 4ac > 0$, but not a perfect square then roots of $ax^2 + bx + c = 0$ are

(a) imaginary (b) rational (c) irrational (d) none of these

(ix) $\frac{1}{\alpha} + \frac{1}{\beta}$ is equal to

(a) $\frac{1}{\alpha}$ (b) $\frac{1}{\alpha} - \frac{1}{\beta}$ (c) $\frac{\alpha - \beta}{\alpha\beta}$ (d) $\frac{\alpha + \beta}{\alpha\beta}$

(x) $\alpha^2 + \beta^2$ is equal to

(a) $\alpha^2 - \beta^2$ (b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

(c) $(\alpha + \beta)^2 - 2\alpha\beta$ (d) $\alpha + \beta$

(xi) Two square roots of unity are

(a) 1, −1 (b) 1, ω (c) 1, −ω (d) ω, ω^2

(xii) Roots of the equation $4x^2 - 4x + 1 = 0$ are

(a) real, equal (b) real, unequal (c) imaginary (d) irrational

(xiii) If α, β are the roots of $px^2 + qx + r = 0$, then sum of the roots 2α and 2β is

(a) $\frac{-q}{p}$ (b) $\frac{r}{p}$ (c) $\frac{-2q}{p}$ (d) $-\frac{q}{2p}$

(xiv) If α, β are the roots of $x^2 - x - 1 = 0$, then product of the roots 2α and 2β is

(a) −2 (b) 2 (c) 4 (d) −4

(xv) The nature of the roots of equation $ax^2 + bx + c = 0$ is determined by

(a) sum of the roots (b) product of the roots

(c) synthetic division (d) discriminant

(xvi) The discriminant of $ax^2 + bx + c = 0$ is

(a) $b^2 - 4ac$ (b) $b^2 + 4ac$ (c) $-b^2 + 4ac$ (d) $-b^2 - 4ac$

2. Write short answers of the following questions.

(i) Discuss the nature of the roots of the following equations.

(a) $x^2 + 3x + 5 = 0$ (b) $2x^2 - 7x + 3 = 0$

(c) $x^2 + 6x - 1 = 0$ (d) $16x^2 - 8x + 1 = 0$

(ii) Find ω^2, if $\omega = \frac{-1 + \sqrt{-3}}{2}$

(iii) Prove that the sum of the all cube roots of unity is zero.

(iv) Find the product of complex cube roots of unity.

(v) Show that $x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y)$

(vi) Evaluate $\omega^{37} + \omega^{38} + 1$

(vii) Evaluate $(1 - \omega + \omega^2)^6$

(viii) If ω is cube root of unity, form an equation whose roots are 3ω and $3\omega^2$.

(ix) Using synthetic division, find the remainder and quotient when $(x^3 + 3x^2 + 2) \div (x - 2)$

(x) Using synthetic division, show that $x - 2$ is the factor of $x^3 + x^2 - 7x + 2$.

(xi) Find the sum and product of the roots of the equation $2px^2 + 3qx - 4r = 0$.

(xii) Find $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ of the roots of the equation $x^2 - 4x + 3 = 0$

(xiii) If α, β are the roots of $4x^2 - 3x + 6 = 0$, find

(a) $\alpha^2 + \beta^2$ (b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (c) $\alpha - \beta$

(xiv) If α, β are the roots of $x^2 - 5x + 7 = 0$, find an equation whose roots are

(a) $-\alpha, -\beta$ (b) $2\alpha, 2\beta$.

3. Fill in the blanks

(i) The discriminant of $ax^2 + bx + c = 0$ is ________.

(ii) If $b^2 - 4ac = 0$, then roots of $ax^2 + bx + c = 0$ are ________.

(iii) If $b^2 - 4ac > 0$, then the roots of $ax^2 + bx + c = 0$ are ________.

(iv) If $b^2 - 4ac < 0$, then the root of $ax^2 + bx + c = 0$ are ________.

(v) If $b^2 - 4ac > 0$ and perfect square, then the roots of $ax^2 + bx + c = 0$ are ________.

(vi) If $b^2 - 4ac > 0$ and not a perfect square, then roots of $ax^2 + bx + c = 0$ are ________.

(vii) If α, β are the roots of $ax^2 + bx + c = 0$, then sum of the roots is ________.

(viii) If α, β are the roots of $ax^2 + bx + c = 0$, then product of the roots is ________.

(ix) If α, β are the roots of $7x^2 - 5x + 3 = 0$, then the sum of the roots is ________.

(x) If α, β are the roots of $5x^2 + 3x - 9 = 0$, then product of the roots is ________.

(xi) For a quadratic equation $ax^2 + bx + c = 0$, $\frac{1}{\alpha\beta}$ is equal to ________.

(xii) Cube roots of unity are ________.

(xiii) Under usual notation sum of the cube roots of unity is ________.

(xiv) If 1, ω, ω^2 are the cube roots of unity, then ω^{-7} is equal to ________.

(xv) If α, β are the roots of the quadratic equation, then the quadratic equation is written as ________.

(xvi) If 2ω and $2\omega^2$ are the roots of an equation, then equation is ________.

SUMMARY

- **Discriminant** of the quadratic expression $ax^2 + bx + c$ is "$b^2 - 4ac$".
- The **cube roots** of unity are 1, $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$.
- **Complex cube roots** of unity are ω and ω^2.
- **Properties of cube roots of unity.**
  (a) The **product** of three cube roots of unity is one. *i.e.*, $(1)(\omega)(\omega^2) = \omega^3 = 1$
  (b) Each of the complex cube roots of unity is **reciprocal** of the other.
  (c) Each of the complex cube roots of unity is the **square** of the other.
  (d) The **sum** of all the cube roots of unity is zero, *i.e.*, $1 + \omega + \omega^2 = 0$

- The **roots** of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are

  $$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

- The **sum** and the **product** of the roots of $ax^2 + bx + c = 0$, $a \neq 0$ are

  $$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \text{ respectively.}$$

- **Symmetric functions** of the roots of a quadratic equation are those functions in which the roots involved are such that the values of the expressions remain unaltered, when roots are interchanged.

- Formation of a quadratic equation if its roots are given;

  x^2 – (sum of the roots) x + product of the roots = 0

  $\Rightarrow \quad \boldsymbol{x^2 - (\alpha + \beta)\, x + \alpha\beta = 0.}$

- **Synthetic division** is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial.

- A system of equations having a common solution is called a system of **simultaneous equations**.