Unit-4

PARTIAL FRACTIONS

In this unit, students will learn how to

- *define proper, improper and rational fraction.*
- *resolve an algebraic fraction into partial fractions when its denominator consists of*
  - *non-repeated linear factors,*
  - *repeated linear factors,*
  - *non-repeated quadratic factors,*
  - *repeated quadratic factors.*

4.1. Fraction

The quotient of two numbers or algebraic expressions is called a **fraction.** The quotient is indicated by a bar (—). We write, the dividend above the bar and the divisor below the bar. For example, $\frac{x^2+2}{x-2}$ is a fraction with $x-2 \neq 0$. If $x-2=0$, then the fraction is not defined because $x-2=0 \Rightarrow x=2$ which makes the denominator of the fraction zero.

4.1.1 Rational Fraction

An expression of the form $\frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials in x with real coefficients and $D(x) \neq 0$, is called a **rational fraction.**

For example, $\frac{x^2+3}{(x+1)^2(x+2)}$ and $\frac{2x}{(x-1)(x+2)}$ are rational fractions.

4.1.2 Proper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ is called a proper fraction if degree of the polynomial $N(x)$ in the numerator is less than the degree of the polynomial $D(x)$ in the denominator. For example, $\frac{2}{x+1}, \frac{2x-3}{x^2+4}$ and $\frac{3x^2}{x^3+1}$ are proper fractions.

4.1.3 Improper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ is called an **improper fraction** if degree of the polynomial $N(x)$ is greater or equal to the degree of the polynomial $D(x)$.

e.g., $\frac{5x}{x+2}, \frac{3x^2+2}{x^2+7x+12}, \frac{6x^4}{x^3+1}$ are improper fractions.

Every improper fraction can be reduced by division to the sum of a polynomial and a proper fraction. This means that if degree of the numerator is greater or equal to the degree of the denominator, then we can divide $N(x)$ by $D(x)$ obtaining a quotient polynomial $Q(x)$ and a remainder polynomial $R(x)$, whose degree is less than the degree of $D(x)$.

Thus $\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$, with $D(x) \neq 0$. Where $Q(x)$ is quotient polynomial and $\frac{R(x)}{D(x)}$ is a proper fraction. For example, $\frac{x^2+1}{x+1}$ is an improper fraction.

$\therefore \quad \frac{x^2+1}{x+1} = (x-1) + \frac{2}{x+1}$ *i.e.,* an improper fraction $\frac{x^2+1}{x+1}$ has been resolved to a quotient polynomial $Q(x) = x-1$ and a proper fraction $\frac{2}{x+1}$

Example 1: Resolve the fraction $\frac{x^3-x^2+x+1}{x^2+5}$ into proper fraction.

Solution: Let $N(x) = x^3-x^2+x+1$ and $D(x) = x^2+5$

By long division, we have

$$\begin{array}{r|l} & x - 1 \\ \hline x^2 + 5 & x^3 - x^2 + x + 1 \\ & -x^3 \quad \pm 5x \\ \hline & -x^2 - 4x + 1 \\ & \mp x^2 \quad \mp 5 \\ \hline & -4x + 6 \end{array}$$

$$\frac{x^3 - x^2 + x + 1}{x^2 + 5} = (x - 1) + \frac{-4x + 6}{x^2 + 5}$$

Activity: Separate proper and improper fractions

(i) $\frac{x^2 + x + 1}{x^2 + 2}$ (ii) $\frac{2x + 5}{(x + 1)(x + 2)}$ (iii) $\frac{x^3 + x^2 + 1}{x^3 - 1}$ (iv) $\frac{2x}{(x - 1)(x - 2)}$

Activity: Convert the following improper fractions into proper fractions.

(i) $\frac{3x^2 - 2x - 1}{x^2 - x + 1}$ (ii) $\frac{6x^3 + 5x^2 - 6}{2x^2 - x - 1}$

4.2 Resolution of Fraction into Partial Fractions

Consider $\frac{1}{x - 1}$, $\frac{-2}{x + 1}$, $\frac{4}{x}$, a set of three fractions each of which is prefixed by a positive or negative sign. It is easy to find a single fraction, which is equal to the sum of these fractions.

Thus $\frac{1}{x - 1} - \frac{2}{x + 1} + \frac{4}{x} = \frac{x(x + 1) - 2x(x - 1) + 4(x - 1)(x + 1)}{x(x - 1)(x + 1)}$

$$= \frac{x^2 + x - 2x^2 + 2x + 4x^2 - 4}{x(x - 1)(x + 1)}$$

$$= \frac{3x^2 + 3x - 4}{x(x - 1)(x + 1)}$$

The single fraction $\frac{3x^2 + 3x - 4}{x(x - 1)(x + 1)}$ is the simplified form of the given fractions and is known as **resultant fraction**. The given fractions $\frac{1}{x - 1}$, $\frac{-2}{x + 1}$ and $\frac{4}{x}$ are called components or **partial fractions**. In this chapter, we shall be given a rational fraction (or resultant fraction) and required to find its partial fractions.

Every proper fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ can be resolved into an algebraic sum of partial fractions as follows:

4.2.1 Resolution of an algebraic fraction into partial fractions, when $D(x)$ consists of non-repeated linear factors.

Rule I: If linear factor $(ax + b)$ occurs as a factor of $D(x)$, then there is a partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to be found.

In $\frac{N(x)}{D(x)}$, the polynomial $D(x)$ may be written as,

$$D(x) = (a_1x + b_1)(a_2x + b_2) \ldots\ldots (a_nx + b_n) \text{ with all factors distinct.}$$

We have, $\frac{N(x)}{D(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \frac{A_3}{a_3x+b_3} + \ldots\ldots + \frac{A_n}{a_nx+b_n}$,

where $A_1\ A_2 \ldots A_n$ are constants to be determined. The following examples illustrate how we can find these constants:

Example 1: Resolve $\frac{5x+4}{(x-4)(x+2)}$ into partial fractions.

Solution: Let $\frac{5x+4}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$ (i)

Multiplying throughout by $(x-4)(x+2)$, we get

$$5x + 4 = A(x+2) + B(x-4) \quad \text{(ii)}$$

Equation (ii) is an identity, which holds good for all values of x and hence for $x = 4$ and $x = -2$.

Put $x - 4 = 0$ *i.e.*, $x = 4$ (factor corresponding to A) on both sides of the equation (ii),

we get $5(4) + 4 = A(4+2) \Rightarrow \boxed{A = 4}$

Put $x + 2 = 0$ *i.e.*, $x = -2$ (factor corresponding to B), we get

$$5(-2) + 4 = B(-2-4) \Rightarrow -6B = -6 \Rightarrow \boxed{B = 1}$$

Thus required partial fractions are $\frac{4}{x-4} + \frac{1}{x+2}$

Hence, $\frac{5x+4}{(x-4)(x+2)} = \frac{4}{x-4} + \frac{1}{x+2}$

This method is called the **zero's method**. This method is especially useful with linear factors in the denominator $D(x)$.

Identity: An identity is an equation, which is satisfied by all the values of the variables involved. For example, $2(x+1) = 2x + 2$ and $\frac{2x^2}{x} = 2x$ are identities, as these equations are satisfied by all values of x.

Example 2: Resolve $\dfrac{1}{3+x-2x^2}$ into partial fractions.

Solution: $\dfrac{1}{3+x-2x^2}$ can be written as for convenience $\dfrac{-1}{2x^2-x-2}$

The denominator $D(x) = 2x^2 - x - 3 = 2x^2 - 3x + 2x - 3$

$$= x(2x-3) + 1(2x-3) = (x+1)(2x-3)$$

Let, $\dfrac{-1}{2x^2-x-3} = \dfrac{-1}{(x+1)(2x-3)} = \dfrac{A}{x+1} + \dfrac{B}{2x-3}$

Multiplying both the sides by $(x+1)(2x-3)$, we get

$$-1 = A(2x-3) + B(x+1)$$

Equating coefficients of x and constants on both sides, we get

$2A + B = 0$ (i) $\qquad -3A + B = -1$ (ii)

Solving (i) and (ii), we get $A = \dfrac{1}{5}$ and $B = \dfrac{-2}{5}$

Thus, $\dfrac{1}{3+x-2x^2} = \dfrac{1}{5(x+1)} - \dfrac{2}{5(2x-3)}$

Note: General method applicable to resolve all rational fractions of the form $\dfrac{N(x)}{D(x)}$ is as follows:

(i) The numerator $N(x)$ must be of lower degree than the denominator $D(x)$.

(ii) If degree of $N(x)$ is greater than the degree of $D(x)$, then division is used and the remainder fraction $R(x)$ can be broken into partial fractions.

(iii) Make substitution of constants accordingly.

(iv) Multiply both the sides by L.C.M.

(v) Arrange the terms on both sides in descending order.

(vi) Equate the coefficients of like powers of x on both sides, we get as many as equations as there are constants in assumption.

(vii) Solving these equations, we can find the values of constants.

EXERCISE 4.1

Resolve into partial fractions.

1. $\dfrac{7x-9}{(x+1)(x-3)}$
2. $\dfrac{x-11}{(x-4)(x+3)}$
3. $\dfrac{3x-1}{x^2-1}$
4. $\dfrac{x-5}{x^2+2x-3}$
5. $\dfrac{3x+3}{(x-1)(x+2)}$
6. $\dfrac{7x-25}{(x-4)(x-3)}$
7. $\dfrac{x^2+2x+1}{(x-2)(x+3)}$
8. $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$

4.2.2 Resolution of a fraction when $D(x)$ consists of repeated linear factors.

Rule II: If a linear factor $(ax + b)$ occurs n times as a factor of $D(x)$, then there are n partial fractions of the form.

$\dfrac{A_1}{(ax+b)} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_n}{(ax+b)^n}$, where $A_1, A_2, \cdots, A_n$ are constants and $n \geq 2$ is a positive integer.

$$\therefore \quad \frac{N(x)}{D(x)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

The method of finding constants and resolving into partial fractions is explained by the following example.

Example: Resolve $\dfrac{1}{(x-1)^2\,(x-2)}$ into partial fractions.

Solution: Let, $\dfrac{1}{(x-1)^2\,(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-2}$

Multiplying both sides by $(x-1)^2\,(x-2)$, we get

$$1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

$$\Rightarrow \quad A(x^2 - 3x + 2) + B(x-2) + C(x^2 - 2x + 1) = 1 \qquad \text{(i)}$$

Since (i) is an identity and is true for all values of x

Put $x - 1 = 0$ or $x = 1$ in (i), we get

$B(1-2) = 1 \quad \Rightarrow \quad -B = 1$ or $B = -1$

Put $x - 2 = 0$ or $x = 2$ in (i), we get

$C(2-1)^2 = 1 \quad \Rightarrow \quad C = 1$

Equating coefficients of x^2 on both the sides of (i)

$A + C = 0 \quad \Rightarrow \quad A = -C$ so $A = -1$

Hence required partial fractions are

$$\frac{-1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{(x-2)}$$

Thus, $\dfrac{1}{(x-1)^2\,(x-2)} = \dfrac{1}{x+2} - \dfrac{1}{(x-1)} - \dfrac{1}{(x-1)^2}$

EXERCISE 4.2

Resolve into partial fractions.

1. $\dfrac{x^2 - 3x + 1}{(x-1)^2\,(x-2)}$
2. $\dfrac{x^2 + 7x + 11}{(x+2)^2\,(x+3)}$
3. $\dfrac{9}{(x-1)(x+2)^2}$
4. $\dfrac{x^4 + 1}{x^2\,(x-1)}$
5. $\dfrac{7x + 4}{(3x+2)(x+1)^2}$
6. $\dfrac{1}{(x-1)^2(x+1)}$
7. $\dfrac{3x^2 + 15x + 16}{(x+2)^2}$
8. $\dfrac{1}{(x^2-1)(x+1)}$

4.2.3 Resolution of fraction when $D(x)$ consists of non-repeated irreducible quadratic factors.

Rule III: If a quadratic factor $(ax^2 + bx + c)$ with $a \neq 0$ occurs once as a factor of $D(x)$, the partial fraction is of the form $\dfrac{Ax + B}{(ax^2 + bx + c)}$, where A and B are constants to be found.

Example: Resolve $\dfrac{11x + 3}{(x - 3)(x^2 + 9)}$ into partial fractions.

Solution: Let $\dfrac{11x + 3}{(x - 3)(x^2 + 9)} = \dfrac{A}{(x - 3)} + \dfrac{Bx + C}{x^2 + 9}$

Multiplying both the sides by $(x - 3)(x^2 + 9)$

$\Rightarrow \quad 11x + 3 = A(x^2 + 9) + (Bx + C)(x - 3)$

$\Rightarrow \quad 11x + 3 = A(x^2 + 9) + B(x^2 - 3x) + C(x - 3) \qquad$ (i)

Since (i) is an identity, we have on substituting $x = 3$

$33 + 3 = A(9 + 9) \quad \Rightarrow \quad 18A = 36 \quad \Rightarrow \quad A = 2$

Comparing the coefficients of x^2 and x on both the sides of (i), we get.

$A + B = 0 \quad \Rightarrow \quad B = -2$

$-3B + C = 11 \quad \Rightarrow \quad -3(-2) + C = 11 \quad \Rightarrow \quad C = 5$

Therefore, the partial fractions are $\dfrac{2}{x - 3} + \dfrac{-2x + 5}{x^2 + 9}$

Thus, $\dfrac{11x + 3}{(x - 3)(x^2 + 9)} = \dfrac{2}{x - 3} + \dfrac{-2x + 5}{x^2 + 9}$

EXERCISE 4.3

Resolve into partial fractions.

1. $\dfrac{3x - 11}{(x + 3)(x^2 + 1)}$
2. $\dfrac{3x + 7}{(x^2 + 1)(x + 3)}$
3. $\dfrac{1}{(x + 1)(x^2 + 1)}$
4. $\dfrac{9x - 7}{(x + 3)(x^2 + 1)}$
5. $\dfrac{3x + 7}{(x + 3)(x^2 + 4)}$
6. $\dfrac{x^2}{(x + 2)(x^2 + 4)}$
7. $\dfrac{1}{x^3 + 1}$ $\quad \left[\text{Hint: } \dfrac{1}{x^3 + 1} = \dfrac{1}{(x + 1)(x^2 - x + 1)}\right]$
8. $\dfrac{x^2 + 1}{x^3 + 1}$

4.2.4 Resolution of a fraction when $D(x)$ has repeated irreducible quadratic factors.

Rule IV: If a quadratic factor $(ax^2 + bx + c)$ with $a \neq 0$, occurs twice in the denominator, the corresponding partial fractions are

$$\frac{Ax + B}{(ax^2 + bx + c)} + \frac{Cx + D}{(ax^2 + bx + c)^2}$$

The constants A, B, C and D are found in the usual way.

Example 1: Resolve $\dfrac{x^3 - 2x^2 - 2}{(x^2+1)^2}$ into partial fractions.

Solution: $\dfrac{x^3 - 2x^2 - 2}{(x^2+1)^2}$ is a proper fraction as degree of numerator is less than the degree of denominator.

Let $\dfrac{x^3 - 2x^2 - 2}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

Multiplying both the sides by $(x^2+1)^2$, we have

$$x^3 - 2x^2 - 2 = (Ax+B)(x^2+1) + Cx + D$$

$$x^3 - 2x^2 - 2 = A(x^3 + x) + B(x^2+1) + Cx + D \qquad \text{(i)}$$

Equating the coefficients of x^3, x^2, x and constant on both the sides of (i).

Coefficients of x^3: $A = 1$

Coefficients of x^2: $B = -2$

Coefficients of x: $A + C = 0 \quad \Rightarrow \quad C = -1$

Constants: $B + D = -2$

$D = -2 - B = -2 - (-2) = -2 + 2 = 0 \quad \Rightarrow \quad D = 0$

Thus $\dfrac{x^3 - 2x^2 - 2}{(x^2+1)^2} = \dfrac{x-2}{x^2+1} + \dfrac{-x+0}{(x^2+1)^2} = \dfrac{x-2}{x^2+1} - \dfrac{x}{(x^2+1)^2}$

Example 2: Resolve $\dfrac{2x+1}{(x-1)(x^2+1)^2}$ into partial fractions.

Solution: Assume that $\dfrac{2x+1}{(x-1)(x^2+1)^2} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

Multiplying both the sides by $(x-1)(x^2+1)^2$

$$2x + 1 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1) \qquad \text{(i)}$$

Now we use zeros' method. Put $x - 1 = 0$ or $x = 1$ in (i), we get

$$3 = A(1+1)^2 \quad \Rightarrow \quad A = \frac{3}{4}$$

Now writing terms of (i) in descending order.

$2x + 1 = A(x^4 + 2x^2 + 1) + Bx(x^3 - x^2 + x - 1) + C(x^3 - x^2 + x - 1) + D(x^2 - x) + E(x-1)$

or $2x + 1 = A(x^4 + 2x^2 + 1) + B(x^4 - x^3 + x^2 - x) + C(x^3 - x^2 + x - 1) + D(x^2 - x) + E(x-1)$

Equating coefficients of x^4, x^3, x^2, and x on both the sides.

Coefficients of x^4: $A + B = 0 \quad \Rightarrow \quad B = \dfrac{-3}{4}$

Coefficients of x^3: $-B + C = 0 \quad \Rightarrow \quad C = \dfrac{-3}{4}$

Coefficients of x^2: $2A + B - C + D = 0 \quad \Rightarrow \quad D = \dfrac{-3}{2}$

Coefficients of x: $-B + C - D + E = 2$

$\dfrac{3}{4} - \dfrac{3}{4} + \dfrac{3}{2} + E = 2 \quad \Rightarrow \quad E = 2 - \dfrac{3}{2} = \dfrac{1}{2}$

Thus required partial fractions are $\frac{3}{4(x-1)} + \frac{\frac{-3}{4}x - \frac{3}{4}}{x^2+1} + \frac{\frac{-3}{2}x + \frac{1}{2}}{(x^2+1)^2}$

$$\therefore \quad \frac{2x+1}{(x-1)(x^2+1)^2} = \frac{3}{4(x-1)} - \frac{3(x+1)}{4(x^2+1)} - \frac{(3x-1)}{2(x^2+1)^2}$$

EXERCISE 4.4

Resolve into partial fractions.

1. $\frac{x^3}{(x^2+4)^2}$
2. $\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}$
3. $\frac{x^2}{(x+1)(x^2+1)^2}$
4. $\frac{x^2}{(x-1)(x^2+1)^2}$
5. $\frac{x^4}{(x^2+2)^2}$
6. $\frac{x^5}{(x^2+1)^2}$

MISCELLANEOUS EXERCISE - 4

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) The identity $(5x+4)^2 = 25x^2 + 40x + 16$ is true for

(a) one value of x (b) two values of x
(c) all values of x (d) none of these

(ii) A function of the form $f(x) = \frac{N(x)}{D(x)}$, with $D(x) \neq 0$, where $N(x)$ and $D(x)$ are polynomials in x is called

(a) an identity (b) an equation
(c) a fraction (d) none of these

(iii) A fraction in which the degree of the numerator is greater or equal to the degree of denominator is called

(a) a proper fraction (b) an improper fraction
(c) an equation (d) algebraic relation

(iv) A fraction in which the degree of numerator is less than the degree of the denominator is called

(a) an equation (b) an improper fraction
(c) an identity (d) a proper fraction

(v) $\frac{2x+1}{(x+1)(x-1)}$ is:

(a) an improper fraction (b) an equation
(c) a proper fraction (d) none of these

(vi) $(x+3)^2 = x^2 + 6x + 9$ is

(*a*) a linear equation (*b*) an equation

(*c*) an identity (*d*) none of these

(vii) $\frac{x^3+1}{(x-1)(x+2)}$ is

(*a*) a proper fraction (*b*) an improper fraction

(*c*) an identity (*d*) a constant term

(viii) Partial fractions of $\frac{x-2}{(x-1)(x+2)}$ are of the form

(*a*) $\frac{A}{x-1} + \frac{B}{x+2}$ (*b*) $\frac{Ax}{x-1} + \frac{B}{x+2}$

(*c*) $\frac{A}{x-1} + \frac{Bx+C}{x+2}$ (*d*) $\frac{Ax+B}{x-1} + \frac{C}{x+2}$

(ix) Partial fractions of $\frac{x+2}{(x+1)(x^2+2)}$ are of the form

(*a*) $\frac{A}{x+1} + \frac{B}{x^2+2}$ (*b*) $\frac{A}{x+1} + \frac{Bx+C}{x^2+2}$

(*c*) $\frac{Ax+B}{x+1} + \frac{C}{x^2+2}$ (*d*) $\frac{A}{x+1} + \frac{Bx}{x^2+2}$

(x) Partial fractions of $\frac{x^2+1}{(x+1)(x-1)}$ are of the form

(*a*) $\frac{A}{x+1} + \frac{B}{x-1}$ (*b*) $1 + \frac{A}{x+1} + \frac{Bx+C}{x-1}$

(*c*) $1 + \frac{A}{x+1} + \frac{B}{x-1}$ (*d*) $\frac{Ax+B}{(x+1)} + \frac{C}{x-1}$

2. Write short answers of the following questions.

(i) Define a rational fraction.

(ii) What is a proper fraction?

(iii) What is an improper fraction?

(iv) What are partial fractions?

(v) How can we make partial fractions of $\frac{x-2}{(x+2)\,(x+3)}$?

(vi) Resolve $\frac{1}{x^2-1}$ into partial fractions.

(vii) Find partial fractions of $\frac{3}{(x+1)(x-1)}$.

(viii) Resolve $\frac{x}{(x-3)^2}$ into partial fractions.

(ix) How we can make the partial fractions of $\frac{x}{(x+a)(x-a)}$?

(x) Whether $(x+3)^2 = x^2 + 6x + 9$ is an identity?

SUMMARY

- A **fraction** is an indicated quotient of two numbers or algebraic expressions.
- An expression of the form $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ and $N(x)$ and $D(x)$ are polynomials in x with real coefficients, is called a **rational fraction.** Every fractional expression can be expressed as a quotient of two polynomials.
- A rational fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ is called a **proper fraction** if degree of the polynomial $N(x)$, in the numerator is less than the degree of the polynomial $D(x)$, in the denominator.
- A rational fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ is called an **improper fraction** if degree of the polynomial $N(x)$ is greater or equal to the degree of the polynomial $D(x)$.
- **Partial fractions:** Decomposition of resultant fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$, when

(a) $D(x)$ consists of non-repeated linear factors.
(b) $D(x)$ consists of repeated linear factors.
(c) $D(x)$ consists of non-repeated irreducible quadratic factors.
(d) $D(x)$ consists of repeated irreducible quadratic factors.