

## PROJECTION OF A SIDE OF A TRIANGLE

*In this unit, students will learn how to*

*Prove the following theorems along with corollaries and apply them to solve appropriate problems.*

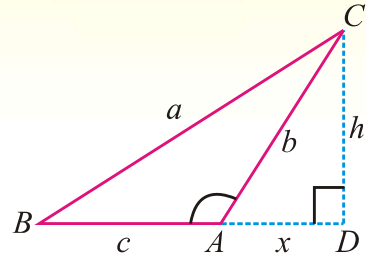
- ✎ *In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.*
- ✎ *In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.*
- ✎ *In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' Theorem).*

## THEOREM 1

**8.1(i)** In an obtuse angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.

**Given:**  $ABC$  is a triangle having an obtuse angle  $BAC$  at  $A$ . Draw  $\overline{CD}$  perpendicular on  $\overline{BA}$  produced. So that  $\overline{AD}$  is the projection of  $\overline{AC}$  on  $\overline{BA}$  produced.

Take  $m\overline{BC} = a$ ,  $m\overline{CA} = b$ ,  $m\overline{AB} = c$ ,  
 $m\overline{AD} = x$  and  $m\overline{CD} = h$ .



**To prove:**  $(BC)^2 = (AC)^2 + (AB)^2 + 2(m\overline{AB})(m\overline{AD})$   
*i.e.*,  $a^2 = b^2 + c^2 + 2cx$

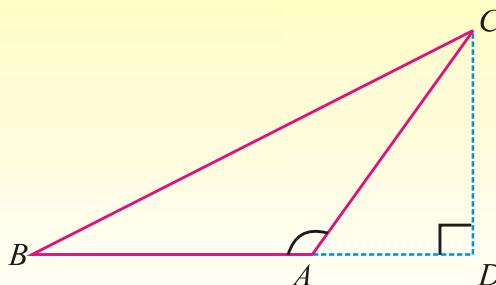
**Proof:**

Statements	Reasons
In $\angle rt\Delta CDA$ ,	
$m\angle CDA = 90^\circ$	Given
$\therefore (AC)^2 = (AD)^2 + (CD)^2$	Pythagoras Theorem
or $b^2 = x^2 + h^2$ (i)	
In $\angle rt\Delta CDB$ ,	
$m\angle CDB = 90^\circ$	Given
$\therefore (BC)^2 = (BD)^2 + (CD)^2$	Pythagoras Theorem
or $a^2 = (c + x)^2 + h^2$	$\overline{BD} = \overline{BA} + \overline{AD}$
$= c^2 + 2cx + x^2 + h^2$ (ii)	
Hence $a^2 = c^2 + 2cx + b^2$	Using (i) and (ii)
<i>i.e.</i> , $a^2 = b^2 + c^2 + 2cx$	
or $(BC)^2 = (AC)^2 + (AB)^2 + 2(m\overline{AB})(m\overline{AD})$	

**Example:** In a  $\triangle ABC$  with obtuse angle at  $A$ , if  $\overline{CD}$  is an altitude on  $\overline{BA}$  produced and  $m\overline{AC} = m\overline{AB}$

Then prove that  $(BC)^2 = 2(AB)(BD)$

**Given:** In a  $\triangle ABC$ ,  $m\angle A$  is obtuse  $m\overline{AC} = m\overline{AB}$  and  $\overline{CD}$  being altitude on  $\overline{BA}$  produced.



**To prove:**  $(BC)^2 = 2(m\overline{AB})(m\overline{BD})$

**Proof:** In a  $\triangle ABC$ , having obtuse angle  $BAC$  at  $A$ .

Statements	Reasons
$(BC)^2 = (BA)^2 + (AC)^2 + 2(m\overline{BA})(m\overline{AD})$	By Theorem 1
$= (AB)^2 + (AB)^2 + 2(m\overline{AB})(m\overline{AD})$	Given
$= 2(AB)^2 + 2(m\overline{AB})(m\overline{AD})$	
$(BC)^2 = 2m\overline{AB}(m\overline{AB} + m\overline{AD})$	On the line segment $BD$ ,
$= 2m\overline{AB}(m\overline{BD})$	Point $A$ is between $B$ and $D$

### EXERCISE 8.1

1. Given  $m\overline{AC} = 1\text{cm}$ ,  $m\overline{BC} = 2\text{cm}$ ,  $m\angle C = 120^\circ$ .

Compute the length  $AB$  and the area of  $\triangle ABC$ .

Hint:  $(AB)^2 = (AC)^2 + (BC)^2 + 2m\overline{AC} \cdot m\overline{CD}$

where  $(m\overline{CD}) = (m\overline{BC}) \cos(180^\circ - m\angle C)$  (Use theorem 1).

2. Find  $m\overline{AC}$  if in  $\triangle ABC$   $m\overline{BC} = 6\text{cm}$ ,  $m\overline{AB} = 4\sqrt{2}\text{cm}$  and  $m\angle ABC = 135^\circ$ .

## THEOREM 2

**8.1(ii)** In any triangle, the square on the side opposite to acute angle is equal to sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.

**Given:**  $\triangle ABC$  with an acute angle  $CAB$  at  $A$ .

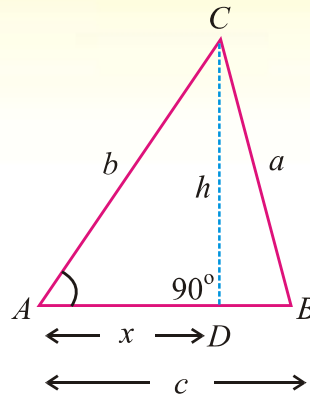
Take  $m\overline{BC} = a$ ,  $m\overline{CA} = b$  and  $m\overline{AB} = c$

Draw  $\overline{CD} \perp \overline{AB}$  so that  $\overline{AD}$  is projection of  $\overline{AC}$  on  $\overline{AB}$

Also,  $m\overline{AD} = x$  and  $m\overline{CD} = h$

**To prove:**  $(BC)^2 = (AC)^2 + (AB)^2 - 2(AB)(AD)$

i.e.,  $a^2 = b^2 + c^2 - 2cx$



**Proof:**

Statements	Reasons
In $\angle rt \triangle CDA$	
$m\angle CDA = 90^\circ$	Given
$(AC)^2 = (AD)^2 + (CD)^2$	Pythagoras Theorem
i.e., $b^2 = x^2 + h^2$ (i)	
In $\angle rt \triangle CDB$ ,	
$m\angle CDB = 90^\circ$	Given
$(BC)^2 = (BD)^2 + (CD)^2$	Pythagoras Theorem
$a^2 = (c-x)^2 + h^2$	From the figure
or $a^2 = c^2 - 2cx + x^2 + h^2$ (ii)	
$a^2 = c^2 - 2cx + b^2$	Using (i) and (ii)
Hence, $a^2 = b^2 + c^2 - 2cx$	
i.e., $(BC)^2 = (AC)^2 + (AB)^2 - 2(AB)(AD)$	

### THEOREM 3

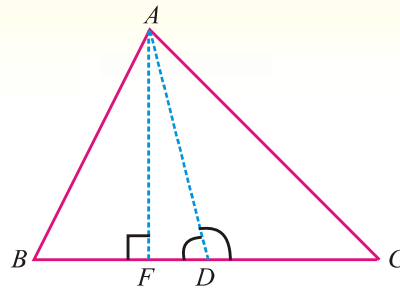
#### (APOLLONIUS' THEOREM)

8.1.(iii) In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.

**Given:** In a  $\triangle ABC$ , the median  $\overline{AD}$  bisects  $\overline{BC}$ .  
*i.e.,  $m\overline{BD} = m\overline{CD}$*

**To prove:**  $(AB)^2 + (AC)^2 = 2(BD)^2 + 2(AD)^2$

**Construction:** Draw  $\overline{AF} \perp \overline{BC}$



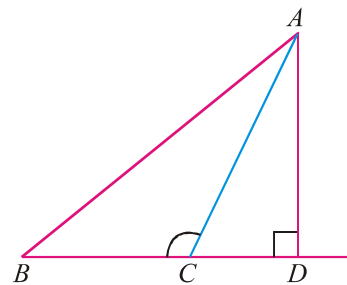
**Proof:**

Statements	Reasons
In $\triangle ADB$ Since $\angle ADB$ is acute at $D$	
$\therefore (AB)^2 = (BD)^2 + (AD)^2 - 2 m\overline{BD} \cdot m\overline{FD}$ (i)	Using Theorem 2
Now in $\triangle ADC$ since $\angle ADC$ is obtuse at $D$	
$\therefore (AC)^2 = (CD)^2 + (AD)^2 + 2 m\overline{CD} \cdot m\overline{FD}$ $= (BD)^2 + (AD)^2 + 2 m\overline{BD} \cdot m\overline{FD}$ (ii)	Using Theorem 1
Thus $(AB)^2 + (AC)^2 = 2(BD)^2 + 2(AD)^2$	Adding (i) and (ii)

**Example 1:** In  $\triangle ABC$ ,  $\angle C$  is obtuse,  $\overline{AD} \perp \overline{BC}$  produced, whereas  $\overline{BD}$  is projection of  $\overline{AB}$  on  $\overline{BC}$ . Prove that  $(AC)^2 = (AB)^2 + (BC)^2 - 2m\overline{BC} \cdot m\overline{BD}$

**Given:** In a  $\triangle ABC$ ,  $\angle BCA$  is obtuse so that  $\angle B$  is acute,  $\overline{AD} \perp \overline{BC}$  produced, whereas  $\overline{BD}$  is projection of  $\overline{AB}$  on  $\overline{BC}$  produced.

**To prove:**  $(AC)^2 = (AB)^2 + (BC)^2 - 2m\overline{BC} \cdot m\overline{BD}$



**Proof:**

Statements	Reasons
In $\angle rt \triangle ABD$ $(AB)^2 = (AD)^2 + (BD)^2$ (i)	Pythagoras Theorem
In $\angle rt \triangle ACD$ $(AC)^2 = (AD)^2 + (CD)^2$ (ii)	Pythagoras Theorem
or $(AC)^2 = (AD)^2 + (BD - BC)^2$ $(AC)^2 = (AD)^2 + (BD)^2 + (BC)^2 - 2BC \cdot BD$ (iii)	$m\overline{BC} + m\overline{CD} = m\overline{BD}$
$(AC)^2 = (AB)^2 + (BC)^2 - 2BC \cdot BD$	Using (i) and (iii)

**Example 2:** In an Isosceles  $\triangle ABC$ , if  $m\overline{AB} = m\overline{AC}$  and  $\overline{BE} \perp \overline{AC}$ , then prove that

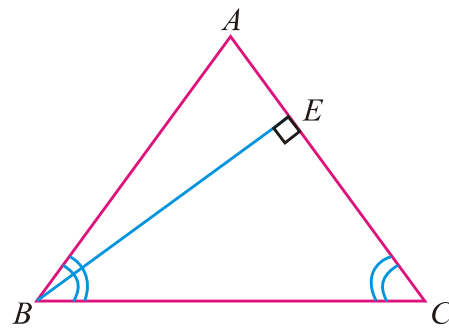
$$(BC)^2 = 2m\overline{AC} \cdot m\overline{CE}$$

**Given:** In an Isosceles  $\triangle ABC$

$$m\overline{AB} = m\overline{AC} \text{ and } \overline{BE} \perp \overline{AC}$$

whereas  $\overline{CE}$  is the projection of  $\overline{BC}$  upon on  $\overline{AC}$ .

**To prove:**  $(BC)^2 = 2m\overline{AC} \cdot m\overline{CE}$

**Proof:**

Statements	Reasons
In an isosceles $\triangle ABC$ with $m\overline{AB} = m\overline{AC}$ . If $\angle C$ is acute, then $(AB)^2 = (AC)^2 + (BC)^2 - 2m\overline{AC} \cdot m\overline{CE}$ $(AC)^2 = (AC)^2 + (BC)^2 - 2m\overline{AC} \cdot m\overline{CE}$ $\Rightarrow (BC)^2 - 2m\overline{AC} \cdot m\overline{CE} = 0$ or $(BC)^2 = 2m\overline{AC} \cdot m\overline{CE}$	By Theorem 2 Given $m\overline{AB} = m\overline{AC}$ Cancel $(\overline{AC})^2$ on both sides

## EXERCISE 8.2

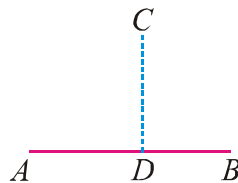
- In a  $\triangle ABC$  calculate  $m\overline{BC}$  when  $m\overline{AB} = 6\text{cm}$ ,  $m\overline{AC} = 4\text{cm}$  and  $m\angle A = 60^\circ$ .
- In a  $\triangle ABC$ ,  $m\overline{AB} = 6\text{ cm}$ ,  $m\overline{BC} = 8\text{ cm}$ ,  $m\overline{AC} = 9\text{ cm}$  and  $D$  is the mid point of side  $\overline{AC}$ . Find length of the median  $\overline{BD}$ .
- In a parallelogram  $ABCD$  prove that  $(AC)^2 + (BD)^2 = 2[(AB)^2 + (BC)^2]$

## MISCELLANEOUS EXERCISE 8

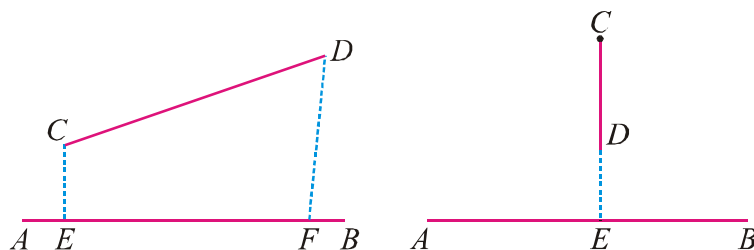
- In a  $\triangle ABC$ ,  $m\angle A = 60^\circ$ , prove that  $(BC)^2 = (AB)^2 + (AC)^2 - m\overline{AB} \cdot m\overline{AC}$ .
- In a  $\triangle ABC$ ,  $m\angle A = 45^\circ$ , prove that  $(BC)^2 = (AB)^2 + (AC)^2 - \sqrt{2} m\overline{AB} \cdot m\overline{AC}$ .
- In a  $\triangle ABC$ , calculate  $m\overline{BC}$  when  $m\overline{AB} = 5$  cm,  $m\overline{AC} = 4$  cm,  $m\angle A = 60^\circ$ .
- In a  $\triangle ABC$ , calculate  $m\overline{AC}$  when  $m\overline{AB} = 5$  cm,  $m\overline{BC} = 4\sqrt{2}$  cm,  $m\angle B = 45^\circ$ .
- In a triangle  $ABC$ ,  $m\overline{BC} = 21$  cm,  $m\overline{AC} = 17$  cm,  $m\overline{AB} = 10$  cm.  
Measure the length of projection of  $\overline{AC}$  upon  $\overline{BC}$ .
- In a triangle  $ABC$ ,  $m\overline{BC} = 21$  cm,  $m\overline{AC} = 17$  cm,  $m\overline{AB} = 10$  cm.  
Calculate the projection of  $\overline{AB}$  upon  $\overline{BC}$ .
- In a  $\triangle ABC$ ,  $a = 17$  cm,  $b = 15$  cm and  $c = 8$  cm. Find  $m\angle A$ .
- In a  $\triangle ABC$ ,  $a = 17$  cm,  $b = 15$  cm and  $c = 8$  cm find  $m\angle B$ .
- Whether the triangle with sides 5 cm, 7 cm, 8 cm is acute, obtuse or right angled.
- Whether the triangle with sides 8 cm, 15 cm, 17 cm is acute, obtuse or right angled.

## SUMMARY

- The **projection** of a given point on a line segment is the foot of  $\perp$  drawn from the point on that line segment. If  $\overline{CD} \perp \overline{AB}$ , then evidently  $D$  is the foot of perpendicular  $CD$  from the point  $C$  on the line segment  $AB$ .



- The projection of a line segment  $\overline{CD}$  on a line segment  $\overline{AB}$  is the portion  $\overline{EF}$  of the latter intercepted between feet of the perpendiculars drawn from  $C$  and  $D$ . However projection of a vertical line segment  $\overline{CD}$  on a line segment  $\overline{AB}$  is a point on  $\overline{AB}$  which is of **zero dimension**.



- In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' Theorem).