Chapter 2

VECTORS AND EQUILIBRIUM

Learning Objectives

At the end of this chapter the students will be able to:

1. Understand and use rectangular coordinate system.
2. Understand the idea of unit vector, null vector and position vector.
3. Represent a vector as two perpendicular components (rectangular components).
4. Understand the rule of vector addition and extend it to add vectors using rectangular components.
5. Understand multiplication of vectors and solve problems.
6. Define the moment of force or torque.
7. Appreciate the use of the torque due to a force.
8. Show an understanding that when there is no resultant force and no resultant torque, a system is in equilibrium.
9. Appreciate the applications of the principle of moments.
10. Apply the knowledge gained to solve problems on statics.

Physical quantities that have both numerical and directional properties are called vectors. This chapter is concerned with the vector algebra and its applications in problems of equilibrium of forces and equilibrium of torques.

2.1 BASIC CONCEPTS OF VECTORS

(i) Vectors

As we have studied in school physics, there are some physical quantities which require both magnitude and direction for their complete description, such as velocity, acceleration and force. They are called vectors. In books, vectors are usually denoted by bold face characters such as **A**, **d**, **r** and **v** while in handwriting, we put an arrowhead over the letter e.g. $\vec{d}$. If we wish to refer only to the magnitude of a vector **d** we use light face type such as *d*.

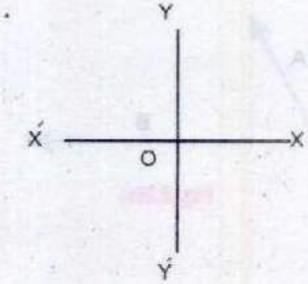


Fig. 2.1(a)

A vector is represented graphically by a directed line segment with an arrowhead. The length of the line segment, according to a chosen scale, corresponds to the magnitude of the vector.

(ii) Rectangular coordinate system

Two reference lines drawn at right angles to each other as shown in Fig. 2.1 (a) are known as coordinate axes and their point of intersection is known as origin. This system of coordinate axes is called Cartesian or rectangular coordinate system.

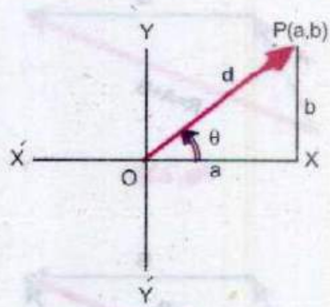


Fig. 2.1(b)

One of the lines is named as x-axis, and the other the y-axis. Usually the x-axis is taken as the horizontal axis, with the positive direction to the right, and the y-axis as the vertical axis with the positive direction upward.

The direction of a vector in a plane is denoted by the angle which the representative line of the vector makes with positive x-axis in the anti-clock wise direction, as shown in Fig 2.1 (b). The point P shown in Fig 2.1 (b) has coordinates (a,b). This notation means that if we start at the origin, we can reach P by moving 'a' units along the positive x-axis and then 'b' units along the positive y-axis.

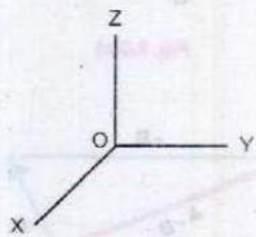


Fig. 2.2(a)

The direction of a vector in space requires another axis which is at right angle to both x and y axes, as shown in Fig 2.2 (a). The third axis is called z-axis.

The direction of a vector in space is specified by the three angles which the representative line of the vector makes with x, y and z axes respectively as shown in Fig 2.2 (b). The point P of a vector **A** is thus denoted by three coordinates (a, b, c).

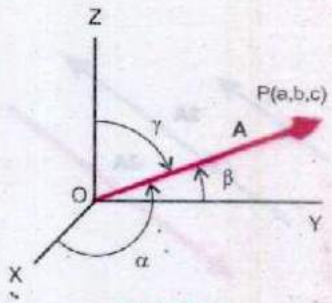


Fig. 2.2(b)

(iii) Addition of Vectors

Given two vectors **A** and **B** as shown in Fig 2.3 (a), their sum is obtained by drawing their representative lines in such a way that tail of vector **B** coincides with the head of the vector **A**. Now if we join the tail of **A** to the head of **B**, as shown in

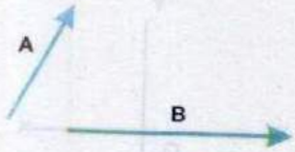


Fig. 2.3(a)

the Fig. 2.3(b), the line joining the tail of **A** to the head of **B** will represent the vector sum (**A+B**) in magnitude and direction. The vector sum is also called resultant and is indicated by **R**. Thus **R** = **A+B**. This is known as head to tail rule of vector addition. This rule can be extended to find the sum of any number of vectors. Similarly the sum **B + A** is illustrated by black lines in Fig 2.3 (c). The answer is same resultant **R** as indicated by the red line. Therefore, we can say that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad \text{..........} \quad (2.1)$$

Fig. 2.3(b)

So the vector addition is said to be commutative. It means that when vectors are added, the result is the same for any order of addition.

(iv) Resultant Vector

The resultant of a number of vectors of the same kind –force vectors for example, is that single vector which would have the same effect as all the original vectors taken together.

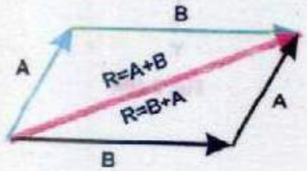


Fig. 2.3(c)

(v) Vector Subtraction

The subtraction of a vector is equivalent to the addition of the same vector with its direction reversed. Thus, to subtract vector **B** from vector **A**, reverse the direction of **B** and add it to **A**, as shown in Fig. 2.3 (d).

A - **B** = **A** + (-**B**) where (-**B**) is negative vector of **B**

(vi) Multiplication of a Vector by a Scalar

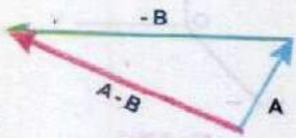


Fig. 2.3(d)

The product of a vector **A** and a number $n > 0$ is defined to be a new vector n**A** having the same direction as **A** but a magnitude n times the magnitude of **A** as illustrated in Fig. 2.4. If the vector is multiplied by a negative number, then its direction is reversed.

In the event that n represents a scalar quantity, the product n**A** will correspond to a new physical quantity and the dimensions of the resulting vector will be the product of the dimensions of the two quantities which were multiplied together. For example, when velocity is multiplied by scalar mass *m*, the product is a new vector quantity called momentum having the dimensions as those of mass and velocity.

Fig. 2.4

(vii) Unit Vector

A unit vector in a given direction is a vector with magnitude one in that direction. It is used to represent the direction of a vector.

A unit vector in the direction of **A** is written as $\hat{\mathbf{A}}$, which we read as 'A hat', thus

$$\mathbf{A} = A\hat{\mathbf{A}}$$

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{A} \quad \text{..........} \quad (2.2)$$

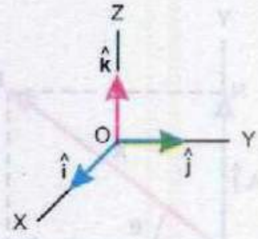


Fig. 2.5(a)

The direction along x, y and z axes are generally represented by unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ respectively (Fig. 2.5 a). The use of unit vectors is not restricted to Cartesian coordinate system only. Unit vectors may be defined for any direction. Two of the more frequently used unit vectors are the vector $\hat{\mathbf{r}}$ which represents the direction of the vector **r** (Fig. 2.5 b) and the vector $\hat{\mathbf{n}}$ which represents the direction of a normal drawn on a specified surface as shown in Fig 2.5 (c).

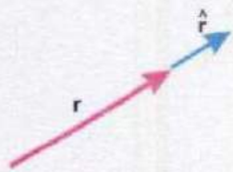


Fig. 2.5(b)

(viii) Null Vector

Null vector is a vector of zero magnitude and arbitrary direction. Forexample, the sum of a vector and its negative vector is a null vector.

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0} \quad \text{..........} \quad (2.3)$$

(ix) Equal Vectors

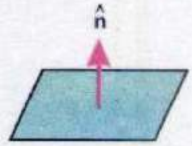


Fig. 2.5(c)

Two vectors **A** and **B** are said to be equal if they have the same magnitude and direction, regardless of the position of their initial points.

This means that parallel vectors of the same magnitude are equal to each other.

(x) Rectangular Components of a Vector

A component of a vector is its effective value in a given direction. A vector may be considered as the resultant of its component vectors along the specified directions. It is usually convenient to resolve a vector into components along mutually perpendicular directions. Such components are called rectangular components.

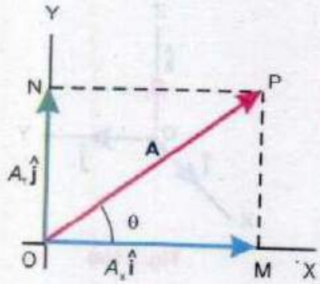


Fig. 2.6

Let there be a vector **A** represented by OP making angle θ with the x-axis. Draw projection OM of vector OP on x-axis and projection ON of vector OP on y-axis as shown in Fig.2.6. Projection OM being along x-direction is represented by $A_x\hat{\mathbf{i}}$ and projection ON = MP along y-direction is represented by $A_y\hat{\mathbf{j}}$. By head and tail rule

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \quad \dots\dots\dots \quad (2.4)$$

Thus $A_x\hat{\mathbf{i}}$ and $A_y\hat{\mathbf{j}}$ are the components of vector **A**. Since these are at right angle to each other, hence, they are called rectangular components of **A**. Considering the right angled triangle OMP, the magnitude of $A_x\hat{\mathbf{i}}$ or x-component of **A** is

$$A_x = A\cos\theta \quad \dots\dots\dots \quad (2.5)$$

And that of $A_y\hat{\mathbf{j}}$ or y-component of **A** is

$$A_y = A\sin\theta \quad \dots\dots\dots \quad (2.6)$$

(xi) Determination of a Vector from its Rectangular Components

If the rectangular components of a vector, as shown in Fig. 2.6, are given, we can find out the magnitude of the vector by using Pythagorean theorem.

In the right angled Δ OMP,

$$OP^2 = OM^2 + MP^2$$

or

$$A^2 = A_x^{\,2} + A_y^{\,2} \quad \dots\dots\dots \quad (2.7)$$

or

$$A = \sqrt{A_x^{\,2} + A_y^{\,2}}$$

and direction θ is given by $\tan\theta = \dfrac{MP}{OM} = \dfrac{A_y}{A_x}$

or

$$\theta = \tan^{-1}\frac{A_y}{A_x} \quad \dots\dots\dots \quad (2.8)$$

(xii) Position Vector

The position vector **r** is a vector that describes the location of a point with respect to the origin. It is represented by a straight line drawn in such a way that its tail coincides with the origin and the head with point P (a,b) as shown in Fig. 2.7(a). The projections of position vector **r** on the x and y axes are the coordinates a and b and they are the rectangular components of the vector **r**. Hence

$$\mathbf{r} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} \quad \text{and} \quad r = \sqrt{a^2 + b^2} \quad \ldots\ldots\ldots \quad (2.9)$$

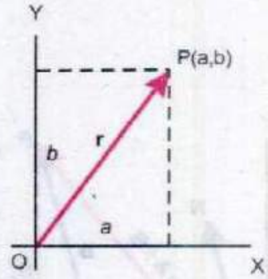


Fig. 2.7(a)

In three dimensional space, the position vector of a point P (a,b,c) is shown in Fig. 2.7 (b) and is represented by

$$\mathbf{r} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}} \quad \text{and} \quad r = \sqrt{a^2 + b^2 + c^2} \quad \ldots\ldots\ldots \quad (2.10)$$

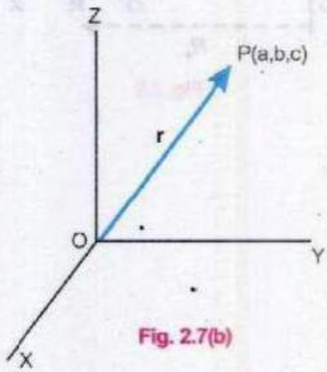


Fig. 2.7(b)

Example 2.1: The positions of two aeroplanes at any instant are represented by two points A (2, 3, 4) and B(5, 6, 7) from an origin O in km as shown in Fig. 2.8.

(i) What are their position vectors?

(ii) Calculate the distance between the two aeroplanes.

Solution: (i) A position vector **r** is given by

$$\mathbf{r} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$$

Thus position vector of first aeroplane A is

$$\mathbf{OA} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

And position vector of the second aeroplane B is

$$\mathbf{OB} = 5\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$$

By head and tail rule

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB}$$

Therefore, the distance between two aeroplanes is given by

$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = (5\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 7\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$$

$$= (3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

Fig. 2.8

Magnitude of vector **AB** is the distance between the position of two aeroplanes which is then:

$$AB = \sqrt{(3\text{km})^2 + (3\text{km})^2 + (3\text{km})^2} = 5.2\,\text{km}$$

2.2 VECTOR ADDITION BY RECTANGULAR COMPONENTS

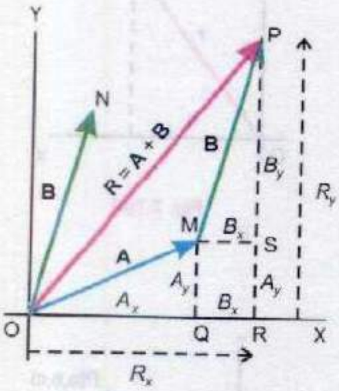


Fig. 2.9

Let **A** and **B** be two vectors which are represented by two directed lines OM and ON respectively. The vector **B** is added to **A** by the head to tail rule of vector addition (Fig 2.9). Thus the resultant vector **R** = **A** + **B** is given, in direction and magnitude, by the vector **OP**.

In the Fig 2.9 $\mathbf{A}_x$, $\mathbf{B}_x$ and $\mathbf{R}_x$ are the x components of the vectors **A**, **B** and **R** and their magnitudes are given by the lines OQ, MS, and OR respectively. But

$$OR = OQ + QR$$

or $$OR = OQ + MS$$

or $$R_x = A_x + B_x \quad \text{..........} \quad (2.11)$$

which means that the sum of the magnitudes of x-components of two vectors which are to be added, is equal to the x-component of the resultant. Similarly the sum of the magnitudes of y-components of two vectors is equal to the magnitude of y-component of the resultant, that is

$$R_y = A_y + B_y \quad \text{..........} \quad (2.12)$$

Since R_x and R_y are the rectangular components of the resultant vector **R**, hence

$$\mathbf{R} = R_x\hat{i} + R_y\hat{j}$$

or $$\mathbf{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

The magnitude of the resultant vector **R** is thus given as

$$R = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2} \quad \text{.........} \quad (2.13)$$

and the direction of the resultant vector is determined from

$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{(A_y + B_y)}{(A_x + B_x)}$$

and $$\theta = \tan^{-1}\frac{(A_y + B_y)}{(A_x + B_x)} \quad .... \quad (2.14)$$

Similarly for any number of coplanar vectors **A**, **B**, **C**...., we can write

$$R = \sqrt{(A_x + B_x + C_x + ...)^2 + (A_y + B_y + C_y + ...)^2} \quad ...... \quad (2.15)$$

and $$\theta = \tan^{-1}\frac{(A_y + B_y + C_y + ...)}{(A_x + B_x + C_x + ...)} \quad .... \quad (2.16)$$

The vector addition by rectangular components consists of the following steps.

i) Find x and y components of all given vectors.

ii) Find x-component R_x of the resultant vector by adding the x-components of all the vectors.

iii) Find y-component R_y of the resultant vector by adding the y-components of all the vectors.

iv) Find the magnitude of resultant vector **R** using

$$R = \sqrt{R_x^{\,2} + R_y^{\,2}}$$

v) Find the direction of resultant vector **R** by using

$$\theta = \tan^{-1}\frac{R_y}{R_x}$$

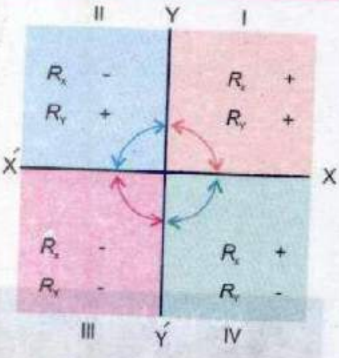
where θ is the angle, which the resultant vector makes with positive x-axis. The signs of R_x and R_y determine the quadrant in which resultant vector lies. For that purpose proceed as given below.

Irrespective of the sign of R_x and R_y, determine the value of $\tan^{-1}\frac{R_y}{R_x} = \phi$ from the calculator or by consulting trigonometric tables. Knowing the value of ϕ, angle θ is determined as follows.

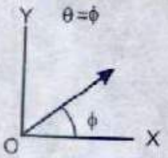
Do You Know?

The Chinese acrobats in this incredible balancing act are in equilibrium.

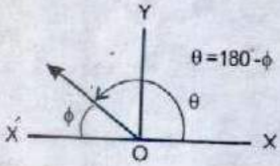
Table 2.1



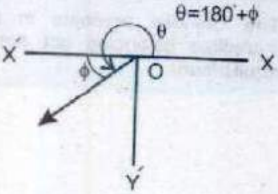
Ist quadrant



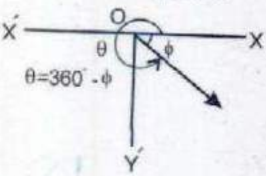
2nd quadrant



3rd quadrant



4th quadrant



a) If both R_x and R_y are positive, then the resultant lies in the first quadrant and its direction is $\theta = \phi$.

b) If R_x is −ive and R_y is +ive, the resultant lies in the second quadrant and its direction is $\theta = 180^\circ - \phi$.

c) If both R_x and R_y are −ive, the resultant lies in the third quadrant and its direction is $\theta = 180^\circ + \phi$.

d) If R_x is positive and R_y is negative, the resultant lies in the fourth quadrant and its direction is $\theta = 360^\circ - \phi$.

Example 2.2: Two forces of magnitude 10 N and 20 N act on a body in directions making angles 30° and 60° respectively with x-axis. Find the resultant force.

Solution:

Step (i) **x-components**

The x-component of the first force $= F_{1x} = F_1 \cos 30^\circ$

$= 10\text{ N} \times 0.866 = 8.66\text{ N}$

The x-component of second force $= F_{2x} = F_2 \cos 60^\circ$

$= 20\text{ N} \times 0.5 = 10\text{ N}$

y-components

The y-component of the first force $= F_{1y} = F_1 \sin 30^\circ$

$= 10\text{ N} \times 0.5 = 5\text{ N}$

The y-component of second force $= F_{2y} = F_2 \sin 60^\circ$

$= 20\text{ N} \times 0.866 = 17.32\text{ N}$

Step (ii)

The magnitude of x component F_x of the resultant force **F**

$$F_x = F_{1x} + F_{2x}$$

$$F_x = 8.66\text{ N} + 10\text{ N} = 18.66\text{ N}$$

Step (iii)

The magnitude of y component F_y of the resultant force **F**

$$F_y = F_{1y} + F_{2y}$$

$$F_y = 5\text{ N} + 17.32\text{ N} = 22.32\text{ N}$$

Step (iv)

The magnitude *F* of the resultant force **F**

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(18.66\,\text{N})^2 + (22.32\,\text{N})^2} = 29\ \text{N}$$

Step (v)

If the resultant force **F** makes an angle θ with the x-axis then

$$\theta = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\frac{22.32\text{N}}{18.66\text{N}} = \tan^{-1} 1.196 = 50^\circ.$$

Example 2.3: Find the angle between two forces of equal magnitude when the magnitude of their resultant is also equal to the magnitude of either of these forces.

Solution: Let θ be the angle between two forces $\mathbf{F}_1$ and $\mathbf{F}_2$, where $\mathbf{F}_1$ is along x-axis. Then x-component of their resultant will be

$$R_x = F_1 \cos 0^\circ + F_2 \cos\theta$$

$$R_x = F_1 + F_2 \cos\theta$$

And y-component of their resultant is

$$R_y = F_1 \sin 0^\circ + F_2 \sin\theta$$

$$R_y = F_2 \sin\theta$$

The resultant R is given by $R^2 = R_x^2 + R_y^2$

As $R = F_1 = F_2 = F$

Hence $F^2 = (F + F\cos\theta)^2 + (F\sin\theta)^2$

Or $0 = 2F^2\cos\theta + F^2(\cos^2\theta + \sin^2\theta)$

Or $0 = 2F^2\cos\theta + F^2$

Or $\cos\theta = -0.5$

Or $\theta = \cos^{-1}(-0.5) = 120^\circ$

Point to Ponder

Why do you keep your legs far apart when you have to stand in the aisle of a bumpy-riding bus?

2.3 PRODUCT OF TWO VECTORS

There are two types of vector multiplications. The product of these two types are known as scalar product and vector product. As the name implies, scalar product of two vector quantities is a scalar quantity, while vector product of two vector quantities is a vector quantity.

Scalar or Dot Product

The scalar product of two vectors **A** and **B** is written as **A** . **B** and is defined as

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta \qquad \text{.........} \qquad (2.17)$$

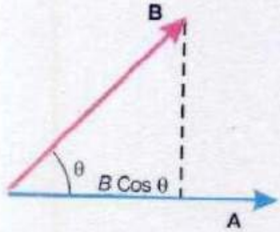


Fig. 2.10 (a)

where A and B are the magnitudes of vectors **A** and **B** and θ is the angle between them.

For physical interpretation of dot product of two vectors **A** and **B**, these are first brought to a common origin (Fig. 2.10 a),

then, $\mathbf{A.B} = (A)$ (projection of **B** on **A**)

or

$\mathbf{A.B} = A$ (magnitude of component of **B** in the direction of **A**)

$= A\,(B \cos\theta) = AB \cos\theta$

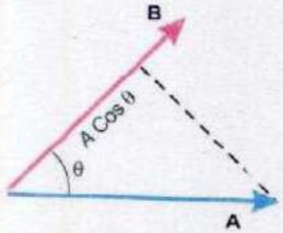


Fig. 2.10 (b)

Similarly $\mathbf{B.A} = B\,(A \cos\theta) = BA \cos\theta$

We come across this type of product when we consider the work done by a force **F** whose point of application moves a distance d in a direction making an angle θ with the line of action of **F**, as shown in Fig. 2.11.

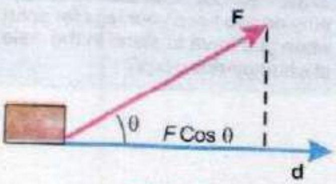


Fig. 2.11

Work done = (effective component of force in the direction of motion) x distance moved

$= (F \cos\theta)\, d = Fd \cos\theta$

Using vector notation

$$\mathbf{F.d} = Fd \cos\theta = \text{work done}$$

Characteristics of Scalar Product

1. Since $\mathbf{A.B} = AB\cos\theta$ and $\mathbf{B.A} = BA\cos\theta = AB\cos\theta$, hence, $\mathbf{A.B} = \mathbf{B.A}$. The order of multiplication is irrelevant. In other words, scalar product is commutative.

2. The scalar product of two mutually perpendicular vectors is zero. $\mathbf{A.B} = AB\cos 90^\circ = 0$

   In case of unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, since they are mutually perpendicular, therefore,

$$\hat{i}\,.\,\hat{j} = \hat{j}\,.\,\hat{k} = \hat{k}\,.\,\hat{i} = 0 \quad ....\ (2.18)$$

3. The scalar product of two parallel vectors is equal to the product of their magnitudes. Thus for parallel vectors ($\theta = 0^\circ$)

$$\mathbf{A.B} = AB\cos 0^\circ = AB$$

In case of unit vectors

$$\hat{i}\,.\,\hat{i} = \hat{j}\,.\,\hat{j} = \hat{k}\,.\,\hat{k} = 1 \quad ....\ (2.19)$$

and for antiparallel vectors ($\theta = 180^\circ$)

$$\mathbf{A.B} = AB\cos 180^\circ = -AB$$

4. The self product of a vector **A** is equal to square of its magnitude.

$$\mathbf{A.A} = AA\cos 0^\circ = A^2$$

5. Scalar product of two vectors **A** and **B** in terms of their rectangular components

$$\mathbf{A.B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}).(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

or

$$\mathbf{A.B} = A_xB_x + A_yB_y + A_zB_z \quad ....\ (2.20)$$

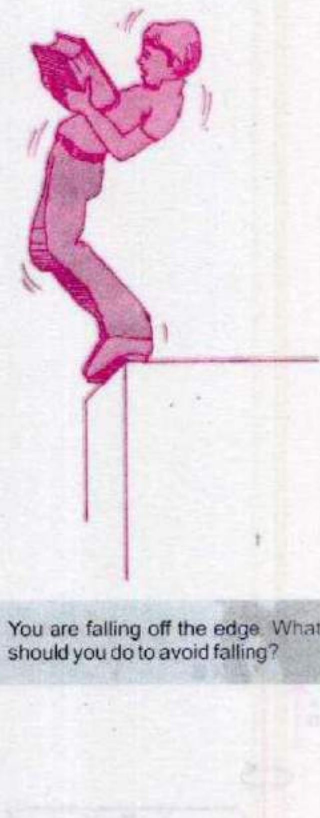
Equation 2.17 can be used to find the angle between two vectors: Since,

$$\mathbf{A.B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

Therefore,

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{AB} \quad .....\ (2.21)$$

What should You do?



You are falling off the edge. What should you do to avoid falling?

Example 2.4: A force $\mathbf{F} = 2\hat{i} + 3\hat{j}$ units, has its point of application moved from point A (1,3) to the point B (5,7). Find the work done.

Solution: Position vector of point A is $\mathbf{r}_A = \hat{i} + 3\hat{j}$ and that of point B is $\mathbf{r}_B = 5\hat{i} + 7\hat{j}$

Displacement $\mathbf{d} = \mathbf{r}_B - \mathbf{r}_A = (5-1)\hat{i} + (7-3)\hat{j} = 4\hat{i} + 4\hat{j}$

$$\text{Work done} = \mathbf{F.d} = (2\hat{i} + 3\hat{j}).(4\hat{i} + 4\hat{j})$$

$$= 8 + 12 = 20 \text{ units}$$

Example 2.5: Find the projection of vector $\mathbf{A} = 2\hat{i} - 8\hat{j} + \hat{k}$ in the direction of the vector $\mathbf{B} = 3\hat{i} - 4\hat{j} - 12\hat{k}$.

Solution: If θ is the angle between **A** and **B**, then $A\cos\theta$ is the required projection.

By definition $\quad \mathbf{A.B} = AB\cos\theta$

$$A\cos\theta = \frac{\mathbf{A.B}}{B} = \mathbf{A}.\hat{\mathbf{B}}$$

Where $\hat{\mathbf{B}}$ is the unit vector in the direction of **B**

Now $\quad B = \sqrt{3^2 + (-4)^2 + (-12)^2} = 13$

Therefore, $\quad \hat{\mathbf{B}} = \frac{(3\hat{i} - 4\hat{j} - 12\hat{k})}{13}$

The projection of **A** on $\mathbf{B} = (2\hat{i} - 8\hat{j} + \hat{k}).\frac{(3\hat{i} - 4\hat{j} - 12\hat{k})}{13}$

$$= \frac{(2)(3) + (-8)(-4) + 1(-12)}{13} = \frac{26}{13} = 2$$

Vector or Cross Product

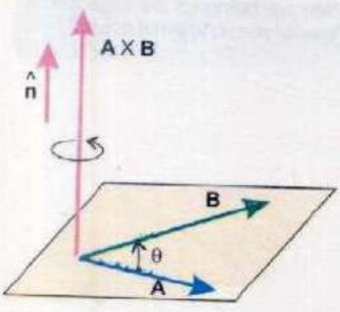


Fig. 2.12 (a)

The vector product of two vectors **A** and **B**, is a vector which is defined as

$$\mathbf{A} \times \mathbf{B} = AB\sin\theta\,\hat{\mathbf{n}} \quad \ldots\ldots\ldots \quad (2.22)$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane containing **A** and **B** as shown in Fig. 2.12 (a). Its direction can be determined by right hand rule. For that purpose, place together the tails of vectors **A** and **B** to define the plane of vectors **A** and **B.** The direction of the product vector is perpendicular to this plane. Rotate the first vector **A** into **B** through the smaller of the two possible angles and curl the fingers of the right hand in the direction of rotation, keeping the thumb erect. The direction of the product vector will be along the erect thumb, as shown in the Fig 2.12 (b). Because of this direction rule, **B x A** is a vector opposite in sign to **A x B**. Hence,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad \ldots\ldots\ldots \quad (2.23)$$

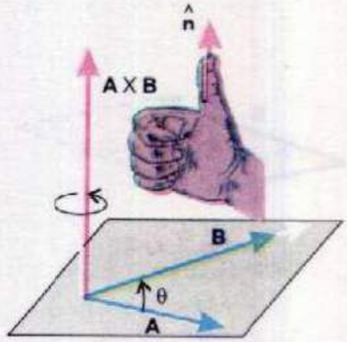


Fig. 2.12(b)

Characteristics of Cross Product

1. Since **A x B** is not the same as **B x A**, the cross product is non commutative.

2. The cross product of two perpendicular vectors has maximum magnitude $\mathbf{A} \times \mathbf{B} = AB \sin 90^\circ \hat{\mathbf{n}} = AB\, \hat{\mathbf{n}}$

   In case of unit vectors, since they form a right handed system and are mutually perpendicular Fig. 2.5 (a)

   $$\hat{i} \times \hat{j} = \hat{k}\ ,\ \hat{j} \times \hat{k} = \hat{i}\ ,\ \hat{k} \times \hat{i} = \hat{j}$$

3. The cross product of two parallel vectors is null vector, because for such vectors $\theta = 0^\circ$ or 180°. Hence

   $$\mathbf{A} \times \mathbf{B} = AB \sin 0^\circ \hat{\mathbf{n}} = AB \sin 180^\circ \hat{\mathbf{n}} = \mathbf{0}$$

   As a consequence $\quad \mathbf{A} \times \mathbf{A} = \mathbf{0}$

   Also $\quad \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \mathbf{0} \quad \ldots\ldots\ldots \quad (2.24)$

Fig. 2.12(c)

4. Cross product of two vectors **A** and **B** in terms of their rectangular components is :

   $$\mathbf{A} \times \mathbf{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\ \hat{i} + (A_zB_x - A_xB_z)\ \hat{j} + (A_xB_y - A_yB_x)\ \hat{k} \quad \ldots\ldots\ldots \quad (2.25)$$

The result obtained can be expressed for memory in determinant form as below:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Fig. 2.12(d)

5. The magnitude of **A x B** is equal to the area of the parallelogram formed with **A** and **B** as two adjacent sides (Fig. 2.12 d).

Examples of Vector Product

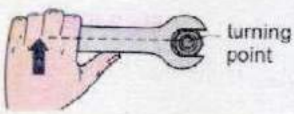
i. When a force **F** is applied on a rigid body at a point whose position vector is **r** from any point of the axis about which the body rotates, then the turning effect of the force, called the torque **τ** is given by the vector product of **r** and **F**.

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

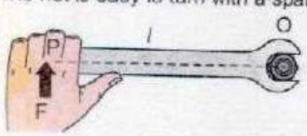
ii. The force on a particle of charge q and velocity **v** in a magnetic field of strength **B** is given by vector product.

$$\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B})$$

2.4 TORQUE



The nut is easy to turn with a spanner.



It is easier still if the spanner has a long handle.

Fig. 2.13

We have already studied in school physics that a turning effect is produced when a nut is tightened with a spanner (Fig. 2.13). The turning effect increases when you push harder on the spanner. It also depends on the length of the spanner: the longer the handle of the spanner, the greater is the turning effect of an applied force. The turning effect of a force is called its moment or torque and its magnitude is defined as the product of force **F** and the perpendicular distance from its line of action to the pivot which is the point O around which the body (spanner) rotates. This distance OP is called moment arm l. Thus the magnitude of torque represented by τ is

$$\tau = l\,F \quad \text{..........} \quad (2.26)$$

When the line of action of the applied force passes through the pivot point, the value of moment arm $l = 0$, so in this case torque is zero.

We now consider the torque due to a force **F** acting on a rigid body. Let the force **F** acts on rigid body at point P whose position vector relative to pivot O is **r**. The force **F** can be resolved into two rectangular components, $F \sin\theta$ perpendicular to **r** and $F \cos\theta$ along the direction of **r** (Fig. 2.14 a). The torque due to $F \cos\theta$ about pivot O is zero as its line of action passes through point O. Therefore, the magnitude of torque due to **F** is equal to the torque due to $F \sin\theta$ only about O. It is given by

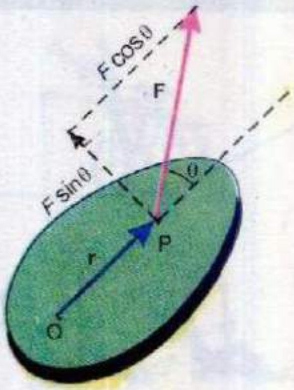


Fig. 2.14(a)

$$\tau = (F \sin\theta)\, r = r F \sin\theta \quad \ldots\ldots\ldots \quad (2.27)$$

Alternatively the moment arm l is equal to the magnitude of the component of **r** perpendicular to the line of action of **F** as illustrated in Fig. 2.14 (b). Thus

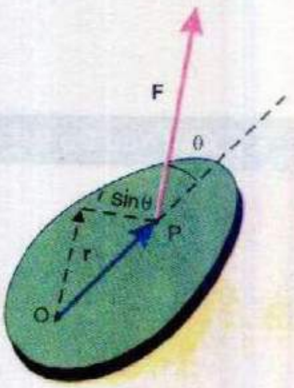


Fig. 2.14(b)

$$\tau = (r \sin\theta) F = rF \sin\theta \quad \ldots\ldots\ldots \quad (2.28)$$

where θ is the angle between **r** and **F**

From Eq. 2.27 and Eq. 2.28 it can be seen that the torque can be defined by the vector product of position vector **r** and the force **F**, so

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

or $$\boldsymbol{\tau} = (r F \sin\theta)\, \hat{\mathbf{n}} \quad \ldots\ldots\ldots \quad (2.29)$$

Where $(rF \sin\theta)$ is the magnitude of the torque. The direction of the torque represented by $\hat{\mathbf{n}}$ is perpendicular to the plane containing **r** and **F** given by right hand rule for the vector product of two vectors.

The SI unit for torque is newton metre (N m).

Just as force determines the linear acceleration produced in a body, the torque acting on a body determines its angular acceleration. Torque is the analogous of force for rotational motion. If the body is at rest or rotating with uniform angular velocity, the angular acceleration will be zero. In this case the torque acting on the body will be zero.

Point to Ponder

Do you think the rider in the above figure is really in danger? What if people below were removed?

Can You Do ?

Stand with one arm and the side of one foot pressed against a wall. Can you raise the other leg side ways? If not, then why not?

Example 2.6: The line of action of a force **F** passes through a point P of a body whose position vector in metre is $\hat{i}-2\hat{j}+\hat{k}$. If $\mathbf{F} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ (in newton), determine the torque about the point 'A' whose position vector (in metre) is $2\hat{i}+\hat{j}+\hat{k}$

Solution:

The position vector of point A $= \mathbf{r}_1 = 2\hat{i}+\hat{j}+\hat{k}$

The position vector of point P $= \mathbf{r}_2 = \hat{i}-2\hat{j}+\hat{k}$ relative to O,

The position vector of P relative to A is

$$\mathbf{AP} = \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\mathbf{AP} = (\hat{i}-2\hat{j}+\hat{k}) - (2\hat{i}+\hat{j}+\hat{k}) = -\hat{i}-3\hat{j}$$

The torque about A $= \mathbf{r} \times \mathbf{F}$

$$= (-\hat{i}-3\hat{j}) \times (2\hat{i}-3\hat{j}+4\hat{k})$$

$$= -12\hat{i}+4\hat{j}+9\hat{k} \text{ N m}$$

2.5 EQUILIBRIUM OF FORCES

We have studied in school physics that if a body, under the action of a number of forces, is at rest or moving with uniform velocity, it is said to be in equilibrium.

First Condition of Equilibrium

A body at rest or moving with uniform velocity has zero acceleration. From Newton's Law of motion the vector sum of all forces acting on it must be zero.

This is known as the first condition of equilibrium. Using the mathematical symbol $\Sigma\mathbf{F}$ for the sum of all forces we can write

$$\Sigma\mathbf{F} = 0 \quad \text{.........} \quad (2.30)$$

In case of coplanar forces, this condition is expressed usually in terms of x and y components of the forces. We have studied that x-component of the resultant force **F** equals the sum of x-directed or x-components of all the forces acting on the body. Hence

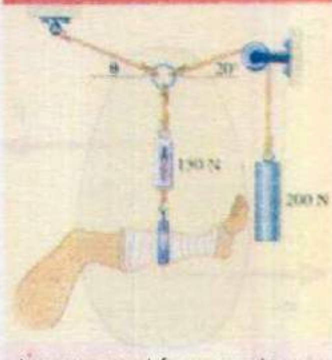
$$\Sigma \mathbf{F}_x = 0 \quad .... \quad (2.31)$$

Similarly for the y-directed forces, the resultant of y-directed forces should be zero. Hence

$$\Sigma \mathbf{F}_y = 0 \quad .... \quad (2.32)$$

It may be noted that if the rightward forces are taken as positive then leftward forces are taken as negative. Similarly if upward forces are taken as positive then downward forces are taken as negative.

Interesting Application



A concurrent force system in equilibrium. The tension applied can be adjusted as desired.

Example 2.7: A load is suspended by two cords as shown in Fig. 2.15. Determine the maximum load that can be suspended at P, if maximum breaking tension of the cord used is 50 N.

Solution: For using conditions of equilibrium, all the forces acting at point P are shown by a force diagram as illustrated in Fig. 2.16 where **w** is assumed to be the maximum weight which can be suspended. The inclined forces can now be easily resolved along x and y directions.

Applying $\Sigma F_x = 0$

$$T_2 \cos 20^\circ - T_1 \cos 60^\circ = 0$$

Or $T_1 = 1.88\ T_2$

As $T_1 > T_2$ $\therefore$ T_1 has the maximum tension

If $T_1 = 50$ N, then $T_2 = 26.6$ N

Now applying $\Sigma F_y = 0$

$$T_1 \sin 60^\circ + T_2 \sin 20^\circ - w = 0$$

Putting the values

$$50\text{ N} \times 0.866 + 26.6\text{ N} \times 0.34 = w$$

or $w = 52$ N

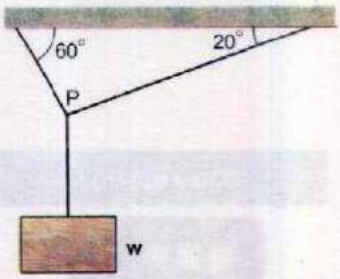


Fig. 2.15

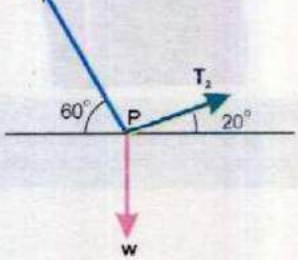


Fig. 2.16

2.6 EQUILIBRIUM OF TORQUES

Second Condition of Equilibrium

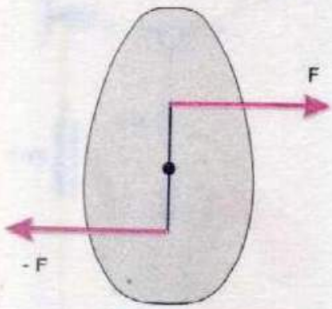


Fig. 2.17

Let two equal and opposite forces act on a rigid body as shown in Fig. 2.17. Although the first condition of equilibrium is satisfied, yet it may rotate having clockwise turning effect. As discussed earlier, for angular acceleration to be zero, the net torque acting on the body should be zero. Thus for a body in equilibrium, the vector sum of all the torques acting on it about any arbitrary axis should be zero. This is known as second condition of equilibrium. Mathematically it is written as

$$\Sigma\tau = 0 \qquad \ldots\ldots\ldots \qquad (2.33)$$

By convention, the counter clockwise torques are taken as positive and clockwise torques as negative. An axis is chosen for calculating the torques. The position of the axis is quite arbitrary. Axis can be chosen anywhere which is convenient in applying the torque equation. A most helpful point of rotation is the one through which lines of action of several forces pass.

We are now in a position to state the complete requirements for a body to be in equilibrium, which are

(1) $\Sigma \mathbf{F} = 0$ i.e $\Sigma \mathbf{F}_x = 0$ and $\Sigma \mathbf{F}_y = 0$

(2) $\Sigma\tau = 0$

When 1st condition is satisfied, there is no linear acceleration and body will be in translational equilibrium. When 2nd condition is satisfied, there is no angular acceleration and body will be in rotational equilibrium.

For a body to be in complete equilibrium, both conditions should be satisfied, i.e., both linear acceleration and angular acceleration should be zero.

If a body is at rest, it is said to be in static equilibrium but if the body is moving with uniform velocity, it is said to be in dynamic equilibrium.

We will restrict the applications of above mentioned conditions of equilibrium to situations in which all the forces lie in a common plane. Such forces are said to be

Can You Do?

With your nose touching the end of the door, put your feet astride the door and try to rise up on your toes.

coplanar. We will also assume that these forces lie in the xy-plane.

If there are more than one object in equilibrium in a given problem, one object is selected at a time to apply the conditions of equilibrium.

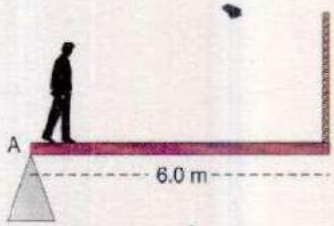


Fig. 2.18

Example 2.8: A uniform beam of 200 N is supported horizontally as shown. If the breaking tension of the rope is 400N, how far can the man of weight 400 N walk from point A on the beam as shown in Fig. 2.18?

Solution: Let breaking point be at a distance *d* from the pivot A. The force diagram of the situation is given in Fig 2.19. By applying 2nd condition of equilibrium about point A

$$\Sigma\tau = 0$$

$$400\ \text{N} \times 6\ \text{m} - 400\ \text{N} \times d - 200\ \text{N} \times 3\ \text{m} = 0$$

or $$400\ \text{N} \times d = 2400\ \text{Nm} - 600\ \text{Nm} = 1800\ \text{Nm}$$

$$d = 4.5\ \text{m}$$

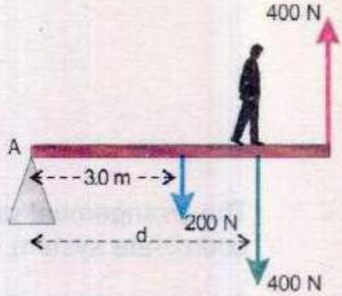


Fig. 2.19

Example 2.9: A boy weighing 300 N is standing at the edge of a uniform diving board 4.0m in length. The weight of the board is 200 N (Fig. 2.20 a). Find the forces exerted by pedestals on the board.

Fig. 2.20(a)

Solution: We isolate the diving board which is in equilibrium under the action of forces shown in the force diagram (Fig. 2.20 b). Note that the weight 200 N of the uniform diving board is shown to act at point C, the centre of gravity which is taken as the mid-point of the board, R_1 and R_2 are the reaction forces exerted by the pedestals on the board. A little consideration will show that R_1 is in the wrong direction, because the board must be actually pressed down in order to keep it in equilibrium. We shall see that this assumption will be automatically corrected by calculations.

Let us now apply conditions of equilibrium

$$\Sigma F_x = 0 \qquad \text{(No x-directed forces)}$$

$$\Sigma F_y = 0 \qquad R_1 + R_2 - 300 - 200 = 0$$

$$R_1 + R_2 = 500\ \text{N} \quad \dots \text{(i)}$$

$$\Sigma\tau = 0 \qquad \text{(pivot at point D)}$$

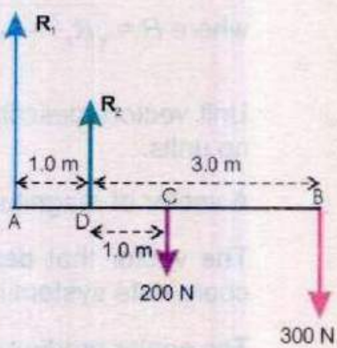


Fig. 2.20(b)

$$-R_1 \times AD - 300\ N \times DB - 200\ N \times DC = 0$$

$$-R_1 \times 1m - 300\ N \times 3m - 200\ N \times 1m = 0$$

$$R_1 = -1100\ N = -1.1\ kN$$

Substituting the value of R_1 in Eq. (i). we have

$$-1100 + R_2 = 500$$

$$R_2 = 1600\ N = 1.6\ kN$$

The negative sign of R_1 shows that it is directed downward. Thus the result has corrected the mistake of our initial assumption.

SUMMARY

- The arrangement of mutually perpendicular axes is called rectangular or Cartesian coordinate system.
- A scalar is a quantity that has magnitude only, whereas a vector is a quantity that has both direction and magnitude.
- The sum vector of two or more vectors is called resultant vector.
- Graphically the vectors are added by drawing them to a common scale and placing them head to tail, the vector connecting the tail of the first to the head of the last vector is the resultant vector.
- Vector addition can be carried out by using rectangular components of vectors. If A_x and A_y are the rectangular components of **A** and B_x and B_y are that of vector **B**, then the sum **R** = **A** + **B** is given by

$$R_x = A_x + B_x \quad , \quad R_y = A_y + B_y$$

where $R = \sqrt{R_x^2 + R_y^2}$ and direction $\theta = \tan^{-1}\frac{R_y}{R_x}$

- Unit vectors describe directions in space. A unit vector has a magnitude of 1 with no units.
- A vector of magnitude zero without any specific direction is called null vector.
- The vector that describes the location of a particle with respect to the origin of coordinate system is known as position vector.
- The scalar product of two vectors **A** and **B** is a scalar quantity, defined as :

$$\mathbf{A.B} = AB\cos\theta$$

- The vector product of two vectors **A** and **B** is another vector **C** whose magnitude is given by : $C = AB\sin\theta$

  Its direction is perpendicular to the plane of the two vectors being multiplied, as given by the right hand rule.
- A body is said to be in equilibrium under the action of several forces if the body has zero translational acceleration and no angular acceleration.
- For a body to be in translational equilibrium the vector sum of all the forces acting on the body must be zero.
- The torque is defined as the product of the force and the moment arm.
- The moment arm is the perpendicular distance from the axis of rotation to the direction of line of action of the force.
- For a body to be in rotational equilibrium, the sum of torques on the body about any axis must be equal to zero.

QUESTIONS

2.1 Define the terms (i) unit vector (ii) Position vector and (iii) Components of a vector.

2.2 The vector sum of three vectors gives a zero resultant. What can be the orientation of the vectors?

2.3 Vector **A** lies in the xy plane. For what orientation will both of its rectangular components be negative ? For what orientation will its components have opposite signs?

2.4 If one of the rectangular components of a vector is not zero, can its magnitude be zero ? Explain.

2.5 Can a vector have a component greater than the vector's magnitude?

2.6 Can the magnitude of a vector have a negative value?

2.7 If **A** + **B** = **0**, What can you say about the components of the two vectors?

2.8 Under what circumstances would a vector have components that are equal in magnitude?

2.9 Is it possible to add a vector quantity to a scalar quantity? Explain.

2.10 Can you add zero to a null vector?

2.11 Two vectors have unequal magnitudes. Can their sum be zero? Explain.

2.12 Show that the sum and difference of two perpendicular vectors of equal lengths are also perpendicular and of the same length.

2.13 How would the two vectors of the same magnitude have to be oriented, if they were to be combined to give a resultant equal to a vector of the same magnitude?

2.14 The two vectors to be combined have magnitudes 60 N and 35 N. Pick the correct answer from those given below and tell why is it the only one of the three that is correct.

i) 100 N ii) 70 N iii) 20 N

2.15 Suppose the sides of a closed polygon represent vector arranged head to tail. What is the sum of these vectors?

2.16 Identify the correct answer.

i) Two ships X and Y are travelling in different directions at equal speeds. The actual direction of motion of X is due north but to an observer on Y, the apparent direction of motion of X is north-east. The actual direction of motion of Y as observed from the shore will be

(A) East (B) West (C) south-east (D) south-west

ii) A horizontal force F is applied to a small object P of mass *m* at rest on a smooth plane inclined at an angle θ to the horizontal as shown in Fig. 2.21. The magnitude of the resultant force acting up and along the surface of the plane, on the object is

a) $F\cos\theta - mg\sin\theta$

b) $F\sin\theta - mg\cos\theta$

c) $F\cos\theta + mg\cos\theta$

d) $F\sin\theta + mg\sin\theta$

e) $mg\tan\theta$

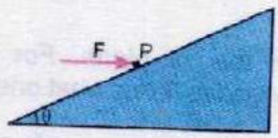


Fig. 2.21

2.17 If all the components of the vectors, $\mathbf{A}_1$ and $\mathbf{A}_2$ were reversed, how would this alter $\mathbf{A}_1 \times \mathbf{A}_2$?

2.18 Name the three different conditions that could make $\mathbf{A}_1 \times \mathbf{A}_2 = 0$.

2.19 Identify true or false statements and explain the reason.

a) A body in equilibrium implies that it is not moving nor rotating.

b) If coplanar forces acting on a body form a closed polygon, then the body is said to be in equilibrium.

2.20 A picture is suspended from a wall by two strings. Show by diagram the configuration of the strings for which the tension in the strings will be minimum.

2.21 Can a body rotate about its centre of gravity under the action of its weight?

NUMERICAL PROBLEMS

2.1. Suppose, in a rectangular coordinate system, a vector **A** has its tail at the point P (-2, -3) and its tip at Q (3,9). Determine the distance between these two points.

(Ans: 13 Units)

2.2. A certain corner of a room is selected as the origin of a rectangular coordinate system. If an insect is sitting on an adjacent wall at a point having coordinates (2,1), where the units are in metres, what is the distance of the insect from this corner of the room?

(Ans: 2.2 m)

2.3. What is the unit vector in the direction of the vector $\mathbf{A}=4\hat{i}+3\hat{j}$?

(Ans: $\frac{(4\hat{i}+3\hat{j})}{5}$)

2.4. Two particles are located at $\mathbf{r}_1 = 3\hat{i} + 7\hat{j}$ and $\mathbf{r}_2 = -2\hat{i} + 3\hat{j}$ respectively. Find both the magnitude of the vector $(\mathbf{r}_2 - \mathbf{r}_1)$ and its orientation with respect to the x-axis.

[Ans: 6.4, 219°]

2.5. If a vector **B** is added to vector **A**, the result is $6\hat{i} + \hat{j}$. If **B** is subtracted from **A**, the result is $-4\hat{i} + 7\hat{j}$. What is the magnitude of vector **A**?

(Ans: 4.1)

2.6. Given that $\mathbf{A} = 2\hat{i} + 3\hat{j}$ and $\mathbf{B} = 3\hat{i} - 4\hat{j}$, find the magnitude and angle of (a) $\mathbf{C} = \mathbf{A} + \mathbf{B}$, and (b) $\mathbf{D} = 3\mathbf{A} - 2\mathbf{B}$.

(Ans: 5.1, 349°; 17, 90°)

2.7. Find the angle between the two vectors, $\mathbf{A} = 5\hat{i} + \hat{j}$ and $\mathbf{B} = 2\hat{i} + 4\hat{j}$.

(Ans: 52°)

2.8. Find the work done when the point of application of the force $3\hat{i} + 2\hat{j}$ moves in a straight line from the point (2,-1) to the point (6,4).

(Ans: 22 units)

2.9. Show that the three vectors $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}-3\hat{j}+\hat{k}$ and $4\hat{i}+\hat{j}-5\hat{k}$ are mutually perpendicular.

2.10. Given that $\mathbf{A}=\hat{i}-2\hat{j}+3\hat{k}$ and $\mathbf{B}=3\hat{i}-4\hat{k}$, find the projection of **A** on **B**.

(Ans: $-\frac{9}{5}$)

2.11. Vectors **A**, **B** and **C** are 4 units north, 3 units west and 8 units east, respectively. Describe carefully (a) **A** x **B** (b) **A** x **C** (c) **B** x **C**

[Ans: (a) 12 units vertically up (b) 32 units vertically down (c) Zero]

2.12. The torque or turning effect of force about a given point is given by **r** x **F** where **r** is the vector from the given point to the point of application of **F**. Consider a force $\mathbf{F}=-3\hat{i}+\hat{j}+5\hat{k}$ (newton) acting on the point $7\hat{i}+3\hat{j}+\hat{k}$ (m). What is the torque in N m about the origin?

[Ans: $14\hat{i}-38\hat{j}+16\hat{k}$ Nm]

2.13. The line of action of force, $\mathbf{F}=\hat{i}-2\hat{j}$, passes through a point whose position vector is $(-\hat{j}+\hat{k})$. Find (a) the moment of **F** about the origin, (b) the moment of **F** about the point of which the position vector is $\hat{i}+\hat{k}$.

[Ans: (a) $2\hat{i}+\hat{j}+\hat{k}$, (b) $3\hat{k}$]

2.14. The magnitude of dot and cross products of two vectors are $6\sqrt{3}$ and 6 respectively. Find the angle between the vectors

(Ans: 30°)

2.15. A load of 10.0 N is suspended from a clothes line. This distorts the line so that it makes an angle of 15° with the horizontal at each end. Find the tension in the clothes line.

[Ans: 19.3N]

2.16 A tractor of weight 15,000 N crosses a single span bridge of weight 8000N and of length 21.0 m. The bridge span is supported half a metre from either end. The tractor's front wheels take 1/3 of the total weight of the tractor, and the rear wheels are 3 m behind the front wheels. Calculate the force on the bridge supports when the rear wheels are at the middle of the bridge span.

Fig. 2.22

(Ans: 12.25 kN, 10.75 kN)

2.17 A spherical ball of weight 50N is to be lifted over the step as shown in the Fig. 2.23. Calculate the minimum force needed just to lift it above the floor.

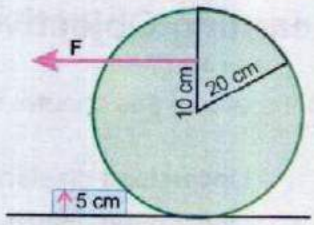


Fig. 2.23

(Ans: 26 N)

2.18 A uniform sphere of weight 10.0 N is held by a string attached to a frictionless wall so that the string makes an angle of 30° with the wall as shown in Fig. 2.24. Find the tension in the string and the force exerted on the sphere by the wall.

Fig. 2.24

(Ans: 11.6 N, 5.77 N)