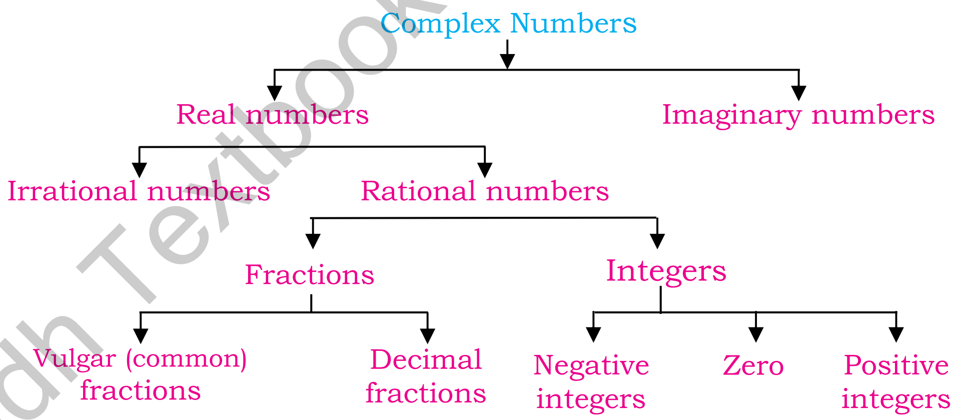
بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ

Complex Numbers

Unit 1

1.1 Complex Numbers and Geometrical Representation of Complex Number

We are already familiar with the system of real numbers. But the system of real numbers is not sufficient to solve all algebraic equations. Thus, real numbers provide inadequate solutions when we have to find the solution of the algebraic equations like $x^2 = -1, x^4 + 4 = 0$, etc., because no real number satisfies these equations. Similarly, there are so many other equations like $x^2 + x + 1 = 0$, $x^2 + 5x + 10 = 0$ which have no real roots. To overcome this inadequacy of real numbers, imaginary numbers were introduced. Later on, complex numbers were defined. The relationship of numbers is shown in the following diagram.



1.1.1 Recall complex number z represented by an expression of the form $z = a + ib$ or of the form (a, b) where a and b are real numbers and $i = \sqrt{-1}$

A complex number is the sum of a real number and an imaginary number. It is represented by an expression of the form $a + ib$ or (a, b), where a and b are real numbers, and 'i' is called imaginary unit and $i = \sqrt{-1}$. Complex number is usually denoted by z.

Note: The set of complex numbers is denoted by $\mathbb{C}$ i.e., $\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}$.

1.1.2 Recognize a as real part of z and b as imaginary part of z

As we have mentioned (a, b) is a complex number. In this complex number a is called real part and b is called imaginary part. Real and imaginary parts of complex number z are denoted by $\mathrm{R}_e(z)$ and $\mathrm{I}_m(z)$ respectively.

For example, in the complex number $z = (3, 2)$, 3 is real part and 2 is imaginary part.

1.1.3 Know the condition for equality of complex numbers

Two complex numbers are said to be equal if and only if they have same real parts and same imaginary parts.

i.e., Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal iff $a_1 = a_2$ and $b_1 = b_2$.

Example: Which of the following pairs of complex numbers are equal.

(i) $z_1 = (6 - 1) - (2 \times 3)i$ and $z_2 = (7 - 2) + 6(\sqrt{-1})$

(ii) $z_1 = 2^3 - (2^3 - 1)i$ and $z_2 = (10 - 2) - (3^2 - 2)i$

Solution: (i) $z_1 = (6 - 1) - (2 \times 3)i$ and $z_2 = (7 - 2) + 6(\sqrt{-1})$

or $z_1 = 5 - 6i$ and $z_2 = 5 + 6i$

Here $z_1 \neq z_2$ because imaginary parts are not equal.

(ii) $z_1 = 2^3 - (2^3 - 1)i$ and $z_2 = (10 - 2) - (3^2 - 2)i$

or $z_1 = 8 - (8 - 1)i = 8 - 7i$ and $z_2 = 8 - (9 - 2)i = 8 - 7i$

Here $z_1 = z_2$ because real and imaginary parts are equal.

1.1.4 Carryout basic operations on complex numbers

Basic operations on complex numbers are addition, subtraction, multiplication and division.

(i) Addition of Complex Numbers

Sum of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is obtained by adding their real and imaginary parts respectively.

i.e. $z_1 + z_2 = (a + c) + i(b + d) = (a + c, \ b + d)$

Hence $(a, b) + (c, d) = (a + c, \ b + d)$

Example 1.

Simplify: $(3 + 7i) + (6 + 9i)$

Solution:

$(3 + 7i) + (6 + 9i) = 9 + 16i$

Example 2.

Simplify: $(2, 3) + (1, -6)$

Solution:

$(2, 3) + (1, -6) = (3, -3)$

(ii) Subtraction of Complex Numbers

Difference of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is obtained by subtracting their real and imaginary parts respectively.

i.e., $z_1 - z_2 = (a + ib) - (c + id)$

$= (a - c) + i(b - d) = (a - c, b - d)$

Hence $(\boldsymbol{a}, \boldsymbol{b}) - (\boldsymbol{c}, \boldsymbol{d}) = (\boldsymbol{a} - \boldsymbol{c}, \boldsymbol{b} - \boldsymbol{d})$.

Example 1. Simplify: $(6 + 5i) - (4 + 3i)$

Solution: $(6 + 5i) - (4 + 3i)$

$= 6 + 5i - 4 - 3i$

$= 2 + 2i$

Example 2. Simplify: $(7, 8) - (5, 6)$

Solution: $(7, 8) - (5, 6)$

$= (2, 2)$

(iii) Multiplication of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$, then

$z_1 z_2 = (a + ib)(c + id)$

$= ac + i^2 bd + iad + ibc$

$= (ac - bd) + i(ad + bc) = (ac - bd, ad + bc)$

Hence $(a, b)(c, d) = (ac - bd, ad + bc)$

Example 1. Find $z_1\, z_2$, where $z_1 = (4,5)$ and $z_2 = (6,7)$

Solution:

$\because \quad z_1 = (4,5)$ and $z_2 = (6,7)$

$\therefore \quad z_1 z_2 = (4 \cdot 6 - 5 \cdot 7,\ 4 \cdot 7 + 5 \cdot 6)$

$= (24 - 35, \quad 28 + 30)$

$= (-11, 58)$

Example 2. Find $z_1\, z_2$, where $z_1 = 1 + 5i$ and $z_2 = 4 + 3i$

Solution:

$z_1\, z_2 = (1 + 5i)(4 + 3i)$

$= 4 + 3i + 20i + 15i^2$

$= 4 + 23i - 15 \qquad (\because i^2 = -1)$

$= -11 + 23i$

(iv) Division of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ where $z_2 \neq (0, 0)$

Then, $\dfrac{z_1}{z_2} = \dfrac{a+ib}{c+id} = \dfrac{a+ib}{c+id} \times \dfrac{c-id}{c-id}$

$= \dfrac{a(c - id) + ib(c - id)}{(c + id)(c - id)}$

$$= \frac{ac - iad + ibc - i^2bd}{c^2 - i^2d^2}$$

$$= \frac{ac - iad + ibc + bd}{c^2 + d^2} \qquad (\because\ i^2 = -1)$$

$$= \frac{ac + bd + i(bc - ad)}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2} = \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right)$$

Hence $(a, b) \div (c, d) = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$

Example 1. Find $\frac{z_1}{z_2}$ where $z_1 = 3 - 7i$ and $z_2 = 2 + 6i$

Solution:

Here, $\frac{z_1}{z_2} = \frac{3-7i}{2+6i} = \frac{3-7i}{2+6i} \times \frac{2-6i}{2-6i}$

$$= \frac{6 - 18i - 14i + 42i^2}{4 - 36i^2}$$

$$= \frac{(6 - 42) + i(-14 - 18)}{4 + 36} \qquad [\because\ i^2 = -1]$$

$$= \frac{-9}{10} - i\ \frac{4}{5}$$

Example 2. Find $\frac{z_1}{z_2}$ when $z_1 = (1, -3)$ and $z_2 = (2,5)$

Solution:

$$\frac{z_1}{z_2} = \frac{(1, -3)}{(2, 5)}$$

$$= \left(\frac{(1)(2) + (-3)(5)}{(2)^2 + (5)^2}, \frac{(-3)(2) - (1)(5)}{(2)^2 + (5)^2}\right)$$

$$= \left(\frac{-13}{29}, \frac{-11}{29}\right)$$

1.1.5 Define $\bar{z} = a - ib$ as the complex conjugate of $z = a + ib$

If $z = a + ib$ then the conjugate of z, denoted as $\bar{z}$, is defined by $\bar{z} = a - ib$.

Example:

(i) The conjugate of $z = 5 + 2i$ is $\bar{z} = 5 - 2i$

(ii) The conjugate of $z = (7, -9)$ is $\bar{z} = (7,9)$

(iii) The conjugate of $z = (3,0)$ is $\bar{z} = (3,0)$

1.1.6 Define $|z| = \sqrt{a^2 + b^2}$ as the absolute value or modulus of a complex number $z = a + ib$

The absolute value or modulus of a complex number $z = a + ib$ is denoted by mod (z) or $|z|$ or $|a + ib|$ and is defined as

$$|z| = |a + ib| = \sqrt{a^2 + b^2}$$

Example: Find modulus of the following complex numbers:

$$6 + 8i \text{ and } 5 - i\sqrt{7}$$

(i) $\because z = 6 + 8i$

$\therefore |z| = \sqrt{6^2 + 8^2}$
$= \sqrt{36 + 64}$
$= \sqrt{100}$
$= 10$

(ii) $\because z = 5 - i\sqrt{7}$

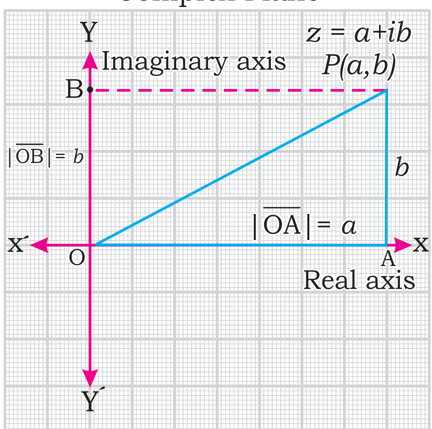
$\therefore |z| = \sqrt{5^2 + (-\sqrt{7})^2}$
$= \sqrt{25 + 7}$
$= \sqrt{32}$
$= 4\sqrt{2}$

1.1.7 Geometrical representation of complex number z as a pair of real numbers (a, b).

The complex number $z = a + ib$ can be represented geometrically by the point whose cartesian coordinates are (a, b) in a plane where real part of z is taken along x-axis (real axis) of the plane and imaginary part of z is taken along y-axis (imaginary axis). This plane is called Argand diagram or complex plane.

In the figure 1.1, the complex number $z = a + ib$ is represented by the point $P(a, b)$ where $a = |\overline{OA}|$ and $b = |\overline{OB}|$.

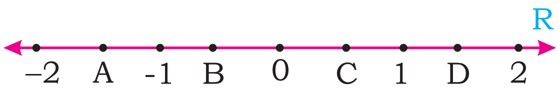
Complex Plane



(Fig. 1.1)

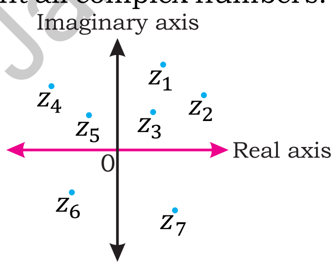
1.1.8 The order relation of complex numbers

Real numbers can be represented in either increasing or decreasing order. Numbers on the right side are greater than those on the left on a number line. Let A,B,C and D are the points representing real numbers on number line as shown in the figure (1.2a). They have increasing and decreasing order.



(Fig 1.2a)

On the other hand, number line cannot represent all complex numbers. The complex numbers $z_1, z_2, z_3, z_4, z_5, z_6, z_7$ etc are represented on the plane but cannot be written in increasing or decreasing order because all of them do not lie on the same line as shown in figure (1.2b).

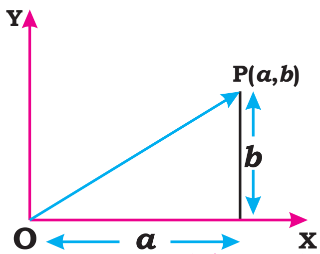


(Fig. 1.2b)

So, the complex numbers cannot be arranged in an order but moduli of complex numbers are real numbers and can be written as increasing and decreasing order on a number line. Hence there is no order relation for all complex number but their moduli follow order relation.

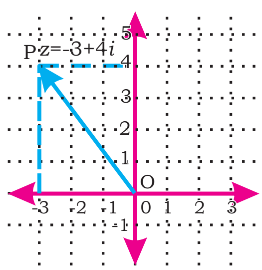
1.1.9 Vector representation of complex numbers

We represented a complex number $z = a + ib$ as the point $P(a, b)$ in the complex plane. The complex number (a, b) is interpreted as vector whose initial point is O and terminal point P as shown in Fig 1.3a. The length of the vector $\overrightarrow{OP}$ is the distance from the tail O of the vector to the tip P.



(Fig. 1.3a)

The vector representing the complex number $z = -3 + 4i$ is shown in Fig 1.3b.



(Fig. 1.3b)

Example: What are the lengths of the vectors representing the complex numbers

$z_1 = -3 + 4i$ and $z_2 = 2 - 7i$?

Solution:

Here $|Z_1| = \sqrt{9+16} = 5$

and $|Z_2| = \sqrt{4+49} = \sqrt{53} = 7.28$

The lengths of the vectors that represent z_1 and z_2 are 5 and 7.28 units respectively.

Exercise 1.1

1. Evaluate: (i) $i^{10} \cdot (-i^{12})$ (ii) $(-i)^{16} \cdot (-i)^{6}$ (iii) $(i)^{11} \cdot (-i)^{14}$

2. For what value of n, i^n is equal to 1, i, -1 or $-i$, where $1 \le n \le 4$

3. Simplify the following:

(i) $(-6,3) \cdot (4,-2)$ (ii) $(8,-4) \div (-2,2)$

(iii) $(2,3) \div (4,5)$ (iv) $(6+5i) - (4+3i)$

(v) $(5-6i) + (3+4i) - (5i-7)$ (vi) $(4+5i)(6+7i)$

(vii) $(7+4i) \div (8+5i)$ (viii) $(5,-6) + (4,8) - (3,-2)$

4. Simplify:

(i) $(2-i)^4$ (ii) $(-1-i\sqrt{3})^2$ (iii) $(-1+i\sqrt{3})^2$ (iv) $(1+i)^3$

(v) $\frac{\sqrt{2}+i}{\sqrt{2}-i}$ (vi) $\frac{1+i}{1-i} \cdot \frac{2-i}{1-i}$ (vii) $\frac{(2+i)^2}{3-4i}$ (viii) $\frac{1}{(2-i)^2}$

5. Show that $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$

6. Find the conjugate and absolute value of the following:

(i) $4+5i$ (ii) $-1+7i$ (iii) $\sqrt{3}i$

(iv) $\sqrt{7} - 3i$ (v) $-3-4i$ (vi) $(5-4i)^2$

(vii) $\frac{2}{3} - \sqrt{\frac{-9}{16}}$ (viii) $\frac{(1+i)(1+2i)}{3+i}$

7. Find real and imaginary parts of:

(i) $2i(3-5i)$ (ii) $\frac{\sqrt{5}+i}{\sqrt{5}-i}$

8. If $z = x + iy$ where $Re(z) = 0$ and $|z| = 2$ then find Z.

9. Solve the following complex equations:

(i) $(x, y)(2,3) = (-4,7)$ (ii) $(x + 3i) = 2yi$

10. Represent the following complex numbers on complex plane.

(i) $(2,-3)$ (ii) $(3,4)$ (iii) $(-5,7)$

(iv) $(-6,-2)$ (v) $(0,6)$ (vi) $(-5,0)$

11. Find the length of vector representing the complex numbers.

(i) $-5 + 2i$ (ii) $\frac{7}{3} + \frac{8}{3}i$ (iii) $\frac{1+i}{\sqrt{2}}$ (iv) $\frac{1+\sqrt{3}i}{2}$

1.2 Properties of Complex Numbers

1.2.1 Describe algebraic properties of complex numbers (e.g. commutative, associative and distributive) with respect to '+' and '×'

(i) Commutative property w.r.t. addition

Let z_1 and z_2 are two complex numbers then commutative property with respect to addition is defined as $z_1 + z_2 = z_2 + z_1$.

Example: Let $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$, then verify commutative property with respect to addition.

Verification: We have to verify that $z_1 + z_2 = z_2 + z_1$

L.H.S $= z_1 + z_2$

$= 2 + 3i + 4 + 5i$

$= 6 + 8i$

R.H.S = $\quad + z$

$= 4 + 5i + 2 + 3i$

$= 6 + 8i$

$\because$ L.H.S = R.H.S

$\therefore$ commutative property w.r.t addition is verified.

(ii) Commutative Property w.r.t. Multiplication

Let z_1 and z_2, are two complex numbers then commutative property under multiplication is defined as $z_1z_2 = z_2z_1$.

Example: Let $z_1 = 4 + 5i$ and $z_2 = 3 + 2i$ then verify commutative property with respect to multiplication.

Verification: We have to verify that $z_1z_2 = z_2z_1$

L.H.S = $z_1z_2 = (4,5)\,(3,2) = (4 \times 3 - 5 \times 2, 4 \times 2 + 5 \times 3) = (2,23)$

R.H.S = $z_2z_1 = (3,2)\,(4,5) = (3\times4-2\times5, 2\times4+3\times5) = (2,23)$
∵ L.H.S = R.H.S
∴ commutative property w.r.t. multiplication is verified.

(iii) Associative Property w.r.t. Addition

Let z_1, z_2 and z_3 are three complex numbers, then associative property under addition is defined as:

$$(z_1+z_2)+z_3 = z_1+(z_2+z_3)$$

Example: Let $z_1 = 4+5i$, $z_2 = 3+2i$ and $z_3 = 2+7i$ then verify associative property w.r.t addition.

Verification: We have to verify that $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$

L.H.S $= (z_1+z_2)+z_3$
$= \{(4+5i)+(3+2i)\}+(2+7i)$
$= 7+7i+2+7i$
$= 9+14i$

R.H.S $= z_1+(z_2+z_3)$
$= (4+5i)+\{(3+2i)+(2+7i)\}$
$= 4+5i+5+9i$
$= 9+14i$

∵ L.H.S = R.H.S
∴ associative property of addition is verified.

(iv) Associative property w.r.t. Multiplication

Let z_1, z_2 and z_3 are three complex numbers, then associative property under multiplication is defined as:

$$(z_1z_2)z_3 = z_1(z_2z_3)$$

Example: Let $z_1 = 2+3i$, $z_2 = 4+5i$ and $z_3 = 1+i$ then verify associative property w.r.t multiplication.

Verification: We have to verify that $(z_1z_2)z_3 = z_1(z_2z_3)$

L.H.S $= (z_1z_2)z_3$
$= \{(2+3i)\cdot(4+5i)\}\cdot(1+i)$
$= (8+10i+12i-15)(1+i)$
$= (-7+22i)\cdot(1+i)$
$= -7-7i+22i-22$
$= -29+15i$

R.H.S $= z_1(z_2z_3)$
$= (2+3i)\cdot\{(4+5i)\cdot(1+i)\}$
$= (2+3i)(4+4i+5i-5)$
$= (2+3i)\cdot(-1+9i)$
$= -2+18i-3i-27$

$= -29 + 15i$

$\because$ L.H.S = R.H.S

$\therefore$ associative property of multiplication is verified.

(v) Distributive Property of multiplication over addition

Let z_1, z_2 and z_3 are three complex numbers, then distributive property of multiplication over addition is defined as:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

Example: Let, $z_1 = 2 + 3i$, $z_2 = 4 + 5i$ and $z_3 = 1 + i$ then verify distributive property of multiplication over addition.

Verification: We have to verify that $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

L.H.S $= z_1(z_2 + z_3)$

$= (2 + 3i)\{(4 + 5i) + (1 + i)\}$

$= (2 + 3i)(5 + 6i)$

$= 10 + 12i + 15i - 18$

$= -8 + 27i$

R.H.S $= z_1 z_2 + z_1 z_3$

$= \{(2 + 3i) \cdot (4 + 5i)\} + \{(2 + 3i) \cdot (1 + i)\}$

$= (8 + 10i + 12i - 15) + (2 + 2i + 3i - 3)$

$= (-7 + 22i) + (-1 + 5i)$

$= -8 + 27i$

$\because$ L.H.S = R.H.S

$\therefore$ distributive property of multiplication over addition is verified.

1.2.2 Know additive identity and multiplicative identity for the set of complex numbers

(i) Additive identity

Let $z = (a, b)$ be any complex number then there exists a complex number $(0, 0)$ such that,

$$z + (0, 0) = (a, b) + (0,0) = (a, b) = z$$

and $$(0,0) + z = (0, 0) + (a, b) = (a, b) = z$$

Thus, (0,0) is additive identity in set of complex numbers.

(ii) Multiplicative identity

Let $z = (a, b)$ be any complex number then there exists a complex number $(1, 0)$ such that,

$$z\,(1, 0) = (a, b)\,(1, 0) = (a, b) = z$$

and $(1, 0)\, z = (1, 0)(a, b) = (a, b) = z$

Thus, $(1, 0)$ is multiplicative identity in set of complex numbers.

1.2.3 Find additive inverse and multiplicative inverse of a Complex number z

(i) Additive Inverse of Complex Numbers

A complex number (c, d) is called the additive inverse of the complex number (a, b) if $(a, b) + (c, d) = (0, 0)$.

$$\because \quad (a, b) + (c, d) = (0, 0)$$
$$\therefore \quad (a + c, b + d) = (0, 0)$$
$$\Rightarrow \quad a + c = 0 \quad \text{and} \quad b + d = 0$$
$$\Rightarrow \quad c = -a \quad \text{and} \quad d = -b$$

Therefore, $(c, d) = (-a, -b)$

So, the additive inverse of $a + ib$ is $-a - ib$

Note: (i) Additive inverse of (0,0) is (0,0).

(ii) Additive inverse of (a, b) is $(-a, -b)$.

Example:

(i) The additive inverse of the complex number $6 - 4i$ is $-6 + 4i$.

(ii) The additive inverse of the complex number $-14i$ is $14i$.

(ii) Multiplicative Inverse of a complex number:

The multiplicative inverse of a non-zero complex number $\boldsymbol{z}$, denoted as z^{-1} or $\frac{1}{z}$, is a complex number such that

$$zz^{-1} = z^{-1}z = (1, 0)$$

As the multiplicative inverse of a complex number z is $\frac{1}{z}$ where $z \neq (0, 0)$

So, $\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib}$ (By Rationalizing the denominator)

$$= \frac{a - ib}{(a + ib)(a - ib)}$$

$$= \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{ib}{a^2 + b^2}$$

Hence the multiplicative inverse of a complex number (a, b) is $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

Example 1.

Find the multiplicative inverse of a complex number $z = 5 + 3i$

Solution: $\because \quad z = 5 + 3i$

$$\therefore z^{-1} = \left(\frac{5}{(5)^2 + (3)^2}, \frac{-3}{(5)^2 + (3)^2}\right) = \left(\frac{5}{25 + 9}, \frac{-3}{25 + 9}\right) = \left(\frac{5}{34}, \frac{-3}{34}\right)$$

Example 2. Find the multiplicative inverse of complex number $z = 4$

Solution:

$$\because \quad z = 4 \text{ or } (4,0)$$

$$\therefore \quad z^{-1} = \left(\frac{4}{(4)^2+(0)^2}, \frac{0}{(4)^2+(0)^2}\right) = \left(\frac{1}{4}, 0\right)$$

Example 3. Prove that $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ is the multiplicative inverse of (a,b) where $(a,b) \neq (0,0)$.

Proof:

$$(a,b)\cdot\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = \left(\frac{a^2}{a^2+b^2} - \frac{-b^2}{a^2+b^2}, \frac{-ab}{a^2+b^2} + \frac{ab}{a^2+b^2}\right) = \left(\frac{a^2+b^2}{a^2+b^2}, 0\right) = (1,0)$$

i.e., $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)\cdot(a,b) = (1,0)$, hence proved.

Note: The multiplicative inverse of (0,0) does not exist.

1.2.4 Demonstrate the following properties

(i) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$,

(ii) $\bar{\bar{z}} = z, z\bar{z} = |z|^2, \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

(iii) Triangle inequality of complex numbers

(iv) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

Property (i) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$

Proof: Let $z = a + ib$ then $|z| = |a+ib| = \sqrt{(a)^2+(b)^2} = \sqrt{a^2+b^2}$ ...(a)

and $|-z| = |-a-ib| = \sqrt{(-a)^2+(-b)^2} = \sqrt{a^2+b^2}$ ... (b)

Aslo, $|\bar{z}| = |a-ib| = \sqrt{(a)^2+(-b)^2} = \sqrt{a^2+b^2}$ ... (c)

and $|-\bar{z}| = |-a+ib| = \sqrt{(-a)^2+(b)^2} = \sqrt{a^2+b^2}$ ... (d)

From the results (a), (b), (c) and (d), we get

$|z| = |-z| = |\bar{z}| = |-\bar{z}|$, hence proved.

Property (ii) (a) $\bar{\bar{z}} = z$, (b) $z\bar{z} = |z|^2$, (c) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(a) $\bar{\bar{z}} = z$

Proof:

Let $z = a + ib$ then $\bar{z} = a - ib$

Now, $\bar{\bar{z}} = \overline{(\bar{z})} = \overline{(a-ib)} = a + ib = z$

Thus it is proved that $\bar{\bar{z}} = z$

(b) $z\bar{z} = |z|^2$

Proof: Let $z = a + ib$ then $\bar{z} = a - ib$

Now, $z\bar{z} = (a+ib)(a-ib) = a^2 - i^2b^2 = a^2 + b^2$ ...(i) and $|z| = |a + ib| = \sqrt{a^2 + b^2}$

So, $|z|^2 = |a + ib|^2 = a^2 + b^2$ ...(ii)

From (i) and (ii) we have,

$z\bar{z} = |z|^2$, hence proved. ...(b)

(c) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Proof: Let $z_1 = a + ib$ and $z_2 = c + id$

L.H.S $= \overline{z_1 + z_2} = \overline{(a + ib) + (c + id)} = \overline{(a + c) + i(b + d)}$

$= (a + c) - i(b + d)$

Thus $\overline{z_1 + z_2} = (a - ib) + (c - id)$ ...(i)

Now, R.H.S $= \overline{z_1} + \overline{z_2} = \overline{(a + ib)} + \overline{(c + id)}$

Thus $\overline{z_1} + \overline{z_2} = (a - ib) + (c - id)$ ...(ii)

From (i) and (ii), we have $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, hence proved.

Property (iii)

Triangle Inequality of complex numbers:

If $z_1, z_2 \in \mathbb{C}$, then $|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|$, which is called Triangle inequality.

Geometrical Proof

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be represented by vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ respectively as shown in (Fig 1.4)

Considering the figure, by completing the parallelogram $OP_1 P_3 P_2$.

In ΔOP_1P_3, $\overrightarrow{OP_3} = z_1 + z_2$

Since sum of the lengths of any two sides of a triangle is greater than the length of the third side.

$$|\overrightarrow{OP_1}| + |\overrightarrow{OP_2}| \ge |\overrightarrow{OP_3}| \qquad (\because \overrightarrow{P_1P_3} = \overrightarrow{OP_2})$$

So $|z_1| + |z_2| \ge |z_1 + z_2|$

...(i)

Again, in ΔOP_1P_2

$$|\overrightarrow{OP_2}| + |\overrightarrow{P_2P_1}| \ge |\overrightarrow{OP_1}|$$

$\Rightarrow$ $|\overrightarrow{P_2P_1}| \ge |\overrightarrow{OP_1}| - |\overrightarrow{OP_2}|$

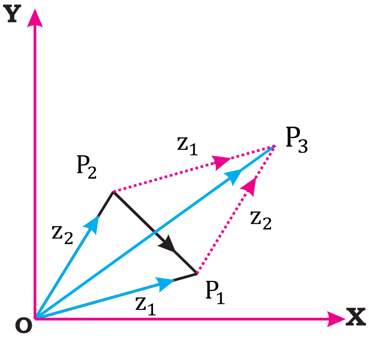
$\Rightarrow$ $|z_1 - z_2| \ge |z_1| - |z_2|$

$\Rightarrow$ $|z_1| - |z_2| \le |z_1 - z_2|$

Replacing z_2 by $-z_2$ we get

$|z_1| - |-z_2| \le |z_1 - (-z_2)|$

$\Rightarrow$ $|z_1| - |z_2| \le |z_1 + z_2|$ ...(ii)



(Fig. 1.4)

Combining inequalities (i) and (ii), and rearranging

$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|$, $\forall z_1, z_2 \in \mathbb{C}$. Hence proved.

Property (iv)

(a) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ (b) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

Proof: (a) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

Let $z_1 = a + ib$ and $z_2 = c + id$
then $z_1 z_2 = (a + ib)(c + id)$
or $z_1 z_2 = (ac - bd) + (bc + ad)i$
L.H.S $= \overline{z_1 z_2} = \overline{(ac - bd) + (bc + ad)i} = (ac - bd) - (bc + ad)i$ ... (i)
R.H.S $= \overline{z_1}\,\overline{z_2} = (a - ib)(c - id) = (ac - bd) - (bc + ad)i$ ...(ii)

From the results (i) and (ii), we get $\overline{z_1 z_2} = \overline{z_1}.\overline{z_2}$, hence proved.

Proof: (b) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$ where, $z_1 = a + ib$ and $z_2 = c + id$.

Now, $\frac{z_1}{z_2} = \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id} = \frac{(a + ib)(c - id)}{c^2 + d^2}$

Thus $\frac{z_1}{z_2} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

L.H.S $= \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}} = \frac{(ac+bd)}{c^2+d^2} - i\frac{(bc-ad)}{c^2+d^2}$ ... (i)

R.H.S $= \frac{\overline{z_1}}{\overline{z_2}} = \frac{a-ib}{c-id} = \frac{a-ib}{c-id} \times \frac{c+id}{c+id}$

$= \frac{(a - ib)(c + id)}{c^2+d^2} = \frac{ac + iad - ibc - i^2bd}{c^2+d^2}$

$= \frac{(ac + bd)}{c^2+d^2} + i\frac{(ad - bc)}{c^2+d^2} = \frac{(ac + bd)}{c^2+d^2} - \frac{i(bc - ad)}{c^2+d^2}$ ... (ii)

From (i) and (ii), we get $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$. Hence proved.

1.2.5 Find real and imaginary parts of the following type of complex numbers

Type I: $(x + iy)^n$,

Type II: $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$, $x_2 + iy_2 \neq 0$ where, $n = \pm 1$ and ± 2

Type: (I) $(x + iy)^n$ where $n = \pm 1$ and ± 2
(a) Let, $n = 1$
then $(x + iy)^n = (x + iy)^1 = x + iy$
Here, x is real part and y is imaginary part.

(b) Let, $n = -1$

then $(x+iy)^n = (x+iy)^{-1} = \frac{1}{x+iy} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2-i^2y^2} = \frac{x-iy}{x^2+y^2}$

Thus, $(x+iy)^{-1} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$

Here, $\frac{x}{x^2+y^2}$ is real part and $-\frac{y}{x^2+y^2}$ is imaginary part

(c) Let, $n = 2$

then $(x+iy)^n = (x+iy)^2 = x^2 + 2xyi + i^2y^2 = (x^2 - y^2) + 2xyi$

Here, real part is $x^2 - y^2$ and $2xy$ is imaginary part.

(d) Let, $n = -2$

$$(x+iy)^{-2} = \frac{1}{(x+iy)^2} = \frac{1}{x^2 + 2xyi + i^2y^2}$$

$$= \frac{1}{(x^2-y^2)+2xyi} = \frac{\{(x^2-y^2)-2xyi\}}{\{(x^2-y^2)+2xyi\}\times\{(x^2-y^2)-2xyi\}}$$

$$= \frac{(x^2-y^2)-2xyi}{(x^2-y^2)^2 + 4x^2y^2i^2} = \frac{(x^2-y^2)-2xyi}{(x^2+y^2)^2}$$

Thus, $(x+iy)^{-2} = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{2xyi}{(x^2+y^2)^2}$

Here, $\frac{x^2-y^2}{(x^2+y^2)^2}$ is real part and $-\frac{2xy}{(x^2+y^2)^2}$ is imaginary part.

Example: Find the real and imaginary parts of the following complex numbers:

(i) $(2-5i)^2$ (ii) $(3-4i)^{-1}$

(i) Solution:

$$(2-5i)^2 = (2)^2 - 2(2)(5i) + (5i)^2 = -21 - 20i$$

Therefore, real part is -21 and imaginary part is -20.

(ii) Solution:

$$(3-4i)^{-1} = \frac{1}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i}{25} = \frac{3}{25} + \frac{4}{25}i$$

we have real part $\frac{3}{25}$ and imaginary part $\frac{4}{25}$.

Note: We can solve the above example directly by using the derived formula as done below.

(i) real part $= x^2 - y^2 = (2)^2 - (-5)^2 = 4 - 25 = -21$

and imaginary part $= 2xy = 2(2)(-5) = -20$.

(ii) real part $= \frac{x}{x^2+y^2} = \frac{3}{(3)^2+(-4)^2} = \frac{3}{9+16} = \frac{3}{25}$

and imaginary part $= \frac{-y}{x^2+y^2} = \frac{-(-4)}{(3)^2+(-4)^2} = \frac{4}{9+16} = \frac{4}{25}$

Type: (II) $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$, $x_2 + iy_2 \neq 0$ where $n = \pm 1$ and ± 2

For $n = 1$

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{x_1+iy_1}{x_2+iy_2}\right)^1$$

$$= \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2)^2 - i^2 y_2^2}$$

$$= \frac{x_1x_2 - (x_1y_2 - x_2y_1)i - (-1)y_1y_2}{x_2^2+y_2^2} = \frac{(x_1x_2+y_1y_2) - (x_1y_2 - x_2y_1)i}{x_2^2+y_2^2}$$

Here, $\frac{(x_1x_2+y_1y_2)}{x_2^2+y_2^2}$ and $\frac{-(x_1y_2-x_2y_1)}{x_2^2+y_2^2}$ are the real and imaginary parts respectively.

For $n = -1$,

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{x_1+iy_1}{x_2+iy_2}\right)^{-1} = \frac{1}{\frac{x_1+iy_1}{x_2+iy_2}} = \frac{x_2+iy_2}{x_1+iy_1} = \frac{x_2+iy_2}{x_1+iy_1} \times \frac{x_1-iy_1}{x_1-iy_1}$$

$$= \frac{x_1x_2 + (x_1y_2 - x_2y_1)i - (-1)y_1y_2}{(x_1)^2 - i^2 y_1^2} = \frac{x_1x_2+y_1y_2}{x_1^2+y_1^2} + \frac{(x_1y_2-x_2y_1)i}{x_1^2+y_1^2}$$

Here, $\frac{x_1x_2+y_1y_2}{x_1^2+y_1^2}$ is real part and $\frac{(x_1y_2-x_2y_1)}{x_1^2+y_1^2}$ is imaginary part.

For $n = 2$

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{x_1+iy_1}{x_2+iy_2}\right)^2 = \frac{(x_1+iy_1)^2}{(x_2+iy_2)^2} = \frac{(x_1)^2 + 2x_1y_1i + i^2y_1^2}{(x_2)^2 + 2x_2y_2i + i^2y_2^2} = \frac{x_1^2 + 2x_1y_1i + (-1)y_1^2}{(x_2)^2 + 2x_2y_2i + (-1)y_2^2}$$

$$= \frac{(x_1^2-y_1^2) + 2x_1y_1i}{(x_2^2-y_2^2) + 2x_2y_2i} = \frac{\{(x_1^2-y_1^2) + 2x_1y_1i\}\{(x_2^2-y_2^2) - 2x_2y_2i\}}{\{(x_2^2-y_2^2) + 2x_2y_2i\}\{(x_2^2-y_2^2) - 2x_2y_2i\}}$$

$$= \frac{\{(x_1^2-y_1^2)(x_2^2-y_2^2) + 4x_1y_1x_2y_2\} + \{2x_1y_1(x_2^2-y_2^2) - 2x_2y_2(x_1^2-y_1^2)\}i}{x_2^4 + y_2^4 - 2x_2^2y_2^2 + 4x_2^2y_2^2}$$

Therefore, the real part is $\frac{(x_1^2-y_1^2)(x_2^2-y_2^2)+4x_1y_1x_2y_2}{(x_2^2+y_2^2)^2}$

and the imaginary part is $\frac{2x_1y_1(x_2^2-y_2^2)-2x_2y_2(x_1^2-y_1^2)}{(x_2^2+y_2^2)^2}$

For $n = -2$

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{x_1+iy_1}{x_2+iy_2}\right)^{-2} = \frac{(x_1+iy_1)^{-2}}{(x_2+iy_2)^{-2}} = \frac{(x_2+iy_2)^2}{(x_1+iy_1)^2} = \frac{(x_2)^2 + 2x_2y_2i + i^2y_2^2}{(x_1)^2 + 2x_1y_1i + i^2y_1^2}$$

$$= \frac{(x_2)^2 + 2x_2y_2i + (-1)y_2^2}{(x_1)^2 + 2x_1y_1i + (-1)y_1^2} = \frac{(x_2^2 - y_2^2) + 2x_2y_2i}{(x_1^2 - y_1^2) + 2x_1y_1i}$$

$$= \frac{\{(x_2^2 - y_2^2) + 2x_2y_2i\}\{(x_1^2 - y_1^2) - 2x_1y_1i\}}{\{(x_1^2 - y_1^2) + 2x_1y_1i\}\{(x_1^2 - y_1^2) - 2x_1y_1i\}}$$

$$= \frac{\{(x_1^2 - y_1^2)(x_2^2 - y_2^2) + 4x_1y_1x_2y_2\} + \{2x_2y_2(x_1^2 - y_1^2) - 2x_1y_1(x_2^2 - y_2^2)\}i}{(x_1^4 + y_1^4 - 2x_1^2y_1^2 + 4x_1^2y_1^2)}$$

Therefore, the real part is $\frac{(x_1^2 - y_1^2)(x_2^2 - y_2^2) + 4x_1y_1x_2y_2}{(x_1^2 + y_1^2)^2}$

and the imaginary part is $\frac{2x_2y_2(x_1^2 - y_1^2) - 2x_1y_1(x_2^2 - y_2^2)}{(x_1^2 + y_1^2)^2}$

Example: Find the real and imaginary parts of the following.

(i) $\left(\frac{2-i}{4-3i}\right)^2$ (ii) $\left(\frac{1-i}{2-i}\right)^{-2}$

(i) Solution: $\left(\frac{2-i}{4-3i}\right)^2$

$$= \frac{4 - 4i + i^2}{16 - 24i + 9i^2}$$

$$= \frac{3 - 4i}{7 - 24i}$$

$$= \frac{3 - 4i}{7 - 24i} \times \frac{7 + 24i}{7 + 24i}$$

$$= \frac{21 + 72i - 28i + 96}{49 + 576}$$

$$= \frac{117 + 44i}{625}$$

$$= \frac{117}{625} + \frac{44}{625}i$$

So, real part $= \frac{117}{625}$ and imaginary part $= \frac{44}{625}$.

(ii) Solution: $\left(\frac{1-i}{2-i}\right)^{-2}$

$$= \left(\frac{2 - i}{1 - i}\right)^2$$

$$= \frac{4 - 4i + i^2}{1 - 2i + i^2}$$

$$= \frac{3 - 4i}{-2i}$$

$$= \frac{3-4i}{-2i} \times \frac{i}{i}$$
$$= \frac{4+3i}{2}$$
$$= 2 + \frac{3}{2}i$$

So, real part = 2 and imaginary part = $\frac{3}{2}$.

Note: We can also solve the above example by using the derived formulas.

Exercise 1.2

1. Let $z_1 = 3 - 4i, \quad z_2 = 4 + 5i$ and $z_3 = -5 + 6i$

Verify the following:

(i) Addition of complex numbers is commutative

(ii) Multiplication of complex numbers is commutative

(iii) Addition of complex numbers is associative

(iv) Multiplication of complex numbers is associative

(v) Multiplication of complex numbers is distributive over addition.

2. If z, z_1 and z_2 are complex numbers then prove that:

(a) $z + \bar{z}$ is real. (b) $z - \bar{z}$ is imaginary.

(c) $|z_1 \cdot z_2| = |z_1||z_2|$

3. Find the additive inverses of the following complex numbers:

(i) $3 + 5i$ (ii) $6 - 5i$ (iii) $(11, 0)$ (iv) $\left(\frac{2}{3}, 6\right)$

(v) $\left(\frac{8}{9}, \frac{-4}{5}\right)$ (vi) $\left(0, \frac{3}{8}\right)$

4. Find the multiplicative inverses of the following complex numbers.

(i) $3 + 5i$ (ii) $8i$ (iii) $(10, 4)$ (iv) $(12, -7)$

(v) $(-8, 0)$ (vi) $\left(0, \frac{-3}{4}\right)$

5. Find the additive and multiplicative inverses by definition of the following:

(i) $(2, 3)$ (ii) $(-4, 5)$

6. Verify the following properties with $z = 4 - 3i, z_1 = 3 - 2i$ and $z_2 = 2 + 3i$

(i) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$ (ii) $\bar{\bar{z}} = z$ (iii) $z\,\bar{z} = |z|^2$

(iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (v) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ (vi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

7. If $z_1 = 3 - 2i$ and $z_2 = 2 - 3i$ then express the following in the form of $a + ib$.

(i) $\frac{z_1\overline{z_2}}{z_1}$ (ii) $\frac{\overline{z_1}z_2}{z_2}$ (iii) $\frac{\overline{z_1 z_2}}{z_1}$

Also verify Triangle inequality of complex numbers.

8. Find real and imaginary parts of each of the following by using any method:

(i) $(\sqrt{2} + i)^{-2}$ (ii) $(1 - \sqrt{5}i)^{-1}$ (iii) $(\sqrt{3} - i)^{-2}$ (iv) $(2i - \sqrt{3})^{-1}$

(v) $\left(\frac{1}{4i-5}\right)^{-1}$ (vi) $\left(\frac{3+4i}{5i-4}\right)^{-2}$ (vii) $\left(\frac{3i-2}{2-3i}\right)^{-2}$

1.3 Solution of complex equations

In this section, we shall find the solution of equations with complex coefficients by using different methods.

1.3.1 Solve the simultaneous linear equations with complex coefficients. For example,

$$5z - (3 + i)w = 7 - i$$
$$(2 - i)z + 2iw = -1 + i$$

Example: Solve $5z - (3 + i)w = 7 - i$

$(2 - i)z + 2iw = -1 + i$

Where w and z are complex numbers.

Solution:

Here, Linear equations are

$$5z - (3 + i)w = 7 - i \quad \ldots(i)$$
$$(2 - i)z + 2iw = -1 + i \quad \ldots(ii)$$

In order to equate the coefficients of z, we multiply equation (i) by $(2 - i)$ and equation (ii) by 5.

$$(2 - i)5z - (2 - i)(3 + i)w = (2 - i)(7 - i) \quad \ldots(iii)$$
$$5(2 - i)z + 5 \times 2iw = 5(-1 + i) \quad \ldots(iv)$$

Subtracting equation (iv) from equation (iii), we get

$$-w(7 + 9i) = 18 - 14i$$

or $$w = \frac{18-14i}{-(7+9i)} = \frac{(-18+14i)\times(7-9i)}{(7+9i)(7-9i)} = \frac{-126+162i+98i+126}{130} = 2i$$

Now substituting $w = 2i$ in equation (i),

we get $$5z - (3 + i)2i = 7 - i$$

$$\Rightarrow \qquad 5z - 6i + 2 = 7 - i$$

$$\Rightarrow \qquad z = \frac{5+5i}{5}$$

$$\Rightarrow \qquad z = 1 + i$$

Thus, $w = 2i$ and $z = 1 + i$.

1.3.2 Represent Polynomial P(z) as a product of linear factors

For example: (a) $z^2 + a^2 = (z + ia)(z - ia)$

(b) $z^3 - 3z^2 + z + 5 = (z + 1)(z - 2 - i)(z - 2 + i)$

(a) $z^2 + a^2 = z^2 - (-1)a^2 = z^2 - i^2a^2 = (z)^2 - (ia)^2$

$= (z - ia)(z + ia)$

Thus, factors of $z^2 + a^2$ are $(z - ia)$ and $(z + ia)$.

(b) Let $p(z) = z^3 - 3z^2 + z + 5$ is a polynomial.

The factors of 5 are ±1 and ±5.

Here, $p(z) = z^3 - 3z^2 + z + 5$

For $z = 1$, $p(1) = (1)^3 - 3(1)^2 + (1) + 5 \neq 0$

For $z = -1$, $p(-1) = (-1)^3 - 3(-1)^2 + (-1) + 5 = 0$

As, $z = -1$, so, $(z + 1)$ is a factor of $p(z) = z^3 - 3z^2 + z + 5$.

By synthetic division

−1	1	−3	1	5
		−1	4	−5
	1	−4	5	0

we get $(z^3 - 3z^2 + z + 5) \div (z + 1) = (z^2 - 4z + 5)$

Hence, $(z^3 - 3z^2 + z + 5) = (z + 1)(z^2 - 4z + 5)$

Now we can easily find the factors of $z^2 - 4z + 5$ by using quadratic formula.

In order to find the factors of $z^2 - 4z + 5$, let $z^2 - 4z + 5 = 0$.

Here, $a = 1$, $b = -4$ and $c = 5$, by using quadratic formula

$$z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

Thus, $z = 2 + i$ and $z = 2 - i$ or $z - 2 - i = 0$ and $z - 2 + i = 0$

Hence the three factors of the given polynomial are $(z + 1), (z - 2 - i)$ and $(z - 2 + i)$

1.3.3 Solve quadratic equation of the form $pz^2 + qz + r = 0$ by completing the squares, where p, q and r are real numbers and z a complex number

For example: Solve $z^2 - 2z + 5 = 0$

$\Rightarrow (z - 1 - 2i)(z - 1 + 2i) = 0$

$\Rightarrow z = 1 + 2i, 1 - 2i$

We recall the method of completing the squares by solving the following standard form of quadratic equation.

The quadratic equation in standard form is:

$$pz^2 + qz + r = 0 \qquad \forall\, p, q, r \in \mathbb{R}$$

$$\Rightarrow \quad pz^2 + qz = -r \qquad \text{(Shifting constant on R.H.S)}$$

$$\Rightarrow \quad z^2 + \frac{q}{p}z = -\frac{r}{p} \qquad \text{(Dividing by the coefficient of } z^2\text{)}$$

$$\Rightarrow \quad (z)^2 + 2\left(\frac{q}{2p}\right)z + \left(\frac{q}{2p}\right)^2 = -\frac{r}{p} + \left(\frac{q}{2p}\right)^2 \qquad \left[\text{Adding } \left(\frac{q}{2p}\right)^2 \text{ to both sides}\right]$$

$$\Rightarrow \quad \left(z + \frac{q}{2p}\right)^2 = -\frac{r}{p} + \frac{q^2}{4p^2} = \frac{q^2 - 4pr}{4p^2}$$

$$z + \frac{q}{2p} = \pm\sqrt{\frac{q^2 - 4pr}{4p^2}} = \pm\frac{\sqrt{q^2 - 4pr}}{2p} \qquad \text{(Taking square root)}$$

$$\Rightarrow \quad z = \pm\frac{\sqrt{q^2 - 4pr}}{2p} - \frac{q}{2p} = \frac{-q \pm \sqrt{q^2 - 4pr}}{2p}$$

Example: Solve $z^2 - 2z + 5 = 0$ by completing square method.

Solution: We have $z^2 - 2z + 5 = 0$

or $\quad z^2 - 2z = -5$

By adding 1 on both sides

we get $\quad z^2 - 2z + 1 = -5 + 1$

$$\Rightarrow \quad (z - 1)^2 = -4$$

By taking square root of both sides

$$z - 1 = \pm\sqrt{-4}$$

$$z - 1 = \pm 2i$$

$$z = 1 \pm 2i$$

Exercise 1.3

1. Solve the following simultaneous linear equations with complex coefficients for w and z.

(i) $3z + (2+i)w = 11 - i$
$(2-i)z - w = -1 + i$

(ii) $2z + (3+i)w = 9 - i$
$(3-i)z - iw = -1 + i$

2. Write the following polynomials as the product of linear factors:

(i) $z^2 + 81b^2$ (ii) $z^3 + 3z^2 + z - 5$ (iii) $4z^2 + 9b^2$

(iv) $z^3 + 3z^2 + 4z + 2$ (v) $z^3 - 7z^2 + 19z - 13$

(vi) $z^3 + 3z^2 + 19z + 17$ (vii) $z^3 - 3z^2 + 4z - 2$

3. Solve the following quadratic equations by completing the squares, where z is a complex number.

(i) $z^2 - 4z + 5 = 0$ (ii) $z^2 + 12z + 52 = 0$

(iii) $34z^2 - 6z = -1$ (iv) $z^2 - 6z + 34 = 0$

(v) $z^2 - 6z = -13$ (vi) $z^2 + 64 = 0$

Review Exercise 1

1. Select the correct option.

i. For any complex number z, $|z|$ is equal to

(a) $|\bar{z}|$ (b) $|-z|$ (c) $|-\bar{z}|$ (d) all of these

ii. If z_1 and z_2 are any two complex numbers, then

(a) $|z_1 + z_2| < |z_1| + |z_2|$ (b) $|z_1 + z_2| \leq |z_1| + |z_2|$

(c) $|z_1 - z_2| < |z_1| - |z_2|$ (d) $|z_1 + z_2| \geq |z_1| + |z_2|$

iii. If $z = 3i - 4$, then $z + \bar{z}$ = --------- (a) 8 (b) $-3i$ (c) -8 (d) $3i$-8

iv. If $a > 0$ and $b < 0$, then (a)$ab > 0$ (b)$ab < 0$ (c) $ab = 0$ (d) all of these

v. If n is an even integer, then $(i)^n$ is equal to:

(a) i (b)$-i$ (c) 1 or -1 (d) i or $-i$

vi. If z is any real number x, then its conjugate is:

(a) x (b) $-x$ (c) xi (d) $-xi$

vii. $(-1)^{\frac{-21}{2}}$ is equal to: (a) i (b) $-i$ (c) 1 (d) -1

viii. If $z = (0, 1)$ then z^2 is: (a) 1 (b) -1 (c) i (d) $-i$

ix. The product of two conjugate complex numbers is:

(a) A real number (b) An imaginary number

(c) Always zero (d) Not defined

x. Real part of $\frac{2+i}{i}$ is equal to: (a) 1 (b) 2 (c) -1 (d) $\frac{1}{2}$

xi. The plane on which complex numbers are shown is called:
(a) Coordinate plane (b) Complex plane
(c) Cartesian plane (d) Real plane

xii. The modulus of a complex number $z = x + iy$ is the distance of P(x, y) from:
(a) x – axis (b) y – axis (c) Origin (d) (x, y)

xiii. If $z_1 = 2 + 3i$, $z_2 = 1 - i$ then $|z_1 z_2| =$ --------------
(a) $\sqrt{13}$ (b) $\sqrt{26}$ (c) $\sqrt{15}$ (d) 26

xiv. If $|x + 5i| = 3$, then x is equal to :(a) ± 4 (b)$\pm 4i$ (c)$\pm 22i$ (d)None of these

xv. Multiplicative inverse of (3,4) is:
(a) $\left(\frac{3}{25}, \frac{4}{25}\right)$ (b) $\left(\frac{3}{25}, \frac{-4}{25}\right)$ (c) $\left(\frac{-3}{25}, \frac{-4}{25}\right)$ (d) (1,0)

xvi. If $z = -3i + 4$, then $\overline{z} =$ --------------
(a) $-3i - 4$ (b)$-3i + 4$ (c) $3i + 4$ (d) $3i - 4$

xvii. Real and imaginary part of $i(3 - 2i)$ are respectively:
(a) $2 \& -3$ (b) $2 \& 3$ (c) $3 \& -2$ (d)$-3 \& -2$

xviii. If $z = a + ib$, then $|z| =$: --------------
(a) $\sqrt{a - b}$ (b) $\sqrt{a^2 - b^2}$ (c) $\sqrt{a^2 + b^2}$ (d) $\sqrt{a - b}$

xix. $\left(\frac{1+i}{1-i}\right)^{12} =$ --------------
(a) -1 (b) 1 (c) i (d) $-i$

xx. The value of i^{-7} is: --------------
(a) 1 (b) i (c) -1 (d) $-i$

2. Simplify:
(i) $(5 - 6i) + (5i) + (7 + 6i)$ (ii) $(4 - i) + (-9 + 6i)$
(iii) $(9 + 11i) - (3 + 5i)$ (iv) $(-2 - 15i) - (-12 + 13i)$
(v) $(-3 + 2i)(3 - 8i)$ (vi) $(3 - i)(4 + 3i)(5 - 2i)$
(vii) $\frac{4-i}{6-3i}$ (viii) $\frac{8+6i}{6-2i}$

3. **(a)** Find each of the following:
(i) $\overline{z}$ for $z = 3 - 15i$ (ii) $\overline{z_1 - z_2}$ for $z_1 = 5 + i$ and $z_2 = -8 + 3i$
(iii)$\overline{z_1 z_2}$ for $z_1 = -10 + 5i, z_2 = 5 - 10i$
(iv) $\frac{z}{\overline{z}}$ for $z = 15 - 3i$ (v) $\overline{z_1} - \overline{z_2}$ for $z_1 = 5 + 4i$ and $z_2 = -8 + 5i$

(b) Find absolute value of the following complex numbers:
(i) $-3 - 6i$ (ii) $-6 + 3i$ (iii) $\frac{6+3i}{10+2i}$ (iv) $\frac{1+i}{1-i}$

4. Find the values of x and y if:

(i) $5x + 3iy = -x + 2iy$ (ii) $x^2 - 7x + 9iy = iy^2 + 20i - 12$

5. Find real and imaginary parts of the following:

(i) $z = i - 42i - 3$ (ii) $\frac{3z+i}{z+4}$ for $z = 3 + 2i$

6. Find the additive and multiplicative inverse of each of the following:

(i) $3 - 7i$ (ii) $-2 + i$ (iii) $-2 - 5i$

(iv) $\frac{1}{2} + \frac{3}{2}i$ (v) $-4 + \sqrt{7}i$ (vi) $3 + 4i$

7. Solve the following equation:

$$z^2 - (2 - 3i)z - 5 + i = 0$$

8. Verify the following:

(i) $(2 + 3i)^2 = -5 + 12i$ (ii) $2(5 - 2i)^2 = 42 - 40i$

(iii) $z^3 + i = (z - i)(z^2 + iz - 1)$.

9. Solve the quadratic equations by completing the squares.

(i) $z^2 - 5z + 7 = 0$ (ii) $2z^2 - 6z + 9 = 0$ (iii) $4z^2 + 25 = 0$

(iv) $z^2 - 10z + 41 = 0$ (v) $4z^2 + 4z + 5 = 0$