# TRIGONOMETRIC IDENTITIES OF SUM AND DIFFERENCE OF ANGLES

Unit 10

## 10.1 Fundamental Law of Trigonometry

### 10.1.1 Recall trigonometric ratios

We have already studied trigonometric ratios in pervious classes. Let us recall. "The ratios of the lengths of sides of a right-angled triangle, are called trigonometric ratios."

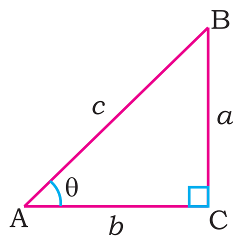

**Fig. 10.1**

There are six trigonometric ratios, namely sine, cosine, tangent, cotangent, secant and cosecant. These six trigonometric ratios are abbreviated as sin, cos, tan, cot, sec, and cosec.

Consider a right-angled triangle ABC in which $m < \mathrm{C} = 90°$ and $m < A = \theta$ as shown in Fig. 10.1.

Trigonometric ratios for acute angle θ, are defined as under.

1. $\sin\theta = \dfrac{\text{length of opposite side of } \theta}{\text{length of hypotenuse}} = \dfrac{a}{c}$
2. $\cos\theta = \dfrac{\text{length of adjacent side of } \theta}{\text{length of hypotenuse}} = \dfrac{b}{c}$
3. $\tan\theta = \dfrac{\text{length of opposite side of } \theta}{\text{length of adjacent side of } \theta} = \dfrac{a}{b}$
4. $\cot\theta = \dfrac{\text{length of adjacent side of } \theta}{\text{length of opposite side of } \theta} = \dfrac{b}{a}$
5. $\sec\theta = \dfrac{\text{length of hypotenuse}}{\text{length of adjacent side of } \theta} = \dfrac{c}{b}$
6. $\operatorname{cosec}\theta = \dfrac{\text{length of hypotenuse}}{\text{length of opposite side of } \theta} = \dfrac{c}{a}$

We have also studied in previous class that these ratios become trigonometric functions when θ is any real number representing measure (in radians) of an angle in standard position in unit circle and P$(x, y)$ be any point on the circle as shown in Fig. 10.2.

Trigonometric functions for any angle θ are defined as under.

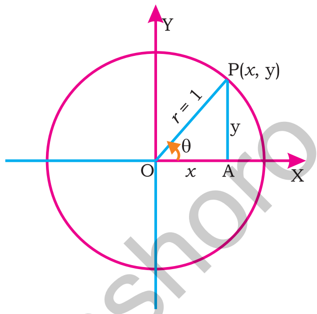

Fig. 10.2

$\sin\theta = y$ , $\quad\cos\theta = x$ ,

$\tan\theta = \dfrac{y}{x}$ , $\quad\cot\theta = \dfrac{x}{y}$ ,

$\sec\theta = \dfrac{1}{x}$ , $\quad\text{cosec}\,\theta = \dfrac{1}{y}$ .

Where $m\overline{OA} = x$ and $m\overline{AP} = y$ in right angled $\Delta$OAP.

### 10.1.2 Use distance formula to establish fundamental law of trigonometry

- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ **and deduce that**
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Recall that the distance between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is found by the formula known as distance formula:

$$|\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

### Fundamental Law $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Consider a unit circle with centre at O(0,0) as shown in Fig 10.3.

Let P$(\cos\beta, \sin\beta)$ and Q$(\cos\alpha, \sin\alpha)$ be any two points on the unit circle.

Then by using distance formula

$$|\overline{PQ}| = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} \quad \ldots\text{(i)}$$

Now rotate the axes so that the positive direction of the $x$-axis passes through the point P. Then with respect to this coordinate system, the coordinates of P and Q, respectively become (1,0) and $(\cos(\alpha - \beta), \sin(\alpha - \beta))$ as shown in Fig 10.4.

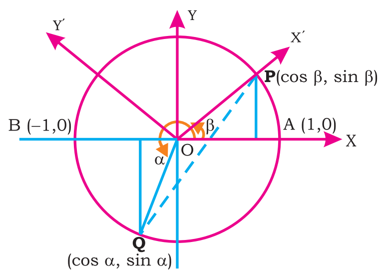

Fig. 10.3

So,

$|\overline{PQ}| = \sqrt{[\cos(\alpha-\beta)-1]^2+[\sin(\alpha-\beta)-0]^2}$ ...(ii)

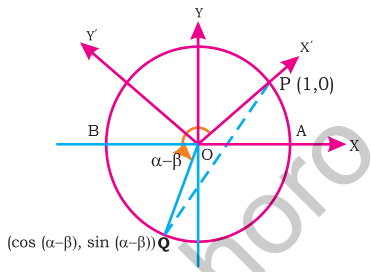


**Fig. 10.4**

Hence, by comparing equation (i) and (ii)

we get, $\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2}$

$= \sqrt{[\cos(\alpha-\beta)-1]^2+[\sin(\alpha-\beta)-0]^2}$

Squaring both sides

$(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2$

$= [\cos(\alpha-\beta)-1]^2+\sin^2(\alpha-\beta)$

or $(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)-2(\cos\alpha\cos\beta+\sin\alpha\sin\beta) =$
$[\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta)]-2\cos(\alpha-\beta)+1$

$\Rightarrow$ $2-2(\cos\alpha\cos\beta+\sin\alpha\sin\beta) = 2-2\cos(\alpha-\beta)$

So, $\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta}$ ... **I**

This law is called fundamental law of trigonometry.

**Example:** If $\cos\alpha = \frac{1}{2}, \cos\beta = \frac{\sqrt{3}}{2}$ and $\alpha$, $\beta$ are in the first quadrant. Find $\cos(\alpha-\beta)$.

**Solution:** We know $\cos^2\alpha+\sin^2\alpha = 1$

$$\Rightarrow \quad \sin\alpha = \pm\sqrt{1-\cos^2\alpha}$$

$$= \pm\sqrt{1-\left(\frac{1}{2}\right)^2} = \pm\frac{\sqrt{3}}{2}$$

$\sin\alpha = \frac{\sqrt{3}}{2}$ (since $\alpha$ is in the first quadrant).

Again from $\cos^2\beta+\sin^2\beta = 1$,

we have $\sin\beta = \pm\sqrt{1-\cos^2\beta}$

$$= \pm\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2} = \pm\frac{1}{2}.$$

$\sin\beta = \frac{1}{2}.$ (since $\beta$ is in the first quadrant)

Now, $\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta$

$$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{\sqrt{3}}{2}$$

## Deductions from the Fundamental Law

Using the above identity (I), we deduce some other identities.

Taking $\alpha = 0$ in (I)

We get, $\cos(0-\beta) = \cos 0\cos\beta+\sin 0\sin\beta$

$$\Rightarrow \quad \boxed{\cos(-\beta) = \cos\beta} \quad [\because \cos 0 = 1, \sin 0 = 0] \quad \ldots \textbf{i}$$

Again, putting $\alpha = \frac{\pi}{2}$, we have

$$\cos\left(\frac{\pi}{2} - \beta\right) = \cos\frac{\pi}{2}\cos\beta + \sin\frac{\pi}{2}\sin\beta$$

So, $\boxed{\cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta}$ $\left[\because \cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1\right]$ ...**ii**

Taking $-\beta$ for $\beta$ in (ii), we have

$$\cos\left(\frac{\pi}{2} - (-\beta)\right) = \sin(-\beta)$$

$$\Rightarrow \quad \cos\left(\beta - \left(-\frac{\pi}{2}\right)\right) = \sin(-\beta)$$

$$\Rightarrow \quad \cos\beta\cos\left(-\frac{\pi}{2}\right) + \sin\beta\sin\left(-\frac{\pi}{2}\right) = \sin(-\beta)$$

$$\Rightarrow \quad \cos\beta\,(0) + \sin\beta\,(-1) = \sin(-\beta)$$

So, $\boxed{\sin(-\beta) = -\sin\beta}$ ... **iii**

Now, $\tan(-\theta) = \dfrac{\sin(-\theta)}{\cos(-\theta)} = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$

So, $\boxed{\tan(-\theta) = -\tan\theta}$ ...**iv**

Again, $\cot(-\theta) = \dfrac{\cos(-\theta)}{\sin(-\theta)} = \dfrac{\cos\theta}{-\sin\theta} = -\cot\theta$

So, $\boxed{\cot(-\theta) = -\cot\theta}$ ...**v**

We know that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

So, $\cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right] = \sin\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow$ $\boxed{\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta}$ ...**vi**

Now, $\tan\left(\frac{\pi}{2} - \theta\right) = \dfrac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} = \dfrac{\cos\theta}{\sin\theta} = \cot\theta.$

So, $\boxed{\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta}$ ...**vii**

Also, $\cot\left(\frac{\pi}{2} - \theta\right) = \dfrac{\cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right)} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

So, $\boxed{\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta}$ ...**viii**

Similarly, $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$
and $\sec(-\theta) = \sec\theta$
Also $\operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta$
and $\sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec}\theta$

## Deduction of $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

By Fundamental law

We have $\quad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

and by putting $\quad -\beta$ for $\beta$, we have

$$\cos(\alpha+\beta) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

But $\quad \sin(-\beta) = -\sin\beta \quad$ and $\quad \cos(-\beta) = \cos\beta$ .

So, $\quad \boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta} \quad$ ...**II**

## Deduction of $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

Since $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, therefore by putting $(\alpha+\beta)$ for $\theta$,

we have $\quad \sin(\alpha+\beta) = \cos\left[\frac{\pi}{2}-(\alpha+\beta)\right] = \cos\left\{\left(\frac{\pi}{2}-\alpha\right)-\beta\right\}$

$$= \cos\left(\frac{\pi}{2}-\alpha\right).\cos\beta + \sin\left(\frac{\pi}{2}-\alpha\right).\sin\beta$$

So, $\quad \boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta} \quad$ ... **III**

By putting $-\beta$ for $\beta$ in (III),

we have $\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$

i.e., $\quad \boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta} \quad$ ...**IV**

**Example 1.** Find the value of $\sin\frac{7}{12}\pi$ without using tables or calculator.

**Solution:** $\quad \sin\frac{7}{12}\pi = \sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

By using values,

We get, $\quad \sin\frac{7}{12}\pi = \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

**Example 2.** Without using tables or calculator, find $\sin 15^\circ$.

**Solution:** $\quad \sin 15^\circ = \sin(60^\circ - 45^\circ) = \sin 60^\circ\cos 45^\circ - \cos 60^\circ\sin 45^\circ$

By using values, $\quad \sin 15^\circ = \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

**Example 3.** Prove that: $\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$; $(\cos\alpha \neq 0, \cos\beta \neq 0)$.

**Proof:**

$$\text{L.H.S} = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$
$$= \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}$$
$$= \tan\alpha + \tan\beta$$
$$= \text{R.H.S}$$

$\because$ L.H.S = R.H.S

$$\therefore \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$$

Hence proved.

**Example 4.** Show that:

$$\sin(180^\circ + \theta) = -\sin\theta.$$

**Solution:**

$$\text{L.H.S} = \sin(180^\circ + \theta)$$
$$= \sin 180^\circ \cos\theta + \cos 180^\circ \sin\theta$$
$$= (0)\cos\theta + (-1)\sin\theta$$
$$= -\sin\theta = \text{R.H.S}$$

So, L.H.S = R.H.S, hence shown.

## Deduction of $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

We know that $\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

Dividing the numerator and the denominator by $\cos\alpha\cos\beta \neq 0$,

we have $\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$ ... **V**

Again putting $-\beta$ for $\beta$ in (V),

We get, $\tan(\alpha-\beta) = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

So, $$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \quad \ldots \textbf{VI}$$

Now, $$\cot(\alpha+\beta)=\frac{1}{\tan(\alpha+\beta)}$$

$$=\frac{1}{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$

$$=\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$

$$=\frac{1-\frac{1}{\cot\alpha}\cdot\frac{1}{\cot\beta}}{\frac{1}{\cot\alpha}+\frac{1}{\cot\beta}}=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

So, $$\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta} \quad \ldots \textbf{VII}$$

Again putting $-\beta$ for $\beta$ in (VII), we have

$$\cot(\alpha-\beta)=\frac{\cot\alpha\cot(-\beta)-1}{\cot\alpha+\cot(-\beta)}$$

$$=\frac{-\cot\alpha\cot\beta-1}{\cot\alpha-\cot\beta}$$

So, $$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha} \quad \ldots \textbf{VIII}$$

**Example 1.** Prove that $\tan(270°-\theta)=\cot\theta$

**Proof:** L.H.S $=\tan(270°-\theta)$

$$=\frac{\sin(270°-\theta)}{\cos(270°-\theta)}$$

$$=\frac{\sin 270°\cos\theta-\cos 270°\sin\theta}{\cos 270°\cos\theta+\sin 270°\sin\theta}$$

$$=\frac{(-1)\cos\theta-(0)\sin\theta}{(0)\cos\theta+(-1)\sin\theta}$$

$$=\frac{-\cos\theta}{-\sin\theta}$$

$$=\cot\theta$$

$$=\text{R.H.S}$$

**Example 2.** Without using tables or calculator, find the value of tan75°.

**Solution:**

Here, $\sin 75°=\sin(30°+45°)=\sin 30°\cos 45°+\cos 30°\sin 45°$

By using values

We get, $\sin 75° = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$

$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

and $\cos 75° = \cos(30° + 45°) = \cos 30° \cos 45° - \sin 30° \sin 45°$

By using values,

We get, $\cos 75° = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Now, $\tan 75° = \frac{\sin 75°}{\cos 75°}$

$$= \frac{\frac{1+\sqrt{3}}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{\sqrt{3}+1}{2\sqrt{2}} \cdot \frac{2\sqrt{2}}{\sqrt{3}-1}$$

So, $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

**Example 3.** Find the value of $\cot\frac{1}{12}\pi$ without using tables or calculator.

**Solution:** $\cot\frac{1}{12}\pi = \cot\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\cot\frac{\pi}{3}\cot\frac{\pi}{4}+1}{\cot\frac{\pi}{4} - \cot\frac{\pi}{3}}$

By using values, $\cot\frac{\pi}{12} = \frac{\frac{1}{\sqrt{3}}.1+1}{1-\frac{1}{\sqrt{3}}}$

$$= \frac{1+\sqrt{3}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}-1}$$

So, $\cot\frac{1}{12}\pi = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

## Exercise 10.1

**1.** Prove that:

(i) $\cos(360° + \theta) = \cos\theta$

(ii) $\sin(180° + \theta) = -\sin\theta$

(iii) $\tan(180° - \theta) = -\tan\theta$

(iv) $\sin(270° - \theta) = -\cos\theta$

(v) $\cot(270° + \theta) - \tan\theta$ (vi) $\cot(270° - \theta) = \tan\theta$
(vii) $\operatorname{cosec}(360° - \theta) = -\operatorname{cosec}\theta$ (viii) $\sec(360° - \theta) = \sec\theta$

**2.** Evaluate the following.
(i) $\sin 150° \cos 300° + \sin 300° \cos 150°$
(ii) $\cos 19° \cos 11° - \sin 19° \sin 11$

**3.** Verify that:
(i) $\cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\cos(180° + 45°)\cos(180° - 45°) = \frac{1}{2}$

**4.** Without using tables or calculator. Find the values of:
(i) $\sin 75°$ (ii) $\tan 105°$ (iii) $\cos 165°$ (iv) $\sin 255°$
(v) $\tan\frac{5\pi}{12}$ (vi) $\cos\frac{13\pi}{12}$ (vii) $\sin\frac{23\pi}{12}$ (viii) $\cos\frac{25\pi}{12}$

**5.** Prove that:
(i) $\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha$ (ii) $\frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{\sin(\alpha+\beta)+\sin(\alpha-\beta)} = \cot\alpha$

**6.** Prove that:
(i) $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$
(ii) $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta$

**7.** Prove that: $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$

**8.** If $\sin\alpha = \frac{12}{13}$ and $\sin\beta = \frac{3}{5}$ where $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$ then find:
(i) $\cos(\alpha + \beta)$ (ii) $\tan(\alpha + \beta)$ (iii) $\cot(\alpha - \beta)$ (iv) $\cos(\alpha - \beta)$
(v) $\sin(\alpha + \beta)$ (vi) $\sin(\alpha - \beta)$ (vii) $\cot(\alpha + \beta)$ (viii) $\tan(\alpha - \beta)$
Also find quadrants of angle $(\alpha + \beta)$ and $(\alpha - \beta)$

## 10.2 Trigonometric Ratios of Allied Angles

### 10.2.1 Define allied angles

A group of angles is said to be allied angles of basic agnel $\theta$, if sum or difference of any two of them gives the integral multiple of 90° or $\frac{\pi}{2}$ radian.

So, the angles of measure of $90° \pm \theta, 180° \pm \theta, 270° \pm \theta, 360° \pm \theta$ are allied angles to basic angle $\theta$.

**Note:** The General formula to produce allied angles is $n(90°) \pm \theta°$ or in radians $n\left(\frac{\pi}{2}\right) \pm \theta$ where $n \in \mathbb{Z}$ and $0 < \theta < \frac{\pi}{2}$

**Example:** Generate all allied angles if $\theta = 30°$.

**Solution:** We use $n(90°) \pm \theta$, to generate allied angles for all integers.

If $n$ is non-negative integer

For $n = 0$, we have $0 \pm 30° = \pm 30°$
For $n = 1$, we have $90° \pm 30° = 60°, 120°$
For $n = 2$, we have $180° \pm 30° = 150°, 210°$ and so on
If $n$ is negative integer.
For $n = -1$, we have $-90° \pm 30° = -60°, -120°$
For $n = -2$, we have $-180° \pm 30° = -150°, -210°$ and so on.
Hence all allied angles are $\pm 30°, \pm 60°, \pm 120°, \pm 150°, \pm 210°, \ldots$

### 10.2.2 Use fundamental law and its deductions to derive trigonometric ratios of allied angles

### Express $a\sin\theta + b\cos\theta$ in the form $r\sin(\theta + \emptyset)$ where $a = r\cos\emptyset$ and $b = r\sin\emptyset$

Trigonometric ratios of allied angles in all four quadrants are given in the following table. Some of which are obtained from section 10.1.2 and the following solved examples. Remaining proofs are left as an exercise.

**Example 1.** Prove that $\sin(90° + \theta) = \cos\theta$

**Proof:**

$$\begin{aligned} \text{L.H.S} &= \sin(90° + \theta) \\ &= \sin 90° \cos\theta + \cos 90° \sin\theta \\ &= (1)\cos\theta + (0)\sin\theta \\ &= \cos\theta \\ &= \text{R.H.S} \end{aligned}$$

**Example 2.** Prove that $\cos(180° - \theta) = -\cos\theta$

**Proof:**

$$\begin{aligned} \text{L.H.S} &= \cos(180° - \theta) \\ &= \cos 180° \cos\theta + \sin 180° \sin\theta \\ &= (-1)\cos\theta + (0)\sin\theta \\ &= -\cos\theta \\ &= \text{R.H.S} \end{aligned}$$

**Example 3.** Prove that $\tan(360° - \theta) = -\tan\theta$

**Proof:**

$$\begin{aligned} \text{L.H.S} &= \tan(360° - \theta) \\ &= \frac{\tan 360° - \tan\theta}{1 + \tan 360° \tan\theta} \\ &= \frac{0 - \tan\theta}{1 + 0 \times \tan\theta} \\ &= -\tan\theta \\ &= \text{R.H.S} \end{aligned}$$

## Trigonometric ratios of Allied angles in all the Four Quadrants

| $-\theta$ (4th Quadrant) | | $90^\circ-\theta$ (1st Quadrant) | | $90^\circ+\theta$ (2nd Quadrant) | |
|---|---|---|---|---|---|
| $\sin(-\theta)$ | $-\sin\theta$ | $\sin(90^\circ-\theta)$ | $\cos\theta$ | $\sin(90^\circ+\theta)$ | $\cos\theta$ |
| $\cos(-\theta)$ | $\cos\theta$ | $\cos(90^\circ-\theta)$ | $\sin\theta$ | $\cos(90^\circ+\theta)$ | $-\sin\theta$ |
| $\tan(-\theta)$ | $-\tan\theta$ | $\tan(90^\circ-\theta)$ | $\cot\theta$ | $\tan(90^\circ+\theta)$ | $-\cot\theta$ |
| $\operatorname{cosec}(-\theta)$ | $-\operatorname{cosec}\theta$ | $\operatorname{cosec}(90^\circ-\theta)$ | $\sec\theta$ | $\operatorname{cosec}(90^\circ+\theta)$ | $\sec\theta$ |
| $\sec(-\theta)$ | $\sec\theta$ | $\sec(90^\circ-\theta)$ | $\operatorname{cosec}\theta$ | $\sec(90^\circ+\theta)$ | $-\operatorname{cosec}\theta$ |
| $\cot(-\theta)$ | $-\cot\theta$ | $\cot(90^\circ-\theta)$ | $\tan\theta$ | $\cot(90^\circ+\theta)$ | $-\tan\theta$ |

| $180^\circ-\theta$ (2nd Quadrant) | | $180^\circ+\theta$ (3rd Quadrant) | | $270^\circ-\theta$ (3rd Quadrant) | |
|---|---|---|---|---|---|
| $\sin(180^\circ-\theta)$ | $\sin\theta$ | $\sin(180^\circ+\theta)$ | $-\sin\theta$ | $\sin(270^\circ-\theta)$ | $-\cos\theta$ |
| $\cos(180^\circ-\theta)$ | $-\cos\theta$ | $\cos(180^\circ+\theta)$ | $-\cos\theta$ | $\cos(270^\circ-\theta)$ | $-\sin\theta$ |
| $\tan(180^\circ-\theta)$ | $-\tan\theta$ | $\tan(180^\circ+\theta)$ | $\tan\theta$ | $\tan(270^\circ-\theta)$ | $\cot\theta$ |
| $\operatorname{cosec}(180^\circ-\theta)$ | $\operatorname{cosec}\theta$ | $\operatorname{cosec}(180^\circ+\theta)$ | $-\operatorname{cosec}\theta$ | $\operatorname{cosec}(270^\circ-\theta)$ | $-\sec\theta$ |
| $\sec(180^\circ-\theta)$ | $-\sec\theta$ | $\sec(180^\circ+\theta)$ | $-\sec\theta$ | $\sec(270^\circ-\theta)$ | $-\operatorname{cosec}\theta$ |
| $\cot(180^\circ-\theta)$ | $-\cot\theta$ | $\cot(180^\circ+\theta)$ | $\cot\theta$ | $\cot(270^\circ-\theta)$ | $\tan\theta$ |

| $270^\circ+\theta$ (4th Quadrant) | | $360^\circ-\theta$ (4th Quadrant) | | $360^\circ+\theta$ (1st Quadrant) | |
|---|---|---|---|---|---|
| $\sin(270^\circ+\theta)$ | $-\cos\theta$ | $\sin(360^\circ-\theta)$ | $-\sin\theta$ | $\sin(360^\circ+\theta)$ | $\sin\theta$ |
| $\cos(270^\circ+\theta)$ | $\sin\theta$ | $\cos(360^\circ-\theta)$ | $\cos\theta$ | $\cos(360^\circ+\theta)$ | $\cos\theta$ |
| $\tan(270^\circ+\theta)$ | $-\cot\theta$ | $\tan(360^\circ-\theta)$ | $-\tan\theta$ | $\tan(360^\circ+\theta)$ | $\tan\theta$ |
| $\operatorname{cosec}(270^\circ+\theta)$ | $-\sec\theta$ | $\operatorname{cosec}(360^\circ-\theta)$ | $-\operatorname{cosec}\theta$ | $\operatorname{cosec}(360^\circ+\theta)$ | $\operatorname{cosec}\theta$ |
| $\sec(270^\circ+\theta)$ | $\operatorname{cosec}\theta$ | $\sec(360^\circ-\theta)$ | $\sec\theta$ | $\sec(360^\circ+\theta)$ | $\sec\theta$ |
| $\cot(270^\circ+\theta)$ | $-\tan\theta$ | $\cot(360^\circ-\theta)$ | $-\cot\theta$ | $\cot(360^\circ+\theta)$ | $\cot\theta$ |

**Example 1.** Find the values of

(i) $\cos 495^\circ$ (ii) $\sin 1230^\circ$ (iii) $\tan(-1590^\circ)$.

**Solution:**

**(i)**
$$\begin{aligned}\cos 495^\circ &= \cos(360^\circ+135^\circ)\\ &= \cos 135^\circ\\ &= \cos(90^\circ+45^\circ)\\ &= -\sin 45^\circ\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

**(ii)**
$$\begin{aligned}\sin 1230^\circ &= \sin(3\text{x}\,360+150)^\circ\\ &= \sin 150^\circ\\ &= \sin(180^\circ-30^\circ)\\ &= \sin 30^\circ = \frac{1}{2}\end{aligned}$$

**(iii)**
$$\begin{aligned}\tan(-1590^\circ) &= -\tan 1590^\circ\\ &= -\tan(4\times 360+150)^\circ\\ &= -\tan 150^\circ\end{aligned}$$

$$= -\tan(180° - 30°)$$
$$= -(-\tan 30°)$$
$$= \tan 30° = \frac{1}{\sqrt{3}}$$

**Example 2.** Without using the tables or calculator, find the values of:

(i) $\operatorname{cosec}(-870°)$ (ii) $\frac{6\cot 62°}{\tan 28°} + \frac{\tan 70°}{2\cot 20°}$

**Solution:**

**(i)** $\operatorname{cosec}(-870°) = \frac{1}{\sin(-870°)} = \frac{1}{-\sin 870°}$

$$= \frac{1}{-\sin(720°+150°)}$$
$$= \frac{1}{-\sin 150°}$$
$$= \frac{1}{-\sin(90°+60°)}$$
$$= \frac{1}{-\cos 60°}$$
$$= -2$$

**(ii)** $\frac{6\cot 62°}{\tan 28°} + \frac{\tan 70°}{2\cot 20°} = \frac{6\cot(90°-28°)}{\tan 28°} + \frac{\tan(90°-20°)}{2\cot 20°}$

$= 6\frac{\tan 28°}{\tan 28°} + \frac{\cot 20°}{2\cot 20°}$ [$\because \tan(90° - \theta) = \cot\theta$]

$= 6 + \frac{1}{2} = \frac{13}{2}$ [$\because \cot(90° - \theta) = \tan\theta$]

**Example 3.** Prove that: $\frac{3\cot(90°-\theta)}{\tan\theta} - \frac{2\sin\theta}{\cos(90°-\theta)} = 1$

**Solution:**

L.H.S $= \frac{3\cot(90°-\theta)}{\tan\theta} - \frac{2\sin\theta}{\cos(90°-\theta)} = 3\frac{\tan\theta}{\tan\theta} - 2\frac{\sin\theta}{\sin\theta}$ [$\because \cos(90° - \theta) = \sin\theta$]

$= 3 - 2 = 1 =$ R.H.S [$\because \cot(90° - \theta) = \tan\theta$]

## Express $a\sin\theta + b\cos\theta$ in the form $r\sin(\theta + \emptyset)$ where $a = r\cos\emptyset$ and $b = r\sin\emptyset$

Let $a = r\cos\emptyset$ and $b = r\sin\emptyset$.

Now by putting values of $a$ and b in $a\sin\theta + b\cos\theta$

we have, $a\sin\theta + b\cos\theta = r\cos\emptyset.\sin\theta + r\sin\emptyset.\cos\theta$

$= r(\cos\emptyset.\sin\theta + \sin\emptyset.\cos\theta)$

$= r\sin(\theta + \emptyset)$ [By using identity]

Thus $a\sin\theta + b\cos\theta = r\sin(\theta + \emptyset)$

**Example:** Express $4\sin\theta + 3\cos\theta$ in the form $r\sin(\theta + \emptyset)$, where $\theta$ and $\emptyset$ are in the first quadrant.

**Solution:**

Since, $\theta$ and $\emptyset$ are in the first quadrant, therefore, all trigonometric functions have positive values.

Let $\quad 4 = r\cos\emptyset$ ...(1)

and $\quad 3 = r\sin\emptyset$ ...(2)

Squaring both sides of equations (1) and (2), and then adding

We get $\quad 4^2 + 3^2 = r^2(\cos^2\emptyset + \sin^2\emptyset)$

$\Rightarrow \quad 25 = r^2 \times 1$

$\Rightarrow \quad r = 5$

Therefore, $\quad \cos\emptyset = \frac{4}{r} = \frac{4}{5}$ and $\sin\emptyset = \frac{3}{r} = \frac{3}{5}$

Now, $\quad 4\sin\theta + 3\cos\theta = r\cos\emptyset\sin\theta + r\sin\emptyset\cos\theta$

$= r(\sin\theta\cos\emptyset + \cos\theta\sin\theta)$

$= 5\sin(\theta + \emptyset)$; $\quad$ where, $\cos\emptyset = \frac{4}{5}$ and $\sin\emptyset = \frac{3}{5}$

## Exercise 10.2

**1.** Generate all allied angles if

(i) $\theta = 60°$ $\quad$ (ii) $\theta = 50°$

**2.** Show that: $\dfrac{\sin(90°-\theta)}{\operatorname{cosec}(90°-\theta)} + \dfrac{\cos(90°-\theta)}{\sec(90°-\theta)} = 1$

**3.** Evaluate the following without using calculator.

(i) $\sin(675°)$ $\quad$ (ii) $\tan\left(\frac{25\pi}{6}\right)$ $\quad$ (iii) $\operatorname{cosec}(2130°)$

(iv) $\dfrac{\sec 55°}{\operatorname{cosec} 35°} + \dfrac{\sin 48°}{\cos 42°}$ $\quad$ (v) $2\dfrac{\sin 56°}{\cos 34°} - \dfrac{\cot 37°}{\tan 53°} - \sqrt{2}\sin 45°$

**4.** Show that:

$$\cos(45° - \theta) = \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)$$

**5.** Express the following in the form $r\sin(\theta + \emptyset)$ where $\theta$ and $\emptyset$ are in the first quadrant.

(i) $15\sin\theta + 8\cos\theta$ $\quad$ (ii) $\dfrac{\sqrt{3}}{2}\sin\theta + \dfrac{1}{2}\cos\theta$

(iii) $\sin\theta + \cos\theta$

**6.** Prove that: $\dfrac{\tan\alpha}{\cot(90°-\alpha)} + \dfrac{\cos(90°-\alpha)}{\sin\alpha} = 2$

**7.** If $A + B + C = 180°$ then prove that

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

## 10.3 Double, Half and Triple Angle Identities

### 10.3.1 Derive double angle, half angle and triple angle identities from fundamental law and its deductions

#### (a) Double angle identities for $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$

We know that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Taking $\alpha = \beta = \theta$, we get

$\sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta$

i.e., $\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$ ... (1)

We know that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Taking $\alpha = \beta = \theta$

We get, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

By using $\sin^2\theta + \cos^2\theta = 1$, we get

$$\cos 2\theta = 2\cos^2\theta - 1$$
or
$$\cos 2\theta = 1 - 2\sin^2\theta \quad \text{... (2)}$$

Again, we know that $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

Taking $\alpha = \beta = \theta$, we have

$$\tan 2\theta = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

i.e., $\boxed{\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}}$ ... (3)

**Example 1.** Prove that: $\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \sec\theta$

**Solution:**

$$\text{L.H.S} = \frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta} - \frac{2\cos^2\theta - 1}{\cos\theta} \quad \text{[By using (1) and (2)]}$$

$$= 2\cos\theta - \frac{2\cos^2\theta - 1}{\cos\theta}$$

$$= \frac{2\cos^2\theta - 2\cos^2\theta + 1}{\cos\theta}$$

$$= \frac{1}{\cos\theta} = \sec\theta = \text{R.H.S}$$

**Example 2.** Prove that: $\sec 4\theta = \dfrac{\tan 2\theta + \cot 2\theta}{\cot 2\theta - \tan 2\theta}$

**Solution:** R.H.S $= \dfrac{\tan 2\theta + \cot 2\theta}{\cot 2\theta - \tan 2\theta}$

$$= \frac{\frac{\sin 2\theta}{\cos 2\theta} + \frac{\cos 2\theta}{\sin 2\theta}}{\frac{\cos 2\theta}{\sin 2\theta} - \frac{\sin 2\theta}{\cos 2\theta}}$$

$$= \frac{\frac{\sin^2 2\theta + \cos^2 2\theta}{\sin 2\theta \cos 2\theta}}{\frac{\cos^2 2\theta - \sin^2 2\theta}{\sin 2\theta \cos 2\theta}}$$

$$= \frac{1}{\cos^2 2\theta - \sin^2 \theta}$$

$$= \frac{1}{\cos^2 2\theta - \sin^2 2\theta}$$

$$= \frac{1}{\cos 4\theta} = \sec 4\theta = \text{L.H.S}$$

### (b) Half angle identities for $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$ and $\tan\frac{\theta}{2}$

We know that $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$

Putting $\frac{\theta}{2}$ for $\theta$, we have,

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$\Rightarrow \quad 2\sin^2\frac{\theta}{2} = 1 - \cos\theta$$

$$\Rightarrow \quad \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

$$\Rightarrow \quad \boxed{\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}} \qquad \ldots (1)$$

We know that $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$

Putting $\frac{\theta}{2}$ for $\theta$, we have,

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$\Rightarrow \quad 2\cos^2\frac{\theta}{2} = 1 + \cos\theta$$

$$\Rightarrow \quad \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

So,
$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \ldots (2)$$

From (1) and (2), by division, we have, $\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

or $\tan\frac{\theta}{2} = \left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)\left(\sqrt{\frac{1+\cos\theta}{1+\cos\theta}}\right)$

$$= \frac{\sqrt{1-\cos^2\theta}}{1+\cos\theta}$$

$$= \frac{\sin\theta}{1+\cos\theta}$$

So,
$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} \qquad (\text{where } \cos\theta \neq -1) \qquad \ldots(3a)$$

or $\tan\frac{\theta}{2} = \dfrac{\sin^2\theta}{\sin\theta\,(1+\cos\theta)}$

$$= \frac{1-\cos^2\theta}{\sin\theta\,(1+\cos\theta)}$$

$$= \frac{1-\cos\theta}{\sin\theta}$$

So, $$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} \qquad (\text{where } \sin\theta \neq 0) \qquad \ldots (3b)$$

**Example 1.** Show that $\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2 = 1 - \sin\theta$

**Solution:**

$$\text{L.H.S} = \left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2 = \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 1 - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 1 - \sin\theta = \text{R.H.S}$$

**Example 2.** If $\sin\theta = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$ then, find the values of

(i) $\cos\frac{\theta}{2}$ (ii) $\sin\frac{\theta}{2}$ (iii) $\tan\frac{\theta}{2}$

**Solution:**

$\because$ $\theta$ is in $1^{st}$ quadrant

$\therefore \cos\theta = \sqrt{1-\sin^2\theta}$

$$\Rightarrow \cos\theta = \sqrt{1-\frac{9}{25}}$$
$$= \sqrt{\frac{16}{25}}$$
$$= \frac{4}{5}$$

**(i)** $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$ $\quad\left(\because 0<\theta<\frac{\pi}{2}\right)$
$$= \sqrt{\frac{1+\frac{4}{5}}{2}}$$
$$= \sqrt{\frac{9}{10}}$$
$$= \frac{3}{\sqrt{10}}$$

**(ii)** $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$ $\quad\left(\because 0<\theta<\frac{\pi}{2}\right)$
$$= \sqrt{\frac{1-\frac{4}{5}}{2}}$$
$$= \sqrt{\frac{1}{10}}$$
$$= \frac{1}{\sqrt{10}}$$

**(iii)** $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$
$$= \frac{\frac{3}{5}}{1+\frac{4}{5}}$$
$$= \frac{3}{5}\times\frac{5}{9}$$
$$= \frac{1}{3}$$

**Example 3.** Find the value of $\sin 15°$.

**Solution:** $\sin 15° = \sin\frac{30°}{2}$
$$= \sqrt{\frac{1-\cos 30°}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{4}}$$

$$= \frac{\sqrt{2-\sqrt{3}}}{2}$$

### (c) Triple angle identities $\sin 3\theta$, $\cos 3\theta$ and $\tan 3\theta$

We know that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

By using $\alpha = 2\theta$ and $\beta = \theta$, we get

$$\sin(2\theta+\theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$$
$$= (2\sin\theta\cos\theta)\cos\theta + (1-2\sin^2\theta)\sin\theta$$
$$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$$
$$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$$
$$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$

$$\boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta} \quad \ldots (1)$$

Similarly, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

By using, $\alpha = 2\theta$ and $\beta = \theta$, we get

$$\cos 3\theta = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$
$$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$

$$\boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta} \quad \ldots (2)$$

We also know that, $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

By using, $\alpha = 2\theta$ and $\beta = \theta$, we get

$$\tan 3\theta = \frac{\tan 2\theta + \tan\theta}{1-\tan 2\theta\tan\theta}$$

$$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan\theta}{1-\tan^2\theta}\tan\theta}$$

$$= \frac{\dfrac{2\tan\theta+\tan\theta(1-\tan^2\theta)}{1-\tan^2\theta}}{\dfrac{(1-\tan^2\theta)-2\tan^2\theta}{1-\tan^2\theta}}$$

$$\boxed{\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}} \quad ....(3)$$

**Example:** Show that $\cos5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

**Solution:**

L.H.S $= \cos5\theta = \cos(2\theta + 3\theta)$

$= \cos 2\theta \cos 3\theta - \sin2\theta\sin3\theta$

$= (2\cos^2\theta - 1)(4\cos^3\theta - 3\cos\theta) - (2\sin\theta\cos\theta)(3\sin\theta - 4\sin^3\theta)$

$= (8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta) - \cos\theta(6\sin^2\theta - 8\sin^4\theta)$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - 2\cos\theta\sin^2\theta(3 - 4\sin^2\theta)$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - 2\cos\theta(1 - \cos^2\theta)\{3 - 4(1 - \cos^2\theta)\}$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - (2\cos\theta - 2\cos^3\theta)(3 - 4 + 4\cos^2\theta)$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - (2\cos\theta - 2\cos^3\theta)(4\cos^2\theta - 1)$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - (8\cos^3\theta - 2\cos\theta - 8\cos^5\theta + 2\cos^3\theta)$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 10\cos^3\theta + 2\cos\theta + 8\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ = R.H.S

## Exercise 10.3

Prove the following identities:

**1.** $\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \sec\theta$

**2.** $\cot 2\theta = \frac{\cot\theta - \tan\theta}{2}$

**3.** $\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$

**4.** $\tan^2\frac{\theta}{2} = \frac{\tan\theta-\sin\theta}{\tan\theta+\sin\theta}$

**5.** $\operatorname{cosec}^2\frac{\theta}{2} = \frac{2\sec\theta}{\sec\theta-1}$

**6.** $\sin 4\theta = 4\sin\theta\cos\theta\cos 2\theta$

**7.** $\cos 2\theta = \frac{\cot^2\theta-1}{\cot^2\theta+1}$

**8.** $\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2 = 1 - \sin\theta$

**9.** $\cos 4\theta = 8\cos^4\theta - 8\cos\theta^2 + 1$

**10.** $\tan\frac{\alpha}{2} = \operatorname{cosec}\alpha - \cot\alpha$

**11.** $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$

**12.** $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

**13.** $\tan 4\theta = \frac{4\tan\theta(1-\tan^2\theta)}{\tan^4\theta-6\tan^2\theta+1}$

**14.** $\sec\theta = \frac{\tan\frac{\theta}{2}+\cot\frac{\theta}{2}}{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}}$

**15.** $\dfrac{\sin 3\alpha}{\sin\alpha} - \dfrac{\cos 3\alpha}{\cos\alpha} = 2$

**16.** $\dfrac{\sin 4\theta}{\sin 2\theta} - \dfrac{\cos 4\theta}{\cos 2\theta} = \sec 2\theta$

**17.** Evaluate the following

(i) $\tan 22.5°$ (ii) $\cos 15°$

**18.** If $\cos\theta = \frac{-4}{5}$ and $\frac{\pi}{2} < \theta < \pi$ then find the following:

(i) $\sin\frac{\theta}{2}$ (ii) $\cos\frac{\theta}{2}$ (iii) $\tan\frac{\theta}{2}$ (iv) $\sin 2\theta$ (v) $\cos 2\theta$ (vi) $\tan 2\theta$

**19.** If $\sin\theta = \frac{12}{13}$ , where $0 < \theta < \frac{\pi}{2}$ then find the following:

(i) $\sin\frac{\theta}{2}$ (ii) $\cos\frac{\theta}{2}$ (iii) $\tan\frac{\theta}{2}$ (iv) $\sin 2\theta$ (v) $\cos 2\theta$ (vi) $\tan 2\theta$

**20.** If $\cos\theta = \frac{3}{5}$ , where $0 < \theta < \frac{\pi}{2}$, then find the following:

(i) $\sin 3\theta$ (ii) $\cos 3\theta$ (iii) $\tan 3\theta$

## 10.4 Sum, Difference and Product of sine and cosine

### 10.4.1 Express the product (of sines and cosines) as sums or differences (of sines and cosines)

We know that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Then by adding and subtracting, we get

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \quad \ldots(i)$$

and

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)] \quad \ldots(ii)$$

Similarly, from $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

by adding and subtracting, we get

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)] \quad \ldots(iii)$$

and

$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)] \quad \ldots(iv)$$

**Example 1.** Express $2\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$ as sum or difference.

**Solution:**

Here $\alpha = \frac{3\theta}{2}$ and $\beta = \frac{\theta}{2}$

Using $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$,

so we get, $2\sin\frac{3\theta}{2}\cos\frac{\theta}{2} = \sin\left(\frac{3\theta}{2}+\frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2}-\frac{\theta}{2}\right)$

$$= \sin\frac{4\theta}{2} + \sin\frac{2\theta}{2}$$
$$= \sin 2\theta + \sin\theta$$

**Example 2.** Express $\cos(y+z)\cos(y-z)$ as sum or difference.

**Solution:**

Here, $\alpha = y + z$ and $\beta = y - z$

Using $\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$,

We get, $\cos(y+z)\cos(y-z) = \frac{1}{2}[\cos(y+z+y-z) + \cos(y+z-y+z)]$

$$= \frac{1}{2}(\cos 2y + \cos 2z)$$

**Example 3.** Verify that: $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}$

**Solution:** L.H.S $= \sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$

$$= \sin 10^\circ \sin 70^\circ \sin 30^\circ \sin 50^\circ$$
$$= \frac{-1}{4}\sin 10^\circ(-2\sin 50^\circ \sin 70^\circ)$$
$$= \frac{-1}{4}\sin 10^\circ [\cos(50^\circ + 70^\circ) - \cos(50^\circ - 70^\circ)]$$
$$= \frac{-1}{4}\sin 10^\circ [\cos 120^\circ - \cos(-20^\circ)]$$
$$= \frac{-1}{4}\sin 10^\circ [\cos(90+30)^\circ - \cos 20^\circ] \qquad [\because \cos(-\theta) = \cos\theta]$$
$$= \frac{-1}{4}\sin 10^\circ [-\sin 30^\circ - \cos 20^\circ] \qquad \left[\because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right]$$
$$= \frac{-1}{4}\sin 10^\circ \left[-\frac{1}{2} - \cos 20^\circ\right]$$
$$= \frac{1}{8}\sin 10^\circ + \frac{1}{4}[\sin 10^\circ \cos 20^\circ]$$
$$= \frac{1}{8}\sin 10^\circ + \frac{1}{8}[2\sin 10^\circ \cos 20^\circ]$$
$$= \frac{1}{8}\sin 10^\circ + \frac{1}{8}[\sin(10^\circ + 20^\circ) + \sin(10^\circ - 20^\circ)]$$
$$= \frac{1}{8}\sin 10^\circ + \frac{1}{8}[\sin 30^\circ + \sin(-10^\circ)]$$
$$= \frac{1}{8}\sin 10^\circ + \frac{1}{8}\left[\frac{1}{2} - \sin 10^\circ\right]$$
$$= \frac{1}{8}\sin 10^\circ + \frac{1}{16} - \frac{1}{8}\sin 10^\circ$$
$$= \frac{1}{16} = \text{R.H.S}$$

## 10.4.2 Express the sums or differences (of sines and cosines) as products (of sines and cosines)

From (i), we have

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

or $$2\sin\alpha\cos\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$$

Putting $\alpha+\beta = u$ and $\alpha-\beta = v$, also $\alpha = \frac{u+v}{2}$ and $\beta = \frac{u-v}{2}$.

we have $$2\sin\frac{u+v}{2}\cos\frac{u-v}{2} = \sin u+\sin v$$

i.e., $$\sin u+\sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2} \quad \ldots (v)$$

Similarly, from (ii), we have

$$\sin u-\sin v = 2\cos\frac{u+v}{2}\sin\frac{u-v}{2} \quad \ldots (vi)$$

Again from (iii), we have $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$

or $$2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$$

Putting $\alpha+\beta = u$ and $\alpha-\beta = v$, also $\alpha = \frac{u+v}{2}$ and $\beta = \frac{u-v}{2}$.

we have $$2\cos\frac{u+v}{2}\cos\frac{u-v}{2} = \cos u+\cos v$$

i.e., $$\cos u+\cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2} \quad \ldots (vii)$$

Similarly from (iv), we have

$$\cos u-\cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2} \quad \ldots (viii)$$

**Example 1.** Express the sum $\cos 20°+\cos 10°$ in the product form.

**Solution:** We know that $\cos u+\cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2}$

So, $$\cos 20°+\cos 10° = 2\cos\frac{20°+10°}{2}\cos\frac{20°-10°}{2}$$

$$= 2\cos 15°\cos 5°$$

**Example 2.** Prove that $\frac{\cos 6\theta-\cos 2\theta}{\sin 3\theta+\sin\theta} = \frac{-\sin 4\theta}{\cos\theta}$

**Solution:** By using $$\cos u-\cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

and $$\sin u+\sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

So, $$\text{L.H.S} = \frac{\cos 6\theta-\cos 2\theta}{\sin 3\theta+\sin\theta} = \frac{-2\sin\left(\frac{6\theta+2\theta}{2}\right)\sin\left(\frac{6\theta-2\theta}{2}\right)}{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)}$$

$$= \frac{-\sin\frac{8\theta}{2}\ \sin\frac{4\theta}{2}}{\sin\frac{4\theta}{2}\ \cos\frac{2\theta}{2}}$$

$$= -\frac{\sin 4\theta\ \sin 2\theta}{\sin 2\theta \cos\theta}$$

$$= \frac{-\sin 4\theta}{\cos\theta}$$

$$= \text{R. H. S}$$

## Exercise 10.4

**1.** Express the following products as sums or differences:

(i) $2\sin 6\theta \cos 3\theta$ (ii) $\cos 3\theta \sin 6\theta$

(iii) $2\sin(\alpha-\beta)\cos(\alpha+\beta)$ (iv) $-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

(v) $\sin(\theta+45^\circ)\sin(\theta-45^\circ)$ (vi) $2\cos(2\theta+60^\circ)\cos(2\theta-60^\circ)$

**2.** Express the following sums or differences as products.

(i) $\sin 2\alpha - \sin 2\beta$ (ii) $\cos\frac{5\theta}{2}+\cos\frac{3\theta}{2}$

(iii) $\cos 25^\circ + \cos 65^\circ$ (iv) $\cos(\theta+30^\circ)-\cos(\theta-30^\circ)$

(v) $\sin\frac{\pi}{2}-\sin\frac{\pi}{4}$ (vi) $\sin 2(\theta+40^\circ)+\sin 2(\theta-40^\circ)$

**3.** Prove the following identities.

(i) $\frac{\sin\alpha-\sin\beta}{\cos\alpha+\cos\beta} = \tan\left(\frac{\alpha-\beta}{2}\right)$

(ii) $\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta} = \tan\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$

(iii) $\frac{\sin 6\theta+\sin 4\theta}{\cos 6\theta+\cos 4\theta} = \tan 5\theta$

(iv) $\frac{\sin 2\theta+\sin 4\theta+\sin 6\theta+\sin 8\theta}{\cos 2\theta+\cos 4\theta+\cos 6\theta+\cos 8\theta} = \tan 5\theta$

**4.** Evaluate:

(i) $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$ (ii) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

**5.** Show that:

(i) $\frac{\cos 8^\circ-\sin 8^\circ}{\cos 8^\circ+\sin 8^\circ} = \tan 37^\circ$ (ii) $\frac{\cos 11^\circ+\sin 11^\circ}{\cos 11^\circ-\sin 11^\circ} = \tan 56^\circ$

# Review Exercise 10

**1. Select the correct option.**

(i) $\cos(\alpha - \beta)$ is equal to:

(a) $\cos\alpha\cos\beta + \sin\alpha\sin\beta$ (b) $\cos\alpha\cos\beta - \sin\alpha\sin\beta$
(c) $\sin\alpha\cos\beta + \cos\alpha\sin\beta$ (d) $\sin\alpha\cos\beta - \cos\alpha\sin\beta$

(ii) Angles associated with basic angles of measure $\theta$ to a right angle or its multiple are called:

(a) Coterminal angles (b) Angles in standard positions
(c) Allied angles (d) Obtuse angles

(iii) $\sin\left(\frac{\pi}{2} + \theta\right)$ is equal to:

(a) $\cos\theta$ (b) $\sin\theta$ (c) $-\cos\theta$ (d) $-\sin\theta$

(iv) $\cos(\pi - \theta)$ is equal to:

(a) $\sin\theta$ (b) $\cos\theta$ (c) $-\sin\theta$ (d) $-\cos\theta$

(v) $\tan(\pi + \theta)$ is equal to:

(a) $\tan\theta$ (b) $-\cot\theta$ (c) $-\tan\theta$ (d) $\cot\theta$

(vi) $\cos(2\pi + \theta)$ is equal to:

(a) $\sin\theta$ (b) $\cos\theta$ (c) $-\sin\theta$ (d) $-\cos\theta$

(vii) $\sin 540°$ is equal to:

(a) 1 (b) 0 (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

(viii) $\sin(-300°)$ is equal to:

(a) 1 (b) $\frac{\sqrt{3}}{2}$ (c) 0 (d) $-1$

(ix) If $\alpha, \beta$ and $\gamma$ are the angles of $\Delta$ ABC, then $\sin(\alpha + \beta)$ is equal to:

(a) $\sin\gamma$ (b) $-\sin\gamma$ (c) $\cos\gamma$ (d) $-\cos\gamma$

(x) $\cos 2\alpha$ is equal to:

(a) $\cos^2\alpha - \sin^2\alpha$ (b) $2\cos^2\alpha - 1$ (c) $1 - 2\sin^2\alpha$ (d) all of these

(xi) $\sin\frac{\alpha}{2}$ is equal to:

(a) $\pm\sqrt{\frac{1+\sin\alpha}{2}}$ (b) $\pm\sqrt{\frac{1-\cos\alpha}{2}}$ (c) $\pm\sqrt{\frac{1+\cos\alpha}{2}}$ (d) $\pm\sqrt{\frac{1-\sin\alpha}{2}}$

(xii) $\cos 3\alpha$ is equal to: (a) $3\cos\alpha - 4\cos\alpha$ (b) $3\cos^3\alpha + 4\cos\alpha$
(c) $4\cos^3\alpha - 3\cos\alpha$ (d) $4\cos^3\alpha + 4\cos\alpha$

(xiii) $\cos\alpha - \cos\beta =$ (a) $2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ (b) $2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
(c) $2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ (d) $-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

(xiv) $2\sin 7\theta \sin 2\theta$ is equal to: (a) $\cos 5\theta - \cos 9\theta$ (b) $\cos 9\theta - \cos 5\theta$ (c) $\sin 9\theta + \sin 5\theta$ (d) $\sin 9\theta - \sin 5\theta$

(xv) An allied angle to $\theta$ is ________.
(a) $270° + \theta$ (b) $60° + \theta$ (c) $45° + \theta$ (d) $30° + \theta$

(xvi) The value of $\cos(\alpha - 2\pi)$ is equal to:
(a) $-\cos\alpha$ (b) $-\sin\alpha$ (c) $\cos\alpha$ (d) $\sin\alpha$

(xvii) The value of $\sin 7\pi$ is equal to:
(a) 0 (b) 1 (c) $-1$ (d) $\frac{1}{2}$

**2.** Prove the following:

(i) $\cos\left(\frac{\pi}{4} - \alpha\right)\cos\left(\frac{\pi}{4} - \beta\right) - \sin\left(\frac{\pi}{4} - \alpha\right)\sin\left(\frac{\pi}{4} - \beta\right) = \sin(\alpha + \beta)$

(ii) $\cos\left(\frac{3\pi}{4} + \theta\right) - \cos\left(\frac{3\pi}{4} - \theta\right) = -\sqrt{2}\sin\theta$

(iii) $\sin^2 6\theta - \sin^2 4\theta = 4\sin 5\theta \cos 5\theta \sin\theta \cos\theta$

(iv) $\frac{\sin 5\theta + \sin 3\theta}{\cos 5\theta + \cos 3\theta} = \tan 4\theta$

(v) $\frac{\cos 4\theta + \cos 3\theta + \cos 2\theta}{\sin 4\theta + \sin 3\theta + \sin 2\theta} = \cot 3\theta$

(vi) $\sin 5\theta - \sin 3\theta + \sin 2\theta = 4\sin\theta \cos\frac{3\theta}{2}\cos\frac{5\theta}{2}$