Application of Trigonometry

Unit 11

11.1 Solving Triangles

We know that a triangle has six elements which are three sides and three angles. If measures of any three of them including the measure of at least one side are known, then we can find the measures of other sides and angles.

11.1.1 Solve right angled triangle when measures of:
(i) two sides are given
(ii) one side and one angle are given

(i) When two sides are given

The following example is suitable to understand the method of solving a right-angled triangle, when measures of two sides are given.

Example:

Solve the right triangle ABC with $\beta = 90°$, $a = 8.6$ cm and $b = 11.4$ cm.

Solution: Given and unknown elements are mentioned in Fig. 11.1.

We have to find α, γ and c.

Here, $\sin \alpha = \frac{a}{b} = \frac{8.6}{11.4} = 0.754$

$\Rightarrow \quad \alpha = \sin^{-1} 0.754$

$\Rightarrow \quad \alpha = 49°$

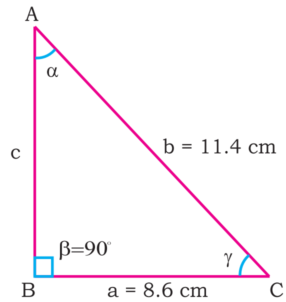
and $\quad \gamma = 180° - \alpha - \beta = 180° - 49° - 90° = 41°$

By Pythagoras theorem,

$$b^2 = a^2 + c^2$$

$\Rightarrow \quad c^2 = b^2 - a^2 = (11.4)^2 - (8.6)^2$

$\Rightarrow \quad c = 7.5$ cm



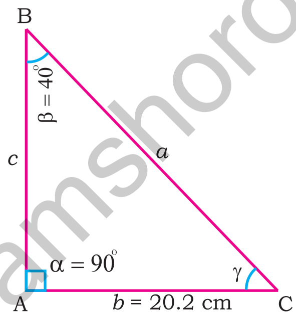
(Fig. 11.1)

(ii) When one side and one angle are given

If we are given a right-angled triangle, in which one side and one angle are given, we can find measures of its unknown sides and angles as explained in the following examples.

Example 1. Solve the right-angled triangle ABC with $\alpha = 90°, \beta = 40°$ and $b = 20.2$ cm.

Solution: Given and unknown elements are mentioned in Fig. 11.2, we have to find γ, c and a.



(Fig. 11.2)

As $\quad \alpha + \beta + \gamma = 180°$

so, $\quad \gamma = 180° - \alpha - \beta \quad = 180° - 90° - 40°$

$\gamma = 50°.$

Now, $\quad \tan\beta = \dfrac{b}{c}$

$\Rightarrow \quad \tan 40° = \dfrac{20.2}{c}$

$\Rightarrow \quad c = \dfrac{20.2}{\tan 40°} \quad \Rightarrow \quad c = 24.0734 \text{ cm}$

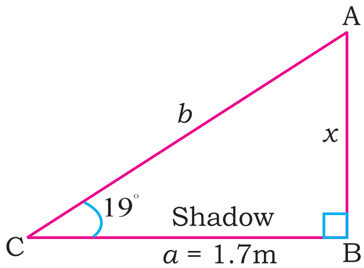
By Pythagoras theorem

$$a^2 = b^2 + c^2 = (20.2)^2 + (24.0762)^2 = 987.7$$

Therefore, $\quad a = 31.42$ cm.

Example 2. Find the height of an object if the angle of elevation of the sun is 19° and the length of the shadow of the object is 1.7 meters.

Solution: Let x be the height of the object. Its shadow has length $a = 1.7$ m as shown in Fig. 11.3.



(Fig. 11.3)

Then, in ΔABC, $\quad \tan 19° = \dfrac{x}{a}$

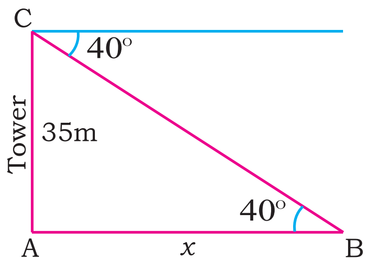
$\Rightarrow \quad x = 1.7 \tan 19°$

$\Rightarrow \quad x = 0.585$

So, the height of the object is 0.585 meters

Example 3. From the top of a tower, the angle of depression to the ship at its waterline is 40°. If the height of the tower is 35m, find the distance between the ship and the foot of the tower.

Solution: Let x be the distance of the ship B from the foot A of the tower $\overline{AC}$ as shown in Fig. 11.4.



(Fig. 11.4)

Then, in ΔABC, $\quad \tan 40° = \dfrac{35}{x}$

$$x = \frac{35}{\tan 40°}$$

$$x = 41.711$$

Thus, the distance of ship from the foot of the tower is 41.711 meters.

Exercise 11.1

1. Solve the right angled triangle ABC in which $\gamma = 90°$

(i) $a = 4\text{m}$, $b = 3\text{m}$ (ii) $a = 12\text{m}$, $b = 5\text{m}$

(iii) $a = 8\text{m}$, $b = 6\text{m}$ (iv) $c = 8\text{m}$, $\beta = 42°\,30'$

(v) $a = 140\text{m}$, $\alpha = 38°\,20'$ (vi) $b = 30.8m$, $\alpha = 41°\,50'$

2. A vertical stick 16m long, casts a 12cm long shadow. Find the angle of elevation of the sun.

3. Find the distance of a man from the foot a tower 169m high. if the angle of depression of the man from its top is 48° 30′.

4. A string of a flying kite is 200 meters long and its angle of elevation is 58° 30′. Find height of the kite when the string is fully stretched.

5. A man 1.8m tall observes that the angle of elevation of the top of a tree at a distance of 12m from him is 31°45′. What is the height of tree?

6. From the top of a cliff 80m high, the angle of depression of a boat is 12°30′. How far is the boat from the cliff?

7. Two masts are 20m and 12m high. If the line joining their tops makes an angle of 35° with the horizontal; find distance between them.

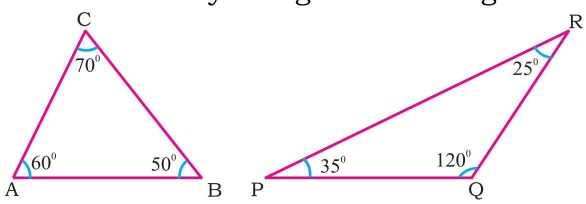
8. A window washer is working in a hotel building. An observer at a distance of 20 m from the building finds the angle of elevation of the worker to be of 30°. The worker climbs up 12m and the observer moves 4m further away from the building. Find the new angle of elevation of the worker.

11.1.2 Define an oblique triangle and prove

- **The laws of sines**
- **The laws of cosines**
- **The laws of tangents**

and deduce respective half angle formulae

An oblique triangle is a triangle with no right angle. It has either three acute angles, or one obtuse and two acute angles. In Fig 11.5, ΔABC and ΔPQR are oblique triangles. An oblique triangle can be solved, if a side and any two other elements are known by using the following laws.



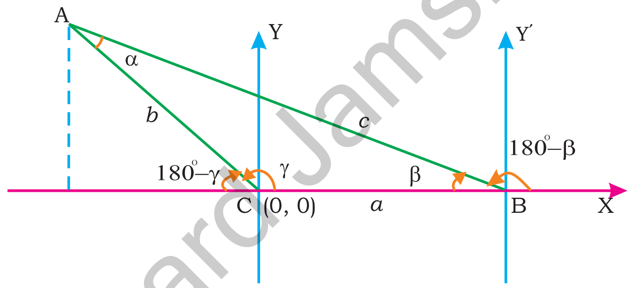
(Fig. 11.5)

(i) The law of sines

Consider a triangle ABC, in which a, b, c are the measures of sides opposite to the angles α, β and γ respectively as shown in Fig.11.6. Take a rectangular coordinate system, in order to place the point C in the standard position. The coordinates of point A will be $(b\cos\gamma\ ,\ b\sin\gamma)$.

If the point B is taken as the origin and measure of the angle $XBA = 180° - \beta$ then the point A will have the coordinates $\left(c\cos(180°-\beta), c\sin(180°-\beta)\right)$.

Since y – coordinate is the same in both the cases; we have



(Fig. 11.6)

$$b\sin\gamma = c\sin(180°-\beta)$$

$$\Rightarrow \quad b\sin\gamma = c\sin\beta$$

$$\Rightarrow \quad \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \qquad \ldots \text{(i)}$$

Similarly, we have

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \qquad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}} \qquad \ldots \text{(iii)}$$

This is called the law of sines, which states that the measures of the sides of the triangle are proportional to the sines of the measures of the opposite angles. This rule is due to a muslim mathematician Al-Beruni.

(ii) The law of cosines

Consider Fig.11.6, the vertices of the triangle ABC are $A(b\cos\gamma, b\sin\gamma)$, $B(a, 0)$ and C(0,0). By using distance formula,

$$|\overline{AB}| = \sqrt{(b\cos\gamma - a)^2 + (b\sin\gamma - 0)^2}.$$

So,

$$c = \sqrt{(b^2\cos^2\gamma - 2ab\cos\gamma + a^2 + b^2\sin^2\gamma}$$

$$\Rightarrow \quad c = \sqrt{a^2 + b^2 - 2ab\cos\gamma}$$

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\gamma} \qquad \ldots (1)$$

Similarly, we can prove that

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos\alpha} \qquad \ldots (2)$$

and

$$\boxed{b^2 = a^2 + c^2 - 2ac\cos\beta} \qquad \ldots (3)$$

The above rules (1), (2) and (3) are called the laws of cosines.
The laws can also be written as:

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc},\ \cos\beta = \frac{a^2+c^2-b^2}{2ac},\ \cos\gamma = \frac{a^2+b^2-c^2}{2ab}.$$

(iii) The laws of tangents

By the law of sines, $\frac{\sin\alpha}{\sin\beta} = \frac{a}{b} \Rightarrow \frac{\sin\alpha}{\sin\beta} - 1 = \frac{a}{b} - 1$

$$\Rightarrow \quad \frac{\sin\alpha - \sin\beta}{\sin\beta} = \frac{a-b}{b} \qquad \text{... (1)}$$

Again

$$\frac{\sin\alpha}{\sin\beta} + 1 = \frac{a}{b} + 1$$

$$\Rightarrow \quad \frac{\sin\alpha + \sin\beta}{\sin\beta} = \frac{a+b}{b} \qquad \text{... (2)}$$

From (1) and (2), by division, we get

$$\frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \frac{a-b}{a+b}$$

$$\Rightarrow \quad \frac{2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}} = \frac{a-b}{a+b}$$

$$\Rightarrow \quad \cot\frac{\alpha+\beta}{2}\tan\frac{\alpha-\beta}{2} = \frac{a-b}{a+b}$$

$$\Rightarrow \quad \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}} = \frac{a-b}{a+b} \qquad \text{... (3)}$$

Similarly,

$$\frac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}} = \frac{b-c}{b+c} \qquad \text{... (4)}$$

and

$$\frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}} = \frac{c-a}{c+a} \qquad \text{... (5)}$$

The above relations (3), (4) and (5) are called the laws of tangents.

(iv) Deduction of half angle formulae of Sine, Cosine and Tangent in terms of lengths of the sides of a triangle:

By the law of cosines, $a^2 = b^2 + c^2 - 2bc\cos\alpha$

or $\cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \quad 1 - \cos\alpha = 1 - \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \quad 1 - \cos\alpha = \dfrac{a^2 - b^2 - c^2 + 2bc}{2bc}$

$\Rightarrow \quad 2\sin^2\dfrac{\alpha}{2} = \dfrac{a^2 - b^2 - c^2 + 2bc}{2bc}$

$= \dfrac{a^2 - (b^2 + c^2 - 2bc)}{2bc}$

$= \dfrac{a^2 - (b - c)^2}{2bc}$

So, $2\sin^2\dfrac{\alpha}{2} = \dfrac{(a+b-c)(a-b+c)}{2bc}$ ... (i)

Again, we take $\cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \quad 1 + \cos\alpha = 1 + \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \quad 2\cos^2\dfrac{\alpha}{2} = \dfrac{b^2 + c^2 - a^2 + 2bc}{2bc}$

$= \dfrac{(b^2 + c^2 + 2bc) - a^2}{2bc}$

$= \dfrac{(b + c)^2 - a^2}{2bc}$

So, $2\cos^2\dfrac{\alpha}{2} = \dfrac{(a+b+c)(b+c-a)}{2bc}$ ... (ii)

By dividing equation (i) by equation (ii),

We get, $\dfrac{2\sin^2\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \dfrac{(a+b-c)(a-b+c)}{(a+b+c)(b+c-a)}$

or $\tan^2\dfrac{\alpha}{2} = \dfrac{(a+b-c)(a-b+c)}{(a+b+c)(b+c-a)}$ ... (iii)

If we set, $s = \dfrac{1}{2}(a + b + c).$

then we have, $a + b = 2s - c$

or $a + b - c = 2s - 2c$

Similarly, $a + c - b = 2s - 2b$

and $b + c - a = 2s - 2a$

Half angle formulae of sine in terms of sides of triangle:

From equation (i), $$2\sin^2\frac{\alpha}{2}=\frac{(a+b-c)(a-b+c)}{2bc}$$

or $$2\sin^2\frac{\alpha}{2}=\frac{(2s-2c)(2s-2b)}{2bc}$$

$$\sin^2\frac{\alpha}{2}=\frac{(s-b)(s-c)}{b\,c}$$

Thus, $$\boxed{\sin\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{b\,c}}}$$

Similarly, $$\sin\frac{\beta}{2}=\sqrt{\frac{(s-a)(s-c)}{a\,c}}$$

and $$\sin\frac{\gamma}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$$

Half angle formulae of cosine in terms of sides of triangle:

From equation (ii),

$$2\cos^2\frac{\alpha}{2}=\frac{(a+b+c)(b+c-a)}{2bc}$$

or $$2\cos^2\frac{\alpha}{2}=\frac{2s(2s-2a)}{2bc}$$

$$\Rightarrow \quad \cos^2\frac{\alpha}{2}=\frac{s(s-a)}{b\,c}$$

$$\Rightarrow \quad \boxed{\cos\frac{\alpha}{2}=\sqrt{\frac{s(s-a)}{b\,c}}}$$

Similarly, $$\cos\frac{\beta}{2}=\sqrt{\frac{s(s-b)}{a\,c}}$$

and $$\cos\frac{\gamma}{2}=\sqrt{\frac{s(s-c)}{ab}}$$

Half angle formulae of tangent in terms of sides of triangle:

From equation (iii), $$\tan^2\frac{\alpha}{2}=\frac{(a+b-c)(a-b+c)}{(a+b+c)(b+c-a)}=\frac{(s-c)(s-b)}{s(s-a)}$$

$$\Rightarrow \quad \boxed{\tan\frac{\alpha}{2}=\sqrt{\frac{(s-c)(s-b)}{s(s-a)}}}$$

Similarly, $$\tan\frac{\beta}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

and $$\tan\frac{\gamma}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

If we put $r^2 = \frac{(s-a)(s-b)(s-c)}{s}$,

then, $\tan\frac{\alpha}{2} = \frac{r}{s-a}$

Similarly, $\tan\frac{\beta}{2} = \frac{r}{s-b}$

and $\tan\frac{\gamma}{2} = \frac{r}{s-c}$

11.1.3 Apply above laws to solve oblique triangles

If we are given any type of triangle, we can find its unknown sides and angles using sine, cosine and tangent laws and also with the help of their half angle formulas.

Example 1. Solve the triangle ABC in which $\alpha = 49°, \beta = 60°$ and $c = 39\ cm$.

Solution: Here, we have to find, γ, a and b.

Since $\alpha + \beta + \gamma = 180°$

$$\gamma = 180° - \alpha - \beta$$
$$= 180° - 49 - 60$$
$$= 71°$$

By the law of sines, $\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$

$$\Rightarrow \frac{a}{\sin 49°} = \frac{39}{\sin 71°}$$
$$\Rightarrow a = \frac{39 \sin 49°}{\sin 71°}$$
$$\Rightarrow a = 31.13\text{cm}$$

Again by the law of sines, $\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

$$\Rightarrow \frac{b}{\sin 60°} = \frac{39}{\sin 71°}$$
$$\Rightarrow \text{b} = \frac{39 \sin 60°}{\sin 71°}$$
$$\Rightarrow \text{b} = 35.72\text{ cm.}$$

Example 2. Solve the triangle ABC in which $a = 70$ cm, $c = 58$ cm and $\beta = 55°$

Solution: Here, we have to find b, α and γ

By using the law of tangents.

$$\frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}} = \frac{c-a}{c+a} = \frac{58-70}{58+70} = \frac{-12}{128} = -0.09375 \qquad \ldots(i)$$

Now, $\gamma + \alpha = 180° - \beta = 180° - 55° = 125°$

$$\Rightarrow \quad \frac{\gamma+\alpha}{2} = \frac{125°}{2} = 62.5°$$

So, from equation (i),

we have $$\frac{\tan\frac{\gamma-\alpha}{2}}{\tan 62.5°} = -0.09375$$

$$\tan\frac{\gamma-\alpha}{2} = -0.09375 \tan 62.5°$$

$$\Rightarrow \quad \tan\frac{\gamma-\alpha}{2} = -0.1801$$

$$\Rightarrow \quad \frac{\gamma-\alpha}{2} = -10.21°$$

or $\gamma - \alpha = -20.42°$ But, $\gamma + \alpha = 125°$

So, $2\gamma = 104.58°$

or $\gamma = 52.29°$

Now, $\alpha = 180° - \beta - \gamma = 180° - 55° - 52.29° = 72.71°$

Now, by the law of sines, $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$

$$\frac{70}{\sin 72.71°} = \frac{b}{\sin 55°} \quad \Rightarrow \quad b = \frac{70 \sin 55°}{\sin 72.71°}$$

$$\Rightarrow \quad b = 60.05 \text{ cm.}$$

Note: The above example may also be solved by using laws of cosines.

Example 3. Find the measure of the largest angle in the triangle ABC with $a = 10$cm, $b = 20$cm and $c = 26$cm.

Solution: Here $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(10+20+26) = 28$

The side c is greater than the sides a and b, so γ will be the largest angle in the given triangle.

By using, $$\sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \sqrt{\frac{(18)(8)}{(10)(20)}}$$

$$= 0.8485$$

$$\Rightarrow \quad \frac{\gamma}{2} = \sin^{-1}(0.8485)$$

$$\Rightarrow \quad \frac{\gamma}{2} = 58°$$

$\Rightarrow$ $\gamma = 116°$ is the largest angle of the triangle

Exercise 11.2

1. Solve the following triangles by using law of sines.

(i) $a = 70,\ b = 34,\ \alpha = 79°$ (ii) $b = 136,\ \gamma = 104.2°,\ \alpha = 43.1°$

(iii) $\alpha = 48°45',\ \beta = 69°15',\ c = 40.5$

2. Solve the following triangles by using laws of cosines.

(i) $\alpha = 78.2°,\ b = 62.4,\ c = 150$ (ii) $a = 48,\ b = 33.3,\ \gamma = 41°30'$

(iii) $a = 73,\ c = 40,\ \beta = 76°45'$

3. Solve by using laws of tangents.

(i) $a = 432,\ c = 325,\ \beta = 42°$ (ii) $b = 39,\ c = 35,\ \alpha = 75°$

(iii) $a = 39.14,\ b = 34.21,\ \gamma = 78°10'$

4. Solve the triangle by using suitable laws.

(i) $a = 9,\ b = 7,\ c = 5$ (ii) $b = 35,\ a = 37, \alpha = 23°25'$

5. Solve the following triangles by using half angle formula of sines.

(i) $a = 55,\ c = 65,\ b = 85$

(ii) $b = 15,\ a = 20,\ c = 14$

(iii) $c = 1.3,\ b = 2.3,\ a = 2.7$

6. Solve the following triangles by using half angle formula of cosines.

(i) $a = 95,\ b = 76,\ c = 85$

(ii) $a = 10,\ b = 7,\ c = 5$

(iii) $c = 10,\ a = 15,\ b = 7$

7. Solve the following by using half angle formula of tangent.

(i) $c = 23,\ a = 13,\ b = 16$

(ii) $a = 25,\ b = 20,\ c = 18$

(iii) $a = 16,\ b = 11,\ c = 13$

8. Find the largest angle of ΔABC, when
a = 6cm, b = 8 cm and c = 9.4 cm

9. Find the smallest angle in ΔABC, when
a = 25 cm, b = 18 cm and c = 21 cm

10. Find the length of the third side of a triangular building that faces 13.6 meters along one street and 13 meters along another street. The angle of intersection between the streets measures72°.

11. If one side of a triangle is y units long, another side is 3 times as long and the angle between the two sides measure 35°, find the measures of other two angles and the third side.

12. If the length of larger side of a parallelogram is 55 cm and one diagonal of the parallelogram makes angles of measure 30° and 50° with a pair of adjacent sides, find the length of the diagonal.

13. The sides of a parallelogram are 25cm and 35cm long and one of its angles is 36°. Find the lengths of its diagonals.

14. Two hikers start from the same point; one walks 9 km heading east, the other one 10 km heading 55° north east. How far apart are they at the end of their walks?

15. Two planes start from Karachi International Airport at the same time and fly in directions that make an angle of 127° with each other. Their speeds are 525km/h. How far apart they are at the end of 2 hours of flying time?

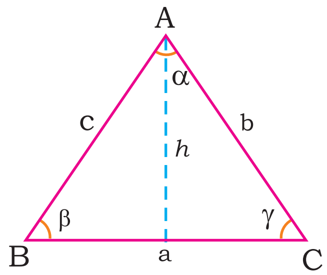
11.2 Area of a Triangle

11.2.1 Derive the formulae to find the area of a triangle in terms of the measures of

- two sides and their included angle,
- one side and two angles,
- three sides (Hero's formula)

(i) Two sides and their included angle

Consider a triangle ABC, in which h is its altitude as shown in Fig. 11.7. We know, from elementary geometry, that



(Fig. 11.7)

Area of a triangle $= \frac{1}{2} \times \text{base} \times \text{altitude}$

or $\qquad \blacktriangle = \frac{1}{2}\, a\, h$

i.e., $\qquad \boxed{\blacktriangle = \frac{1}{2} ab \sin\gamma} \qquad (\because h = b\sin\gamma)$

where ▲ denotes the area of the triangle.

Similarly, $\qquad \blacktriangle = \frac{1}{2} ac \sin\beta$ and $\blacktriangle = \frac{1}{2} bc \sin\alpha$

Example: Find the area of triangle ABC, in which $a = 5.34$ cm, $b = 9.3$ cm and $\gamma = 53° \, 34'$.

Solution: We know that, $\qquad \blacktriangle = \frac{1}{2} ab \sin\gamma$

So, $\qquad \blacktriangle = \frac{1}{2}(5.34)(9.3) \sin 53° \, 34'$

$\Rightarrow \qquad \blacktriangle = \frac{39.918}{2} = 19.98$ sq. cm

(ii) One side and two angles

By the law of sines, $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

$\Rightarrow \qquad a = \dfrac{c\,\sin\alpha}{\sin\gamma}$ and $b = \dfrac{c\,\sin\beta}{\sin\gamma}$ ...(i)

We know that

Area of the triangle $\qquad \blacktriangle = \frac{1}{2}a\,b\sin\gamma$

So, $\qquad \blacktriangle = \frac{1}{2}\frac{c\,\sin\alpha}{\sin\gamma}\cdot\frac{c\,\sin\beta}{\sin\gamma}\sin\gamma \qquad$ (using equation (i))

or $\qquad \boxed{\blacktriangle = \frac{1}{2}c^2\frac{\sin\alpha\sin\beta}{\sin\gamma}}$

Similarly, $\quad \blacktriangle = \frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta}$ and $\blacktriangle = \frac{1}{2}a^2\frac{\sin\gamma\sin\beta}{\sin\alpha}$

Example: Find the area of triangle ABC, with a =36cm, β =46° and γ = 66°.

Solution: Here $\quad \alpha = 180^\circ - \beta - \gamma = 180^\circ - 46^\circ - 66^\circ = 68^\circ$

Now $\qquad \blacktriangle = \frac{1}{2}a^2\frac{\sin\gamma\,\sin\beta}{\sin\alpha}$

$= \frac{1}{2}\frac{(36)^2\times\sin 66^\circ\times\sin 46^\circ}{\sin 68^\circ} = \frac{1}{2}\frac{1296\times 0\cdot 913\times 0\cdot 719}{0\cdot 927}$

$= 459.27$ sq. cm

(iii) Three sides (Hero's formula)

As $\quad$ Area of triangle $\quad \blacktriangle = \frac{1}{2}\,ac\,\sin\beta,$

So, $\qquad \blacktriangle = ac\,\sin\frac{\beta}{2}\cos\frac{\beta}{2} \qquad \left[\because \sin\beta = 2\sin\frac{\beta}{2}\cos\frac{\beta}{2}\right]$

or $\blacktriangle = ac\sqrt{\frac{(s-a)(s-c)}{ac}\cdot\frac{s(s-b)}{ac}}, \qquad \left[\because \sin\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} \text{ and } \cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}\right]$

i.e., $\qquad \boxed{\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)}}$

Example: Find the area of triangle ABC with a = 120 cm, b = 200cm, c = 98 cm.

Solution: Here, $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(120+200+98) = 209$

Then, $s-a = 209-120 = 89, \quad s-b = 9, \quad s-c = 111$

By Hero's formula, $\quad \blacktriangle = \sqrt{s(s-a)(s-b)(s-c)}$

or $\quad \blacktriangle = \sqrt{(209)(89)(9)(111)} = \sqrt{18582399}$

or $\quad \blacktriangle = 4310.7$ sq. cm.

Exercise 11.3

1. Find the area of triangle ABC when

i. $a = 8.6,\ b = 11.4,\ \gamma = 90°$ ii. $b = 63, c = 17,\ \alpha = 120°$

iii. $c = 5.34,\ a = 9.30, \beta = 53°34'$ iv. $c = 36,\ \alpha = 46°,\ \beta = 66°$

v. $a = 54,\ \beta = 37°,\ \gamma = 54°$ vi. $b = 4.8, \alpha = 35°9', \gamma = 85°31'$

vii. $a = 37,\ b = 41,\ c = 37$ viii. $a = 41.34, b = 35.65, c = 56.81$

ix. $a = 98, b = 120, c = 200$

2. The area of triangle is 3.346 square unit. If $\beta = 20.9°, \gamma = 117.2°$. Find a and angle α.

3. The measures of the two sides of a triangle are 4 and 5 units. Find the third side so that area of the triangle is 6 square units.

4. Find area of the equilateral triangle whose each side is x units long.

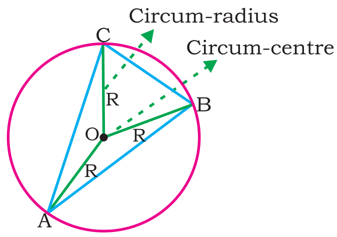
11.3 Circles connected with triangle

There are three types of circles connected with the triangle namely circum-circle, in-circle and escribed-circle.

11.3.1 Define circum-circle, in-circle and escribed-circle

(i) Circum-circle

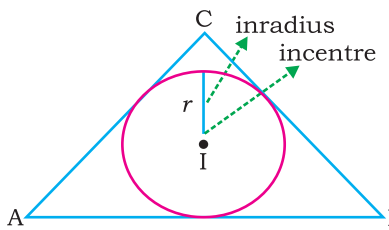
Consider a triangle ABC, (Fig. 11.8). The circle which passes through vertices A, B and C of the triangle is called the *circum-circle* of the triangle. The centre of the circle is called *circum-centre* and the radius is called *circum-radius* and is denoted by R.



(Fig. 11.8)

(ii) in-circle

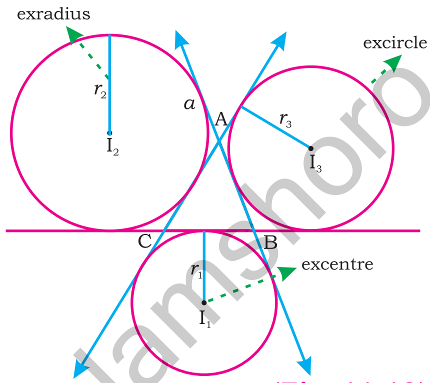
The circle inscribed within a triangle so that it touches all the sides of the triangle is called the in-circle of the triangle (Fig. 11.9). The centre and the radius of this circle are called in-centre and in-radius respectively. Incentre is denoted by I and in-radius by r.



(Fig. 11.9)

(iii) Escribed circle or excircle

A circle which touches one side of the triangle externally and two extended sides internally is called *escribed-circle* or ex-circle or *e-circle*. The centre and radius of excircle opposite to vertex A are denoted by I_1 and r_1 respectively. Similarly the centers and radii of the e-circles opposite to the vertices B and C are denoted by I_2, I_3 and r_2, r_3 respectively as shown in Fig. 11.10.



(Fig. 11.10)

11.3.2 Derive the formulae to find
- **circum-radius,**
- **in-radius,**
- **escribed-radii,**

and apply them to deduce different identities.

(i) Circum-radius of a triangle

Let O be the circum-centre of the circum-circle of triangle ABC. Join O and B. Produce $\overline{BO}$ to D and join C with D, then $m\overline{BD} = 2R$. If the triangle ABC is acute angled triangle (Fig 11.11) then $m\angle A = \alpha = m\angle D$. If the triangle ABC is obtuse angled triangle (Fig 11.12) then $m\angle D = \pi - \alpha$.

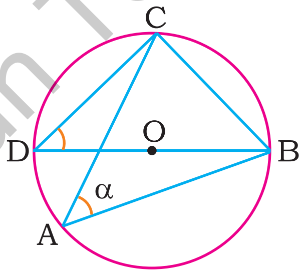


Fig. 11.11

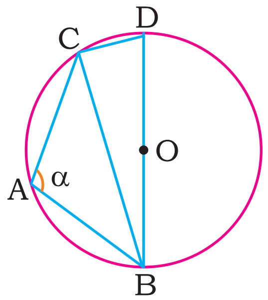


Fig. 11.12

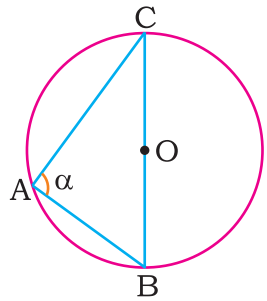


Fig. 11.13

Since, $\sin(\pi - \alpha) = \sin\alpha$,

Therefore, in both cases

$$\sin(m < D) = \sin\alpha.$$

But $\sin(m < D) = \sin\alpha = \dfrac{m\overline{BC}}{m\overline{BD}} = \dfrac{a}{2R}$,

so $R = \dfrac{a}{2\sin\alpha}$

If the triangle is a right triangle (Fig 11.13), then $\sin\alpha = \dfrac{a}{2R} = 1 \quad (\because \alpha = 90°)$

Thus $R = \dfrac{a}{2\sin\alpha}$

Similarly, we can show that $R = \dfrac{b}{2\sin\beta} = \dfrac{c}{2\sin\gamma}$

Hence, we have

$$\frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma} = R$$

Now, $R = \dfrac{a}{2\sin\alpha}$

$= \dfrac{a}{2\left(\frac{2\blacktriangle}{bc}\right)} \qquad \left(\because \blacktriangle = \frac{1}{2}bc\sin\alpha\right)$

Therefore, $R = \dfrac{abc}{4\blacktriangle}$ where, $\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)}$.

Example: Find the value of R in triangle ABC where $a = 10\ cm$, $b = 8\ cm$ and $c = 5\ cm$

Solution:

Here, $s = \dfrac{1}{2}(a+b+c)$

$= \dfrac{1}{2}(10+8+5) = \dfrac{23}{2} = 11.5\text{cm}$

and $s-a = 11.5-10 = 1.5, \quad s-b = 3.5, \quad s-c = 6.5$

Now, $\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{11.5(1.5)(3.5)(6.5)}$

$= \sqrt{392.4375}$

$\blacktriangle = 19.81$ square cm

We know that $R = \dfrac{abc}{4\blacktriangle}$

So, $R = \dfrac{(10)(8)(5)}{4(19.81)} = \dfrac{100}{19.81} = 5.04cm$

(ii) In-Radius of a triangle

Consider the figure (11.14). An in-circle is a circle inscribed in a triangle ABC we find the in-radius for the circle as follows:

Draw the bisectors of angles A, B and C so that they meet at I, the in-centre. Draw perpendiculars $\overline{IF}, \overline{IE}$ and $\overline{ID}$ on $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively.

If r is the in-radius, then we have

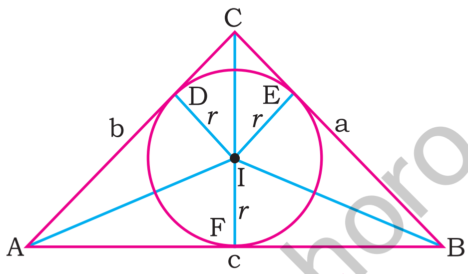
$$m\overline{IF} = m\overline{IE} = m\overline{ID} = r.$$

Since Area of $\Delta ABC = \blacktriangle ABI + \blacktriangle BCI + \blacktriangle CAI$,

$$= \frac{1}{2}\ cr + \frac{1}{2}\ ar + \frac{1}{2}\ br$$

$$= \frac{1}{2}\ r(a + b + c)$$

$$\blacktriangle\ = r.\ s$$



(Fig. 11.14)

Therefore, $$\boxed{r = \frac{\blacktriangle}{s}}$$

Example: Find the value of in- radius of a circle inscribed in a triangle, where $a = \sqrt{2}\ cm,\ and\ \ b = c = 1\ cm$

Solution: We know that $r = \frac{\Delta}{s}$

$$\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)}$$

Hence,

$$s = \frac{1}{2}(a + b + c)$$

$$s = \frac{1}{2}\left(\sqrt{2} + 1 + 1\right) = \frac{1}{2} = (1.41 + 2) = \frac{3.41}{2} = 1.71$$

$$s - a = 1.71 - \sqrt{2} = 1.71 - 1.41 = 0.30$$

$$s - b = 1.71 - 1 = 0.71$$

$$s - c = 1.71 - 1 = 0.71$$

Now, $$\blacktriangle\ = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{1.71(0.30)(0.71)(0.71)}\ \ = \sqrt{2586}$$

$$\blacktriangle\ \ = 0.5085 \text{ square cm}$$

Now, $$r = \frac{\blacktriangle}{s} = \frac{0.5085}{0.71} = 0.29cm$$

(iii) Radii of e-circles of a triangle

Consider a triangle ABC (Fig. 11.15). Let I_1, be the excentre of e-circle opposite to the vertex A of the triangle. Draw perpendiculars $\overline{I_1P_1}$, $\overline{I_1E_1}$ and $\overline{I_1F_1}$ on $\overline{CB}$, $\overrightarrow{AO}$ and $\overrightarrow{AR}$ respectively.

then $m\overline{F_1I_1} = m\overline{E_1I_1} = m\overline{P_1I_1} = r_1$.

where r_1 is the radius of the e-circle, we have

Area of triangle ABC =

Area of triangle I_1CA +Area of triangle I_1BA − Area of triangle I_1BC

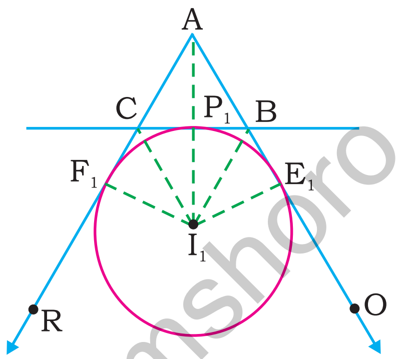
Since, $\blacktriangle = \frac{1}{2}br_1 + \frac{1}{2}cr_1 - \frac{1}{2}ar_1$

Therefore,

$\frac{1}{2}r_1(-a+b+c) = \frac{1}{2}r_1(2s-2a) = r_1(s-a)$

So, $$\boxed{r_1 = \frac{\blacktriangle}{s-a}}$$

Similarly, $r_2 = \frac{\blacktriangle}{s-b}$ and $r_3 = \frac{\blacktriangle}{s-c}$



(Fig. 11.15)

Example: Find the radii r_1, r_2, r_3 of the escribed circles of a triangle ABC in (i) $a = b = 5.5\ cm$ and $c = 9\ cm$ (ii) $a = b = c$

Solution (i):

Here $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(5.5+5.5+9) = 10\text{cm}$

and $s-a = 10-5.5 = 4.5$, $s-b = 10-5.5 = 4.5$, $s-c = 10-9 = 1$

Now, $\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{10(4.5)(4.5)(1)} = 4.5\sqrt{10}$

We know that $r_1 = \frac{\blacktriangle}{s-a} = \frac{4.5\sqrt{10}}{4.5} = 3.16$ cm, $r_2 = 3.16$ cm and $r_3 = \frac{\blacktriangle}{s-c} = 14.23$ cm

Solution (ii):

Here, $a = b = c = x$

So, $S = \frac{1}{2}(a+b+c) = \frac{3x}{2}$

and $s-a = \frac{3x}{2} - x = \frac{x}{2}$, $s-b = s-c = \frac{x}{2}$

Now $\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{3x}{2}\cdot\frac{x}{2}\cdot\frac{x}{2}\cdot\frac{x}{2}} = \frac{x^2\sqrt{3}}{4}$ sq. units

Hence, $r_1 = \frac{\blacktriangle}{s-a} = \frac{\frac{x^2\sqrt{3}}{4}}{\frac{x}{2}} = \frac{x^2\sqrt{3}}{4}.\frac{2}{x} = \frac{\sqrt{3}}{2}x,$

Similarly, $r_2 = r_3 = \frac{\sqrt{3}x}{2}$ units.

Exercise 11.4

1. Find the values of R and r of ΔABC, when

(i) $a = 5cm,$ $b = 10cm,$ $c = 12cm$

(ii) $a = 10.5cm,$ $b = 11.5cm,$ $c = 20.5cm$

(iii) $a = 40cm,$ $b = 12cm,$ $c = 38cm$

2. Find r_1, r_2 and r_3, if the measures of sides of triangle ABC are

(i) $a = 23cm,$ $b = 24cm,$ $c = 28cm$

(ii) $a = 24.4cm,$ $b = 34.8cm,$ $c = 42.5cm$

(iii) $a = 75cm,$ $b = 62cm,$ $c = 53cm$

3. If $a = b = c$, then prove that

$$r_1 : \text{R} : r = 3 : 2 : 1$$

4. Show that:(i) $r_1 r_2 r_3 = rs^2$ (ii) $r = 4\text{R}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$ (iii)$r_1 = \frac{\text{a.}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$

5. Prove that $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\blacktriangle^2}$

6. Show that $\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$

Review Exercise 11

1. Select correct answer.

i. From the top of a cliff 80m high the angle of depression of a boat is α. If the distance between the boat and foot of cliff is $80\sqrt{3}$ m, then angle α is: (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{3\pi}{4}$

ii. To solve an oblique triangle, we use:
(a) law of sines (b) Laws of cosines (c) Laws of tangents (d) All of these

iii. A circle which touches all the sides of a triangle is called __________.
(a) circum-circle (b) in-circle (c) ex-circle (d) tri-circle

iv. In a triangle ABC, if $\beta = 90°$, then $b^2 = c^2 + a^2 - 2ac\cos\beta$ becomes:
(a) Law of sines (b) Law of tangents
(c) Pythagoras theorem (d) None of these

v. In any triangle ABC, $\sqrt{\frac{(s-a)(s-c)}{ac}}$ is equal to:
(a) $\sin\frac{\alpha}{2}$ (b) $\cos\frac{\alpha}{2}$ (c) $\sin\frac{\beta}{2}$ (d) $\sin\frac{\gamma}{2}$

vi. In any triangle ABC, $\cos\frac{\gamma}{2}$ is equal to:
(a) $\sqrt{\frac{s(s-a)}{ab}}$ (b) $\sqrt{\frac{s(s-b)}{ac}}$ (c) $\sqrt{\frac{s(s-a)}{bc}}$ (d) $\sqrt{\frac{s(s-c)}{ab}}$

vii. In any triangle ABC, with usual notations, s is equal to:
(a) $a + b + c$ (b) $\frac{a+b+c}{2}$ (c) $\frac{a+b+c}{3}$ (d) $\frac{abc}{2}$

viii. $\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \cdots$ (a) $\sin\frac{\beta}{2}$ (b) $\cos\frac{\beta}{2}$ (c) $\tan\frac{\beta}{2}$ (d) $\cot\frac{\beta}{2}$

ix. In any triangle ABC, $\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$ is equal to:

(a) $\sin\frac{\gamma}{2}$ (b) $\cos\frac{\gamma}{2}$ (c) $\tan\frac{\gamma}{2}$ (d) $\cot\frac{\gamma}{2}$

x. We can solve an oblique triangle, if:

(a) One side and two angles are known

(b) Three sides are known

(c) Two sides and their included angles are known

(d) All of these

xi. In ΔABC, inradius = ------------------

(a) $\frac{\blacktriangle}{s-a}$ (b) $\frac{\blacktriangle}{s-b}$ (c) $\frac{\blacktriangle}{s-c}$ (d) $\frac{\blacktriangle}{s}$

xii. In any triangle ABC, Area of triangle is:

(a) $bc\sin\alpha$ (b) $\frac{1}{2}ca\sin\alpha$ (c) $\frac{1}{2}ab\sin\gamma$ (d) $\frac{1}{2}ab\sin\beta$

xiii. Hero's formula is:

(a) $\blacktriangle = s(s-a)(s-b)(s-c)$ (b) $\blacktriangle = \sqrt{(s-a)(s-b)(s-c)}$

(c) $\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)}$ (d) $\blacktriangle = \frac{a+b+c}{2}$

xiv. The circle passing through the three vertices of a triangle is called:

(a) Circum-circle (b) In-circle (c) Ex-centre (d) Escribed circle

xv. The point of intersection of the right bisectors of the sides of a triangle is called:

(a) Circum-centre (b)In-centre (c)Escribed centre (d) Ortho-centre

xvi. Radius of the circle which passes through all the vertices of a triangle is:

(a) Circum-radius (b) In-radius (c) e-Radius (d) Diameter

xvii. In any triangle ABC, with usual notations, abc =

(a) R (b) Rs (c) 4R▲ (d) $\frac{\blacktriangle}{s}$

xviii. The point of intersection of the internal bisectors of angles of a triangle is:

(a) In-centre (b) e-centre (c) Circum-centre (d) Ex-centre

xix. In any equilateral triangle ABC, with usual notations, $r:R:r_1$

(a) 1: 2: 3 (b) 3: 2: 1 (c) 1: 3: 2 (d) 1: 1: 1

xx. Circum radius of ΔABC is:

(a) $\frac{\blacktriangle}{S}$ (b) $\frac{\blacktriangle}{S-b}$ (c) $\frac{\blacktriangle}{S-a}$ (d) $\frac{abc}{4\blacktriangle}$

2. Prove that:

(i) $r_2 = \frac{b\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$ (ii) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

3. Find the area of triangle ABC when
(i). $a = 5.2cm$, $\beta = 30°$, $\gamma = 40°$ (ii) $a = 125cm$, $b = 120$, $\gamma = 150°$

4. Find the R and r of ΔABC when:
(i) $a = 18.8cm$, $b = 24.5cm$, $c = 30.2c$ (ii) $a = 13.8cm$, $b = 13.8cm$, $c = 10.4cm$

5. Find r_1, r_2 and r_3, of the escribed circle if the measures of sides of triangle ABC are a = 3cm, b = 4cm and c = 5cm.

6. A hiker walks due east at 4 km per hour and a second hiker, starting at the same point, walks 55° north-east at the rate of 5 km per hour. How far apart will they be after 3 hours?

7. Three points A, B, C form a triangle such that the ratio of the measures of their angles is 1 : 2 : 3. Find the ratio of the lengths of the sides.

8. A piece of plastic strip 1 meter long is bent to form an isosceles triangle with 95° as measure of its largest angle. Find the length of the sides.

9. Two airplanes leave a field at the same time. One flies 30° East of North at 250km/h, the other 45° East of South at 300km/h. How far apart are they at the end of 2 hours?

10. Two men are on the opposite sides of a 100m high tower. If the measures of the angles of elevation of the top of the tower are 18° and 22° respectively. Find the distance between them.

11. A man standing 60m away from a tower notices that the angles of elevation of the top and the bottom of a flagstaff on the top of the tower are 64° and 62° respectively. Find the length of the flagstaff.

12. The angle of elevation of the top of a 60m high tower from a point A, on the same level as the foot of the tower, is 25°. Find the angle of elevation of the top of the tower from a point B, 20m nearer to A from the foot of the tower.

13. Two buildings A and B are 100m apart. The angle of elevation from the top of the building A to the top of the building B is 20°. The angle of elevation from the base of the building B to the top of the building A is 50°. Find the height of the building B.