Graphs of Trigonometric and Inverse Trigonometric Functions and Solution of Trigonometric Equations

Unit 12

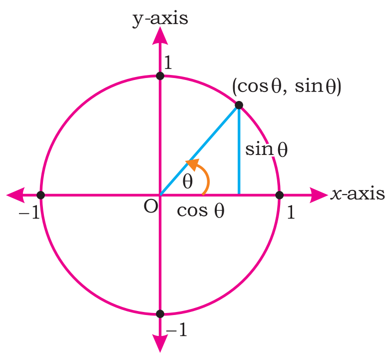
12.1 Period of Trigonometric Functions

12.1.1 Find the domain and range of the trigonometric functions

The method of finding domain and range of a function has already been discussed in section 8.1.1. Here, we discuss domain and range of trigonometric functions.

Domain and Range of sinθ, cosθ and tanθ

(i) Function $y = \sin\theta$ is defined as the ordinate (y-coordinate) of a point on a unit circle that corresponds to an angle of θ radians. Therefore, the domain of this function is the set of all real numbers as θ can be any real number and the range is set of all real numbers from −1 to +1 as the maximum and minimum values of y are 1 and −1 respectively. (Fig 12.1)



(Fig. 12.1)

Example: Find the domain and range of $2\sin 5x$

Solution: Here, $y = 2\sin 5x$

$\because$ Given function is defined for all real numbers.

$\therefore$ Domain of $2\sin 5x = \mathbb{R}$

As Range of $\sin 5x = \{y | y \in \mathbb{R} \wedge -1 \leq y \leq 1\}$

So, Range of $2\sin 5x = \{y | y \in \mathbb{R} \wedge -2 \leq y \leq 2\}$

(ii) Function $y = \cos\theta$ is defined as the abscissa (x-coordinate) of a point on a unit circle that corresponds to an angle of θ radians. Therefore, the domain of this function is the set of all real numbers as θ can be any real number and the range is set of all real numbers from −1 to +1 as the maximum and minimum values of y are 1 and −1 respectively. (Fig 12.1)

Example: Find the domain and range of $\cos 7x$

Solution: Here, $y = \cos 7x$

$\because$ Given function is defined for all real numbers

$\therefore$ Domain of $\cos 7x = \mathbb{R}$

As the range of $\cos 7x$ is same as the range of $\cos x$

So, range of $\cos 7x = \{y | y \in \mathbb{R} \wedge -1 \le y \le 1\}$

(iii) Function $y = \tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is defined when $\cos\theta \ne 0$. The domain of this function is the set of all real numbers except those where $\cos\theta = 0$ and the angles where $y = \tan\theta$ is undefined are $\dfrac{n\pi}{2}$ radians where n is an odd integer. The range is the set of all real numbers between $-\infty$ and $+\infty$ as the minimum and maximum values of $\dfrac{\sin\theta}{\cos\theta}$ are $-\infty$ and $+\infty$ respectively.

Example: Find the domain and range of $\tan 4x$.

Solution: Here $y = \tan 4x$

We know that $\tan 4x = \dfrac{\sin 4x}{\cos 4x}$ is undefined if $\cos 4x = 0$

Let $\cos 4x = 0$

$\Rightarrow \quad 4x = n\dfrac{\pi}{2}$ (where n is an odd integer)

$\Rightarrow \quad x = n\dfrac{\pi}{8}$

So, domain of $\tan 4x = \mathbb{R} - \left\{n\dfrac{\pi}{8} \middle| n \text{ is an odd integer}\right\}$

$\because$ The range of $\tan 4x$ is same as the range of $\tan x$

$\therefore$ Range of $\tan 4x = \mathbb{R}$

For the trigonometric functions $\sin\theta$, $\cos\theta$, $\tan\theta$, $\operatorname{cosec}\theta$, $\sec\theta$, and $\cot\theta$ the domain and range are given in the following table.

Function	Domain	Range
$f(\theta) = \sin\theta$	$\mathbb{R}$	$-1 \le \sin\theta \le 1$
$f(\theta) = \cos\theta$	$\mathbb{R}$	$-1 \le \cos\theta \le 1$
$f(\theta) = \tan\theta$	$\mathbb{R} - \{n\frac{\pi}{2} \mid n \text{ is an odd integer}\}$	$\mathbb{R}$
$f(\theta) = \operatorname{cosec}\theta$	$\mathbb{R} - \{n\frac{\pi}{2} \mid n \text{ is an even integer}\}$	All real numbers ≥ 1 and ≤ -1
$f(\theta) = \sec\theta$	$\mathbb{R} - \{n\frac{\pi}{2} \mid n \text{ is an odd integer}\}$	All real numbers ≥ 1 and ≤ -1
$f(\theta) = \cot\theta$	$\mathbb{R} - \{n\frac{\pi}{2} \mid n \text{ is an even integer}\}$	$\mathbb{R}$

12.1.2 Define even and odd functions

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x in the domain of f and a function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$ for all x in the domain of f.

Example: Decide whether each of the following function is even, odd or neither. In the following functions, check whether the function is even or odd.

(i) $f(x) = \sqrt{x^4 + 5}$ (ii) $h(x) = x^5$ (iii) $k(x) = x + |x|$

Solution:

(i) $f(x) = \sqrt{x^4 + 5}$

$f(-x) = \sqrt{(-x)^4 + 5} = \sqrt{x^4 + 5}$ (Replacing x by $-x$)

$\because$ $f(-x) = f(x)$

$\therefore$ $f(x)$ is an even function.

(ii) $h(x) = x^5$

$h(-x) = (-x)^5 = -x^5$ (Replacing x by $-x$)

$\because$ $h(-x) = -h(x)$

$\therefore$ $h(x)$ is an odd function.

(iii) $k(x) = x + |x|$

$k(-x) = -x + |-x| = -x + |x|$ (Replacing x by $-x$)

i.e., $k(-x) \neq k(x)$ and $k(-x) \neq -k(x)$

Hence, $k(x)$ is neither even nor odd function.

Even and Odd Trigonometric functions

Consider the basic trigonometric functions $\sin x$, $\cos x$, and $\tan x$

(i) $f(x) = \sin x$

Replacing x by $-x$

We get, $f(-x) = \sin(-x)$

$= -\sin(x)$ $(\because \sin(-x) = -\sin x)$

$= -f(x)$

Hence, $\sin x$ is an odd function

(ii) $f(x) = \cos x$

We get, $f(-x) = \cos(-x)$

$= \cos(x)$ $(\because \cos(-x) = \cos x)$

$= f(x)$

Hence, $\cos x$ is even function

(iii) $f(x) = \tan x$

Replacing x by $-x$

We get, $f(-x) = \tan(-x)$

$= -\tan(x)$ $\quad (\because \tan(-x) = -\tan x)$

$= -f(x)$

Hence, $\tan x$ is odd function.

Note: $\cot x$ and $\operatorname{cosec} x$ are odd functions, where $\sec x$ is an even function.

Example: Determine whether the following trigonometric functions are even, odd or neither.

(a) $f(x) = \sec x \tan x$ (b) $g(x) = x^3 \sin x \cos 2x$ (c) $h(x) = \cos x + \sin x$

Solution:

a) $f(x) = \sec x \tan x$

Here, $f(-x) = \sec(-x)\tan(-x)$

$= [\sec x][-\tan x]$ $\quad (\because \sec(-x) = \sec x$ and $\tan(-x) = -\tan x)$

$= -\sec x \tan x$

$= -f(x)$

Hence, $f(x)$ is odd

b) $g(x) = x^3 \sin x \cos 2x$

Here, $g(-x) = (-x)^3 \sin(-x)\cos 2(-x)$

$= (-x^3)(-\sin x)(\cos 2x)$ $\quad (\because \cos(-x) = \cos x$ and $\sin(-x) = -\sin x)$

$= x^3 \sin x \cos 2x$

$= g(x)$

Hence, $g(x)$ is even

c) $h(x) = \cos x + \sin x$

Here $h(-x) = \cos(-x) + \sin(-x)$ $\quad (\because \sin(-x) = -\sin x$ and $\cos(-x) = \cos x)$

$= \cos x + (-\sin x)$

$= \cos x - \sin x$

$\neq -h(x)$ or $h(x)$

Hence, $h(x)$ is neither even nor odd

12.1.3 Discuss the periodicity of trigonometric functions. Find the maximum and minimum value of a given function of the type:

- $a + b\sin\theta$,
- $a + b\cos\theta$,
- $a + b\sin(c\theta + d)$,
- $a + b\cos(c\theta + d)$,
- the reciprocals of above

where a, b, c and d are real numbers.

12.1.3(a) Periodicity of the Trigonometric Functions

Let X and Y be the subsets of set of real numbers. A function $f : \mathrm{X} \to \mathrm{Y}$ is called a periodic function of period p if $f(x + p) = f(x)$, for all $x \in \mathrm{X}$, and p is the smallest positive real number.

Since for any integer n, $\sin(\theta + 2n\pi) = \sin\theta$; $\cos(\theta + 2n\pi) = \cos\theta$ and $\tan(\theta + n\pi) = \tan\theta$. Therefore, sine and cosine are periodic functions of period 2π and tangent is a periodic function of period π.

Similarly, cosecant, secant and cotangent are periodic functions of period 2π, 2π and π respectively.

Since all trigonometric functions are periodic. Therefore, they repeat their values after specific interval and this property is called periodicity of trigonometric functions.

Example 1. Verify that $\sin x, \cos x$ and $\tan x$ have periods 2π, 2π and π respectively.

Solution:

(i) $f(x) = \sin x$

Replacing x by $x + 2\pi$

$$f(x + 2\pi) = \sin(x + 2\pi) = \sin x \cos 2\pi + \cos x \sin 2\pi$$
$$= \sin x(1) + \cos x(0)$$
$$= \sin x$$

i.e., $f(x + 2\pi) = f(x)$

Hence, the period of $\sin x$ is 2π.

(ii) $f(x) = \cos x$

Replacing x by $x + 2\pi$

$$f(x+2\pi) = \cos(x+2\pi) = \cos x \cos 2\pi - \sin x \sin 2\pi$$
$$= \cos x(1) - \sin x(0)$$
$$= \cos x$$

Thus, $f(x+2\pi) = \cos x = f(x)$

Hence, the period of $\cos x$ is 2π.

(iii) $f(x) = \tan x$

Replacing x by $x+\pi$

$$f(x+\pi) = \tan(x+\pi) = \frac{\tan x + \tan\pi}{1 - \tan x \tan\pi}$$
$$= \frac{\tan x + 0}{1 - \tan x\,(0)} \qquad (\because \tan\pi = 0)$$
$$= \tan x$$

Thus, $f(x+\pi) = \tan x = f(x)$

Hence, the period of $\tan x$ is π.

Example 2. Find the periods of the following functions.

i. $5\sin x$ **ii.** $\cos 3x$ **iii.** $\frac{1}{6}\tan\frac{x}{6}$

Solution:

i. Here $f(x) = 5\sin x$

So, $f(x) = 5\sin(x+2\pi) = f(x+2\pi)$

$\because f(x) = f(x+2\pi)$

$\therefore$ period of $f(x)$ is 2π.

ii. Here, $f(x) = \cos 3x$

$$= \cos(3x+2\pi)$$
$$= \cos 3\left(x+\frac{2\pi}{3}\right)$$
$$= f\left(x+\frac{2\pi}{3}\right)$$

$\because f(x) = f\left(x+\frac{2\pi}{3}\right)$

$\therefore f(x)$ has period of $\frac{2\pi}{3}$.

iii. Here, $f(x) = \frac{1}{6}\tan\frac{x}{6}$

$$= \frac{1}{6}\tan\left(\frac{x}{6}+\pi\right)$$
$$= \frac{1}{6}\tan\left(\frac{x+6\pi}{6}\right)$$
$$= f(x+6\pi)$$

$\because f(x) = f(x+6\pi)$

$\therefore$ Period of $f(x)$ is 6π.

12.1.3(b) Find the maximum and minimum value of a given functions of the type

(i) $a + b\sin\theta$, **(ii)** $a + b\cos\theta$

(iii) $a + b\sin(c\theta + d)$, **(iv)** $a + b\cos(c\theta + d)$

(v) **the reciprocals of the above,**

where a, b, c and d are real numbers

Consider types (i) and (ii), the expressions attain their maximum values when both $\sin\theta$ and $\cos\theta$ are at the maximum. i.e., $\sin\theta = 1$ and $\cos\theta = 1$ provided that b is non-negative.

If $b < 0$ then we get maximum values at $\sin\theta = -1$ and $\cos\theta = -1$.

So, maximum value of $(a + b\sin\theta) = a + |b|$ (maximum value of $\sin\theta$)

$= a + |b|(1) = a + |b|$

Thus, $\boxed{\text{Maximum value of } (a + b\sin\theta) = a + |b|}$ ....(i)

Similarly, $\boxed{\text{Maximum value of } (a + b\cos\theta) = a + |b|}$ ....(ii)

Now, the functions attain their minimum values when both $\sin\theta$ and $\cos\theta$ are at the minimum. i.e., $\sin\theta = -1$ and $\cos\theta = -1$ provided that b is non-negative.

If $b < 0$ then we get minimum values at $\sin\theta = 1$ and $\cos\theta = 1$.

So, minimum value of $(a + b\sin\theta) = a + |b|$ (minimum value of $\sin\theta$)

$= a + |b|(-1) = a - |b|$

Thus, $\boxed{\text{Minimum value of } (a + b\sin\theta) = a - |b|}$ ... (iii)

Similarly, $\boxed{\text{Minimum value of } (a + b\cos\theta) = a - |b|}$ ...(iv)

Now, consider the types (iii) and (iv), that is $a + b\sin(c\,\theta + d)$ and $a + b\cos(c\,\theta + \text{d})$. In these types, the values of c and d do not affect the function. So, we get the same results as for types (i) and (ii) which are:

Maximum value of $\big(a + b\sin(c\theta + d)\big) = a + |b|$ ...(v)

Maximum value of $\big(a + b\cos(c\theta + d)\big) = a + |b|$ ...(vi)

Minimum value of $\big(a + b\sin(c\theta + d)\big) = a - |b|$ ...(vii)

Minimum value of $\big(a + b\cos(c\theta + d)\big) = a - |b|$ ...(viii)

Thus, we conclude that, if M and m respectively represent the maximum and minimum values of the expression of types (i), (ii), (iii) and (iv). Then we have the following formulae.

$M = a + |b|$ and $m = a - |b|$

Now, let M' and m' respectively represent the maximum and minimum values of the reciprocals of the expressions.

Then, for $m > 0, M > 0$ and $m < 0, M < 0$, we have $M' = \frac{1}{m}$ and $m' = \frac{1}{M}$

And for $m < 0, M > 0$, we have $M' = \frac{1}{M}$ and $m' = \frac{1}{m}$

Example: Find the maximum and minimum values of the functions.

(i) $y = 3 - 5\sin\theta$ (ii) $y = 2 + 3\cos(5\theta + 10)$ (iii) $y = \frac{1}{4-5\sin(7\theta-8)}$

Solution:

(i) $y = 3 - 5\sin\theta$, Here, $a = 3$ and $b = -5$

Maximum value of $y = a + |b| = 3 + |-5| = 3 + 5 = 8$

Minimum value of $y = a - |b| = 3 - |-5| = 3 - 5 = -2$

(ii) $y = 2 + 3\cos(5\theta + 10)$, Here, $a = 2$ and $b = 3$

Maximum value of $y = M = a + |b| = 2 + |3| = 2 + 3 = 5$

Minimum value of $y = m = a - |b| = 2 - |3| = 2 - 3 = -1$

(iii) $y = \frac{1}{4-5\sin(7\theta-8)}$

Here, $a = 4$ and $b = -5$ then, $M = a + |b| = 4 + |-5| = 4 + 5 = 9$

and $m = a - |b| = 4 - |-5| = 4 - 5 = -1$

Let M' and m' respectively, represent the maximum and minimum value of the reciprocals of the functions

$\because m < 0, M > 0$

$\therefore M' = \frac{1}{M}$ and $m' = \frac{1}{m}$

So, $M' = \frac{1}{9}$ and $m' = \frac{1}{-1} = -1$

Exercise 12.1

1. Find the domain and range of each of the following functions:

(i) $2\sin 3x$ (ii) $5\cos 4x$ (iii) $8\tan 2x$ (iv) $\text{cosec}\, 6x$

(v) $5\cot 2x$ (vi) $\sin\frac{x}{3}$ (vii) $\text{cosec}\,\frac{x}{4}$ (viii) $\tan\frac{x}{7}$

2. Determine whether the following trigonometric functions are even, odd or neither.

i. $f(x) = \sin x \cos x$ ii. $g(x) = \frac{Sin^2 x}{1+\tan x}$

iii. $h(x) = \frac{\tan x}{x+\sin x}$ iv. $k(x) = x^3(\sin x + \cos x)$

3. Find the period of the following functions.

(i) $\sin 3x$ (ii) $\cos 4x$ (iii) $\tan\frac{x}{3}$ (iv) $\sec\frac{x}{5}$

(v) $\text{cosec}\, 8x$ (vi) $\cot\frac{x}{6}$ (vii) $\sqrt{5}\cos\frac{3x}{2}$ (viii) $\cot\sqrt{2}x$

(ix) $\sin\frac{x}{3}$ (x) $\cos\frac{x}{4}$ (xi) $\frac{7}{2}\cot\frac{2\pi x}{3}$ (xii) $-\frac{2}{5}\sec\frac{3x}{\pi}$

4. Find the maximum and minimum values of each of the following functions.

(i) $y = 4 + 3\sin\theta$ (ii) $y = \frac{2}{3} - 4\cos\theta$

(iii) $y = 6 - \frac{1}{3}\sin(3\theta + 2)$ (iv) $y = 8 + 5\cos(\theta - 25)$

(v) $y = \frac{1}{25-12\,sin\,(3\theta-2)}$ (vi) $y = \frac{1}{1+6\,cos\,(5\theta-4)}$

12.2 Graphs of Trigonometric Functions

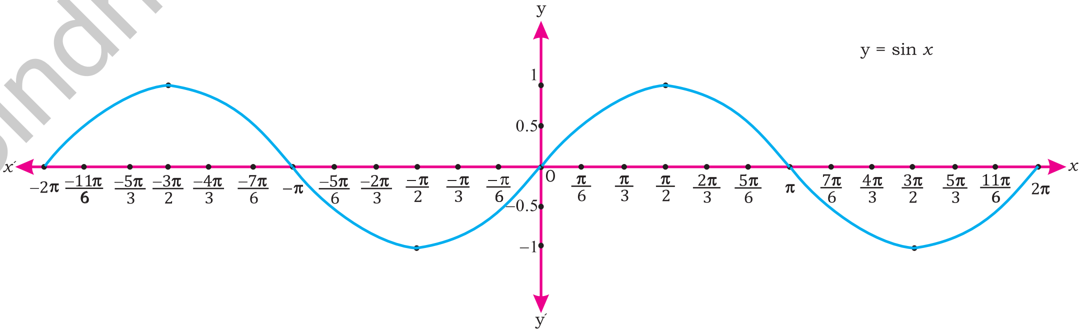
The procedure for plotting the graphs of trigonometric functions is similar as that of graphs of algebraic functions.

In order to draw the graph of trigonometric function we take the angles x on x-axis and the corresponding values of trigonometric functions are taken on y-axis.

12.2.1 Recognize the shapes of the graphs of sine, cosine and tangent for all angles

The graphs of trigonometric functions represent curves and the trigonometric functions are periodic, so their curves repeat after a specific interval. The shapes of the graphs of trigonometric functions sine, cosine and tangent are as follows.

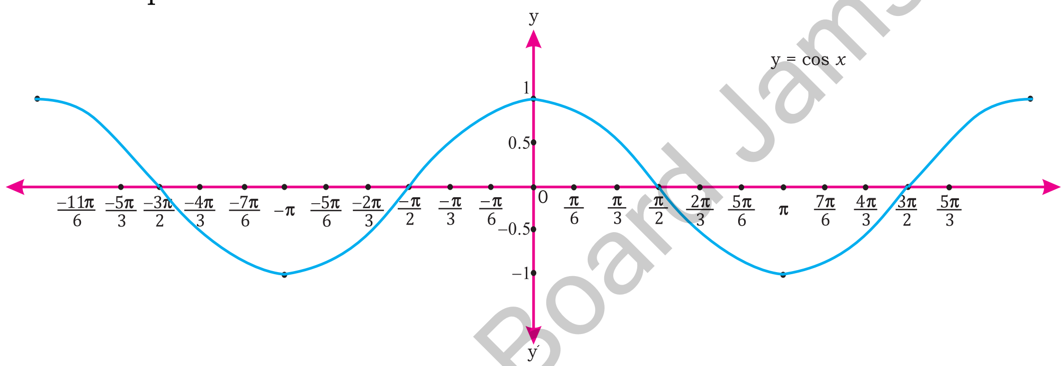
Graph of $\sin x$



(Fig. 12.2)

The shape of sine function is sinusoidal or up-down curve which repeats after every 2π radians. It passes through origin, heads up to 1 by $\frac{\pi}{2}$ radians and then heads down to -1 at $\frac{3\pi}{2}$ radians. Its cycle completes at 2π radians. The cycles repeat after the interval of 2π. The graph is symmetric about the origin.

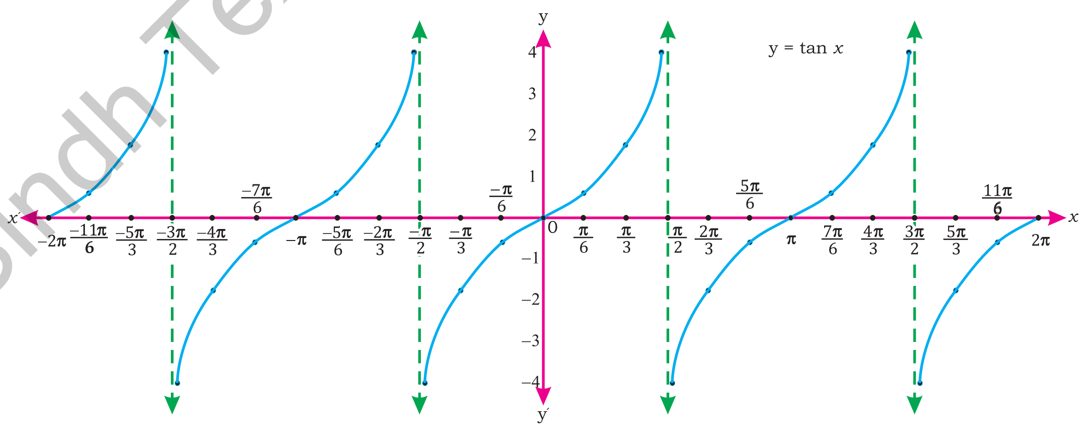
Graph of $\cos x$



(Fig. 12.3)

The shape of cosine function is also sinusoidal or up-down curve which repeats after every 2π radians. It does not pass-through origin. Its maximum value is 1 and minimum value is -1. Its cycle completes in interval of 2π radians. The cycles repeat after interval of a 2π. The graph is symmetric about the y-axis.

Graph of $\tan x$



(Fig. 12.4)

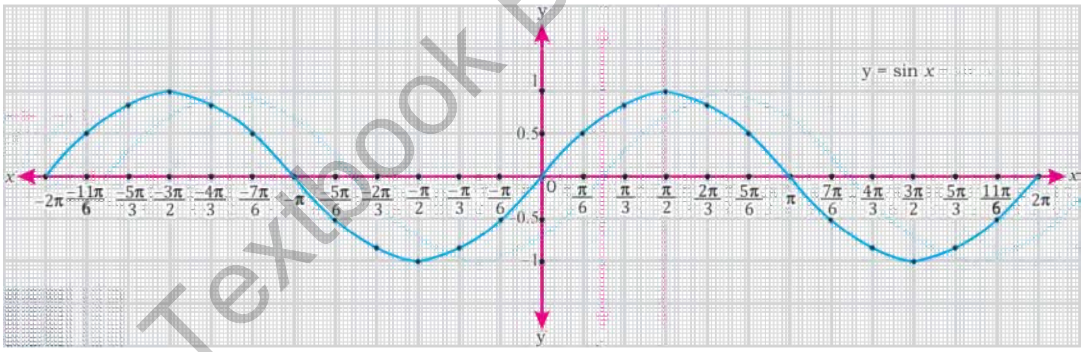
The tangent function has a completely different shape. It goes between $-\infty$ and $+\infty$. It passes through origin. At $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ radians ..., etc., the function is undefined. It is discontinuous curve and is symmetric about origin.

12.2.2 Draw the graphs of the six basic trigonometric functions within the domain from -2π to 2π

(a) Graph of $y = \sin x, -2\pi \le x \le 2\pi$

Table of values of $\sin x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	$-360°$	$-330°$	$-300°$	$-270°$	$-240°$	$-210°$	$-180°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$0°$
$y = \sin x$	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$y = \sin x$	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0



(Fig. 12.5)

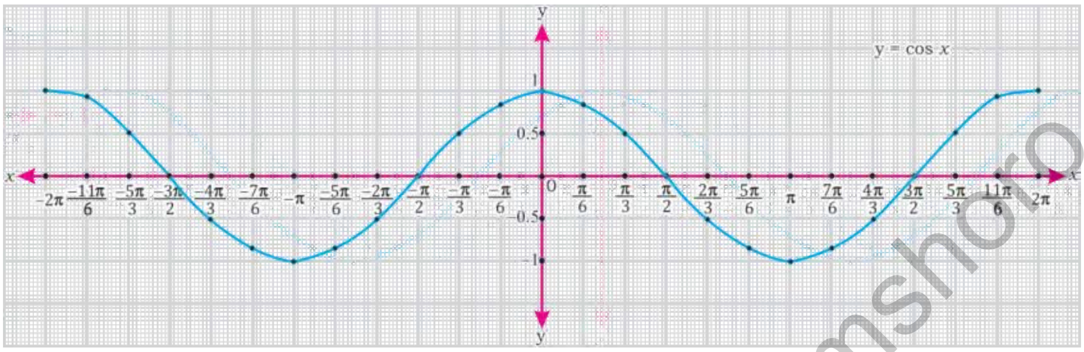
Graph of $y = \sin x, -2\pi \le x \le 2\pi$

The graph of $\sin x$, is sinusoidal curve and it is also called sine wave.

(b) Graph of $y = \cos x, -2\pi \le x \le 2\pi$

Table of values of $\cos x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	$-360°$	$-330°$	$-300°$	$-270°$	$-240°$	$-210°$	$-180°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$0°$
$y = \cos x$	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.87	0.5	1
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$y = \cos x$	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87	1



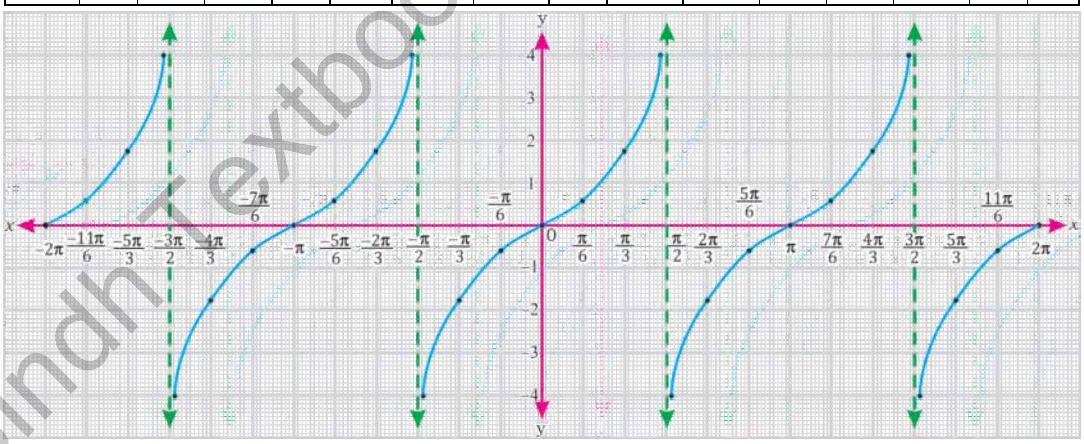
(Fig. 12.6)

Graph of $y = \cos x, -2\pi \leq x \leq 2\pi$

(c) Graph of $y = \tan x, -2\pi \leq x \leq 2\pi$

Table of values of $\tan x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	$-360°$	$-330°$	$-300°$	$-270°$	$-270°$	$-240°$	$-210°$	$-180°$	$-150°$	$-120°$	$-90°$	$-90°$	$-60°$	$-30°$	$0°$
$y = \tan x$	0	0.58	1.73	$+\infty$	$-\infty$	-1.73	-0.58	0	0.58	1.73	$+\infty$	$-\infty$	-1.73	0.58	0
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	$0°$	$30°$	$60°$	$90°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$270°$	$300°$	$330°$	$360°$
$y = \tan x$	0	0.58	1.73	$+\infty$	$-\infty$	-1.73	-0.58	0	0.58	1.73	$+\infty$	$-\infty$	-1.73	-0.58	0



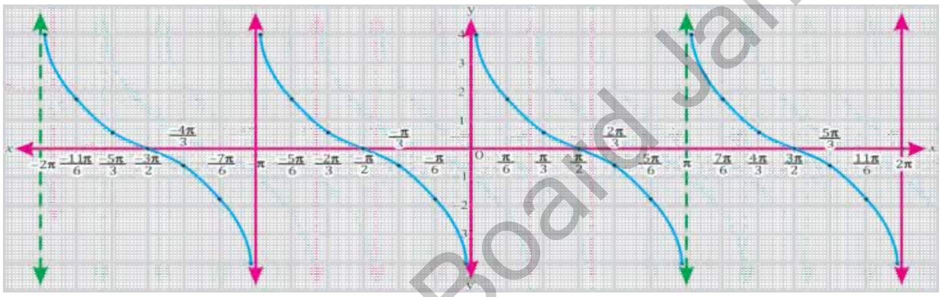
(Fig. 12.7)

Graph of $y = \tan x, -2\pi \leq x \leq 2\pi$

(d) Graph of $y = \cot x, -2\pi \le x \le 2\pi$

Table of values of $\cot x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	$-360°$	$-330°$	$-300°$	$-270°$	$-240°$	$-210°$	$-180°$	$-180°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$0°$
$y = \cot x$	∞	1.73	0.58	0	−0.58	−1.73	$+\infty$	$-\infty$	1.73	0.58	0	−0.58	−1.73	$-\infty$
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$y = \cot x$	∞	1.73	0.58	0	−0.58	−1.73	$+\infty$	$-\infty$	1.73	0.58	0	−0.58	−1.73	$-\infty$



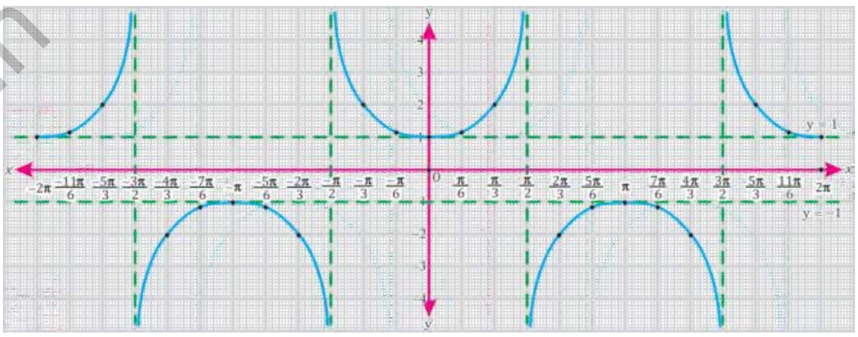
(Fig. 12.8)

Graph of $y = \cot x, -2\pi \le x \le 2\pi$

(e) Graph of $y = \sec x, -2\pi \le x \le 2\pi$

Table of values of $\sec x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	$-360°$	$-330°$	$-300°$	$-270°$	$-270°$	$-240°$	$-210°$	$-180°$	$-150°$	$-120°$	$-90°$	$-90°$	$-60°$	$-30°$	$0°$
$y = \sec x$	1	1.15	2	$+\infty$	$-\infty$	−2	−1.15	−1	−1.15	−2	$+\infty$	$-\infty$	2	1.15	1
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	$0°$	$30°$	$60°$	$90°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$270°$	$300°$	$330°$	$360°$
$y = \sec x$	1	1.15	2	$+\infty$	$-\infty$	−2	−1.15	−1	−1.15	−2	$+\infty$	$-\infty$	2	1.15	1



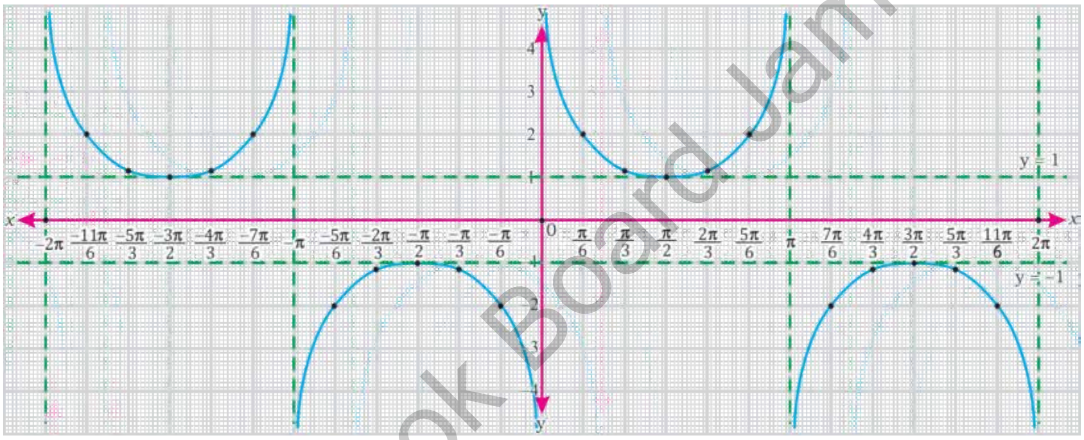
(Fig. 12.9)

Graph of $y = \sec x, -2\pi \le x \le 2\pi$

(f) Graph of $y = \operatorname{cosec} x, -2\pi \le x \le 2\pi$

Table of values of $\operatorname{cosec} x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	−360°	−330°	−300°	−270°	−240°	−210°	−180°	−180°	−150°	−120°	−90°	−60°	−30°	0°
$y = \operatorname{cosec} x$	∞	2	1.15	1	1.15	2	+∞	−∞	−2	−1.15	−1	−1.15	−2	−∞
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	0°	30°	60°	90°	120°	150°	+180°	180°	210°	240°	270°	300°	330°	360°
$y = \operatorname{cosec} x$	∞	2	1.15	1	1.15	2	+∞	−∞	−2	−1.15	−1	−1.15	−2	−∞



(Fig. 12.10)

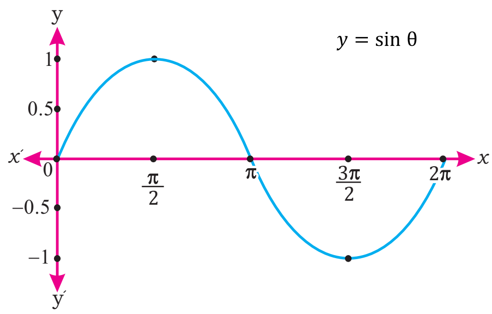
Graph of $y = \operatorname{cosec} x, -2\pi \le x \le 2\pi$

12.2.3 Guess the graphs of $\sin 2\theta, \cos 2\theta, \sin\frac{\theta}{2}, \cos\frac{\theta}{2}$ etc, without actually drawing them

We know that the graph of $y = \sin\theta$ and $y = \cos\theta$ both have period 2π. In $y = \sin n\theta$ and $y = \cos n\theta$, n is constant and indicates the number of cycles in the interval of 0 to 2π, i.e., $0 \le \theta \le 2\pi$

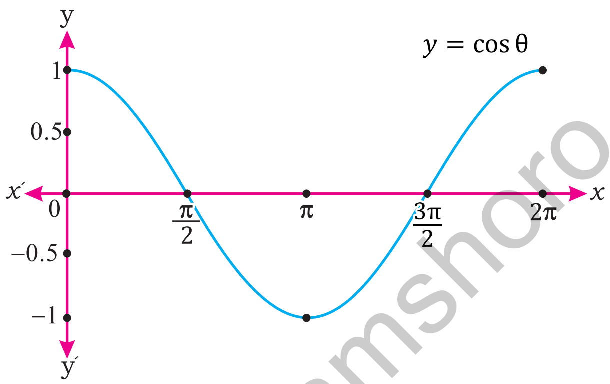
Now, we draw the graphs of $\sin 2\theta$, $\cos 2\theta$, $\sin\frac{\theta}{2}$, and $\cos\frac{\theta}{2}$ without using table. If $n = 1$ in $\sin n\theta$ and $\cos n\theta$, it means that there is only one cycle in the interval: $0 \le \theta \le 2\pi$

Fig 12.11 shows the graph of $\sin\theta$, $0 \le \theta \le 2\pi$ and period of $\sin\theta$ is 2π.



(Fig. 12.11)

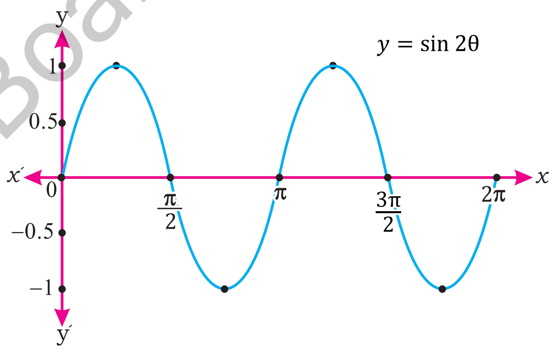
In Fig 12.12, there is one cycle in graph of $y = \cos\theta, 0 \le \theta \le 2\pi$ and period of $\cos\theta$ is 2π.



(Fig. 12.12)

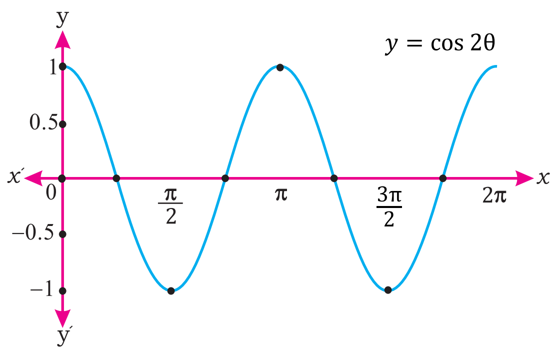
If $n = 2$ then we have, $y = \sin 2\theta$ and $y = \cos 2\theta$ which means that there are two cycles in the interval 0 to 2π. The graphs of $y = \sin 2\theta$ and $y = \cos 2\theta$ are the compressed forms of the graphs of $y = \sin\theta$ and $y = \cos\theta$ respectively. The graphs of $\sin 2\theta, \cos 2\theta$ are as follows:

In Fig 12.13, there are two cycles in the graph of $y = \sin 2\theta,\ 0 \le \theta \le 2\pi$ and period of $\sin 2\theta$ is π.



(Fig. 12.13)

Fig 12.14 shows the graph of $y = \cos 2\theta, 0 \le \theta \le 2\pi$. It has two cycles and period of $\cos 2\theta$ is π.

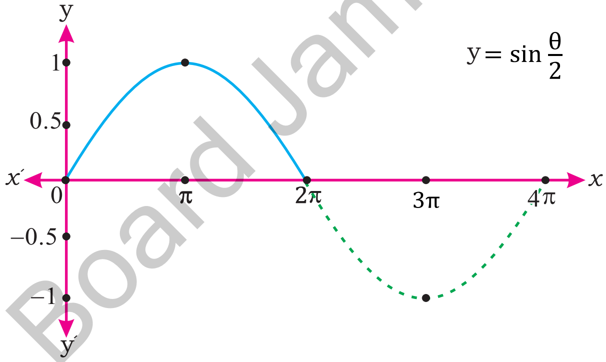


(Fig. 12.14)

Now, if $n = 3$ then we have $y = \sin 3\theta$ and $y = \cos 3\theta$, it means that there are three cycles in the interval 0 to 2π. The graphs of $y = \sin 3\theta$ and $y = \cos 3\theta$ are the compressed forms of the graphs of $y = \sin\theta$ and $y = \cos\theta$ respectively.

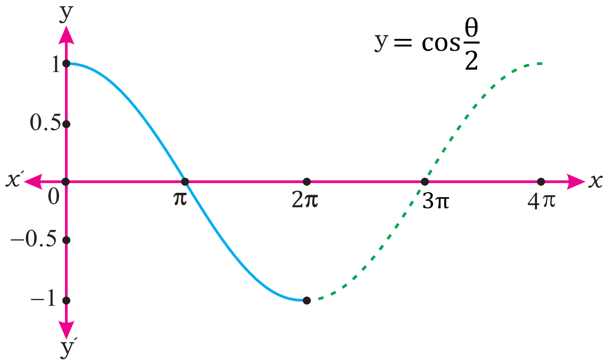
Now, if $n = \frac{1}{2}$ then we have $y = \sin\frac{1}{2}\theta$ and $y = \cos\frac{1}{2}\theta$, it means that there is only half a cycle in the interval 0 to 2π. The graphs of $y = \sin\frac{\theta}{2}$ and $y = \cos\frac{\theta}{2}$ are the expanded forms of the graph $y = \sin\theta$ or $y = \cos\theta$ respectively.

Fig. 12.15 shows the graph of $y = \sin\frac{\theta}{2}$, $0 \leq \theta \leq 2\pi$. It has half cycle in 0 to 2π and period of $\sin\frac{\theta}{2} = 4\pi$.



(Fig. 12.15)

Fig. 12.16 shows the graph of $y = \cos\frac{\theta}{2}, 0 \leq \theta \leq 2\pi$. If has half cycle in the interval 0 to 2π and period of $\cos\frac{\theta}{2} = 4\pi$



(Fig. 12.16)

Thus, we conclude that multiplying θ by a number greater than 1 compresses the graph of sin θ or cos θ, while multiplying θ by a positive number less than 1, expands the graph. In this case the period is given by

$$\boxed{Period = \frac{2\pi}{n}}$$ where, n is the number of cycles.

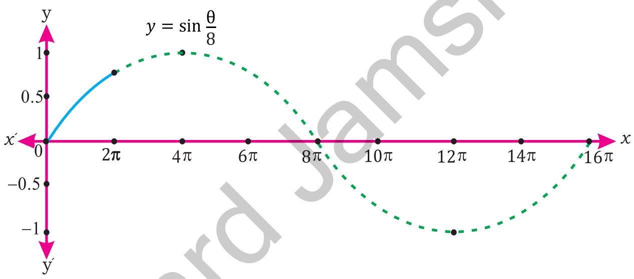
The period of sine and cosine functions can be found by the given formula.

Now, we also define the amplitude as the maximum height of the graph of sine or cosine functions from the horizontal axis. The functions $y = \sin\theta$ and $y = \cos\theta$ have amplitude 1.

Example 1. Guess the graph of $\sin\frac{\theta}{8}$. Also find its period, and amplitude.

Solution: Let $y = \sin\frac{1}{8}\theta$

Here, $n = \frac{1}{8} < 1$, so the graph of $\sin\frac{1}{8}\theta$ is an expanded type of the graph of $y = \sin\theta$, in the interval of 0 to 2π, there is one eighth of a cycle as shown in Fig. 12.17.



(Fig. 12.17)

Now, period of $\sin\frac{1}{8}\theta = \frac{2\pi}{n} = \frac{2\pi}{\frac{1}{8}} = 8\times 2\pi = 16\pi$

The amplitude of the sine function is the coefficient of sine function, which is 1 in this case.

Hence, Amplitude of $\sin\frac{1}{8}\theta = 1$

Example 2. Guess the graph of $2\sin 2\theta$. Also find its period, and amplitude.

Solution: Let $y = 2\sin 2\theta$

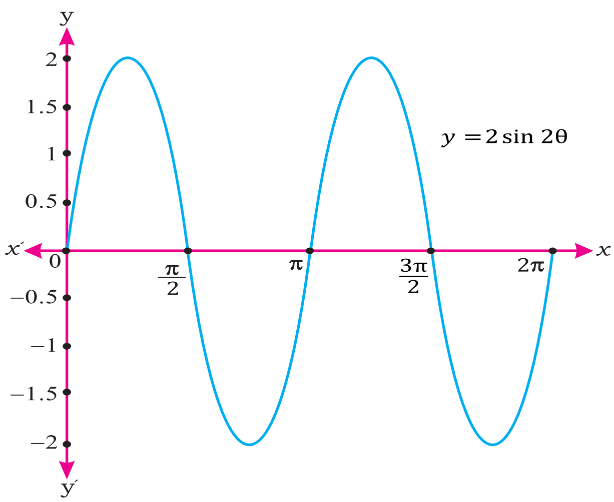
Here, $n = 2 > 1$, so the graph of $2\sin 2\theta$ is a compressed type of the graph of $y = \sin\theta$, in the interval of 0 to 2π, there are two cycles as shown in Fig. 12.18.

Now, Period of $2\sin 2\theta = \frac{2\pi}{2} = \pi$

The amplitude of the sine function is the coefficient of sine, which is 2.

Hence,

Amplitude of $2\sin 2\theta = 2$



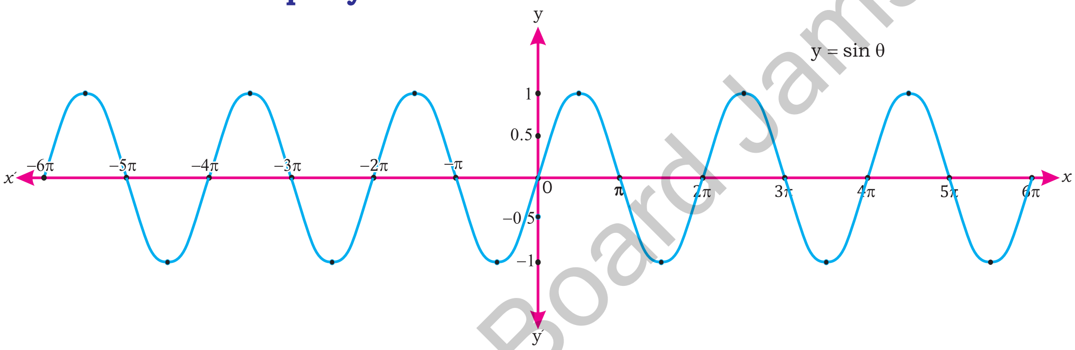
(Fig. 12.18)

12.2.4 Define periodic, even/odd and translation properties of the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$, i.e., $\sin\theta$ has

- **Periodic property** $\sin(\theta \pm 2\pi) = \sin\theta$
- **Odd property** $\sin(-\theta) = -\sin\theta$
- **Translation property** $\begin{cases} \sin(\theta - \pi) = -\sin\theta \\ \sin(\pi - \theta) = \sin\theta \end{cases}$

a) Properties of the graph of $\sin\theta$

i. Periodic Property

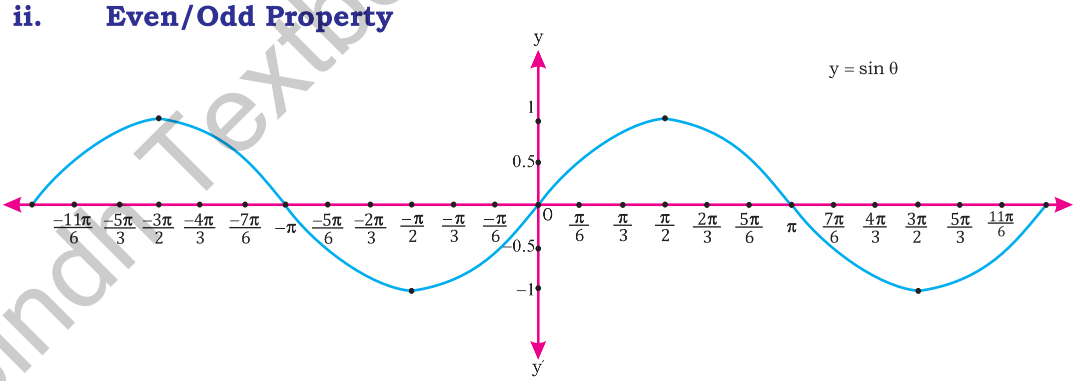


(Fig. 12.19)

We know that the graph of $\sin\theta$ repeats itself after a period of 2π as shown in Fig. 12.19. Therefore, $\sin(\theta \pm 2\pi) = \sin\theta$.

This property of graph of $\sin\theta$ is known as periodic property.

ii. Even/Odd Property



(Fig. 12.20)

We know that the graph of $y = \sin\theta$ is symmetrical about the origin as shown in Fig. 12.20. It means that if θ is replaced by $-\theta$ then the graph will be changed.

Therefore, $\sin(-\theta) = -\sin\theta$

Hence, $\sin\theta$ is an odd function and this property is called the odd property of graph of $\sin\theta$.

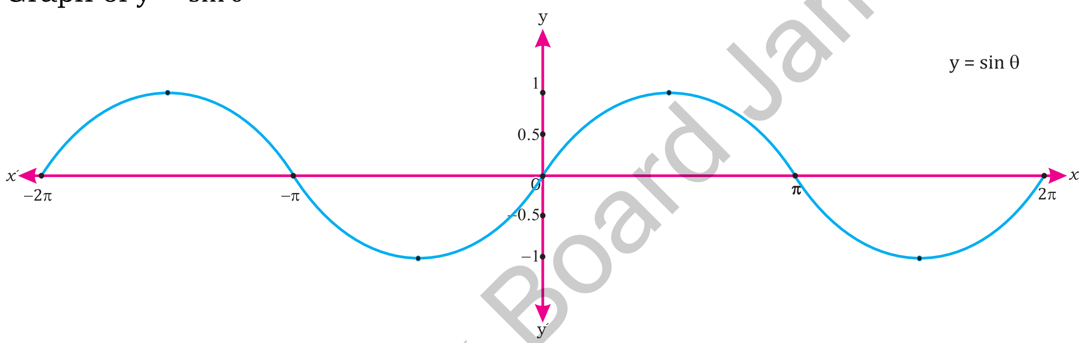
iii. Translation Property:

We know that

$$\left.\begin{aligned} \sin(\theta-\pi) &= -\sin\theta\,; \\ \sin(\pi-\theta) &= \sin\theta \end{aligned}\right\}$$

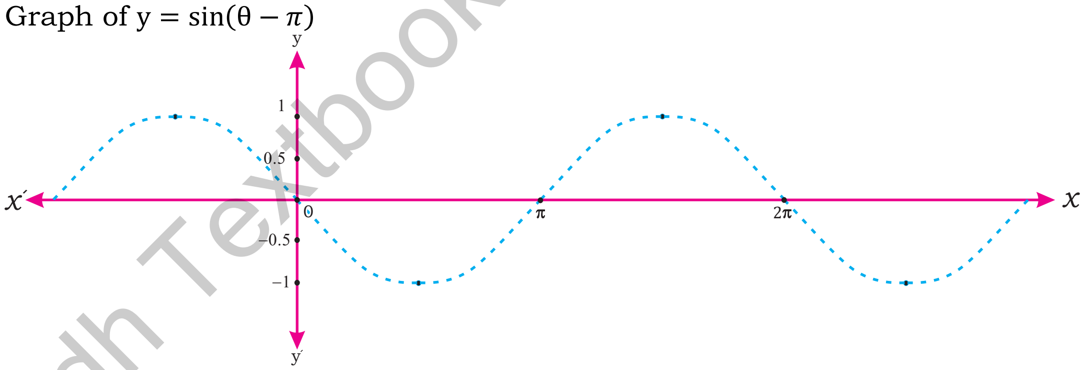
This property is called Translation property of graph of $\sin\theta$ because the graph of $y=\sin(\theta-\pi)$ is similar to the graph of original sine function but translated or shifted horizontally π units right to the graph of $y=\sin\,\theta$ as shown in Fig. 12.21 and Fig. 12.22.

Graph of $y=\sin\theta$



(Fig. 12.21)

Graph of $y=\sin(\theta-\pi)$



(Fig. 12.22)

We observe that graph of $y=\sin(\theta-\pi)$ is reflection of graph of $y=\sin\theta$ about x-axis.

So, the graph of $y=\sin(\theta-\pi)$ is same as the graph $y=-\sin\theta$

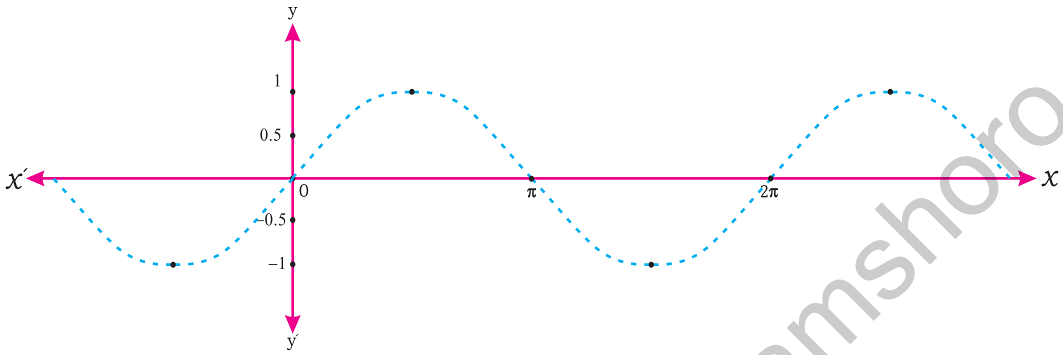
Hence, $\sin(\theta-\pi)=-\sin\theta$

Now, $\sin(\pi-\theta)=\sin[-(\theta-\pi)]$

$=-\sin(\theta-\pi)$ $\quad(\because\ \sin(-\theta)=-\sin\theta)$

So, graph of $\sin(\pi-\theta)$ is reflection of graph of $y=\sin(\theta-\pi)$ about x-axis as shown in Fig. 12.23.

Graph of $y = \sin(\pi - \theta)$



(Fig. 12.23)

We observe that the graph of $y = \sin(\pi - \theta)$ is same as the graph of $y = \sin\theta$

Hence, $\sin(\pi - \theta) = \sin\theta$

Note: (i) The graph of $y = \sin(\theta - k)$ is similar to the graph of original sine function but translated or shifted horizontally k units right to the graph of $y = \sin\theta$.

(ii) The graph of $y = \sin(\theta + k)$ is similar to the graph of original sine function but translated horizontally k units left to the graph of $y = \sin\theta$.

Example: Draw one cycle of the graph of $y = 3\sin(\theta - 3\pi)$

Solution: Here ,

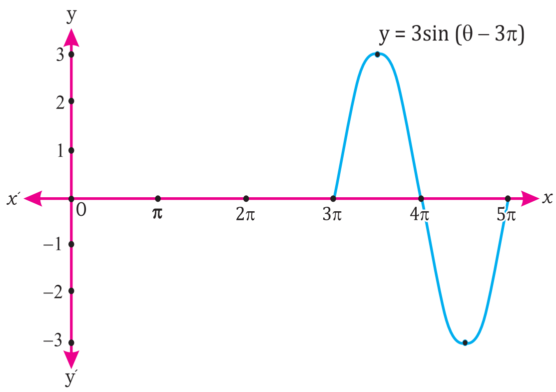
Amplitude = 3

and period = 2π

According to the Translation property, the graph of $3\sin(\theta - 3\pi)$ is similar to graph of $\sin\theta$ but translated 3π units to the right and its amplitude is 3.

Now, for initial point, we take $\theta - 3\pi = 0 \Rightarrow \theta = 3\pi$ and for terminal point, we take

$$\theta - 3\pi = 2\pi \Rightarrow \theta = 5\pi$$



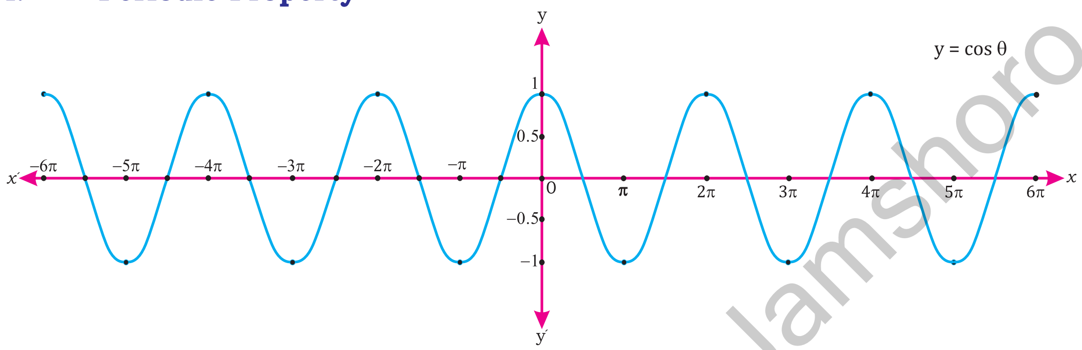
(Fig. 12.24)

Therefore, the interval of the graph is $3\pi \leq \theta \leq 5\pi$

The graph of $y = 3\sin(\theta - 3\pi)$ is shown in Fig. 12.24.

b) Properties of the graph of cosθ

i. Periodic Property

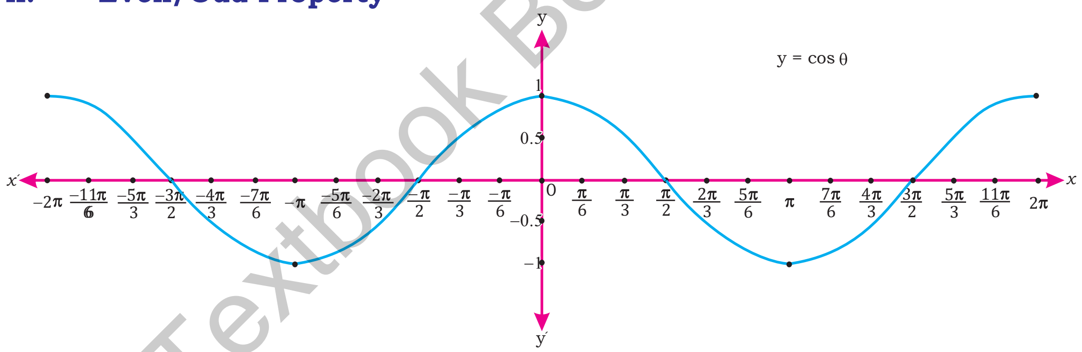


(Fig. 12.25)

We know that the graph of $\cos\theta$ repeats itself after a period of 2π as shown in Fig. 12.25. Therefore, $\cos(\theta \pm 2\pi) = \cos\theta$.

This property of graph of $\cos\theta$ is known as periodic property.

ii. Even/Odd Property



(Fig. 12.26)

We know that the graph of $y = \cos\theta$ is symmetrical about y-axis as shown I Fig. 12.26. It means that if θ is replaced by $-\theta$ then the value will be unchanged.

Therefore, $\qquad \cos(-\theta) = \cos\theta$

Hence, $\cos\theta$ is an even function and this property is called even property of graph of $\cos\theta$.

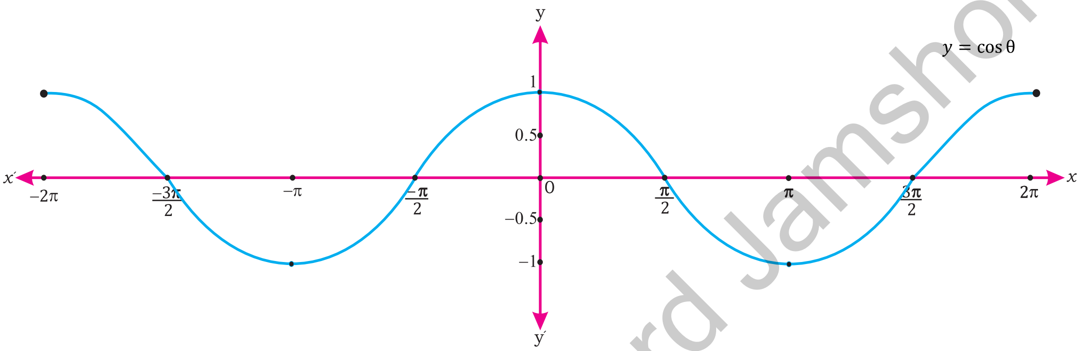
iii. Translation Property:

We know that

$$\left.\begin{aligned}\cos(\theta - \pi) &= -\cos\theta\,;\\ \cos(\pi - \theta) &= -\cos\theta\end{aligned}\right\}$$

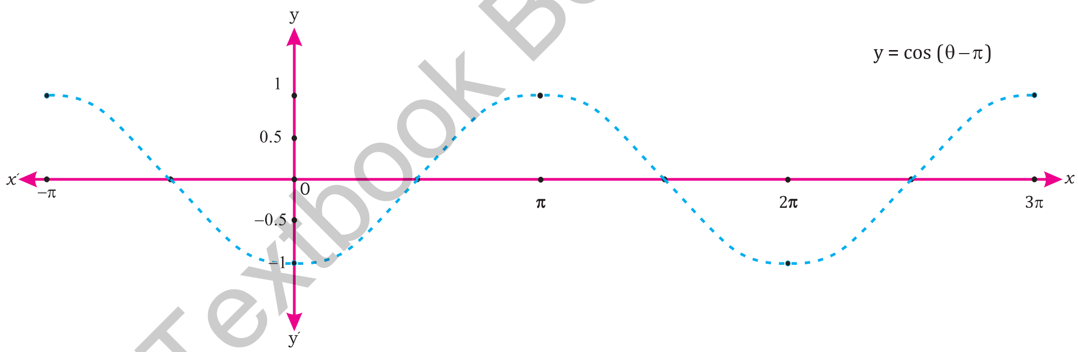
The property is called Translation property of graph of $\cos\theta$ because the graph of $y = \cos(\theta - \pi)$ is similar to the graph of original cosine function but translated or shifted horizontally π units right to the graph of $y = \cos\theta$ as shown in Fig. 12.27 and Fig. 12.28.

Graph of $y = \cos\theta$



(Fig. 12.27)

Graph of $y = \cos(\theta - \pi)$



(Fig. 12.28)

We observe that the graph of $y = \cos(\theta - \pi)$ is reflection of graph of $y = \cos\theta$ about x-axis.

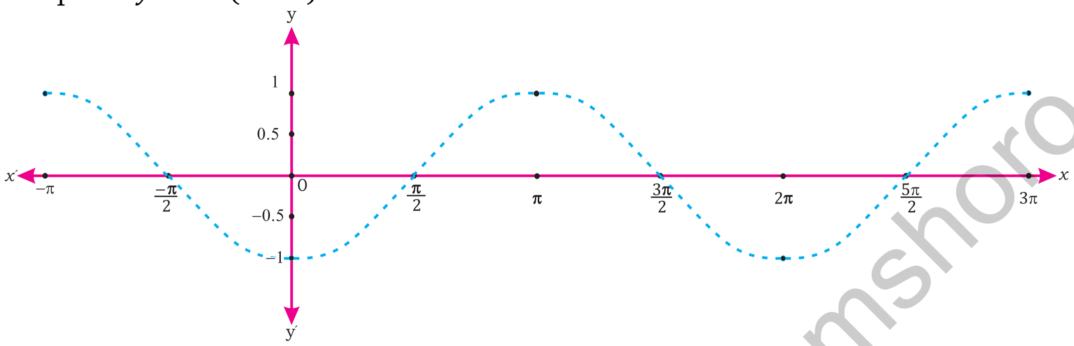
So, the graph of $y = \cos(\theta - \pi)$ is same as the graph of $y = -\cos\theta$.

Hence, $\cos(\theta - \pi) = -\cos\theta$

Since $\quad \cos(\pi - \theta) = \cos(\theta - \pi)$

Therefore, graph of $\cos(\pi - \theta)$ is same as graph of $\cos(\theta - \pi)$ as shown in Fig. 12.29.

Graph of $y = \cos(\pi - \theta)$



(Fig. 12.29)

We observe that the graph of $y = \cos(\pi - \theta)$ is reflection of graph of $y = \cos\theta$ about x-axis, so, the graph of $y = \cos(\pi - \theta)$ is same as the graph $y = -\cos\theta$

Hence, $\cos(\pi - \theta) = -\cos\theta$

Note: (i) If we add constant $k > 0$ to θ as $y = \cos(\theta + k)$ then graph of cosine function will be translated k units to the left.
(ii) If we subtract constant $k > 0$ from θ as $y = \cos(\theta - k)$ then graph of cosine function will be translated k units to the right.

Example: Using graphs and its properties, show that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

Solution:

Let, $y = \sin\left(\frac{\pi}{2} - \theta\right) = -\sin\left(\theta - \frac{\pi}{2}\right)$

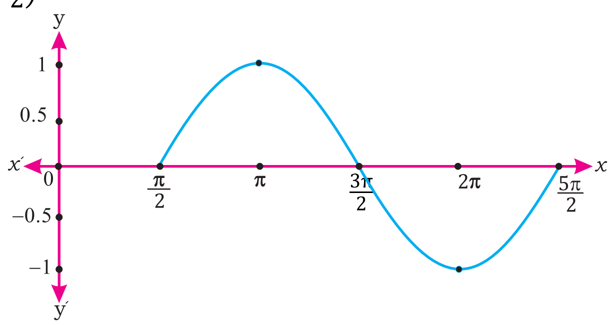
In order to draw graph of $\sin\left(\frac{\pi}{2} - \theta\right)$, we first draw graph of $\sin\left(\theta - \frac{\pi}{2}\right)$ and then reflect it about x-axis.

Here, amplitude $= |-1| = 1$

and period $= 2\pi$

For initial point of the graph:

$$\theta - \frac{\pi}{2} = 0 \implies \theta = \frac{\pi}{2}$$



(Fig. 12.30)

For terminal point of the graph: $\theta - \frac{\pi}{2} = 2\pi \implies \theta = \frac{5\pi}{2}$

Therefore, the interval of the graph is $\frac{\pi}{2} \le \theta \le \frac{5\pi}{2}$

By translation property, graph of $\sin\theta$ will be translated $\frac{\pi}{2}$ units to the right as shown in Fig. 12.30. Now, we reflect it about x-axis to get graph of

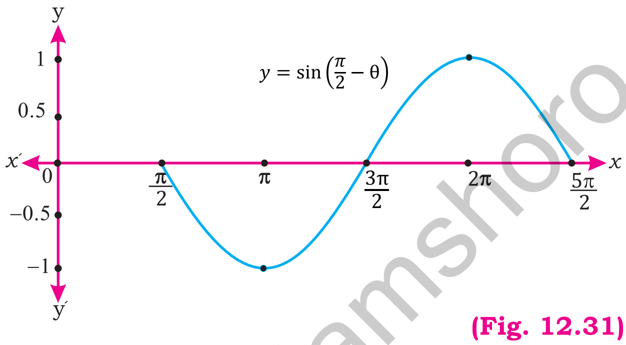
$y = \sin\left(\frac{\pi}{2} - \theta\right)$ as shown in Fig. 12.31.

We observe that the graph of $y = \sin\left(\frac{\pi}{2} - \theta\right)$ is as same as the graph of $y = \cos\theta$.

Hence,

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

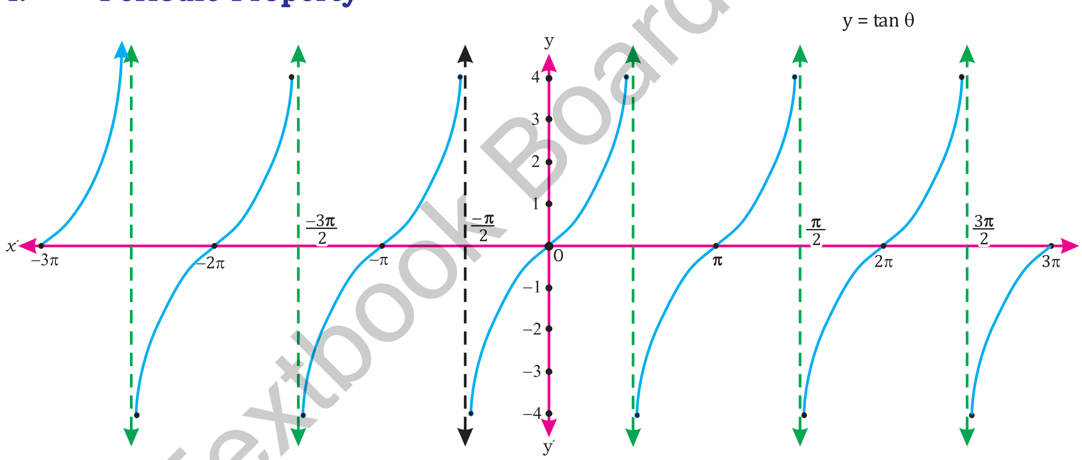
Hence proved.



(Fig. 12.31)

c) Properties of the graph of tanθ

i. Periodic Property



(Fig. 12.32)

We know that the graph of $\tan\theta$ repeats itself after a period of π as shown in Fig. 12.32. Therefore, $\tan(\theta \pm \pi) = \tan\theta$.

where, $\theta \neq \pm\frac{\pi}{2}, \pm3\frac{\pi}{2}, \pm5\frac{\pi}{2}, \pm7\frac{\pi}{2}, \ldots, (2k+1)\frac{\pi}{2}$; k is any integer

At these values of θ the tangent function is undefined.

This property of graph of $\tan\theta$ is known as periodic property.

ii. Even/Odd Property

The graph of $y = \tan\theta$ is symmetrical about origin as shown in Fig.12.32. It means that if θ is replaced by −θ then the graph will be changed.

Therefore, $\tan(-\theta) = -\tan\theta$

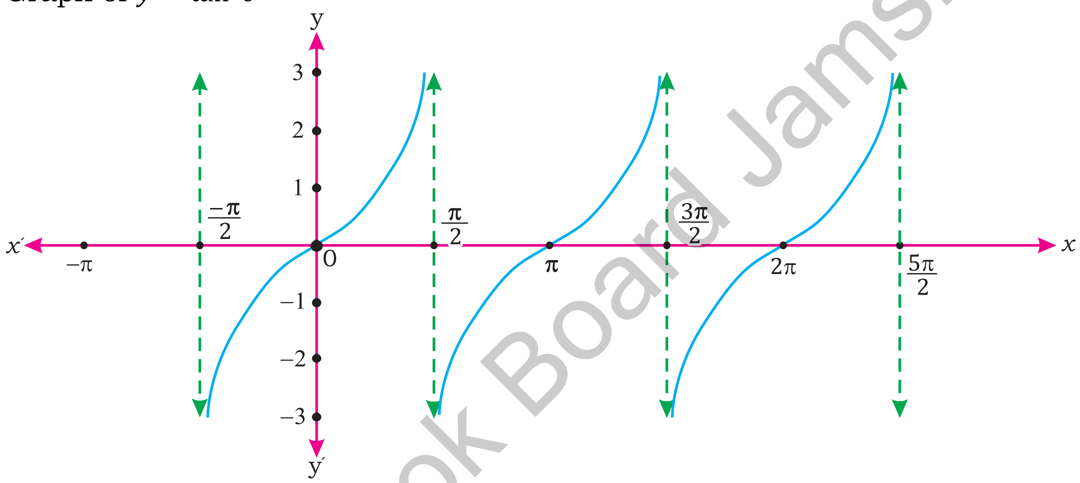
Hence, $y = \tan\theta$ is an odd function and this property of graph of $\tan\theta$ is called odd property.

iii. Translation Property: We know that

$$\left.\begin{aligned}\tan(\theta - \pi) &= \tan\theta\,;\\ \tan(\pi - \theta) &= -\tan\theta\end{aligned}\right\}$$

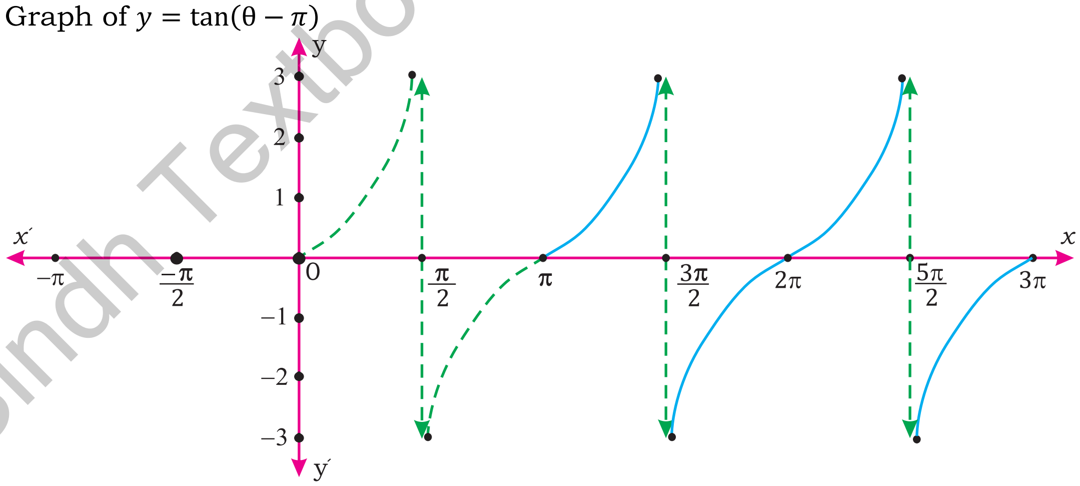
This property is called Translation property of $\tan\theta$ because the graph of $y = \tan(\theta - \pi)$ is similar to the graph of $\tan\theta$ but translated or shifted horizontally π units right to the graph of $y = \tan\,\theta$ as shown in Fig. 12.33 and Fig. 12.34.

Graph of $y = \tan\,\theta$



(Fig. 12.33)

Graph of $y = \tan(\theta - \pi)$



(Fig. 12.34)

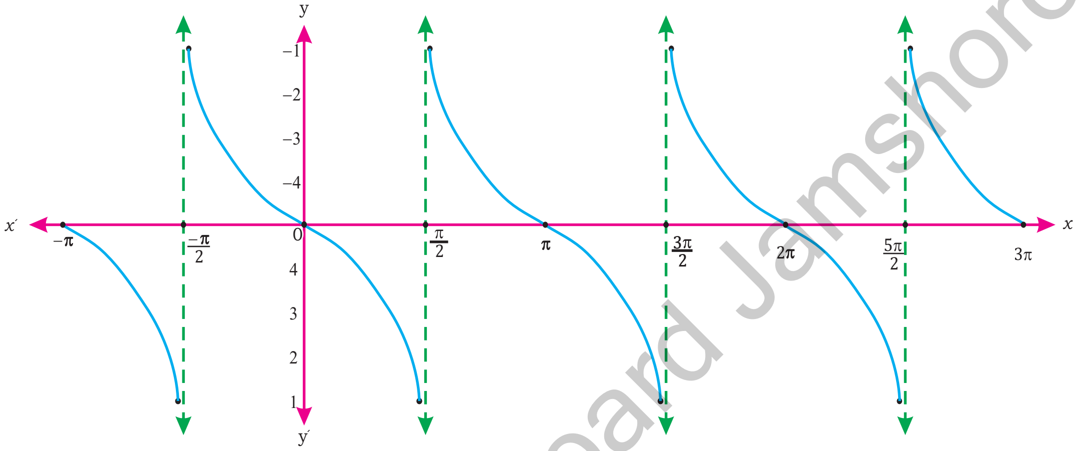
We observe that the graph of $y = \tan(\theta - \pi)$ is same as the graph of $y = \tan\theta$

Hence, $\tan(\theta - \pi) = \tan\theta$

Now, $\tan(\pi - \theta) = -\tan(\theta - \pi)$

So, graph of $y = \tan(\pi - \theta)$ is reflection of graph of $\tan(\theta - \pi)$ about x-axis as shown in Fig. 12.35.

Graph of $y = \tan(\pi - \theta)$



(Fig. 12.35)

We observe that the graph of $y = \tan(\pi - \theta)$ is same as the graph $y = -\tan\theta$

Thus, $\tan(\pi - \theta) = -\tan\theta$

Note: (i) If we add positive real number k to θ as $y = \tan(\theta + k)$, the graph of tangent function will be translated k units to the left.
(ii) If we subtract positive real number k from θ as $y = \tan(\theta - k)$ then the graph of tangent function will be translated k units to the right.

12.2.5 Deduce $\sin(\theta + 2k\pi) = \sin\theta$, where k is an integer

We know that

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

So, $\sin(\theta + 2k\pi) = \sin\theta\cos 2k\pi + \cos\theta\sin 2k\pi$ where, $k \in Z$ ...(i)

$\because$ $\sin\theta$ and $\cos\theta$ are periodic functions of period 2π.

$\therefore$ for, any integer $k = 0, \pm1, \pm2, ...$

$\cos 2k\pi = 1$ and $\sin 2k\pi = 0$

By using $\cos 2k\pi = 1$ and $\sin 2k\pi = 0$ in equation (i)

We get $\sin(\theta + 2k\pi) = \sin\theta(1) + \cos\theta(0)$

$\sin(\theta + 2k\pi) = \sin\theta + 0$

Hence, $\boxed{\sin(\theta + 2k\pi) = \sin\theta}$

Exercise 12.2

Draw the graph of the following trigonometric functions for the given interval.

1. $y = -\sin x$, $0 \le x \le 2\pi$ **2.** $y = \cos x$, $0 \le x \le 2\pi$

3. $y = \sin\frac{x}{2}$, $0 \le x \le 2\pi$ **4.** $y = \tan 2x$, $-\pi \le x \le 2\pi$

5. $y = 5\sec\frac{x}{3}$, $-2\pi \le x \le 2\pi$ **6.** $y = -4\cot x$, $-\pi \le x \le 2\pi$

7. Guess and draw the graph of the following functions without using table of values where $0 \le \theta \le 2\pi$

(i) $y = \cos 4\theta$ (ii) $y = 2\sin\frac{\theta}{2}$ (iii) $y = 3\cos\frac{\theta}{2}$

(iv) $y = 2\sin 3\theta$ (v) $y = \cos\frac{\theta}{3}$ (vi) $y = \sin 2\theta$

8. By using properties of graph of sine, cosine and tangent, show the following.

(i) $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ (ii) $\sin\left(\frac{3\pi}{2} + \theta\right) = \cos\theta$

(iii) $\cos(\pi - \theta) = -\cos\theta$ (iv) $\tan(\pi + \theta) = \tan\theta$

(v) $\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$ (vi) $\cos(2\pi + \theta) = \cos\theta$

9. For any integer k, deduce that:

(i) $\cos(\theta + 2k\pi) = \cos\theta$ (ii) $\operatorname{cosec}(\theta + 2k\pi) = \operatorname{cosec}\theta$

(iii) $\tan(\theta + 2k\pi) = \tan\theta$ (iv) $\cot(\theta + 2k\pi) = \cot\theta$

12.3 Solving Trigonometric Equations Graphically

The equation, containing at least one trigonometric function is called trigonometric equation. For example, $\cos x = \frac{1}{2}$ and $\cos x + \sin x = 1$

Trigonometric equations have an infinite number of solutions due to the periodicity of the trigonometric functions. For example, if $\sin\theta = 0$ then it has infinite number of solutions i.e., $\theta = 0, \pm\pi, \pm 2\pi, \dots$; which can be written as $\theta = n\pi, \forall\, n \in Z$.

12.3.1 Solve trigonometric equations of the type: $\sin\theta = k, \cos\theta = k$ and $\tan\theta = k$ using periodic, even/odd and translation properties

We know that sine and cosine functions are periodic and have period 2π. i.e., they repeat their values after every 2π units. Thus, if we want to find all solutions of the type of $\sin\theta = k$ and $\cos\theta = k$, we simply add and subtract integral multiple of 2π from the solution in the interval $0 \le \theta \le 2\pi$.

Similarly, tangent function is also periodic having period π. Thus, to find all the solution of the equation of the type $\tan\theta = k$, we add and subtract integral multiple of π from the solutions in the interval $0 \le \theta \le 2\pi$.

The method of solving trigonometric equations is explained through the following examples.

Example 1. Solve the equation: $\sin\theta = \frac{1}{2}$

Solution: We know that $\sin\frac{\pi}{6} = \frac{1}{2}$, so the reference angle is $\theta = \frac{\pi}{6}$.

Sine function is positive in 1st and 2nd quadrants, so equation has two solutions in the interval $0 \le \theta \le 2\pi$, one in 1st quadrant and the other in 2nd quadrant.

For 1st quadrant, $\theta = \frac{\pi}{6}$

For 2nd quadrant, $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ [Translation property]

Since, 2π is the period of sinθ.

Thus, $\theta = \frac{\pi}{6} + 2n\pi$ or $\theta = \frac{5\pi}{6} + 2n\pi,\ \forall\, n \in Z$ [Periodic property]

Hence, the solution set of $\sin\theta = \frac{1}{2}$ is $\left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi\right\},\ \forall\, n \in Z$

Example 2. Solve the equation: $\cos\theta = \frac{-1}{\sqrt{2}}$

Solution: We know that $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, so the reference angle is $\theta = \frac{\pi}{4}$.

Cosine function is negative in 2nd and 3rd quadrant.

Thus, the equation has two solutions in the interval $0 \le \theta \le 2\pi$.

For 2nd quadrant, $\theta = \pi - \frac{\pi}{4}$ [Translational property]

$= \frac{3\pi}{4}$

For 3rd quadrant, $\theta = \pi + \frac{\pi}{4}$ [Translational property]

$= \frac{5\pi}{4}$

Since, 2π is the period of cosθ.

Thus, $\theta = \frac{3\pi}{4} + 2n\pi$, or $\theta = \frac{5\pi}{4} + 2n\pi,\ \forall n \in Z$ [Periodic property]

Hence, the solution set of $\cos\theta = \frac{-1}{\sqrt{2}}$ is $\left\{\frac{3\pi}{4} + 2n\pi\right\} \cup \left\{\frac{5\pi}{4} + 2n\pi\right\},\ \forall\, n \in Z$

Example 3. Solve the equation: $\tan\theta = \sqrt{3}$

Solution: We know that $\tan\frac{\pi}{3} = \sqrt{3}$, so the reference angle is $\theta = \frac{\pi}{3}$.

Tangent function is positive in 1st quadrant and 3rd quadrant, so the equation has two solutions in the interval $0 \leq \theta \leq 2\pi$, one in 1st quadrant and the other in 3rd quadrant.

For 1st quadrant, $\theta = \frac{\pi}{3}$

For 3rd quadrant, $\theta = \pi + \frac{\pi}{3}$ [Translational property]

$= \frac{4\pi}{3}$

Since, π is the period of $\tan\theta$.

Thus $\theta = \frac{\pi}{3} + n\pi, \ \forall n \in Z$ [Periodic property]

Hence, the solution set of $\tan\theta = \sqrt{3}$ is $\left\{\frac{\pi}{3} + n\pi\right\}$, $\forall\, n \in Z$

12.3.2 Solve graphically the trigonometric equations of the type (i) $\sin\theta = \frac{\theta}{2}$ (ii) $\cos\theta = \theta$ (iii) $\tan\theta = 2\theta$ when $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

The method of graphical solution of trigonometric equations is illustrated through the following examples.

Example 1. By using graph, solve the equation: $\sin\theta = \frac{\theta}{2}$ for $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Solution: Let $y = \sin\theta$ then $y = \frac{\theta}{2}$

We will draw the graph of two functions on the same graph paper. The point of intersection will be the solution of given equation.

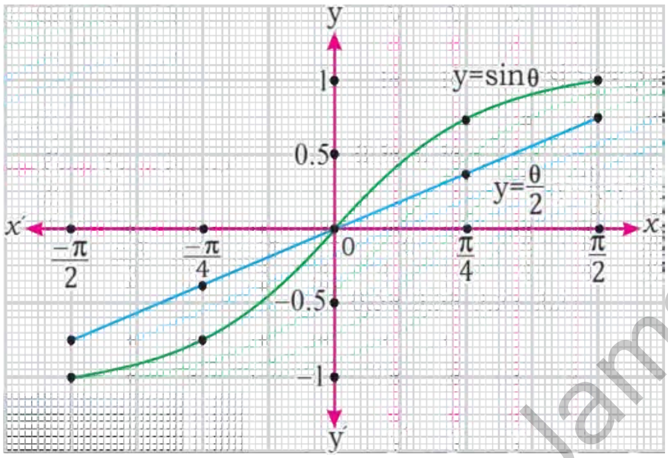
Some values of $y = \sin\theta$ where $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$ are given in the following table.

θ	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$y = \sin\theta$	-1	-0.71	0	0.71	1

and some values of $y = \frac{\theta}{2}$, where $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$ are as under:

θ	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$y = \frac{\theta}{2}$	-0.78	-0.39	0	0.39	0.78

The graphs of the equations are plotted on the same graph paper as shown in Fig. 12.36.



(Fig. 12.36)

From the graph, the point of intersection of these equations is origin. Thus, solution is θ = 0

Example 2. Solve the equation $\cos\theta - \theta = 0$, graphically for interval $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Solution:

Here, $\cos\theta - \theta = 0$

or $\cos\theta = \theta$

Let $y = \cos\theta$ then $y = \theta$

We will draw the graph of these two functions on the same graph paper, the point of intersection will be the solution of given equation.

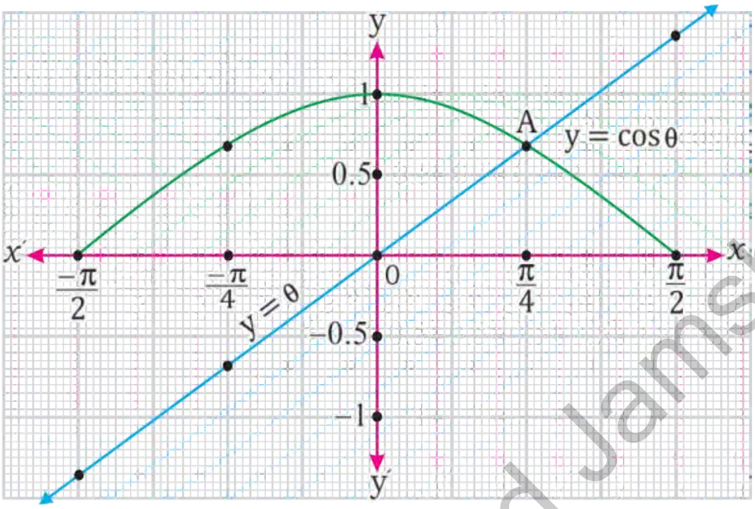
Some values of $y = \cos\theta$, $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$ are given in the following table.

θ	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$y = \cos\theta$	0	0.71	1	0.71	0

Some values of $y = \theta$, $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$ are as under:

θ	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$y = \theta$	−1.57	−0.78	0	0.78	1.57

The graphs of the equations are plotted on the same graph paper as shown in Fig. 12.37.



(Fig. 12.37)

From the graph, we observe that A is the point of intersection of these equations. Thus, the approximate value of abscissa of A is $\frac{\pi}{4}$ i.e., $\theta \approx \frac{\pi}{4}$.

Hence solution is $\theta \approx \frac{\pi}{4}$.

Example 3. Find the solution set of the equation $\tan\theta - 2\theta = 0$, graphically for the interval $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Solution: $\tan\theta - 2\theta = 0$

or $\tan\theta = 2\theta$

Let $y = \tan\theta$ then $y = 2\theta$

We will draw the graph of these two functions on the same graph paper, the point of intersection will be the solution of given equation.

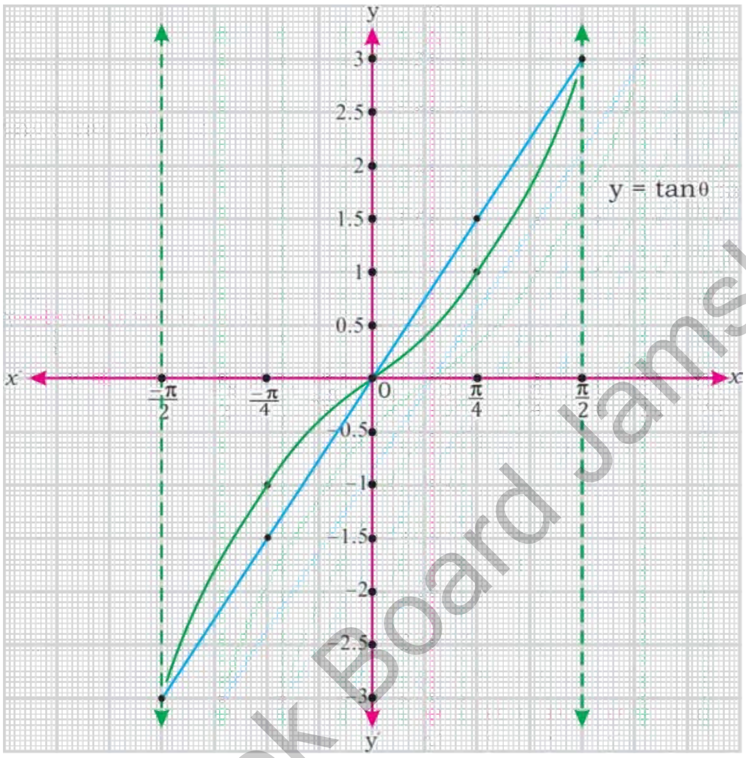
Some values of $y = \tan\theta, \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$ are as under:

θ	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$y = \tan\theta$	$-\infty$	-1	0	1	$+\infty$

Some values of $y = 2\theta$, $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$ are:

θ	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$y = 2\theta$	-3.14	-1.57	0	1.57	3.14

The graphs of the equations are plotted on the same graph as shown in Fig. 12.38.



(Fig. 12.38)

From the graph, we observe that the point of intersection of these equations is origin. Hence, the solution is θ = 0

Exercise 12.3

Find solutions by using properties, when $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$

1. $\sin x - \frac{1}{2} = 0$ **2.** $\cos x - \frac{1}{2} = 0$ **3.** $\cos x + \frac{1}{\sqrt{2}} = 0$

4. $\sin x + \frac{1}{\sqrt{2}} = 0$ **5.** $\tan x - \frac{1}{\sqrt{3}} = 0$ **6.** $\tan x - \sqrt{3} = 0$

7. $\sin x + \frac{\sqrt{3}}{2} = 0$ **8.** $\cos x - 1 = 0$ **9.** $\text{Tan}x + \frac{1}{\sqrt{3}} = 0$

10. $\tan x - 1 = 0$

By using graph, find the solution of the following equations.

11. $\text{Sin}x - \frac{2x}{\pi} = 0$ **12.** $\text{Cos}\,x + \frac{x}{\pi} = 0$ **13.** $\text{Tan}x - \frac{4x}{\pi} = 0$

14. $\text{Sin}x - \frac{3x}{\pi} = 0$ **15.** $\text{Cos}x - \frac{3x}{2\pi} = 0$ **16.** $\text{Tan}x + x = 0$

12.4 Inverse Trigonometric Functions

12.4.1 Define the inverse trigonometric functions and their domain and range

We know that inverse function exists, if the function is bijective. All the trigonometric functions are not bijective. If we restrict the domain of trigonometric functions, then these functions will become bijective and inverse trigonometric functions can be defined within the restricted domain. The inverse of f is denoted by f^{-1}.

and the domain of f = range of f^{-1}

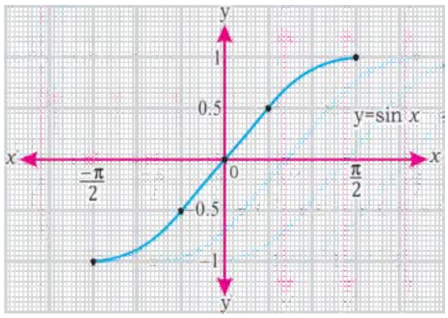
and range of f = domain of f^{-1}

The trigonometric functions with restricted domains where inverse functions exist, are called principal trigonometric functions and are usually denoted by Sin$x$, Cot$x$, Tan$x$ etc.

(a) The Inverse Sine Function

We know that sine function is not one-to-one because a horizontal line will cut its graph at more than one points. If domain of sin$x$ is restricted to the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ then it will be one-to-one and onto (bijective) as shown in Fig. 12.39 and we can find inverse sine function.

The inverse sine function is denoted by $\text{Sin}^{-1} x$ or arcSin x and is defined as $y = \text{Sin}^{-1}x$, if and only if $x = \text{Sin}\, y$,



(Fig. 12.39)

where, Domain of inverse sine function = $\{x | x \in \mathbb{R} \wedge -1 \leq x \leq 1\}$

Range of inverse sine function = $\left\{y | y \in \mathbb{R} \wedge \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}\right\}$

Note: $\sin^{-1}x \neq (\sin x)^{-1}$

Example: Without using calculator, find the value of:

(i) $\text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\text{Sin}^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

Solution:

(i) We want to find the angle y, whose Sine is $\frac{1}{\sqrt{2}}$

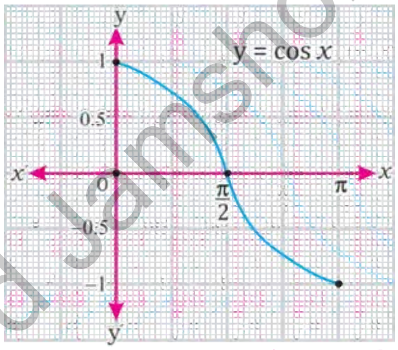
i.e., $\text{Sin} y = \frac{1}{\sqrt{2}} \Rightarrow y = \frac{\pi}{4}$, $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$ Thus, $\text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

(ii) We want to find the angle y whose Sine is $\frac{-\sqrt{3}}{2}$

i.e., $\text{Sin}y = \frac{-\sqrt{3}}{2} \Rightarrow y = \frac{-\pi}{3}, \quad \frac{-\pi}{2} \leq y \leq \frac{\pi}{2} \quad$ Thus, $\text{Sin}^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{-\pi}{3}$

(b) The Inverse Cosine Function

We know that cosine function is also not one-to-one because a horizontal line will cut its graph at more than one points. If domain of $\cos x$ is restricted to the interval $0 \leq x \leq \pi$ then it will be one-to-one and onto (bijective) as shown in Fig. 12.40 and we can find inverse cosine function.



(Fig. 12.40)

The inverse cosine function is denoted by $\text{Cos}^{-1}x$ or arc $\text{Cos}x$ and is defined as $y = \text{Cos}^{-1}x$ if and only if $x = \text{Cos}\, y$,

where, Domain of inverse cosine function $= \{x | x \in \mathbb{R} \wedge -1 \leq x \leq 1\}$

Range of inverse cosine function $= \{y | y \in \mathbb{R} \wedge 0 \leq y \leq \pi\}$

Note: $\cos^{-1}x \neq (\cos x)^{-1}$

Example: Without using calculator, find the value of:

(i) $\text{Cos}^{-1}\left(\frac{1}{2}\right)$ (ii) $\text{Cos}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution:

(i) We want to find the angle y, whose Cosine is $\frac{1}{2}$

i.e., $\text{Cos}\, y = \frac{1}{2} \Rightarrow y = \frac{\pi}{3}, \quad 0 \leq y \leq \pi \quad$ Thus, $\text{Cos}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

(ii) We want to find the angle y, whose Cosine is $\frac{-1}{\sqrt{2}}$

i.e., $\text{Cos}\, y = \frac{-1}{\sqrt{2}} \Rightarrow y = \frac{3\pi}{4}, \quad 0 \leq y \leq \pi.$ Thus, $\text{Cos}^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

(c) The Inverse Tangent Function

Like sine and cosine functions, tangent function is also not bijective.

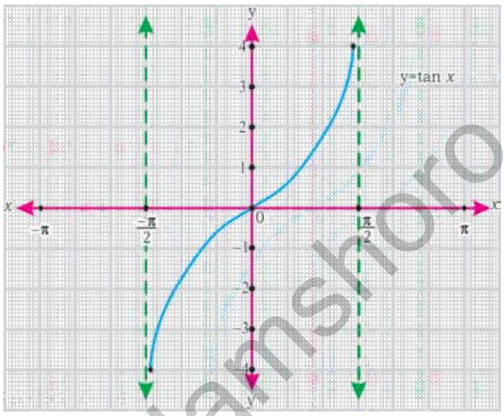
If domain of $\tan x$ is restricted to the interval $\frac{-\pi}{2} < x < \frac{\pi}{2}$, then each horizontal line will intersect the graph only once, so, $y = \text{Tan}\, x$ is one-to-one for the interval $\frac{-\pi}{2} < x < \frac{\pi}{2}$ as shown in Fig. 12.41 and its inverse tangent function is possible.

The inverse tangent function is denoted by $\mathrm{Tan}^{-1}x$ or arc $\mathrm{Tan}\,x$ and is defined as $y = \mathrm{Tan}^{-1}x$ if and only if $x = \mathrm{Tan}\,y$

where,

Domain of inverse tangent function is $\mathbb{R}$ and range of inverse tangent function $= \left\{y|y \in \mathbb{R} \wedge \frac{-\pi}{2} < y < \frac{\pi}{2}\right\}$

Note: $\tan^{-1}x \neq (\tan x)^{-1}$.



(Fig. 12.41)

Example: Without using calculator, find the value of:

(i) $\mathrm{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (ii) $\mathrm{Tan}^{-1}(1)$

Solution:

(i) We want to find the angle y, whose Tangent is $\frac{1}{\sqrt{3}}$

$\mathrm{Tan}\,y = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{\pi}{6}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$ Thus, $\mathrm{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

(ii) We want to find the angle y, whose Tangent is 1

i.e., $\mathrm{Tan}\,y = 1 \Rightarrow y = \frac{\pi}{4}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$ Thus, $\mathrm{Tan}^{-1}(1) = \frac{\pi}{4}$

(d) The Inverse Cotangent Function

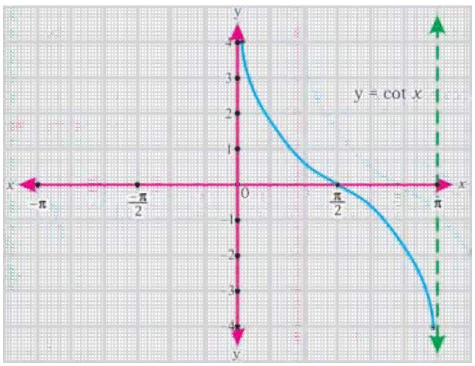
Since, cotangent function is not bijective. Therefore, domain of $\cot x$ is restricted to the interval $0 < x < \pi$, so that each horizontal line intersects the graph only once. so, $y = \mathrm{Cot}\,x$ is bijective for the interval $0 < x < \pi$ as shown in Fig. 12.42 and its inverse function is possible.

The inverse cotangent function is denoted by $\mathrm{Cot}^{-1}x$ or arc $\mathrm{Cot}\,x$ and is defined as $y = \mathrm{Cot}^{-1}x$ if and only if $x = \mathrm{Cot}\,y$

where,

Domain of inverse cotangent function $= \mathbb{R}$

and range of inverse cotangent function $=\{y|y \in \mathbb{R} \wedge 0 < y < \pi\}$



(Fig. 12.42)

Note: $\cot^{-1}x \neq (\cot x)^{-1}$.

Example: Without using calculator, find the value of:

(i) $\text{Cot}^{-1}(\sqrt{3})$ (ii) $\text{Cot}^{-1}(-1)$

Solution:

(i) We want to find the angle y, whose Cotangent is $\sqrt{3}$

i.e., $\text{Cot}\, y = \sqrt{3} \Rightarrow \frac{1}{\text{Tan}\, y} = \frac{1}{\sqrt{3}}, \quad \frac{-\pi}{2} < y < \frac{\pi}{2},$ where $y \neq 0$

$\Rightarrow$ $\text{Tan}\, y = \sqrt{3} \Rightarrow y = \frac{\pi}{3}, \quad \frac{-\pi}{2} < y < \frac{\pi}{2},$ where $y \neq 0$

Thus, $\text{Cot}^{-1}(\sqrt{3}) = \frac{\pi}{3}$

(ii) We want to find the angle y, whose Cotangent is -1

$\text{Cot}\, y = -1 \Rightarrow \frac{1}{\text{Tan}\, y} = -1, \quad \frac{-\pi}{2} < y < \frac{\pi}{2},$ where $y \neq 0$

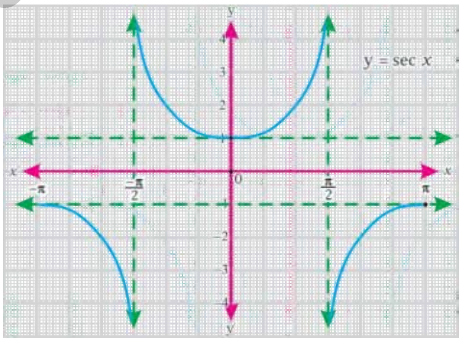
$\Rightarrow \text{Tan}\, y = -1 \Rightarrow y = \frac{-\pi}{4}, \quad \frac{-\pi}{2} < y < \frac{\pi}{2},$ where $y \neq 0$

Thus, $\text{Cot}^{-1}(-1) = \frac{-\pi}{4}$

(e) The Inverse Secant Function

The graph of secant function from $-\pi$ to π is shown in Fig. 12.43. It indicates that every horizontal line intersects the graph at more than one points.

If domain of $\sec x$ is restricted to the interval $0 \leq x \leq \pi$, where $x \neq \frac{\pi}{2}$ then the horizontal line intersects the graph only once, so, this restriction makes the function $y = \text{Sec}\, x$ one-to-one and we can find its inverse.



(Fig. 12.43)

The inverse secant function is denoted by $\text{Sec}^{-1}x$ or arc Sec$x$ and is defined as, $y = \text{Sec}^{-1}x$ if and only if $x = \text{Sec}\, y$

where,

Domain of inverse secant function $= \{x | x \in \mathbb{R} \wedge x \leq -1, x \geq 1\}$

Range of inverse secant function $= \left\{y | y \in \mathbb{R} \wedge 0 \leq y \leq \pi,\ y \neq \frac{\pi}{2}\right\}$

Note: $\sec^{-1}x \neq (\sec x)^{-1}$.

Example: Without using tables / calculator, find the value of:

(i) $\text{Sec}^{-1}(2)$ (ii) $\text{Sec}^{-1}(-1)$

Solution:

(i) We want to find the angle y, whose Secant is 2, $0 \leq y \leq \pi$, where $y \neq \frac{\pi}{2}$

i.e., $\text{Sec}\, y = 2$

or $\quad \dfrac{1}{\text{Cos}\, y} = 2 \Rightarrow \text{Cos}\, y = \dfrac{1}{2} \Rightarrow y = \dfrac{\pi}{3} \quad \left(\because \sec y = \dfrac{1}{\cos y} \text{ and } \text{Cos}\dfrac{\pi}{3} = \dfrac{1}{2}\right)$

Thus, $\text{Sec}^{-1}(2) = \dfrac{\pi}{3}$

(ii) We want to find the angle y, whose Secant is -1, $0 \le y \le \pi$, where $y \ne \dfrac{\pi}{2}$

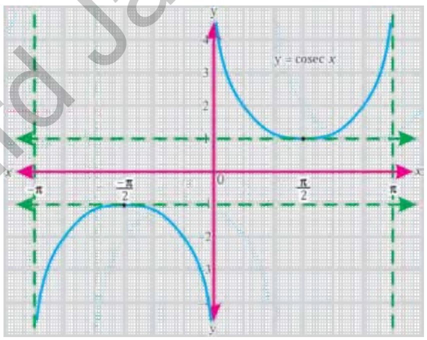
i.e., $\text{Sec}\, y = -1$

or $\quad \dfrac{1}{\text{Cos}\, y} = -1 \Longrightarrow \text{Cos}\, y = -1 \Rightarrow y = \pi \quad \left(\because \sec y = \dfrac{1}{\cos y} \text{ and } \text{Cos}\,\pi = -1\right)$

Thus, $\text{Sec}^{-1}(-1) = \pi$

(f) The inverse Cosecant Function

The graph of cosecant function from $-\pi$ to π is shown in Fig 12.44. It indicates that every horizontal line intersects the graph at more than one points. So, cosecant function is not one-to-one. If domain of $\text{cosec}\, x$ is restricted to the interval $\dfrac{-\pi}{2} \le x \le \dfrac{\pi}{2}$, where $x \ne 0$ then the horizontal line intersects the graph only once, so, this restriction makes the function $y = \text{cosec}\, x$ one-to-one and we can find its inverse.



(Fig. 12.44)

The inverse cosecant function is denoted by $\text{Cosec}^{-1}x$ or arc Cosec x and is defined as, $y = \text{Cosec}^{-1}x$ if and only if $x = \text{Cosec}\, y$

where,

Domain of inverse cosecant function $= \{x | y \in \mathbb{R} \wedge x \le -1, x \ge 1\}$

Range of inverse cosecant function $= \left\{y | y \in \mathbb{R} \wedge \dfrac{-\pi}{2} \le y \le \dfrac{\pi}{2} \text{ where } y \ne 0\right\}$

Note: $\text{Cosec}^{-1}x \ne (\text{cosec}\, x)^{-1}$.

Example: Without using tables / calculator, find the value of:

(i) $\text{Cosec}^{-1}(-2)$ (ii) $\text{Cosec}^{-1}\left(\dfrac{-2}{\sqrt{3}}\right)$

Solution:

(i) We want to find the angle y, whose Cosecant is -2

i.e., $\text{Cosec}\, y = -2$, $\quad \dfrac{-\pi}{2} \le y \le \dfrac{\pi}{2}$, $\quad$ where $y \ne 0$

$\Rightarrow \quad \dfrac{1}{Sin\, y} = -2 \Rightarrow \text{Sin}\, y = \dfrac{-1}{2} \Rightarrow y = \dfrac{-\pi}{6} \quad \left(\because \text{cosec}\, y = \dfrac{1}{\text{Sin}\, y} \text{ and Sin}\left(\dfrac{-\pi}{6}\right) = -\dfrac{1}{2}\right)$

Thus, $\text{Cosec}^{-1}(-2) = \dfrac{-\pi}{6}$

(ii) We want to find the angle y, whose Cosecant is $\dfrac{-2}{\sqrt{3}}$

i.e., $\text{Cosec}\, y = \dfrac{-2}{\sqrt{3}}, \quad \dfrac{-\pi}{2} \leq y \leq \dfrac{\pi}{2}, \quad \text{where } y \neq 0$

$\Rightarrow \dfrac{1}{\text{Sin}\, y} = \dfrac{-2}{\sqrt{3}} \Rightarrow \text{Sin}\, y = \dfrac{-\sqrt{3}}{2} \Rightarrow y = \dfrac{-\pi}{3} \quad \left(\because \text{cosec}\, y = \dfrac{1}{\sin y} \text{ and Sin}\left(\dfrac{-\pi}{3}\right) = \dfrac{-\sqrt{3}}{2}\right)$

Thus, $\text{Cosec}^{-1}\left(\dfrac{-2}{\sqrt{3}}\right) = -\dfrac{\pi}{3}$

12.4.2 Domains and ranges of:
- **principal trigonometric functions**
- **inverse trigonometric functions**

i. Domains and Ranges of principal trigonometric functions

We have studied in section 12.4.1 that general trigonometric functions are not bijective but principal trigonometric functions are bijective because of the restricted domains. We found domains of principal trigonometric functions using graphs in the previous section which are summarized in the following table alongwith their ranges which are same as that of general trigonometric functions.

Function	Domain	Range
$y = \text{Sin}\, x$	$\dfrac{-\pi}{2} \leq x \leq \dfrac{\pi}{2}$	$-1 \leq y \leq 1$
$y = \text{Cos}\, x$	$0 \leq x \leq \pi$	$-1 \leq y \leq 1$
$y = \text{Tan}\, x$	$\dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$	$\mathbb{R}$
$y = \text{Cot}\, x$	$0 < x < \pi$	$\mathbb{R}$
$y = \text{Sec}\, x$	$0 \leq x \leq \pi$, where $x \neq \dfrac{\pi}{2}$	$y \leq -1$ *and* $y \geq 1, \forall\, y \in \mathbb{R}$
$y = \text{Cosec}\, x$	$\dfrac{-\pi}{2} \leq x \leq \dfrac{\pi}{2}$, where $x \neq 0$	$y \leq -1$ *and* $y \geq 1, \forall\, y \in \mathbb{R}$

ii) Domains and Ranges of Inverse trigonometric functions

We know that the domain of inverse trigonometric function is the range of corresponding principal trigonometric function. Similarly, the range of inverse trigonometric function is the domain of corresponding principal trigonometric function.

Domain and ranges of inverse trigonometric functions are given below.

Function	Domain	Range
$y = \text{Sin}^{-1}x$	$-1 \le x \le 1$	$\frac{-\pi}{2} \le y \le \frac{\pi}{2}$
$y = \text{Cos}^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \text{Tan}^{-1}x$	$\mathbb{R}$	$\frac{-\pi}{2} < y < \frac{\pi}{2}$
$y = \text{Cot}^{-1}x$	$\mathbb{R}$	$0 < y < \pi$
$y = \text{Sec}^{-1}x$	$x \le -1$ *and* $x \ge 1,\ \forall\, x \in \mathbb{R}$	$0 \le y \le \pi$, where $y \ne \frac{\pi}{2}$
$y = \text{Cosec}^{-1}x$	$x \le -1$ *and* $x \ge 1, \forall x \in \mathbb{R}$	$\frac{-\pi}{2} \le y \le \frac{\pi}{2}$, where $y \ne 0$

Example 1. Find the value of (i) $\text{Cos}\left(\text{Sin}^{-1}\frac{1}{2}\right)$ (ii) $\text{Tan}\left(\text{Sin}^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right)$

Solution:

(i) Consider $\text{Sin}y = \frac{1}{2} \Rightarrow y = \frac{\pi}{6}$, $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ $\left(\because \text{Sin}\frac{\pi}{6} = \frac{1}{2}\right)$

Now, $\text{Cos}\left(\text{Sin}^{-1}\frac{1}{2}\right) = \text{Cos}\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(ii) Consider $\text{Sin}\,y = \frac{-\sqrt{3}}{2} \Rightarrow y = \frac{-\pi}{3}$, $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ $\left(\because \text{Sin}\left(\frac{-\pi}{3}\right) = \frac{-\sqrt{3}}{2}\right)$

Now, $\text{Tan}\left(\text{Sin}^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right) = \text{Tan}\left(\frac{-\pi}{3}\right) = -\sqrt{3}$

Example 2. Show that $\text{Cos}^{-1}\left(\frac{12}{13}\right) = \text{Sin}^{-1}\left(\frac{5}{13}\right)$

Solution: Let $\text{Cos}^{-1}\left(\frac{12}{13}\right) = \alpha \Rightarrow \text{Cos}\,\alpha = \frac{12}{13}$

Now, $$\text{Sin}\,\alpha = \pm\sqrt{1 - \text{Cos}^2\alpha} = \pm\sqrt{1 - \left(\frac{12}{13}\right)^2}$$

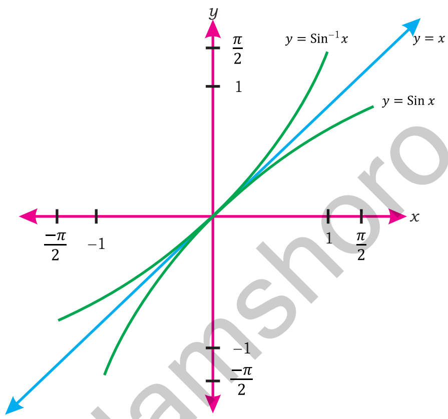
$$= \pm\sqrt{1 - \frac{144}{169}} = \pm\sqrt{\frac{25}{169}} = \pm\frac{5}{13}$$

Since $\sin\alpha$ is positive if $0 \le \alpha \le \frac{\pi}{2}$, in which cosine is also positive.

Thus, $\text{Sin}\,\alpha = \frac{5}{13} \Rightarrow \alpha = \text{Sin}^{-1}\left(\frac{5}{13}\right)$, Hence, $\text{Cos}^{-1}\left(\frac{12}{13}\right) = \text{Sin}^{-1}\left(\frac{5}{13}\right)$

12.4.3 Draw the graphs of inverse trigonometric functions

We draw the graphs of inverse trigonometric functions by plotting few points obtained by some corresponding values of x and y. In drawing graphs of inverse trigonometric functions, we keep in view the fact that each graph of inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$ as shown in Fig. 12.45.

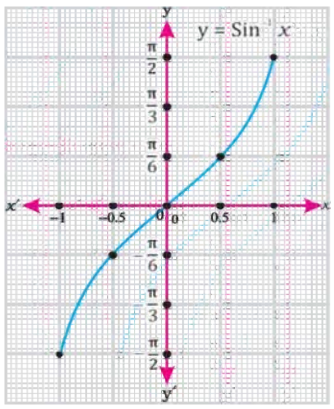


(Fig. 12.45)

Graph of Inverse Sine Function

Here some values of x and their corresponding values of $y = \text{Sin}^{-1}x$ are mentioned in the table and graph is plotted as shown in the Fig. 12.46.

x	-1	-0.5	0	0.5	1
$y = \text{Sin}^{-1}x$	$\frac{-\pi}{2}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$



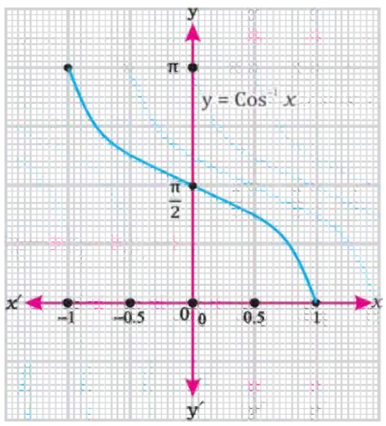
(Fig. 12.46)

Graph of Inverse Cosine Function

Some values of x and their corresponding values of $y = \text{Cos}^{-1}x$ are mentioned in the following table.

x	-1	-0.5	0	0.5	1
$y = \text{Cos}^{-1}x$	π	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	0

Graph is plotted as shown in Fig. 12.47.



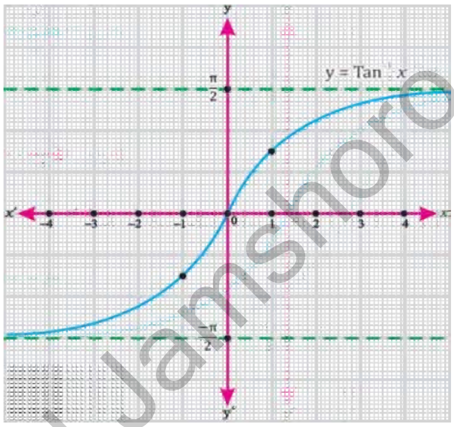
(Fig. 12.47)

Graph of Inverse Tangent Function

Following table shows, some values of x and their corresponding values of $y = \tan^{-1}x$.

x	$-\infty$	-1	0	1	$+\infty$
$y = \text{Tan}^{-1}x$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$

Graph of $y = \tan^{-1}x$ is plotted as shown in Fig. 12.48.



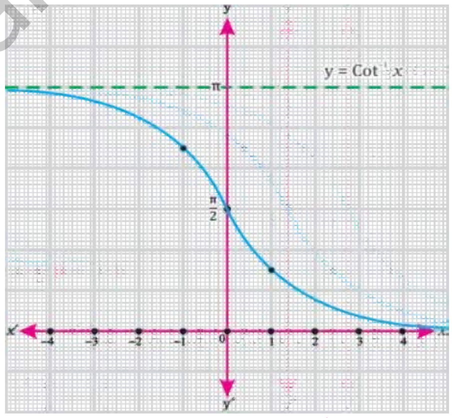
(Fig. 12.48)

Graph of Inverse Cotangent Function

For $y = \cot^{-1}x$, some corresponding values of x and y are mentioned in the following table.

x	$+\infty$	-1	0	1	$-\infty$
$y = \text{Cot}^{-1}x$	0	$\frac{3\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	π

Fig. 12.49 shows the graph of $y = \cot^{-1}x$



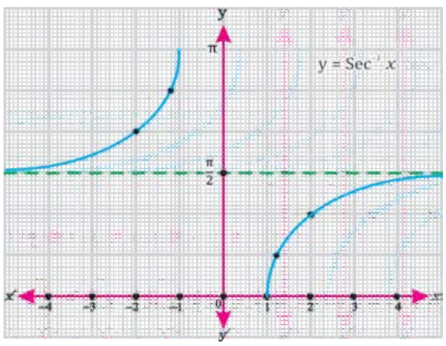
(Fig. 12.49)

Graph of Inverse Secant Function

some corresponding values of x and y are mentioned in the following table for $y = \text{Sec}^{-1}x$

x	1	1.15	2	-2	-1.15	-1	$\pm\infty$
$y = \text{Sec}^{-1}x$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{\pi}{2}$

Fig. 12.50 shows the graph of $y = \sec^{-1}x$



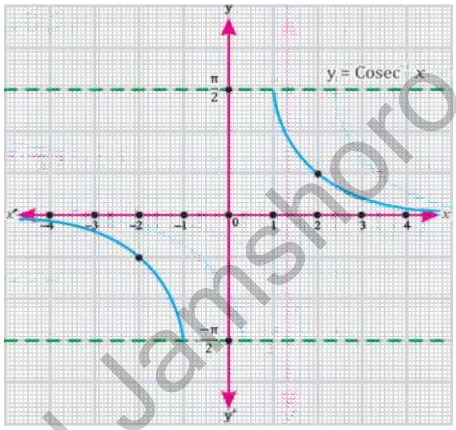
(Fig. 12.50)

Graph of Inverse Cosecant Function

Following table shows some corresponding values of x and y for $y = \text{Cosec}^{-1}x$

x	-2	-1	1	2	$\pm\infty$
$y = \text{Cosec}^{-1}x$	$\frac{-\pi}{6}$	$\frac{-\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{6}$	0

Fig. 12.51 shows the graph of $y = \text{Cosec}^{-1}x$



(Fig. 12.51)

12.4.4 Prove the addition and subtraction formulae of inverse trigonometric functions

1. $\text{Sin}^{-1}A + \text{Sin}^{-1}B = \text{Sin}^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$

Proof: Let $\text{Sin}^{-1}A = x \Rightarrow A = \text{Sin}\,x$ and $\text{Sin}^{-1}B = y \Rightarrow B = \text{Sin}\,y$

Now, $\text{Cos}\,x = \sqrt{1-\text{Sin}^2x} = \sqrt{1-A^2}$ $(\because \text{Sin}^2x + \text{Cos}^2x = 1)$

So, $\text{Cos}\,x = \sqrt{1-A^2}$ Similarly, $\text{Cos}\,y = \sqrt{1-B^2}$

Now, we know that $\text{Sin}(x+y) = \text{Sin}\,x\,\text{Cos}\,y + \text{Cos}\,x\,\text{Sin}\,y$

$$\text{Sin}(x+y) = A\sqrt{1-B^2} + B\sqrt{1-A^2}$$

$$x + y = \text{Sin}^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$$

i.e., $$\boxed{\text{Sin}^{-1}A + \text{Sin}^{-1}B = \text{Sin}^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)}$$

2. $\text{Sin}^{-1}A - \text{Sin}^{-1}B = \text{Sin}^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)$

Proof: Let $\text{Sin}^{-1}A = x \Rightarrow A = \text{Sin}\,x$ and $\text{Sin}^{-1}B = y \Rightarrow B = \text{Sin}\,y$

Now, $\text{Cos}\,x = \sqrt{1-\text{Sin}^2x} = \sqrt{1-A^2}$ $(\because \text{Sin}^2x + \text{Cos}^2x = 1)$

So, $\text{Cos}\,x = \sqrt{1-A^2}$ Similarly, $\text{Cos}\,y = \sqrt{1-B^2}$

Now, we know that $\text{Sin}(x-y) = \text{Sin}\,x\,\text{Cos}\,y - \text{Cos}\,x\,\text{sin}\,y$

$$\text{Sin}(x-y) = A\sqrt{1-B^2} - B\sqrt{1-A^2}$$

$$x - y = \text{Sin}^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)$$

i.e., $$\boxed{\text{Sin}^{-1}A - \text{Sin}^{-1}B = \text{Sin}^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)}$$

3. $\mathbf{Cos^{-1}A + Cos^{-1}B = Cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)}$

Proof: Let $\quad \text{Cos}^{-1}A = x \Rightarrow A = \text{Cos}\,x \quad$ and $\quad \text{Cos}^{-1}B = y \Rightarrow B = \text{Cos}\,y$

Now, $\quad \text{Sin}\,x = \sqrt{1-\text{Cos}^2 x} = \sqrt{1-A^2} \quad (\because \text{Sin}^2 x + \text{Cos}^2 x = 1)$

So, $\quad \text{Sin}\,x = \sqrt{1-A^2} \quad$ Similarly, $\quad \text{Sin}\,y = \sqrt{1-B^2}$

Now, we know that $\quad \text{Cos}(x+y) = \text{Cos}\,x\,\text{Cos}\,y - \text{Sin}\,x\,\text{Sin}\,y$

$$\text{Cos}(x+y) = AB - (\sqrt{1-A^2})(\sqrt{1-B^2})$$

$$x + y = \text{Cos}^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)$$

Hence, $\quad \boxed{\text{Cos}^{-1}A + \text{Cos}^{-1}B = \text{Cos}^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)}$

4. $\mathbf{Cos^{-1}A - Cos^{-1}B = Cos^{-1}\left(AB + \sqrt{(1-A^2)(1-B^2)}\right)}$

Proof: Let $\quad \text{Cos}^{-1}A = x \Rightarrow A = \text{Cos}\,x \quad$ and $\quad \text{Cos}^{-1}B = y \Rightarrow B = \text{Cos}\,y$

Now, $\quad \text{Sin}\,x = \sqrt{1-\text{Cos}^2 x} = \sqrt{1-A^2} \quad (\because \text{Sin}^2 x + \text{Cos}^2 x = 1)$

So, $\quad \text{Sin}\,x = \sqrt{1-A^2} \quad$ Similarly, $\quad \text{Cos}\,y = \sqrt{1-B^2}$

Now, we know that $\quad \text{Cos}(x-y) = \text{Cos}\,x\,\text{Cos}\,y + \text{Sin}\,x\,\text{Sin}\,y$

$$\text{Cos}(x-y) = AB + \sqrt{1-A^2}\sqrt{1-B^2}$$

$$x - y = \text{Cos}^{-1}\left(AB + \sqrt{(1-A^2)(1-B^2)}\right)$$

i.e.,, $\quad \boxed{\text{Cos}^{-1}A - \text{Cos}^{-1}B = \text{Cos}^{-1}\left(AB + \sqrt{(1-A^2)(1-B^2)}\right)}$

5. $\mathbf{Tan^{-1}A + Tan^{-1}B = Tan^{-1}\left(\frac{A+B}{1-AB}\right)}$

Proof: Let $\quad \text{Tan}^{-1}A = x \Rightarrow A = \text{Tan}\,x \quad$ and $\quad \text{Tan}^{-1}B = y \Rightarrow B = \text{Tan}\,y$

Now, we know that $\quad \text{Tan}(x+y) = \frac{\text{Tan}\,x + \text{Tan}\,y}{1 - \text{Tan}\,x\,\text{Tan}\,y}$

$$\text{Tan}(x+y) = \frac{A+B}{1-AB} \quad \Rightarrow \quad x + y = \text{Tan}^{-1}\left(\frac{A+B}{1-AB}\right)$$

i.e., $\quad \boxed{\text{Tan}^{-1}A + \text{Tan}^{-1}B = \text{Tan}^{-1}\left(\frac{A+B}{1-AB}\right)}$

6. $\mathbf{Tan^{-1}A - Tan^{-1}B = Tan^{-1}\left(\frac{A-B}{1+AB}\right)}$

Proof: Let $\quad \text{Tan}^{-1}A = x \Rightarrow A = \text{Tan}\,x \quad$ and $\quad \text{Tan}^{-1}B = y \Rightarrow B = \text{Tan}\,y$

Now, we know that $\quad \text{Tan}(x-y) = \frac{\text{Tan}\,x - \text{Tan}\,y}{1 + \text{Tan}\,x\,\text{Tan}\,y}$

$$\text{Tan}(x-y) = \frac{A-B}{1+AB} \Rightarrow \quad x - y = \text{Tan}^{-1}\left(\frac{A-B}{1+AB}\right)$$

i.e., $\quad \boxed{\text{Tan}^{-1}A - \text{Tan}^{-1}B = \text{Tan}^{-1}\left(\frac{A-B}{1+AB}\right)}$

12.4.5 Apply addition and subtraction formulae of inverse trigonometric functions to verify related identities

In this section, we apply addition and subtraction formulae of inverse trigonometric functions to prove or verify the related identities and some important relations as explained in the following examples.

Example 1. Prove that $\text{Tan}^{-1}(-x) - \text{Tan}^{-1}(-x) = 0$

Proof: L.H.S $= \text{Tan}^{-1}(-x) - \text{Tan}^{-1}(-x)$

$$= \text{Tan}^{-1}\left(\frac{-x-(-x)}{1+(-x)(-x)}\right) \left(\because \text{Tan}^{-1}A - \text{Tan}^{-1}B = \text{Tan}^{-1}\left(\frac{A-B}{1+AB}\right)\right.$$

$$= \text{Tan}^{-1}\left(\frac{-x+x}{1+x^2}\right)$$

$$= \text{Tan}^{-1}(0)$$

$$= 0$$

= R.H.S proved.

Example 2. Show that $\text{Sin}^{-1}\frac{3}{5} + \text{Sin}^{-1}\frac{4}{5} = \frac{\pi}{2}$

Solution:

By using the formula: $\text{Sin}^{-1}A + \text{Sin}^{-1}B = \text{Sin}^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$

We get $$\text{Sin}^{-1}\frac{3}{5} + \text{Sin}^{-1}\frac{4}{5} = \text{Sin}^{-1}\left(\frac{3}{5}\sqrt{1-\left(\frac{4}{5}\right)^2} + \frac{4}{5}\sqrt{1-\left(\frac{3}{5}\right)^2}\right)$$

$$= \text{Sin}^{-1}\left(\frac{3}{5}\sqrt{1-\frac{16}{25}} + \frac{4}{5}\sqrt{1-\frac{9}{25}}\right)$$

$$= \text{Sin}^{-1}\left(\frac{3}{5}\sqrt{\frac{9}{25}} + \frac{4}{5}\sqrt{\frac{16}{25}}\right)$$

$$= \text{Sin}^{-1}\left(\frac{9}{25} + \frac{16}{25}\right) = \text{Sin}^{-1}(1) = \frac{\pi}{2}$$

Hence, $\text{Sin}^{-1}\frac{3}{5} + \text{Sin}^{-1}\frac{4}{5} = \frac{\pi}{2}$

Example 3. Show that $\text{Cos}^{-1}\frac{4}{5} + \text{Cos}^{-1}\frac{3}{5} = \frac{\pi}{2}$

Solution:

By using the formula: $\text{Cos}^{-1}A + \text{Cos}^{-1}B = \text{Cos}^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)$

We get $$\text{Cos}^{-1}\frac{4}{5} + \text{Cos}^{-1}\frac{3}{5} = \text{Cos}^{-1}\left(\frac{4}{5}\cdot\frac{3}{5} - \sqrt{\left[1-\left(\frac{4}{5}\right)^2\right]\left[1-\left(\frac{3}{5}\right)^2\right]}\right)$$

$$= \text{Cos}^{-1}\left(\frac{12}{25} - \sqrt{\frac{9}{25} \times \frac{16}{25}}\right)$$

$$= \text{Cos}^{-1}\left(\frac{12}{25} - \frac{12}{25}\right) = \text{Cos}^{-1}(0) = \frac{\pi}{2}$$

Hence, $\text{Cos}^{-1}\frac{4}{5} + \text{Cos}^{-1}\frac{3}{5} = \frac{\pi}{2}$

Example 4. Show that $\text{Tan}^{-1}\left(\frac{1}{6}\right) - \text{Tan}^{-1}\left(\frac{1}{7}\right) = \text{Tan}^{-1}\left(\frac{1}{43}\right)$

Solution: $\text{L.H.S} = \text{Tan}^{-1}\left(\frac{1}{6}\right) - \text{Tan}^{-1}\left(\frac{1}{7}\right)$

By using the formula: $\text{Tan}^{-1}A - \text{Tan}^{-1}B = \text{Tan}^{-1}\left(\frac{A-B}{1+AB}\right)$

$$\text{Tan}^{-1}\left(\frac{1}{6}\right) - \text{Tan}^{-1}\left(\frac{1}{7}\right) = \text{Tan}^{-1}\left(\frac{\frac{1}{6} - \frac{1}{7}}{1 + \frac{1}{6}\cdot\frac{1}{7}}\right)$$

$$= \text{Tan}^{-1}\left(\frac{\frac{7-6}{42}}{1 + \frac{1}{42}}\right)$$

$$= \text{Tan}^{-1}\left(\frac{1}{42} \div \frac{43}{42}\right)$$

$$= \text{Tan}^{-1}\left(\frac{1}{42}\cdot\frac{42}{43}\right)$$

$$= \text{Tan}^{-1}\frac{1}{43} = \text{R.H.S, Hence shown.}$$

Exercise 12.4

1. Without using calculator, find the value of the following:

(i) $\text{Sin}^{-1}(1)$ (ii) $\text{Sin}^{-1}(-1)$ (iii) $\text{Cos}^{-1}\frac{\sqrt{3}}{2}$

(iv) $\text{Cot}^{-1}(-1)$ (v) $\text{Cos}^{-1}\left(\frac{1}{2}\right)$ (vi) $\text{Tan}^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

(vii) $\text{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (viii) $\text{Cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ (ix) $\text{Sin}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

(x) $\text{Cot}^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (xi) $\text{Cot}^{-1}(-1)$ (xii) $\text{Sec}^{-1}(2)$

(xiii) $\text{Sec}^{-1}(1)$ (xiv) $\text{Cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (xv) $\text{Cosec}^{-1}(1)$

2. Find the value of each of the following

(i) $\text{Cos}\left(\text{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$ (ii) $\text{Cosec}\left(\text{Cos}^{-1}\left(\frac{1}{2}\right)\right)$ (iii) $\text{Tan}\left(\text{Cos}^{-1}(-1)\right)$

(iv) $\text{Sec}\left(\text{Tan}^{-1}(1)\right)$ (v) $\text{Cot}\left(\text{Tan}^{-1}(-\sqrt{3})\right)$ (vi) $\text{Sin}\left(\text{Sec}^{-1}\sqrt{2}\right)$

3. Show that:

(i) $\text{Tan}^{-1}\left(\frac{12}{5}\right) = \text{Cot}^{-1}\left(\frac{5}{12}\right)$ (ii) $\text{Tan}^{-1}(\sqrt{3}) = \text{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(iii) $\text{Cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = \text{Cos}^{-1}\left(\frac{1}{2}\right)$ (iv) $\text{Cos}^{-1}\left(\frac{4}{9}\right) = \text{Cot}^{-1}\left(\frac{4}{\sqrt{65}}\right)$

4. Show that:

(i) $\text{Sin}^{-1}\left(\frac{77}{85}\right) + \text{Sin}^{-1}\left(\frac{3}{5}\right) = \text{Cos}^{-1}\left(\frac{87}{425}\right)$

(ii) $\text{Cos}^{-1}\left(\frac{2}{\sqrt{5}}\right) + \text{Tan}^{-1}\left(\frac{1}{3}\right) = \left(\frac{\pi}{4}\right)$

(iii) $\text{Cos}^{-1}\left(\frac{63}{65}\right) + 2\text{Tan}^{-1}\left(\frac{1}{5}\right) = \text{Cos}^{-1}\left(\frac{4}{5}\right)$

(iv) $\text{Cos}^{-1}\left(\frac{3}{5}\right) - \text{Cos}^{-1}\left(\frac{8}{17}\right) = \text{Cos}^{-1}\left(\frac{84}{85}\right)$

(v) $2\text{Tan}^{-1}\left(\frac{2}{3}\right) = \text{Cos}^{-1}\left(\frac{5}{13}\right)$

(vi) $\text{Sec}^{-1}\left(\frac{\sqrt{5}}{2}\right) + \text{Cot}^{-1}(3) = \frac{\pi}{4}$

(vii) $\text{Tan}^{-1}\left(\frac{3}{4}\right) + \text{Tan}^{-1}\left(\frac{3}{5}\right) - \text{Tan}^{-1}\left(\frac{8}{19}\right) = \frac{\pi}{4}$

(viii) $\text{Tan}^{-1}\left(\frac{1}{3}\right) + \text{Tan}^{-1}\left(\frac{1}{2}\right) = \text{Tan}^{-1}\left(\frac{1}{11}\right) + \text{Tan}^{-1}\left(\frac{5}{6}\right)$

5. Prove that:

(i) $\sin(\text{Cos}^{-1}y) = \sqrt{1-y^2}$ (ii) $\text{Tan}^{-1}(y) + \text{Tan}^{-1}(-y) = 0$

(iii) $\text{Cos}^{-1}y + \text{Cos}^{-1}(-y) = \pi$ (iv) $\sin(2\text{Cos}^{-1}y) = 2y\sqrt{1-y^2}$

12.5 Solving General Trigonometric Equations

12.5.1 Solve trigonometric equations and check their roots by substitution in the given trigonometric equations so as to discard extraneous roots

The method of solving trigonometric equations and checking their roots is explained with the help of the following example.

Example: Solve and verify the equation: $\sin\theta - \cos\theta = 1$, where $0 \le \theta \le 2\pi$

Solution: We have

$$\cos\theta - \sin\theta = 1 \quad \ldots(i)$$

Squaring both sides

$$(\cos\theta - \sin\theta)^2 = (1)^2$$

$\Rightarrow$ $$\cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta = 1$$

$$1 - 2\cos\theta\sin\theta = 1 \quad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$-2\cos\theta\sin\theta = 0$$

Either $\sin\theta = 0$ or $\cos\theta = 0$

if $\sin\theta = 0$ then, $\theta = 0$ or π

if $\cos\theta = 0$ then, $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

Verification:

For, $\theta = 0$, given equation becomes $\cos 0 - \sin 0 = 1$

or $1 = 1$ (True)

For, $\theta = \pi$, given equation becomes $\cos\pi - \sin\pi = 1$

or $-1 = 1$ (False)

For, $\theta = \frac{\pi}{2}$, given equation becomes $\cos\frac{\pi}{2} - \sin\frac{\pi}{2} = 1$

or $-1 = 1$ (False)

For, $\theta = \frac{3\pi}{2}$, given equation becomes $\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2} = 1$

$\Rightarrow$ or $1 = 1$ (True)

$\because$ $\theta = \pi$ and $\theta = \frac{\pi}{2}$ do not satisfy given equation.

$\because$ π and $\frac{\pi}{2}$ are extraneous roots.

Thus, solution set $= \left\{0, \frac{3\pi}{2}\right\}$.

12.5.2 Use the periods of trigonometric functions to find the solution of general trigonometric equations

The method of using the periods of trigonometric function to find solution of general trigonometric equation is explained through the following examples.

Example 1. Solve the equation: $1 - \cos x = \frac{1}{2}$

Solution: We have $1 - \cos x = \frac{1}{2}$

$\Rightarrow$ $\cos x = \frac{1}{2}$

$\Rightarrow$ $x = \cos^{-1}\frac{1}{2}$

$\Rightarrow$ $x = \frac{\pi}{3}, \frac{5\pi}{3}$ in particular ; $0 \le x \le 2\pi$

Verification:

For, $\theta = \frac{\pi}{3}$, we have $1 - \cos\frac{\pi}{3} = \frac{1}{2}$

$\Rightarrow$ $\frac{1}{2} = \frac{1}{2}$ (True)

For, $\theta = \frac{5\pi}{3}$, we have $1 - \cos\frac{5\pi}{3} = \frac{1}{2}$

$$\Rightarrow \quad 1-\frac{1}{2}=\frac{1}{2}$$

$$\text{or} \quad \frac{1}{2}=\frac{1}{2} \text{ (True)}$$

Since 2π is the period of $\cos x$,

Therefore, General solution is $\left\{\frac{\pi}{3}+2n\pi\right\} \cup \left\{\frac{5\pi}{3}+2n\pi\right\}$ where, $n \in Z$.

Example 2. Solve the equation: $3\tan^2 x - 1 = 0$

Solution: We have

$$3\tan^2 x - 1 = 0$$

$$\Rightarrow \quad 3\tan^2 x = 1 \quad \text{or} \quad \tan^2 x = \frac{1}{3}$$

$$\Rightarrow \quad \tan x = \pm\sqrt{\frac{1}{3}}$$

$$\text{or} \quad \tan x = \pm\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \tan x = \frac{1}{\sqrt{3}} \quad \text{and} \quad \tan x = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad \Rightarrow \quad x = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \quad x = \frac{\pi}{6} \quad \Rightarrow \quad x = -\frac{\pi}{6}, \text{ in particular}$$

$\because$ π is period of $\tan x$

$\therefore$ solution set is $= \left\{n\pi \pm \frac{\pi}{6}\right\}, \forall n \in Z$

Example 3. Solve the equation $\sqrt{3}\cot x - \operatorname{cosec} x - 1 = 0$

Solution: We have

$$\sqrt{3}\cot x - \operatorname{cosec} x - 1 = 0$$

$$\Rightarrow \quad \sqrt{3}\cot x = \operatorname{cosec} x + 1$$

Squaring both sides

$$\Rightarrow \quad 3\cot^2 x = \operatorname{cosec}^2 x + 2\operatorname{cosec} x + 1$$

$$\Rightarrow \quad 3(\operatorname{cosec}^2 x - 1) = \operatorname{cosec}^2 x + 2\operatorname{cosec} x + 1 \quad [\because 1+\cot^2 x = \operatorname{cosec}^2 x]$$

$$\Rightarrow \quad 2\operatorname{cosec}^2 x - 2co\sec x - 4 = 0$$

$$\Rightarrow \quad \operatorname{cosec}^2 x - \operatorname{cosec} x - 2 = 0$$

$$\Rightarrow \quad \operatorname{cosec}^2 x - 2\operatorname{cosec} x + \operatorname{cosec} x - 2 = 0$$

$$\Rightarrow \quad \operatorname{cosec} x(\operatorname{cosec} x - 2) + 1(\operatorname{cosec} x - 2) = 0$$

$$\Rightarrow \quad (\operatorname{cosec} x - 2)(\operatorname{cosec} x + 1) = 0$$

$$\Rightarrow \quad \operatorname{cosec} x - 2 = 0 \text{ or } \operatorname{cosec} x + 1 = 0$$

$$\text{or} \quad \operatorname{cosec} x = 2 \text{ or } \operatorname{cosec} x = -1$$

$$\Rightarrow \quad \sin x = \frac{1}{2} \text{ and } \sin x = -1$$

$$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6} \quad \Rightarrow \quad x = \frac{3\pi}{2}, -\frac{\pi}{2} \text{ in particular.}$$

Verification:

For, $\theta = \frac{\pi}{6}$, we have

$$\sqrt{3}\cot\frac{\pi}{6} - \operatorname{cosec}\frac{\pi}{6} - 1 = 0$$

$\Rightarrow$ $$3 - 2 - 1 = 0$$

$\Rightarrow$ $$0 = 0 \text{ (True)}$$

For, $\theta = \frac{5\pi}{6}$, we have

$$\sqrt{3}\cot\frac{5\pi}{6} - \operatorname{cosec}\frac{5\pi}{6} - 1 = 0$$

$\Rightarrow$ $$-3 - 2 - 1 = 0$$

$\Rightarrow$ $$-6 = 0 \text{ (False)}$$

For, $\theta = \frac{3\pi}{2}$, we have

$$\sqrt{3}\cot\frac{3\pi}{2} - \operatorname{cosec}\frac{3\pi}{2} - 1 = 0$$

$\Rightarrow$ $$0 + 1 - 1 = 0$$

$\Rightarrow$ $$0 = 0 \text{ (True)}$$

For, $\theta = \frac{-\pi}{2}$, we have

$$\sqrt{3}\cot\left(\frac{-\pi}{2}\right) - \operatorname{cosec}\left(\frac{-\pi}{2}\right) - 1 = 0$$

$\Rightarrow$ $$0 + 1 - 1 = 0$$

$\Rightarrow$ $$0 = 0 \text{ (True)}$$

Hence, solution set $= \left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{-\frac{\pi}{2} + 2n\pi\right\} \cup \left\{\frac{3\pi}{2} + 2n\pi\right\} \forall n \in Z$

Exercise 12.5

Find general solution of the following trigonometric equations.

1. $\cos x + \frac{1}{2} = 0$ **2.** $\tan x + \frac{1}{\sqrt{3}} = 0$ **3.** $\operatorname{cosec}\frac{3x}{2} + 2 = 0$

4. $\sec^2\theta - \frac{4}{3} = 0$ **5.** $4\sin^2 x = 1$ **6.** $\sec^2 x = 4$

7. $2\cos x + \sin^2 x = 1$ **8.** $4\cos^2 x - 8\sin x + 1 = 0$

9. $3\tan^2 x + 2\sqrt{3}\tan x + 1 = 0$ **10.** $(\operatorname{cosec} x + 2)(2\cos x - 1) = 0$

11. $\tan\theta = 2\sin\theta$ **12.** $4\sin^2\frac{x}{2} - 3 = 0$

13. $\cos 2x = \cos x$ **14.** $\sin 4\theta - \sin 2\theta = \cos 3\theta$

15. $\cos\theta + \cos 3\theta + \cos 5\theta = 0$ **16.** $\text{Sin}x + \sin 3x + \sin 5x + \sin 7x = 0$

17. $\operatorname{cosec}\theta = \sqrt{3} + \cot\theta$ **18.** $2\cos x + \sin^2 x - 1 = 0$

Review Exercise 12

1. Select correct answer.

i. If $y = \operatorname{Sin} x$, then domain is:

(a) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (b) $0 \le x \le \pi$ (c) $[0, x], x \ne \frac{\pi}{2}$ (d) None of these

ii. If $y = \operatorname{Cos} x$, then range is:

(a) $-1 \le y \le 1$ (b) R (c) $y \le -1$ or $y \ge 1$ (d) $y \le -1$ or $y \ge 1$

iii. If $y = \operatorname{Tan}^{-1} x$, then domain is:

(a) $-1 \le x \le 1$ (b) R (c) $x \ge -1$ or $x \le 1$ (d) $x \le -1$ or $x \ge 1$

iv. If $y = \operatorname{Sec}^{-1} x$, then range is:

(a) $0 \le y \le \pi,\ y \ne \frac{\pi}{2}$ (b) $-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$

(c) $0 < y < \pi$ (d) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

v. Inverse function exists if and only if the function is:

(a) bijective (b) (1-1) function

(c) onto function (d) into function

vi. $\operatorname{Cot}^{-1} x =$ ....

(a) $\frac{\pi}{2} - \tan^{-1} x$ (b) $\frac{\pi}{2} - \cot^{-1} x$ (c) $\frac{\pi}{2} + \tan^{-1} x$ (d) $\frac{\pi}{2} + \cot^{-1} x$

vii. $\operatorname{Sec}\left[\operatorname{Sin}^{-1}\left(-\frac{1}{2}\right)\right] =$ ....

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $-\frac{2}{\sqrt{3}}$

viii. $\operatorname{Cos}\left(\operatorname{Sin}^{-1}\frac{1}{\sqrt{2}}\right) =$...... (a) $\frac{2}{\sqrt{3}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

ix. $\cos^{-1}A + \cos^{-1}B =$ ....

(a) $\cos^{-1}\left(AB - \sqrt{(1 - A^2)(1 - B^2)}\right)$ (b) $\cos^{-1}\left(AB + \sqrt{(1 - A^2)(1 - B^2)}\right)$

(c) $\cos^{-1}\left(AB + \sqrt{(1 + A^2)(1 + B^2)}\right)$ (d) $\cos^{-1}\left(AB - \sqrt{(1 + A^2)(1 + B^2)}\right)$

x. $\tan^{-1}(-x) = \cdots$ (a) $-\tan^{-1}x$ (b) $\pi - \tan^{-1}x$ (c) $\cot^{-1}x$ (d) $\tan^{-1}x$

xi. $\operatorname{Tan}^{-1}(-1) =$--------- (a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{2}$

xii. If $\tan 2x = -1$, then solution in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is:

(a) $-\frac{\pi}{8}$ (b) $-\frac{\pi}{4}$ (c) $\frac{3\pi}{8}$ (d) $\frac{3\pi}{4}$

xiii. General solution of $4\sin x - 8 = 0$ is:

(a) $\{\pi + 2n\pi\}, n \in \mathbb{Z}$ (b) $\{\pi + n\pi\}, n \in \mathbb{Z}$

(c) $\{-\pi + n\pi\},\ n \in \mathbb{Z}$ (d) Not possible

xiv. Solution of equation: $2\,\text{Sin}\,x + \sqrt{3} = 0$ in 4th Quadrant is:

(a) $\frac{\pi}{3}$ (b) $\frac{-\pi}{3}$ (c) $\frac{-\pi}{6}$ (d) $\frac{11\pi}{6}$

xv. All trigonometric function are .... function:

(a) periodic (b) even (c) injective (d) bijective

xvi. General solution of every trigonometric equation consists of:

(a) one solution only (b) two solution

(c) infinitely many solutions (d) no real solution

xvii. The period of $\cos\theta$ is:

(a) $\frac{\pi}{2}$ (b) π (c) 2π (d) 4π

xviii. One solution of $\text{Sec}\,x = -2$ is:

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{4\pi}{5}$ (d) $\frac{-\pi}{3}$

xix. If $\cos\theta = -\frac{1}{2}$ and $\sin\theta = \frac{-\sqrt{3}}{2}$, then θ is:

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{5\pi}{3}$

xx. The period of $\sin\frac{x}{2}$ is

(a) 4π (b) π (c) 2π (d) 3π

xxi. The value of $\cos 5\pi$ is

(a) 0 (b) 1 (c) -1 (d) None of these

xxii. The period of $3\sec\frac{x}{3}$ is

(a) 5π (b) π (c) 2π (d) 6π

xxiii. The period of $7\tan(7x)$ is

(a) $\frac{7\pi}{3}$ (b) 7π (c) $\frac{\pi}{7}$ (d) π

2. Find the maximum and minimum value of the each of the following functions.

(i) $y = 5 - 7\cos\theta$ (ii) $y = 4 + 3\sin(2\theta - 5)$

3. By using graph, find the solution of the following equation.

$2\cos\theta - \theta = 0$

4. Without using calculator, show that:

(i) $\text{cosec}^{-1}\left(\frac{13}{12}\right) = \tan^{-1}\left(\frac{12}{15}\right)$ (ii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

5. Show that: $\cos^{-1}\left(\frac{7}{25}\right) - \cos^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{323}{325}\right)$.

6. Find the solution sets of the equation $\sec 3x = \sec x$.