Vectors

Unit 3

3.1 Vectors in Plane

There are many quantities in daily life which need direction for their specification. Let us take an example, in the army, when they are launching missiles, they first need the direction and distance as to know their target and the impact, it is going to cause. This kind of quantity which needs magnitude and direction, is called vector. Furthermore, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors.

3.1.1 Define a scalar and a vector

The quantity that is completely specified only by its magnitude with an appropriate unit, is called scalar quantity or scalar, for example mass, volume, area, distance, energy etc.

The quantity that has direction as well as magnitude with appropriate unit, is called vector quantity, or vector for example weight, force, displacement, velocity etc.

3.1.2 Give geometrical representation of a vector

A vector in a plane is represented by a directed line segment. In Fig 3.1, a vector AB is shown which is denoted as $\overrightarrow{AB}$ or $\vec{u}$. The endpoints of the segment are called the initial point and the terminal point of the vector.

The length of the segment represents the magnitude whereas the arrow indicates the direction of the vector.

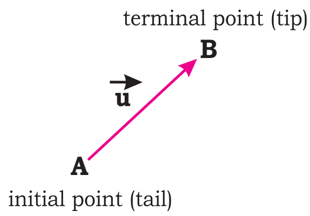


Fig. 3.1

3.1.3 Give the following fundamental definitions using geometrical representation:

i. magnitude of a vector,
ii. equal vectors
iii. negative of a vector,
iv. unit vector,
v. zero/null vector,
vi. position vector,
vii. parallel vectors,
viii. addition and subtraction of vectors,
ix. triangle, parallelogram and polygon laws of addition,
x. scalar multiplication

i. Magnitude of a vector

The length of a vector is called its magnitude. The magnitude of vector $\vec{v}$ is denoted by $|\vec{v}|$ or v as shown in Fig. 3.2.

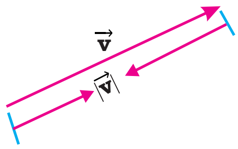


Fig 3.2

ii. Equal vectors

Two vectors $\vec{u}$ and $\vec{v}$ are said to be equal if they have the same magnitude and direction (Fig. 3.3). Symbolically, it is written as $\vec{u} = \vec{v}$.

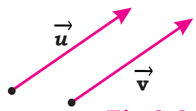


Fig 3.3

iii. Negative of a vector

The negative of a vector $\vec{v}$ is denoted by $-\vec{v}$. It has the same length as $\vec{v}$ but opposite in direction as shown in Fig. 3.4.

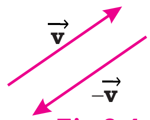


Fig 3.4

iv. Unit vector

A vector whose magnitude is 1 is called a unit vector. The unit vector of $\vec{v}$ is denoted by $\hat{v}$ as shown in Fig. 3.5.

Note: "$\hat{v}$" is pronounced as "v cap".

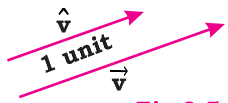


Fig 3.5

v. Zero or Null vector

A vector whose initial point and terminal point are same is called zero or null vector, it is denoted by $\vec{0}$.

vi. Position vector

A vector whose initial point is origin and terminal point is P, is called position vector of point P and it is denoted as $\overrightarrow{OP}$ as shown in Fig. 3.6. Its magnitude represents the distance of the point P from the origin.

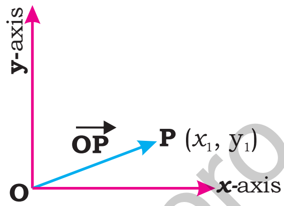


Fig 3.6

vii. Parallel vectors

Two vectors are said to be parallel if they have the same direction. In figure 3.7, $\vec{a}$ and $\vec{b}$ are parallel vectors, we denote as $\vec{a} \parallel \vec{b}$.

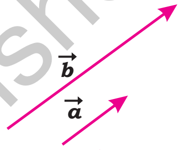


Fig. 3.7

viii. Addition and subtraction of vectors

Addition of vectors:

Consider two vectors $\vec{a}$ and $\vec{b}$ as shown in Fig. 3.8 (a). In order to add these two vectors, we first place $\vec{b}$ in such a way that its tail coincides with tip or head of $\vec{a}$ keeping its length and direction same, as shown in Fig. 3.8 (b).

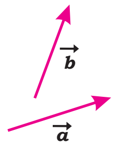


Fig. 3.8(a)

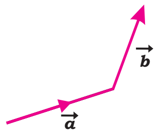


Fig. 3.8(b)

Now, the sum or resultant of $\vec{a}$ and $\vec{b}$, denoted as $\vec{a} + \vec{b}$, is the vector whose initial point is the tail of $\vec{a}$ and terminal point is the head of $\vec{b}$ as shown in Fig. 3.8(c).

The above method of addition of vectors is called Head and Tail rule.

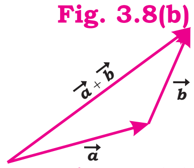


Fig. 3.8(c)

Subtraction of vectors:

Consider two vectors $\vec{a}$ and $\vec{b}$ as shown in Fig. 3.9 (a). The difference $\vec{a} - \vec{b}$ of these two vectors $\vec{a}$ and $\vec{b}$ is defined as the sum of the vectors $\vec{a}$ and $(-\vec{b})$, i.e., $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

Thus, in order to obtain the difference $\vec{a} - \vec{b}$, we just add $\vec{a}$ with the negative vector of $\vec{b}$ by Head and Tail rule as shown in Fig. 3.9(b).

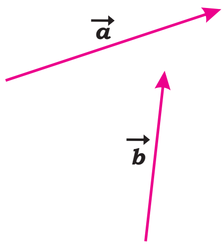


Fig. 3.9a

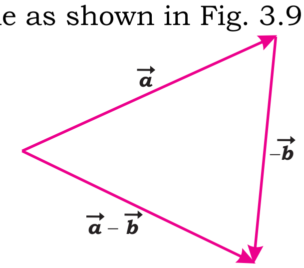


Fig. 3.9b

ix. Triangle, Parallelogram and Polygon Laws of Addition

Triangle law of addition:

Triangle law of vector addition states that when two vectors $\vec{a}$ and $\vec{b}$ are represented by two sides of a triangle in same order, then the third side $\vec{c}$ of the triangle represents the resultant vector of the vectors $\vec{a}$ and $\vec{b}$ taken in the opposite order, as shown in (Fig. 3.10).

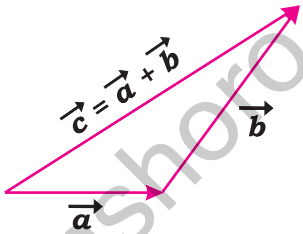


Fig. 3.10

$$\vec{c} = \vec{a} + \vec{b}$$

Parallelogram law of addition:

Parallelogram law of vector addition states that when two vectors $\vec{a}$ and $\vec{b}$ of same initial point represented by two adjacent sides of a parallelogram then the resultant of these vectors is $\vec{c} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$ represented by the diagonal of the parallelogram starting from the same initial point of the vectors $\vec{a}$ and $\vec{b}$, as shown in (Fig. 3.11).

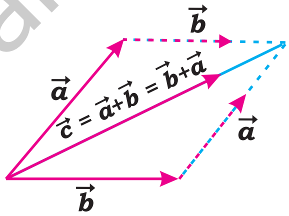


Fig. 3.11

Polygon law of addition:

Polygon law of vector addition states that if a number of vectors are represented in magnitude and direction by the consecutive sides of a polygon taken in the same order, then their resultant is represented by the closing side of the polygon taken in the opposite order. In Fig. 3.12, the sides of given polygon are represented by the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \dots, \vec{a}_{n-1}$, the closing side is represented by vector $\vec{a}_n$ which is the resultant.

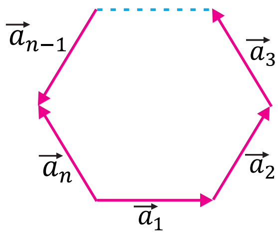


Fig. 3.12

i.e., $\vec{a}_n = \vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \dots + \vec{a}_{n-1}$.

Example: Prove pentagon law of vector addition.

Proof: **Pentagon law of vector addition:**

Pentagon law of vector addition states that, if four vectors are represented by four consecutive sides of a pentagon in same order then the last side is the resultant in opposite order.

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be the four vectors represented by the sides $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{DE}$ respectively of pentagon ABCDE as shown in Fig 3.13. By pentagon law of vectors, we have to prove that $\overrightarrow{AE} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$

In Δ ABC, by triangle law of addition

$\overrightarrow{AC} = \vec{a} + \vec{b}$ ... (i)

In Δ ACD, by triangle law of addition

$\overrightarrow{AD} = \overrightarrow{AC} + \vec{c}$

i.e., $\overrightarrow{AD} = \vec{a} + \vec{b} + \vec{c}$ ... (ii) (Using equation (i))

In Δ ADE, by triangle law of addition

$\overrightarrow{AE} = \overrightarrow{AD} + \vec{d}$

i.e., $\overrightarrow{AE} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$ (Using equation (ii)).

Hence proved.

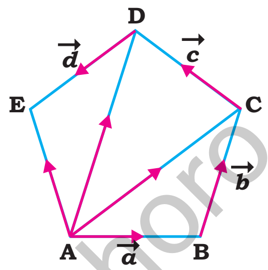


Fig. 3.13

x. Scalar Multiplication of vector

When we multiply a vector $\vec{v}$ by a scalar k, the result is a vector $k\vec{v}$. The scalar multiplication by a positive number other than 1 changes the magnitude of the vector but not its direction. The scalar multiplication by negative number other than -1 changes its magnitude and reverses its direction as shown in the Fig. 3.14, whereas multiplication by -1 only reverses the direction.

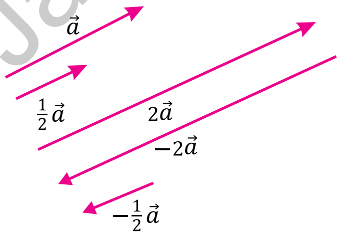


Fig 3.14

3.1.4 Represent a vector in a Cartesian plane by defining fundamental unit vectors *i* and *j*.

The algebra of vectors is based on representing each vector in terms of components parallel to the Cartesian coordinate axes and writing each component as an appropriate multiple of a unit vector along either of axes.

The unit vector along *x*-axis is the vector $\hat{\imath}$ determined by the directed line segment from (0,0) to (1,0). The basic or unit vector along y-axis is the vector $\hat{\jmath}$ determined by the directed line segment from (0,0) to (0,1).

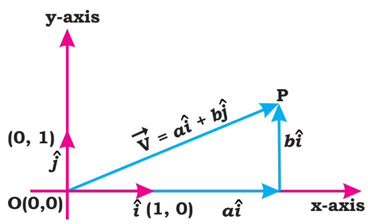


Fig 3.15

Figure 3.15 shows a vector $\vec{v} = \overrightarrow{OP}$ resolved into its $\hat{\imath}$ and $\hat{\jmath}$ components as the sum: $\vec{v} = a\hat{\imath} + b\hat{\jmath}$

Here $a\hat{\imath}$, where a is a scalar, represents a vector of length $|a|$, parallel to the x-axis, pointing to the right if, $a > 0$ and to the left if, $a < 0$. Similarly, $b\hat{\jmath}$ is a vector of length $|b|$, parallel to the y-axis, pointing up if, $b > 0$ and down wards if, $b < 0$.

The numbers 'a' and 'b' are the scalar components of $\vec{v}$ in the directions of $\hat{\imath}$ and $\hat{\jmath}$ respectively.

The vector $\vec{v} = a\hat{\imath} + b\hat{\jmath}$ can be written as $\begin{pmatrix} a \\ b \end{pmatrix}$ or $[a, b]$ or (a, b).

3.1.5 Recognize all above definitions using analytical representation

i. Magnitude of a vector

If a vector $\vec{v}$ is given in terms of components as $\vec{v} = a\hat{\imath} + b\hat{\jmath}$ then its magnitude is obtained by $|\vec{v}| = \sqrt{a^2 + b^2}$.

Example 1. If $\vec{v} = 7\hat{\imath} - \hat{\jmath}$, find $|\vec{v}|$.

Solution: Here $a = 7$ and $b = -1$

Now, $|\vec{v}| = \sqrt{(7)^2 + (-1)^2} = \sqrt{50}$ units

Let P, Q are the points in the xy-plane with Cartesian coordinates (x_1, y_1) and (x_2, y_2) respectively, then the vector from P to Q written as $\overrightarrow{PQ}$, is:

$$\overrightarrow{PQ} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath}$$

By using distance formula between two points.

Magnitude of $\overrightarrow{PQ}$ is: $|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example 2. If P(7,0) and Q(13,8) are two points in Cartesian plane then, find magnitude of vector $\overrightarrow{PQ}$

Solution: Here $(x_1, y_1) = (7, 0)$ and $(x_2, y_2) = (13, 8)$

Magnitude of vector $\overrightarrow{PQ} = |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(13 - 7)^2 + (8 - 0)^2}$$

$$= \sqrt{(6)^2 + (8)^2} = 10 \text{ units}$$

ii. Equal vectors

We know that two vectors $\vec{p}$ and $\vec{q}$ are said to be equal if they have same magnitude and the same direction.

In terms of components if, $\vec{p} = a\hat{\imath} + b\hat{\jmath}$ and $\vec{q} = a'\hat{\imath} + b'\hat{\jmath}$

then $\vec{p} = \vec{q}$ if and only if $a = a'$ and $b = b'$

Example: If $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $\overrightarrow{CD} = \frac{1}{2}\begin{pmatrix} 12 \\ 6 \end{pmatrix}$ then show that $\overrightarrow{AB} = \overrightarrow{CD}$

Solution:

Here, $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 6\hat{\imath} + 3\hat{\jmath}$ and $\overrightarrow{CD} = \frac{1}{2}\begin{pmatrix} 12 \\ 6 \end{pmatrix} = 6\hat{\imath} + 3\hat{\jmath}$.

$\because$ coefficient of $\hat{\imath}$ and $\hat{\jmath}$ are equal.

$\therefore$ $\overrightarrow{AB} = \overrightarrow{CD}$

iii. Negative of a vector

Negative of vector $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath}$ is $-\vec{a} = -a_1\hat{\imath} - a_2\hat{\jmath}$

Let, $\vec{a} = 6\hat{\imath} + \hat{\jmath}$ then $-\vec{a} = -(6\hat{\imath} + \hat{\jmath}) = -6\hat{\imath} - \hat{\jmath}$

iv. Unit Vector

We have already studied that a vector having magnitude 1 is called unit vector.

Let $\vec{v} = a\hat{i} + b\hat{j}$ then, $|\vec{v}| = \sqrt{a^2 + b^2}$

The unit vector of $\vec{v}$ is denoted by $\hat{v}$ and is obtained by

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{a\hat{i} + b\hat{j}}{\sqrt{a^2 + b^2}}$$

Example: If $\vec{v} = \hat{i} - 7\hat{j}$ then find $\hat{v}$.

Solution:

Here $a = 1, \quad b = -7$ then, $|\vec{v}| = \sqrt{(1)^2 + (-7)^2} = 5\sqrt{2}$ units

Now $\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\hat{i} - 7\hat{j}}{5\sqrt{2}} = \frac{1}{5\sqrt{2}}\hat{i} - \frac{7}{5\sqrt{2}}\hat{j}$

v. Null vector or Zero vector

A vector whose magnitude is zero is called Zero vector or Null vector. It is denoted by $\vec{0}$.

Zero vector in plane, in terms of component, is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

vi. Position vector

The vector from origin $\mathbf{O}(0,0)$ to a point $P(x, y)$ is vector $\overrightarrow{OP}$ and is called position vector of the point P.

The position vector of $P(x, y)$ is $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = x\hat{i} + y\hat{j}$

Example: Write down the position vector of P(3,4) and Q(5,2).

Solution: The position vector of P(3,4) is $\overrightarrow{OP} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\hat{i} + 4\hat{j}$.

and the position vector of Q(5,2) is $\overrightarrow{OQ} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 5\hat{i} + 2\hat{j}$.

vii. Parallel vectors

Two vectors $\vec{a}$ and $\vec{b}$ are parallel if there exists a non-zero real number k such that $\vec{b} = k\vec{a}$.

Note: (i) $\vec{b}$ has a magnitude $|k|$ times the magnitude of $\vec{a}$.
(ii) if $k < 0$ then $\vec{a}$ is antiparallel to $\vec{b}$.

Example: If $\vec{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 10 \\ -15 \end{pmatrix}$ are three vectors then show that, $\vec{a}$ is parallel to $\vec{b}$. Also show that $\vec{c}$ is antiparallel to $\vec{a}$.

Solution:

Here, $\vec{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$;

and $\vec{b} = \begin{pmatrix} -6 \\ 9 \end{pmatrix} = 3\begin{pmatrix} -2 \\ 3 \end{pmatrix} = 3\vec{a}$

$\because \quad \vec{b} = 3\vec{a}$

$\therefore \ \vec{b}$ is parallel to $\vec{a}$

and the magnitude of $\vec{b}$ is 3 times the magnitude of $\vec{a}$.

Also $\quad \vec{c} = \begin{pmatrix} 10 \\ -15 \end{pmatrix} = -5\begin{pmatrix} -2 \\ 3 \end{pmatrix} = -5\vec{a}$

$\because \quad \vec{c} = -5\vec{a}$

$\therefore \quad \vec{c}$ is antiparallel to $\vec{a}$.

viii. Addition and subtraction of vectors

Vector Addition:

If $\quad \vec{a} = a_1\hat{\imath} + a_2\hat{\jmath}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath}$

then, $\vec{a} + \vec{b} = (a_1 + b_1)\hat{\imath} + (a_2 + b_2)\,\hat{\jmath}$

For example, if $\vec{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\hat{\imath} + 4\hat{\jmath}$ and $\vec{b} = \begin{pmatrix} 8 \\ -5 \end{pmatrix} = 8\hat{\imath} - 5\hat{\jmath}$

then $\quad \vec{a} + \vec{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \end{pmatrix}$

or $\quad \vec{a} + \vec{b} = 3\hat{\imath} + 4\hat{\jmath} + 8\hat{\imath} - 5\hat{\jmath} = 11\hat{\imath} - \hat{\jmath}$

Vector Subtraction:

If $\vec{a} = a_1 i + a_2 j$ and $\vec{b} = b_1 i + b_2 j$

then $\quad \vec{a} - \vec{b} = (a_1 - b_1)\hat{\imath} + (a_2 - b_2)\,\hat{\jmath}$

For example, if $\vec{a} = \begin{pmatrix} 7 \\ -4 \end{pmatrix} = 7\hat{\imath} - 4\hat{\jmath}$ and $\vec{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} = 5\hat{\imath} + 3\hat{\jmath}$

then, $\quad \vec{a} - \vec{b} = \begin{pmatrix} 7 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$

or $\quad \vec{a} - \vec{b} = 7\hat{\imath} - 4\hat{\jmath} - 5\hat{\imath} - 3\hat{\jmath} = 2\hat{\imath} - 7\hat{\jmath}$

ix. Scalar Multiplication of vector

If k is a scalar and $\vec{v} = a\hat{\imath} + b\hat{\jmath}$ is a vector, then

$$k\vec{v} = k\,(a\hat{\imath} + b\hat{\jmath}) = (ka)\hat{\imath} + (kb)\hat{\jmath}$$

The length of $k\vec{v}$ is $|k|$ times the length of $\vec{v}$:

i.e., $\quad |k\vec{v}| = |(ka)\hat{\imath} + (kb)\hat{\jmath}|$

$$= \sqrt{(ka)^2 + (kb)^2} = \sqrt{k^2(a^2 + b^2)} = |k||\vec{v}|.$$

Example: If $\ k = -2$ and $\vec{v} = -3\hat{\imath} + 4\hat{\jmath}$, Find $|k\vec{v}|$.

Solution:

Here $\vec{v} = -3\hat{\imath} + 4\hat{\jmath}$, so $|\vec{v}| = |-3\hat{\imath} + 4\hat{\jmath}| = \sqrt{(-3)^2 + (4)^2} = 5$

Now, $\quad |k\vec{v}| = |-2\vec{v}| = |-2||\vec{v}|$

$= 2 \times 5$

$= 10$

3.1.6 Find a unit vector in the direction of another given vector

The unit vector in the direction of a given vector $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$ or $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$.

Example: Find unit vector in the direction of $\vec{v} = 3\hat{\imath} - 4\hat{\jmath}$ and verify.

Solution:

Here, $|\vec{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = 5$

Thus, unit vector in the direction of $\vec{v}$ is: $\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{3\hat{\imath} - 4\hat{\jmath}}{5} = \frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}$

Verification:

$$|\hat{v}| = \left|\frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}\right| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$$

Hence verified.

3.1.7 Find the position vector of a point which divides the line segment joining two points in a given ratio

Vector in terms of position vectors

Let X and Y be the end points of a vector XY. Now taking position vectors of points X and Y, we have

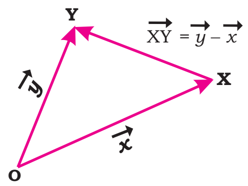
$\overrightarrow{OX} = \vec{x}$ and $\overrightarrow{OY} = \vec{y}$ as shown in the Fig. 3.16.

Now applying triangle law of vector addition, we get

$$\overrightarrow{OX} + \overrightarrow{XY} = \overrightarrow{OY}$$

$$\Rightarrow \quad \overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX}$$

$$\text{or} \quad \overrightarrow{XY} = \vec{y} - \vec{x}$$



(Fig. 3.16)

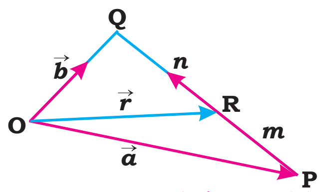
Thus, if $\vec{x}$ and $\vec{y}$ are position vectors of points X and Y then $\overrightarrow{XY} = \vec{y} - \vec{x}$.

Position vector of point of division of line segment

Consider two points P and Q having position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ with respect to origin O. The line segment connecting P and Q is divided by a point R lying on $\overrightarrow{PQ}$. The point R can divide the line segment PQ in two ways, viz. internally and externally as shown in Fig. 3.17 and Fig. 3.18.

(i) Line segment PQ is divided by R internally:

Consider the point R which divides the line segment PQ internally in the ratio $m\!:\!n$, given that m and n are positive scalar quantities (Fig. 3.17)



(Fig. 3.17)

So, $\frac{|\overrightarrow{PR}|}{|\overrightarrow{RQ}|} = \frac{m}{n}$

$$\Rightarrow \quad m|\overrightarrow{RQ}| = n|\overrightarrow{PR}|$$

$$\Rightarrow \quad m\overrightarrow{RQ} = n\overrightarrow{PR} \qquad \ldots\text{(i)}$$

Let $\vec{a}$, $\vec{b}$ and $\vec{r}$ be position vectors of points P, Q and R respectively.

i.e., $\overrightarrow{OP} = \vec{a}$, $\overrightarrow{OQ} = \vec{b}$ and $\overrightarrow{OR} = \vec{r}$

Using equation (i)

We have $m(\vec{b} - \vec{r}) = n(\vec{r} - \vec{a})$

$$\Rightarrow \quad m\vec{b} - m\vec{r} = n\vec{r} - n\vec{a}$$

$$\Rightarrow \quad \vec{r}(m+n) = m\vec{b} + n\vec{a}$$

$$\Rightarrow \quad \vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Hence, the position vector of point R dividing $\overline{PQ}$ internally in the ratio $m:n$ is given by:

$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

If R is the midpoint, then R divides the line segment PQ in the ratio 1:1, i.e., $m:n = 1:1$. Then position vector of point R will be as under

$$\vec{r} = \frac{\vec{b} + \vec{a}}{2}$$

(ii) Line segment PQ which is divided by R externally:

Consider the point R which divides the line segment PQ externally in the ratio $m:n$, given that m and n are positive scalar quantities (Fig. 3.18)

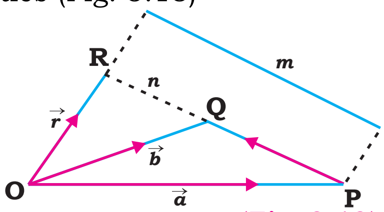
So, $\dfrac{|\overrightarrow{PR}|}{|\overrightarrow{QR}|} = \dfrac{m}{n}$

$$\Rightarrow \quad n|\overrightarrow{PR}| = m|\overrightarrow{QR}|$$

$$\Rightarrow \quad n\overrightarrow{PR} = m\overrightarrow{QR}$$

$$n\overrightarrow{PR} = -m\overrightarrow{RQ}$$

or $$m\overrightarrow{RQ} = -n\overrightarrow{PR} \qquad \ldots\text{(ii)}$$



(Fig. 3.18)

Let $\vec{a}$, $\vec{b}$ and $\vec{r}$ be position vectors of points P, Q and R respectively.

i.e., $\overrightarrow{OP} = \vec{a}$, $\overrightarrow{OQ} = \vec{b}$ and $\overrightarrow{OR} = \vec{r}$

Using equation (ii) $m(\vec{b} - \vec{r}) = -n(\vec{r} - \vec{a})$

$$\Rightarrow \quad m\vec{b} - m\vec{r} = -n\vec{r} + n\vec{a}$$

$$\Rightarrow \quad \vec{r}(m-n) = m\vec{b} - n\vec{a}$$

$$\Rightarrow \quad \vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

Hence, the position vector of point R dividing $\overrightarrow{PQ}$ externally in the ratio $m:n$ is given by:

$$\vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

Example: Find the position vector of the point dividing the join of two points with position vectors $3\hat{\imath} + 4\hat{\jmath}$ and $8\hat{\imath} - 5\hat{\jmath}$ in the ratio of 1:3.

Solution:

Let, $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$; $\vec{b} = 8\hat{\imath} - 5\hat{\jmath}$ and $m : n = 1:3$

Let $\vec{r}$ be the required position vector

$$\text{So,} \quad \vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$
$$= \frac{1(8\hat{\imath} - 5\hat{\jmath}) + 3(3\hat{\imath} + 4\hat{\jmath})}{1+3}$$
$$= \frac{8\hat{\imath} - 5\hat{\jmath} + 9\hat{\imath} + 12\hat{\jmath}}{4}$$
$$\vec{r} = \frac{17}{4}\hat{\imath} + \frac{7}{4}\hat{\jmath}$$

3.1.8 Use vectors to prove simple theorems of descriptive geometry

Theorem 1:

If the position vectors of points X and Y are $\vec{x}$ and $\vec{y}$ respectively and M be the midpoint of $\overline{XY}$ then position vector of point M is $\frac{\vec{x}+\vec{y}}{2}$

Proof:

Since $\vec{x}$ and $\vec{y}$ are the position vectors of X and Y respectively.

Therefore $\overrightarrow{OX} = \vec{x}$ and $\overrightarrow{OY} = \vec{y}$.

Let $\vec{m}$ be the position vector of midpoint M as shown in the Fig. 3.19.

So, $\vec{m} = \overrightarrow{OM}$

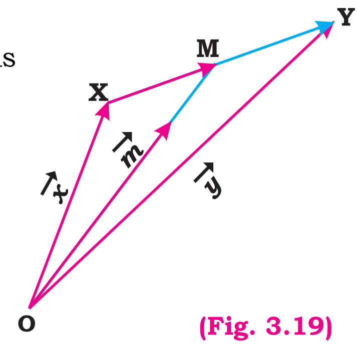
Now applying triangle law of vector addition, we get

$$\overrightarrow{OM} = \overrightarrow{OX} + \overrightarrow{XM} \qquad \ldots\text{(i)}$$

Also, $$\overrightarrow{OM} = \overrightarrow{OY} - \overrightarrow{MY} \qquad \ldots\text{(ii)}$$

By adding eq. (i) and (ii)

$$2\overrightarrow{OM} = \overrightarrow{OX} + \overrightarrow{XM} + \overrightarrow{OY} - \overrightarrow{MY}$$
$$2\vec{m} = \vec{x} + \overrightarrow{XM} + \vec{y} + \overrightarrow{YM}$$



(Fig. 3.19)

Since M is the mid-point of the line segment XY therefore, $\overrightarrow{XM}$ and $\overrightarrow{YM}$ are equal in magnitude but opposite in direction and cancel each other, so we get

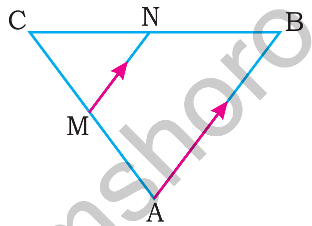
$$2\vec{m} = \vec{x} + \vec{y}$$

i.e., $\vec{m} = \frac{\vec{x}+\vec{y}}{2}$, Hence proved.

Theorem 2:

The line segment joining the midpoints of two sides of a triangle is parallel and half of the length of the third side.

Proof: Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of vertices A, B and C of Δ ABC respectively as shown in Fig 3.20.



(Fig. 3.20)

By using Theorem 1, the position vector of midpoint M of $\overline{AC}$ is $\dfrac{\vec{a}+\vec{c}}{2}$.

Similarly, the position vector of mid-point N of $\overline{BC}$ is $\dfrac{\vec{b}+\vec{c}}{2}$.

Now $\overrightarrow{MN} = \dfrac{\vec{b}+\vec{c}}{2} - \dfrac{\vec{a}+\vec{c}}{2} = \dfrac{1}{2}(\vec{b} - \vec{a}) = \dfrac{1}{2}\overrightarrow{AB}.$

So, $\quad \overrightarrow{MN} || \overrightarrow{AB}$ and $\overrightarrow{MN}$ is half of length of $\overrightarrow{AB}$.

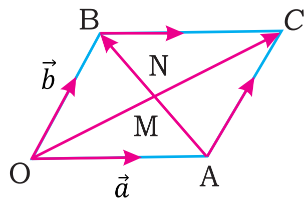
Hence proved.

Theorem 3:

The diagonals of a parallelogram bisect each other

Proof:

Consider a parallelogram OACB as shown in Fig 3.21, where $\vec{a}, \vec{b}$ be the position vectors of the vertices A and B respectively.



(Fig 3.21)

then $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

In ΔOAC, by triangle law of addition of vectors

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \vec{a} + \vec{b} \qquad [\because\ \overrightarrow{AC} = \overrightarrow{OB} = \vec{b}]$$

The midpoint M of the diagonal $\overrightarrow{OC}$ has the position vector

$$\vec{m} = \frac{\overrightarrow{OC}}{2} = \frac{\vec{a}+\vec{b}}{2} \qquad \ldots\text{(i)}$$

Using Theorem 1, the midpoint N of the diagonal $\overrightarrow{AB}$ has the position vector $\dfrac{\vec{a}+\vec{b}}{2}$.

$$\text{i.e.,} \qquad \vec{n} = \frac{\vec{a}+\vec{b}}{2} \qquad \ldots\text{(ii)}$$

From (i) and (ii) we have $\vec{m} = \vec{n}$

Hence the midpoints of both the diagonal are same

Thus, the diagonals of the parallelogram bisects each other. Hence proved.

Exercise 3.1

1. Classify the following quantities as scalars or vectors:

(i)	Force	(ii)	Speed	(iii)	Velocity
(iv)	Area	(v)	Acceleration	(vi)	Work
(vii)	Volume	(viii)	Weight	(ix)	Mass

2. Write the vector $\overrightarrow{PQ}$ in the form $x\hat{\imath} + y\hat{\jmath}$

(i) $P\,(4,6),\ Q(-4,5)$ (ii) $P\left(-\frac{5}{2},9\right), Q(10,8)$

(iii) $P(-3,6),\ Q\,(5,9)$ (iv) $P(-6,2), Q\,(0,-5)$

3. Find the following vectors if $\vec{p} = -5\hat{\imath} + 4\hat{\jmath}$, $\vec{q} = -3\hat{\imath} + 5\hat{\jmath}$ and $\vec{r} = -6\hat{\imath} + 7\hat{\jmath}$.

(i) $2\vec{p} + 3\vec{q}$ (ii) $3\vec{p} + \frac{1}{2}\vec{q} + \vec{r}$

(iii) $3\vec{p} + 5\vec{q} - 4\vec{r}$ (iv) $\frac{1}{2}\vec{p} - 2\vec{r} + 4\vec{q}$

4. $P(-2,5), Q\,(8,0), R\left(\frac{7}{2},6\right)$ and $S\,(-10,7)$ are the given points. Find the sum of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$.

5. Find unit vector in the direction of each of the following vector:

(i) $\vec{p} = -3\hat{\imath} + 8\hat{\jmath}$ (ii) $\vec{q} = 5\hat{\imath} - 9\hat{\jmath}$

(iii) $\vec{r} = 10\hat{\imath} - 12\hat{\jmath}$ (iv) $\vec{s} = 7\hat{\imath} + 5\hat{\jmath}$

6. Find the vectors in the direction of the following vectors with given magnitudes:

(i) $\vec{a} = 2\hat{\imath} - 3\hat{\jmath}$; magnitude 6 units

(ii) $\vec{b} = 10\hat{\imath} - 9\hat{\jmath}$; magnitude 7 units

(iii) $\vec{c} = 5\hat{\imath} + 6\hat{\jmath}$; magnitude 11 units

7. Find the values of unknowns if:

(i) $4\hat{\imath} + 5\hat{\jmath} = p\hat{\imath} + 5\hat{\jmath}$ (ii) $(5,7) = (5,-q)$

(iii) $\frac{7}{2}\hat{\imath} - \frac{3}{5}\hat{\jmath} = p\hat{\imath} - \frac{3}{\sqrt{25}}\hat{\jmath}$ (iv) $\begin{pmatrix} \frac{5}{2} \\ -7 \end{pmatrix} = \begin{pmatrix} p \\ -7 \end{pmatrix}$

8. Find the position vector of point dividing the join of A and B in the given ratio, where position vectors of A and B are given:

(i) $4\hat{\imath} - 5\hat{\jmath}$ and $2\hat{\imath} + 7\hat{\jmath}$ where ratio is $3:5$

(ii) $3\hat{\imath} + 5\hat{\jmath}$ and $11\hat{\imath} + 8\hat{\jmath}$ where ratio is $3:2$

(iii) $2\hat{\imath} - 3\hat{\jmath}$ and $2\hat{\imath} + 6\hat{\jmath}$ where ratio is $4:3$

9. Prove Hexagon law of vector addition.

10. If $A(3,-5), B(6,0)$ and $C(2,4)$ be vertices of a parallelogram, then using vectors find the coordinates of point D if:

(a) ABCD is the parallelogram (b) ADBC is the parallelogram

(c) ABDC is the parallelogram

11. Given that A is the point (1,3). $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are $\binom{4}{1}$ and $\binom{2}{3}$ respectively. Find the coordinates of vertices B, C and D of the parallelogram ABCD.

12. Using vectors, prove that:
   (i) If one pair of opposite sides of a quadrilateral are congruent and parallel then it is a parallelogram.
   (ii) The line segments joining the mid points of the opposite sides of a quadrilateral bisect each other.
   (iii) Medians of a triangle are concurrent.
   (iv) The midpoint of the hypotenuse of a right-angled triangle is equidistant from its vertices.

13. Given that the points (1,1), (5,4), (8,9) and (0,3) represent position vectors of A,B,C and D respectively. Show that ABCD is a trapezium.

14. A, B, C are the points $\vec{a}, \vec{b}$ and $2\vec{a} - \vec{b}$ respectively. D divides $\overrightarrow{AC}$ in 2:3 and E divides $\overrightarrow{BD}$ in 4:1. Find the position vector of E.

15. In a parallelogram ABCD, X is the mid-point of $\overrightarrow{AB}$ and Y divides $\overrightarrow{BC}$ in 1:2. Show that if Z divides $\overrightarrow{DX}$ in 6:1 then it also divides $\overrightarrow{AY}$ in 3:4.

3.2 Vectors in Space

In the previous section we studied vectors in plane, now we study vectors in three-dimensional space.

3.2.1 Recognize rectangular coordinate system in space

In three-dimensional space, the rectangular coordinate system or Cartesian coordinate system is based on three mutually perpendicular lines which are called coordinate axes, viz. the x – axis, y – axis and z – axis as shown in the Fig 3.22 (a).

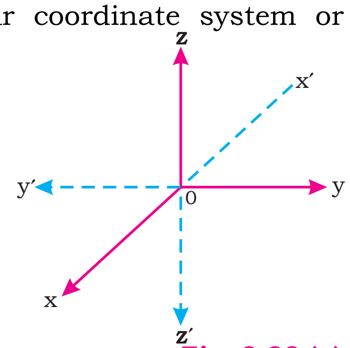


Fig. 3.22 (a)

These axes determine three mutually perpendicular planes which are called coordinate planes (yz –plane, zx –plane and xy –plane). The point of intersection of coordinate axes is called origin.

The origin along with the three axes, is said to form a Cartesian coordinate system for 3-space. It is of two types: left-handed and right-handed system.

A right-handed system has the property that when the fingers of the right-hand are curled from positive x-axis toward the positive y-axis, the thumb points roughly in the direction of the positive z-axis. The system which is not right-handed is called left-handed (Fig 3.22 (b). We shall follow the right-handed system.

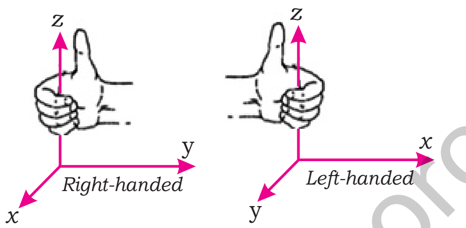


Fig. 3.22 (b)

To locate a point P in space, we draw perpendicular $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$ from P on yz-, zx- and xy -planes respectively. The lengths of these perpendiculars are taken as x, y and z. So coordinates of P are (x, y, z) as shown in Fig. 3.22(c).

It should be noted that

(i) for a point on x -axis, both $y = 0$ and $z = 0$

(ii) for a point on y -axis, both $z = 0$ and $x = 0$

(iii) for a point on z -axis, both $x = 0$ and $y = 0$

(iv) for a point on xy -plane, $z = 0$

(v) for a point on yz -plane, $x = 0$

(vi) for a point on zx -plane, $y = 0$

Furthermore, the coordinate planes divide the space into eight octants.

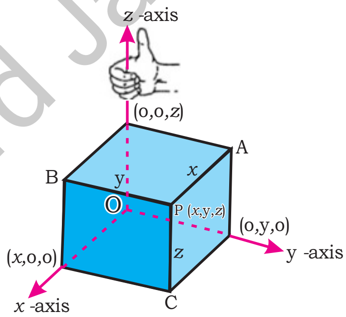


Fig. 3.22 (c)

3.2.2 Define unit vectors *i*, *j* and *k*

The vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the unit vectors in the direction of positive x, y, and z -axis, respectively. In terms of coordinates, we can write them as $\hat{\imath}$= (1,0,0), $\hat{\jmath}$ = (0,1,0), and $\hat{k}$ =(0,0,1).

3.2.3 Recognize components of a vector

In three-dimensional space, vector $\vec{A}$ has vector components, $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ along x, y and z-axis respectively. Vector $\vec{A}$ is equal to the sum of its three component Fig. 3.23.

i.e., $\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$

$\Rightarrow$ $\vec{A} = \vec{A}_x\hat{\imath} + \vec{A}_y\hat{\jmath} + \vec{A}_z\hat{k}$

Where, $\vec{A}_x = \vec{A}_x\hat{\imath}$, $\vec{A}_y = \vec{A}_y\hat{\jmath}$ and $\vec{A}_z = \vec{A}_z\hat{k}$.

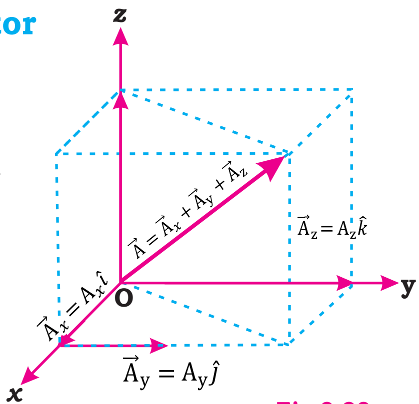


Fig 3.23

3.2.4 Give analytic representation of a vector

Let, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be a vector in 3- space.

Here, a_1, a_2 and a_3 are real numbers. The notation (a_1, a_2, a_3) or $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ or $[a_1, a_2, a_3]$ is called the analytic representation of the vector $\vec{a}$ and the numbers a_1, a_2 and a_3 are called the *x*,y and *z* components of the vector in space respectively.

Example: In the given triangle of vertices P, Q and R with its coordinates P(1, −1, 0), Q(2,1, −1) and R(−1, 1, 2) respectively. Find the components of the vectors $\overrightarrow{PQ}, \overrightarrow{QP}, \overrightarrow{PR}, \overrightarrow{RP}, \overrightarrow{QR}, \overrightarrow{RQ}$

Solution: Here $\overrightarrow{OP} = (1, -1, 0)$, $\overrightarrow{OQ} = (2, 1, -1)$ and $\overrightarrow{OR} = (-1, 1, 2)$ are the position vectors of points P, Q and R respectively.

Now,

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

i.e., $\overrightarrow{PQ} = (2-1, 1+1, -1-0) = (1, 2, -1)$

$\Rightarrow$ $\overrightarrow{QP} = (-1, -2, 1)$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$

i.e., $\overrightarrow{PR} = (-1-1, 1+1, 2-0) = (-2, 2, 2)$

$\Rightarrow$ $\overrightarrow{RP} = (2, -2, -2)$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$

i.e., $\overrightarrow{QR} = (-1-2, 1-1, 2+1) = (-3, 0, 3)$

$\Rightarrow$ $\overrightarrow{RQ} = (3, 0, -3)$

3.2.5 Find magnitude of a vector

We have already studied that the length of a vector is its magnitude.

If $\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$ is a vector in space, then its magnitude $|\vec{V}|$ is obtained by $|\vec{V}| = \sqrt{a^2 + b^2 + c^2}$.

Example 1. Find the magnitude of a vector $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$.

Solution: $\because$ $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\therefore$ $|\vec{a}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$ units.

Example 2. Find the values of t if the vectors $3\hat{i} + 5\hat{j} - 7\hat{k}$ and $5\hat{i} + 2\hat{j} - t\hat{k}$ have same magnitude.

Solution: Let $\vec{a} = 3\hat{i} + 5\hat{j} - 7\hat{k}$ and $\vec{b} = 5\hat{i} + 2\hat{j} - t\hat{k}$

As both vectors have same magnitude i.e., $|\vec{a}| = |\vec{b}|$

$$\Rightarrow \quad \sqrt{(3)^2 + (5)^2 + (-7)^2} = \sqrt{(5)^2 + (2)^2 + (-t)^2}$$

$$\Rightarrow \quad \sqrt{9 + 25 + 49} = \sqrt{25 + 4 + t^2}$$

$$t = \pm 3\sqrt{6}$$

Example 3. Find the vector of magnitude 6 in the direction of vector $5\hat{i} + 2\hat{j} - 7\hat{k}$.

Solution: Here, $\vec{a} = 5\hat{i} + 2\hat{j} - 7\hat{k}$

So, $6\hat{a}$ is the required vector.

Now,
$$6\hat{a} = 6\frac{\vec{a}}{|\vec{a}|}$$
$$= 6\frac{(5\hat{i} + 2\hat{j} - 7\hat{k})}{\sqrt{78}}$$
$$= \frac{30\hat{i}}{\sqrt{78}} + \frac{12\hat{j}}{\sqrt{78}} - \frac{42\hat{k}}{\sqrt{78}}$$

3.2.6 Repeat all fundamental definitions for vectors in space which, in the plane, have already been discussed

(i) Equal vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal vectors if they have same corresponding components.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then $\vec{a} = \vec{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

(ii) Negative of a vector

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then $-\vec{a} = -a_1\hat{i} - a_2\hat{j} - a_3\hat{k}$ is negative of the given vector $\vec{a}$.

(iii) Unit vector

We have already studied that any vector whose length or magnitude is 1 is a unit vector. The vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are basic unit vectors in the direction of x, y and z -axis respectively.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is a vector in space then its unit vector in the direction of $\vec{a}$ is $\hat{a}$ and defined as; $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{a_1\hat{i}+a_2\hat{j}+a_3\hat{k}}{\sqrt{a^2+b^2+c^2}}$

(iv) Zero or Null vector

We know that a vector with magnitude zero, is called zero vector. In space $\begin{pmatrix}0\\0\\0\end{pmatrix}$ is zero vector or null vector.

(v) Position vector

Position vector is the vector which represents the position of a point in space with respect to the origin O.

It also represents the distance and direction of the point from the origin. If P(x, y, z) is the point then its position vector is represented by

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

as shown in Fig. 3.24

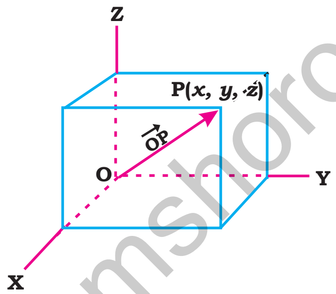


Fig. 3.24

Example:

The position vector of point P is $\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ and the position vector of point Q is $\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$, then find vector $\overrightarrow{PQ}$

Solution:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3-4 \\ -2-2 \\ 7+1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 8 \end{pmatrix}$$

Therefore, $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ -4 \\ 8 \end{pmatrix}$

(vi) Parallel vectors

Two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are parallel if, and only if, they are scalar multiples of one another:

i.e., if $\vec{a} = k\vec{b}$ or $\vec{b} = h\vec{a}$ where k and h are non-zero real numbers.

In other words, if two vectors are parallel, then the ratios of each of their corresponding components are same.

i.e., $$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

(vii) Addition and Subtraction of vectors

Addition of vectors

Two vectors $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are added by adding their corresponding components.

i.e., $$\vec{a} + \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

Subtraction of vectors

Difference of two vectors $\vec{a}$ and $\vec{b}$ is obtained by subtracting their corresponding components.

If $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are two vectors then the difference $\vec{a} - \vec{b}$ is obtained as;

$$\vec{a} - \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$$

Example: If $\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ then find

(i) $\vec{a} + \vec{b}$ (ii) $\vec{a} - \vec{b}$

Solution:

(i) $\vec{a} + \vec{b}$; $\vec{a} + \vec{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ 4 \end{pmatrix}$

(ii) $\vec{a} - \vec{b}$; $\vec{a} - \vec{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

(viii) Scalar Multiplication

If $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ be a vector and $k \in \mathbb{R}$ then; $k\vec{a} = k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$

Example: If $\vec{a} = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$ then find: (i) $-\vec{a}$ (ii) $3\vec{a}$

Solution:

(i) $-\vec{a} = \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix}$

(ii) $3\vec{a} = 3\begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -15 \\ 12 \end{pmatrix}$

Exercise 3.2

1. Let $\vec{u} = \hat{i} + 2\hat{j} + \hat{k}, \vec{v} = 3\hat{i} - 5\hat{j}$ and $\vec{w} = -8\hat{i} + 7\hat{j} - 2\hat{k}$. Find:

(i) $3\vec{u} + 2\,\vec{v}$ (ii) $\vec{v} - 2\,\vec{u}$ (iii) $\vec{u} - 3\,\vec{v} + 2\,\vec{w}$

(iv) $3\vec{u} + \vec{v} - \vec{w}$ (v) $-2\vec{u} + \frac{1}{2}\vec{v} - 3\,\vec{w}$

2. If $\vec{a} = -5\hat{i} + 3\hat{j} - 4\hat{k}$ then find:

(i) $-2\vec{a}$ (ii) $3|\vec{a}|$ (iii) $\frac{4}{5}\vec{a}$ (iv) $-\frac{1}{2}\vec{a}$

3. Let $\vec{A} = \hat{i} + 2\hat{j} - 3\hat{k}, \vec{B} = 2\hat{i} - \hat{j} + 2\hat{k}, \vec{C} = 3\hat{i} - \hat{j} + 5\hat{k}$. Find

(i) $\vec{A} - 2\vec{B}$ (ii) $3\vec{B} + 2\vec{C}$ (iii) $3\vec{A} - (2\vec{B} + \vec{C})$

4. Let $\vec{u} = \hat{i} - 3\hat{j} + 2\hat{k}, \vec{v} = \hat{i} + \hat{j}$ and $\vec{w} = 2\hat{i} + 2\hat{j} - 4\hat{k}$. Find

(i) $|\vec{u} + \vec{v} - \vec{w}|$ (ii) $|\vec{u}| + |\vec{v}|$ (iii) $\left|\frac{1}{|\vec{w}|}\vec{w}\right|$

5. **(a)** The position vector of point P is $\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$ and the position vector of point Q is $\begin{pmatrix} -2 \\ 4 \\ -6 \end{pmatrix}$. Find $4\overrightarrow{PQ}$ and $|5\overrightarrow{PQ}|$.

(b) If $\vec{a} = 3\hat{i} - 5\hat{j} + 2\hat{k}$, $\vec{b} = -\hat{i} - 2\hat{j} + 3\hat{k}$ then find

(i) $\hat{a}$ (ii) $\hat{b}$ (iii) $|\vec{a} + \vec{b}|$ (iv) $|\vec{a} - \vec{b}|$

6. Find vector A_1A_2 when:

(i) $A_1(0,0,0)$, $A_2(-2,5,1)$ (ii) $A_1(2,1,-3)$, $A_2(7,1,-3)$

(iii) $A_1(5,-2,1)$, $A_2(2,4,2)$

7. (i) Find the initial point of the vector $\vec{a} = (-2,1,2)$ if the terminal point `is $(4,0,-1)$.

(ii) Find the terminal point of the vector $\overrightarrow{A_1A_2} = (1,3,-3)$ if the initial point is $(-2,1,4)$.

8. The initial point of a vector $\vec{a}$ of magnitude 5 is $(1,-\sqrt{3},-5)$. Find k if the terminal point is:

(a) $(3,\sqrt{3},k)$ (b) $(2,-3\sqrt{3},-10k)$ (c) $(-2,k,-3)$

9. Find the coordinates of A, if $\overrightarrow{OA}$ is of length 6 units in the direction of $\overrightarrow{OB}$, where B is the point $(2,-1,4)$.

10. P,Q,R,S are the points with position vectors given by $(1,1,-1)$, $(1,-1,2)$, $(0,1,1)$ and $(2,1,0)$ respectively.

(i) Find $|\overrightarrow{PQ}|$ and $|\overrightarrow{QS}|$.

(ii) Find the position vector of the point which:

(a) divides $\overline{\text{QR}}$ internally in the ratio $3:2$

(b) divides $\overline{\text{PR}}$ externally in the ratio $3:2$

(iii) If X and Y are the midpoints of $\overline{\text{PR}}$ and $\overline{\text{RS}}$. Show that $\overrightarrow{\text{XY}} = \frac{1}{2}\overrightarrow{\text{PS}}$.

11. Find the vector $\overrightarrow{\text{OA}}$, where:

(i) $|\overrightarrow{\text{OA}}| = 6$ and $\overrightarrow{\text{OA}}$ is in the direction of the vector $(2, -3, 6)$.

(ii) $|\overrightarrow{\text{OA}}| = 2$ and $\overrightarrow{\text{OA}}$ is in the opposite direction of the vector $(8, 1, -4)$.

12. **(a)** Find the magnitude of a vector $\vec{a} = -2\hat{i} + 3\hat{j} - 4\hat{k}$.

(b) Find the value of t if the vectors $-3\hat{i} + 5\hat{j} - 6\hat{k}$ and $2\hat{i} - 3\hat{j} + t\hat{k}$ have the same magnitude.

(c) Find the vector whose magnitude is 5 times the length of $\vec{a}$ and is in the opposite direction of $\vec{a} = 4\hat{i} - 5\hat{j} + 6\hat{k}$.

(d) Express $\vec{a} = -2\hat{i} + 4\hat{j} - 6\hat{k}$ as a product of its magnitude and direction.

13. Find the length of the median through point O of the triangle OCD, if C and D are $(2, 7, -1)$ and $(4, 1, 2)$ respectively.

14. If S and T are the mid points of $\overline{\text{PR}}$ and $\overline{\text{QR}}$ show that $\overrightarrow{\text{ST}} = \frac{1}{2}\overrightarrow{\text{PQ}}$.

3.3 Properties of Vector Addition

3.3.1 State and prove

i. commutative law for vector addition
ii. associative law for vector addition

(i) Commutative law for vector addition

This law states that the sum of two vectors remains same irrespective of their order.

Let $\vec{A}$ and $\vec{B}$ are two vectors then by commutative law for vector addition

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

Consider two vectors $\vec{A}$ and $\vec{B}$, which represent two adjacent sides of a parallelogram OPQR as shown in Fig 3.25.

In ΔOPQ, by triangle law of vector addition

$$\vec{R} = \vec{B} + \vec{A} \qquad \ldots(i)$$

In ΔOQR, by triangle law of vector addition

$$\vec{R} = \vec{A} + \vec{B} \qquad \ldots(ii)$$

From (i) and (ii), we get $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.

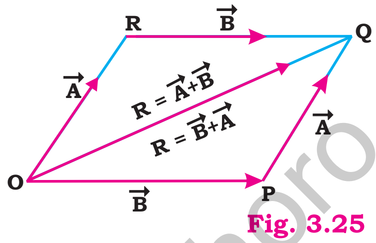


Fig. 3.25

(ii) Associative law for vector addition

This law states that the sum of three vectors remains same irrespective of their order or grouping in which they are arranged.

Let $\vec{A}, \vec{B}$ and $\vec{C}$ are three vectors then by the associative law of vector addition $\quad \vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

Consider three vectors $\vec{A}, \vec{B}$ $\vec{C}$ and their resultant vector $\vec{R}$ represented by line segments $\overline{OP}$, $\overline{PQ}$, $\overline{QR}$ and $\overline{OR}$ respectively as shown in Fig. 3.26.

We apply triangle law of vector addition to obtain $(\vec{A}+\vec{B})$ and $(\vec{B}+\vec{C})$.

Now in ΔOPR; $\quad \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$

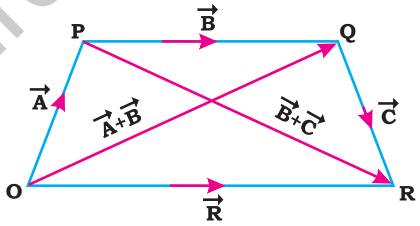
i.e., $\quad \vec{R} = \vec{A} + (\vec{B} + \vec{C}) \qquad \ldots(i)$

And in ΔOQR; $\quad \overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR}$

i.e., $\quad \vec{R} = (\vec{A} + \vec{B}) + \vec{C} \qquad \ldots(ii)$

From (i) and (ii)

We get $\quad \vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$



(Fig. 3.26)

Example: $\vec{A} = 5\hat{i} + \hat{j} - \hat{k}$, $\vec{B} = \hat{i} + 5\hat{j} - 7\hat{k}$ and $\vec{C} = \hat{i} + \hat{j} - \hat{k}$ are three vectors then verify associative law of vector addition.

Verification:

Associative law of vector addition is:

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

$$\text{L.H.S} = \vec{A} + (\vec{B} + \vec{C}) = (5\hat{i} + \hat{j} - \hat{k}) + [(\hat{i} + 5\hat{j} - 7\hat{k}) + (\hat{i} + \hat{j} - \hat{k})]$$
$$= (5\hat{i} + \hat{j} - \hat{k}) + (2\hat{i} + 6\hat{j} - 8\hat{k})$$
$$= (5 + 2)\hat{i} + (1 + 6)\hat{j} + (-1 - 8)\hat{k}$$
$$= 7\hat{i} + 7\hat{j} - 9\hat{k}$$

$$\text{R.H.S} = (\vec{A} + \vec{B}) + \vec{C} = [(5\hat{i} + \hat{j} - \hat{k}) + (\hat{i} + 5\hat{j} - 7\hat{k})] + (\hat{i} + \hat{j} - \hat{k})$$
$$= (6\hat{i} + 6\hat{j} - 8\hat{k}) + (\hat{i} + \hat{j} - \hat{k})$$
$$= (6 + 1)\hat{i} + (6 + 1)\hat{j} + (-8 - 1)\hat{k}$$
$$= 7\hat{i} + 7\hat{j} - 9\hat{k}$$

$\because$ L.H.S = R.H.S

$\therefore$ $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

Hence verified.

3.3.2 Prove that

(i) $\vec{0}$ is the identity for vector addition

(ii) $-\vec{A}$ is the inverse for $\vec{A}$.

(i) $\vec{0}$ is the Identity for vector addition

Let V is the set of all vectors in space, containing vector $\vec{0}$ which satisfies the property: $\vec{0}+\vec{v}=\vec{v}+\vec{0}=\vec{v}$, for any vector $\vec{v}\in V$.

Therefore, $\vec{0}$ is the identity for vector addition.

Proof: Let $\vec{v}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$

then there exist a vector $\vec{0}=0\hat{i}+0\hat{j}+0\hat{k}$

such that $\vec{v}+\vec{0}=(v_1\hat{i}+v_2\hat{j}+v_3\hat{k})+(0\hat{i}+0\hat{j}+0\hat{k})=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$

So, $\vec{v}+\vec{0}=\vec{v}$

and similarly we can prove that $\vec{0}+\vec{v}=\vec{v}$

Hence, $\vec{0}$ is the additive identity or identity for vector addition.

(ii) $-\vec{A}$ is the additive inverse for vector $\vec{A}$.

Let $\vec{A}$ be a vector, the additive inverse $-\vec{A}$ which is also called negative vector of $\vec{A}$, is a vector when added to $\vec{A}$ gives the additive identity

i.e $\vec{A}+(-\vec{A})=\vec{0}$

Proof: Let $\vec{A}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ is a vector in space.

then $-\vec{A}=-a_1\hat{i}-a_2\hat{j}-a_3\hat{k}$

Now, $\vec{A}+(-\vec{A})=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}+(-a_1\hat{i}-a_2\hat{j}-a_3\hat{k})$

$=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}-a_1\hat{i}-a_2\hat{j}-a_3\hat{k}=\vec{0}$

Thus, $-\vec{A}$ is the additive inverse of $\vec{A}$.

Example: If $\vec{A}=4\hat{i}+\hat{j}-\hat{k}$ and $\vec{B}=-4\hat{i}-\hat{j}+\hat{k}$ then show that $\vec{A}$ and $\vec{B}$ are additive inverses of each other.

Solution:

Here, $\vec{A}+\vec{B}=(4\hat{i}+\hat{j}-\hat{k})+(-4\hat{i}-\hat{j}+\hat{k})=(4-4)\hat{i}+(1-1)\hat{j}+(-1+1)\hat{k}=\vec{0}$

$\because$ $\vec{A}+\vec{B}=\vec{0}$

$\therefore$ $\vec{A}$ and $\vec{B}$ are additive inverses of each other. Hence shown.

3.4 Properties of Scalar Multiplication of vectors

3.4.1 State and verify

(i) **commutative law for scalar multiplication:** $m(\vec{a}) = (\vec{a})m$

(ii) **associative law for scalar multiplication:** $m\left(n(\vec{a})\right) = (mn)(\vec{a})$

(iii) **distributive laws for scalar multiplication:**

$m\left(\vec{a}+\vec{b}\right) = m\vec{a} + m\vec{b}$ **and** $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

(i) Commutative law for scalar multiplication

Let $\vec{a}$ be a vector and m is any scalar such that $m \in \mathbb{R}$

Then $$m(\vec{a}) = (\vec{a})m$$

is called commutative law for scalar multiplication.

Proof:

Let $\vec{a} = [x, y, z]$ be a vector

then $m(\vec{a}) = m[x, y, z] = [mx, my, mz]$

and $= [xm, ym, zm]$

$= [x, y, z]m = (\vec{a})m$

Thus, $m(\vec{a}) = (\vec{a})m$. Hence Proved.

(ii) Associative law for scalar multiplication

Let $\vec{a}$ be a vector and m, n are scalars such that $m, n \in \mathbb{R}$

Then $$m(n(\vec{a})) = (mn)\vec{a}$$

is called associative law for scalar multiplication.

Proof:

Let $\vec{a} = [x, y, z]$ be a vector

then $n(\vec{a}) = n[x, y, z] = [nx, ny, nz]$

Now, $m\left(n(\vec{a})\right) = m[nx, ny, nz]$

$= [(mn)x, (mn)y, (mn)z]$

$= (mn)[x, y, z]$

$= (mn)\vec{a}$

Thus, $m\left(n(\vec{a})\right) = (mn)\vec{a}$ Hence proved.

(iii) Distributive laws for scalar multiplication

Let $\vec{a}$ and $\vec{b}$ are two vectors in space and $m, n \in \mathbb{R}$ are scalars then;

(i) $m\left(\vec{a}+\vec{b}\right) = m\vec{a} + m\vec{b}$ (ii) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

are called distributive laws for scalar multiplication.

(i) $m\left(\vec{a}+\vec{b}\right) = m\vec{a} + m\vec{b}$

Proof: Let $\vec{a} = [x, y, z]$ and $\vec{b} = [u, v, w]$ are two vectors in space and $m \in \mathbb{R}$ is scalar.

Then, $m(\vec{a}+\vec{b}) = m\{[x,y,z]+[u,v,w]\} = m[x+u, y+v, z+w]$
$= [mx+mu, my+mv, mz+mw]$
$= [mx,my,mz]+[mu,mv,mw]$
$= m[x,y,z]+m[u,v,w] = m\vec{a}+m\vec{b}$

Thus, $m(\vec{a}+\vec{b}) = m\vec{a}+m\vec{b}$, Hence proved.

(ii) $(m+n)\vec{a} = m\vec{a}+n\vec{a}$

Proof:

Let, $\vec{a} = [x,y,z]$ is a vector in space and $m,n \in \mathbb{R}$ are scalars,

Then, $(m+n)\vec{a} = (m+n)[x,y,z] = [(m+n)x,(m+n)y,(m+n)z]$
$= [(mx+nx),(my+ny),(mz+nz)]$
$= [mx,my,mz]+[nx,ny,nz]$
$= m[x,y,z]+n[x,y,z]$
$= m\vec{a}+n\vec{a}$

Thus, $(m+n)\vec{a} = m\vec{a}+n\vec{a}$, Hence proved .

Exercise 3.3

1. If $\vec{a} = -5\hat{i}+3\hat{j}-4\hat{k}$ and $\vec{b} = 2\hat{i}+4\hat{j}+5\hat{k}$, then:
(i) verify: $\vec{0}+\vec{a} = \vec{a}+\vec{0} = \vec{a}$
(ii) find additive inverses of $\vec{a}$, $\vec{b}$ and $2\vec{a}+3\vec{b}$
(iii) verify commutative law for vector addition

2. Let $\vec{u} = 3\hat{i}+\hat{j}+5\hat{k}$, $\vec{v} = \hat{i}-2\hat{j}$ and $\vec{w} = -\hat{i}+3\hat{j}-\hat{k}$.
Verify associative law of vector addition.

3. Let $\vec{a} = 3\hat{i}+2\hat{j}+5\hat{k}$ is a vector and $m = 5$ is a scalar then verify commutative law of scalar multiplication.

4. Let $\vec{a} = 5\hat{i}+2\hat{j}-3\hat{k}$ is a vector and $m = 2$ and $n = 5$ then verify associative law of scalar multiplication.

5. Let $\vec{a} = 3\hat{i}+\hat{j}+5\hat{k}$ and $\vec{b} = -5\hat{i}+3\hat{j}-4\hat{k}$ are two vectors and $m = 3$ and $n = 7$ are scalars, prove that;
(i) $m(\vec{a}+\vec{b}) = m\vec{a}+m\vec{b}$ (ii) $(m+n)\vec{a} = m\vec{a}+n\vec{a}$

3.5 Dot or Scalar Product

3.5.1 Define dot or scalar product of two vectors and give its geometrical interpretation

Let $\vec{a}$ and $\vec{b}$ are two vectors with magnitudes $|\vec{a}| = a$ and $|\vec{b}| = b$ respectively then their dot or scalar product is denoted by $\vec{a}\cdot\vec{b}$ and is defined as: $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

or $= a\,b\cos\theta$

where θ is the angle between $\vec{a}$ and $\vec{b}$ in the counter clockwise direction with $0 \le \theta \le \pi$

When two non-zero vectors $\vec{a}$ and $\vec{b}$ are placed so that their initial points coincide, they form an angle θ of measure $0 \le \theta \le \pi$ in the counter clockwise direction, whereas $\vec{b}\cos\theta$ is the component of vector $\vec{b}$ along vector $\vec{a}$ and its length $b\cos\theta$ is called projection of $\vec{b}$ onto $\vec{a}$ as shown in Fig.3.27.

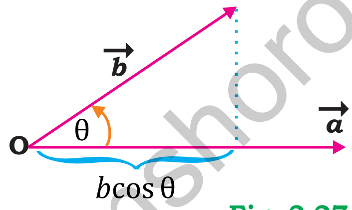


Fig. 3.27

Hence geometrically, the dot or scalar product of vector $\vec{a}$ and $\vec{b}$ is the product of magnitude of $\vec{a}$ with projection of $\vec{b}$ onto $\vec{a}$.

i.e., $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

3.5.2 Prove that:

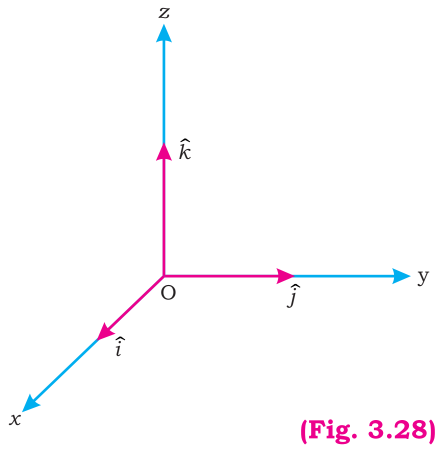
(i) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(ii) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Proof (i) : $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

Since $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along coordinate axes, so they are mutually perpendicular vectors as shown in Fig. 3.28. By definition of dot product

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$



(Fig. 3.28)

If $\vec{a} = \vec{b} = \hat{i}$, then $|\hat{i}| = 1$ and $\theta = 0°$

So, $\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos 0° = 1$ $[\because \cos 0° = 1]$

i.e., $\hat{i}\cdot\hat{i} = 1$

Similarly, we can prove that $\hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

Hence $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

Proof (ii): $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

If $\bar{a} = \hat{i}$ and $\bar{b} = \hat{j}$, then $\theta = 90°$ and $|\hat{i}| = |\hat{j}| = 1$

So, $\hat{i}\cdot\hat{j} = |\hat{i}||\hat{j}|\cos 90° = 0$ $[\because \cos 90° = 0]$

i.e., $\hat{i}\cdot\hat{j} = 0$

Similarly, we can prove that $\hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Hence $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.

3.5.3 Express dot product in terms of components

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be two non-zero vectors. The dot product in term of component is defined as $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Proof:

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Taking scalar product, we have

$$\vec{a} \cdot \vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) \cdot \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$
$$= a_1b_1(\hat{i}\cdot\hat{i}) + a_1b_2(\hat{i}\cdot\hat{j}) + a_1b_3(\hat{i}\cdot\hat{k}) + a_2b_1(\hat{j}\cdot\hat{i}) + a_2b_2(\hat{j}\cdot\hat{j}) + a_2b_3(\hat{j}\cdot\hat{k}) + a_3b_1(\hat{k}\cdot\hat{i}) + a_3b_2(\hat{k}\cdot\hat{j}) + a_3b_3(\hat{k}\cdot\hat{k})$$

Since $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Therefore, $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Hence, the dot product of two vectors is the sum of the product of their corresponding components.

Example: If $\vec{a} = \hat{i} - 4\hat{j} - 7\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + \hat{k}$ then find $\vec{a} \cdot \vec{b}$

Solution:

Here, $\vec{a} = \hat{i} - 4\hat{j} - 7\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + \hat{k}$

then $\vec{a} \cdot \vec{b} = (1)(3) + (-4)(-1) + (-7)(1)$
$= 3 + 4 - 7 = 0$

So, $\vec{a} \cdot \vec{b} = 0$

3.5.4 Find the condition for orthogonality of two vectors

Two non-zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular or orthogonal if the angle between them is $\frac{\pi}{2}$ as shown in Fig. 3.29.

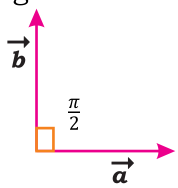


Fig. 3.29

For such vectors, we have $\vec{a}\cdot\vec{b} = 0$ because $\cos\frac{\pi}{2} = 0$

Conversely, if $\vec{a}$ and $\vec{b}$ are non-zero orthogonal vectors with $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 0$, then $\cos\theta = 0$ and $\theta = \cos^{-1}0 = \frac{\pi}{2}$.

Hence the condition of orthogonality of two vectors is $\vec{a}\cdot\vec{b} = 0$.

Example 1. If $\vec{a} = 3\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} - 2\hat{k}$, then show that they are orthogonal.

Solution: Here, $\vec{a} = 3\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} - 2\hat{k}$,

Now, $\vec{a}\cdot\vec{b} = \left(3\hat{i} + \hat{j} + \hat{k}\right)\cdot\left(\hat{i} - \hat{j} - 2\hat{k},\right) = (3)(1) + (1)(-1) + (1)(-2) = 0$

So, the vectors $\vec{a}$ and $\vec{b}$ are orthogonal.

Example 2. If $\vec{a} = 5\hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$, find $\vec{a} \cdot \vec{b}$.
Are these vectors perpendicular to each other?

Solution:

Here, $\vec{a} = 5\hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$

Now, $\vec{a} \cdot \vec{b} = (5\hat{i} + 3\hat{j} - 2\hat{k}) \cdot (3\hat{i} - 2\hat{j} + 7\hat{k}) = (5)(3) + (3)(-2) + (-2)(7)$

$= 15 - 6 - 14 = -5 \neq 0$

$\because$ $\vec{a} \cdot \vec{b} \neq 0$ the, vectors are not perpendicular.

3.5.5 Prove the commutative and distributive laws for dot product

(i) Commutative law for dot product

If $\vec{a}$ and $\vec{b}$ are two vectors, then commutative law for dot product is:

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors,

then $\vec{a} \cdot \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$= a_1b_1(\hat{i} \cdot \hat{i}) + a_2b_2(\hat{j} \cdot \hat{j}) + a_3b_3(\hat{k} \cdot \hat{k})$

$= a_1b_1 + a_2b_2 + a_3b_3$

similarly, $\vec{b} \cdot \vec{a} = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \cdot (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$= b_1\, a_1(\hat{i} \cdot \hat{i}) + b_2a_2(\hat{j} \cdot \hat{j}) + b_3a_3(\hat{k} \cdot \hat{k})$

$= a_1b_1 + a_2b_2 + a_3b_3$

Thus, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Hence dot product obeys the commutative law.

(ii) Distributive law for dot product

If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors, then distribution law for dot product is:

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are three vectors,

then $\vec{a} \cdot (\vec{b} + \vec{c}) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \{(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})\}$

$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k} + c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + c_1\hat{i} + b_2\hat{j} + c_2\hat{j} + b_3\hat{k} + c_3\hat{k})$

$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \{(b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}\}$

$= a_1(b_1 + c_1)(\hat{i} \cdot \hat{i}) + a_2(b_2 + c_2)(\hat{j} \cdot \hat{j}) + a_3(b_3 + c_3)(\hat{k} \cdot \hat{k})$

$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$

$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Thus, $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

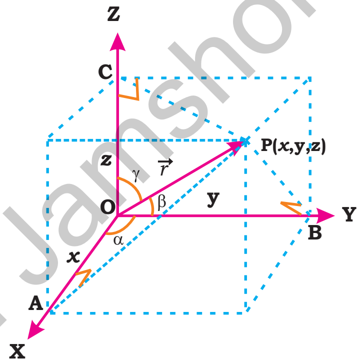
Hence dot product obeys the distributive law.

3.5.6 Explain direction cosines and direction ratios of a vector

Consider a vector $\overrightarrow{OP}$ passing through the origin O makes angles α, β and γ with the positive x, y and z -axis respectively as shown in Fig. 3.30. These angles are referred as the direction angles of $\overrightarrow{OP}$ and the cosines of these angles give us the direction cosines. These direction cosines are usually represented by l, m and n.

i.e., $\cos\alpha = \frac{x}{r} = l$, $\cos\beta = \frac{y}{r} = m$ and $\cos\gamma = \frac{z}{r} = n$,

where $|\overrightarrow{OP}| = r$



(Fig. 3.30)

These are called direction cosines.

Now, $x = r\cos\alpha$, $y = r\cos\beta$ and $z = r\cos\gamma$

These are called the direction ratios of vector $\overrightarrow{OP}$ or $\vec{r}$

where $r = |\overrightarrow{OP}| = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2+y^2+z^2}$

3.5.7 Prove that the sum of the squares of direction cosines is unity

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then the square of direction cosine of $\vec{r} = 1$.

i.e., $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Proof: $\because$ $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$\therefore$ $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$ ...(i)

We know that $\cos\alpha = \frac{x}{r}$, $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$

By squaring and adding above equations, we have

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1$$

Hence, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Example 1. Find the direction cosines of;

(i) $\vec{r} = \hat{i} + 3\hat{j} - 2\hat{k}$ (ii) $\overrightarrow{AB}$; A(1,3,2) and B(5,−7,0)

Solution:

(i) $\vec{r} = \hat{i} + 3\hat{j} - 2\hat{k}$

$\because \quad \vec{r} = \hat{i} + 3\hat{j} - 2\hat{k}$

$\therefore \quad |\vec{r}| = r = \sqrt{(1)^2 + (3)^2 + (-2)^2} = \sqrt{14}$

Now, $\cos\alpha = \dfrac{x}{r} = \dfrac{1}{\sqrt{14}}$;

$\cos\beta = \dfrac{y}{r} = \dfrac{3}{\sqrt{14}}$;

$\cos\gamma = \dfrac{z}{r} = \dfrac{-2}{\sqrt{14}}$

(ii) $\overrightarrow{AB}$; A(1,3,2) and B(5,−7,0)

Here, $\overrightarrow{AB} = (5-1)\hat{i} + (-7-3)\hat{j} + (0-2)\hat{k} = 4\hat{i} - 10\hat{j} - 2\hat{k}$

$|\overrightarrow{AB}| = \sqrt{(4)^2 + (-10)^2 + (-2)^2}\,| = \sqrt{120} = 2\sqrt{30}$

$\cos\alpha = \dfrac{x}{|\overrightarrow{AB}|} = \dfrac{4}{2\sqrt{30}} = \dfrac{2}{\sqrt{30}}$

$\cos\beta = \dfrac{y}{|\overrightarrow{AB}|} = \dfrac{-10}{2\sqrt{30}} = \dfrac{-5}{\sqrt{30}}$

and $\cos\gamma = \dfrac{z}{|AB|} = \dfrac{-2}{2\sqrt{30}} = \dfrac{-1}{\sqrt{30}}$

Example 2. If measures of two direction angles of a vector are $\alpha = 30°$ and $\beta = 60°$. Find measure of third direction angle.

Solution:

Let $\alpha = 30°$, $\beta = 60°$, $\gamma = ?$

We know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\cos^2 30° + \cos^2 60° + \cos^2\gamma = 1$$

$$\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (\cos\gamma)^2 = 1$$

$$(\cos\gamma)^2 = 1 - \frac{3}{4} - \frac{1}{4} = 0$$

$$\cos\gamma = 0$$

$$\gamma = \cos^{-1}(0) = 90° \text{ or } 270°$$

3.5.8 Use dot product to find the angle between two vectors

Let $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ are two vectors

By the definition of scalar product $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\Rightarrow \quad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{(a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k})\cdot(b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k})}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

$$\theta = \cos^{-1}\left(\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

Note:

(i) If $\theta = 0$ or π, then vectors $\vec{a}$ and $\vec{b}$ are collinear.

(ii) If $\theta = 90°$ or $\frac{\pi}{2}$, then vectors $\vec{a}$ and $\vec{b}$ are orthogonal.

Example 1. Find angle 'θ' between the vectors $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

Solution:

As, $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

So, $\vec{a}\cdot\vec{b} = (3\hat{\imath} - 2\hat{\jmath} + \hat{k})\cdot(\hat{\imath} - 2\hat{\jmath} + 3\hat{k}) = (3)(1) + (-2)(-2) + (1)(3) = 10$

Also, $|\vec{a}| = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{14}$

So, $\theta = \cos^{-1}\left(\frac{10}{\sqrt{14}\cdot\sqrt{14}}\right) = \cos^{-1}\left(\frac{10}{14}\right) = 44.4°$

Example 2. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and $\vec{a}\cdot\vec{b} = 1$.

Solution:

$\because \quad \vec{a}\cdot\vec{b} = 1,\ |\vec{a}| = 1$ and $|\vec{b}| = 2$

$\therefore \quad \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = 60°$ or $\frac{\pi}{3}$

3.5.9 Find the projection of a vector along another vector

Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$, and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are two vectors, then the projection of a vector $\vec{a}$ along a vector $\vec{b}$ is $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

Consider two vectors $\vec{a}$ and $\vec{b}$, the angle between them is α. Draw a perpendicular $\overline{AB}$ on vector $\vec{b}$ from $\vec{a}$. The projection of $\vec{a}$ along $\vec{b}$ is

$|\overrightarrow{OB}| = |\vec{a}| \cos\alpha$ as shown in Fig 3.31.

In ΔOBA, we have

$$\frac{|\overrightarrow{OB}|}{|\overrightarrow{OA}|} = \cos\alpha$$

$$\Rightarrow \quad |\overrightarrow{OB}| = |\overrightarrow{OA}| \cos\alpha = |\vec{a}| \cos\alpha = \frac{|\vec{a}||\vec{b}|\cos\alpha}{|\vec{b}|}$$

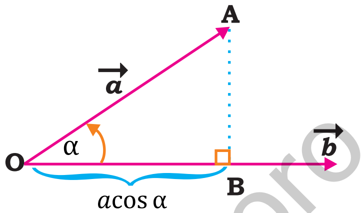


Fig. 3.31

By the definition of scalar product, we have

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\alpha$$

So, $\quad |\overrightarrow{OB}| = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

This shows that the projection of $\vec{a}$ along $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Similarly, in ΔOCB (Fig 3.32), we have

The projection of vector $\vec{b}$ along $\vec{a}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

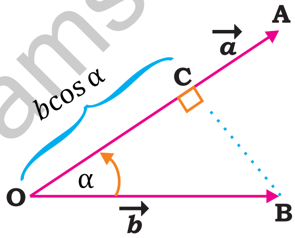


Fig.3.32

Example: Find the projection of $\hat{i}-2\hat{j}+3\hat{k}$ along $3\hat{i}-\hat{j}-5\hat{k}$

Solution:

Let $\bar{a} = \hat{i}-2\hat{j}+3\hat{k}$ and $\bar{b} = 3\hat{i}-\hat{j}-5\hat{k}$ then the projection of $\vec{a}$ along $\vec{b}$ is:

$$\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|} = \frac{(\hat{i}-2\hat{j}+3\hat{k})\cdot(3\hat{i}-\hat{j}-5\hat{k})}{\sqrt{(3)^2+(-1)^2+(-5)^2}} = \frac{(1)(3)+(-2)(-1)+(3)(-5)}{\sqrt{9+1+25}} = \frac{-10}{\sqrt{35}}$$

3.5.10 Find the work done by a constant force in moving an object along a given vector

If a constant force $\vec{F}$ acts on an object for some time and it covers displacement $\vec{d}$, then by the definition, the work done is

$$W = \vec{F}\cdot\vec{d} = Fd\cos\theta\ ,$$

where θ is the angle between the force $\vec{F}$ and displacement $\vec{d}$.

Example: A particle moving in space undergoes a displacement $\vec{d} = 2\hat{i}+2\hat{j}+2\hat{k}$ as a constant force $\vec{F} = 7\hat{i}+9\hat{j}-11\hat{k}$ acts on the particle. Calculate;

(i) magnitude of force and displacement.

(ii) work done by the force.

Solution: Here $\quad \vec{F} = 7\hat{i}+9\hat{j}-11\hat{k} \quad$ and $\quad \vec{d} = 2\hat{i}+2\hat{j}+2\hat{k}$

(i) Magnitude of Force and displacement:

$$|\vec{F}| = \sqrt{(7)^2+(9)^2+(-11)^2} = \sqrt{251}$$

$$|\vec{d}| = \sqrt{(2)^2 + (2)^2 + (2)^2} = 2\sqrt{3}$$

(ii) Work done by the force:

$$W = \vec{F} \cdot \vec{d} = (7\hat{i} + 9\hat{j} - 11\hat{k}) \cdot (2\hat{i} + 2\hat{j} + 2\hat{k}) = 14 + 18 - 22$$

$$W = 10 \text{ Joules}$$

3.5.11 Solve daily life problems based on work done

Example 1. A father pulls a child in a sleigh with a force of 150 N at an angle of 30 degrees with the ground. How much work is done over 2 km walk.

Solution: Work done = (Force).(Displacement)

$$= |\vec{F}|\,|\vec{d}|\cos\theta = (150)(2000)\cos 30^\circ$$

$$= 259807.6 \text{ Nm (Joules)}$$

Example 2. A box is dragged along the floor by the rope which makes an angle 60° with the horizontal. The force applied by the rope is 100 N and the work done by the force is 500 Joules. Find how much distance, the box is dragged.

Solution:

Here force and displacement are at an angle of 60°.

Now, Work done = Dot product of Force and Displacement

$$= |\vec{F}||\vec{d}|\cos\theta$$

$$500 = (100)(d)\cos 60^\circ$$

$$d = \frac{5}{0.5} = 10 \text{ meters.}$$

Exercise 3.4

1. If $\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$ and $\vec{c} = 5\hat{i} + \hat{j} - \hat{k}$ then find:

(i) $\vec{a} \cdot \vec{b}$ (ii) $\vec{a} \cdot (\vec{b} + \vec{c})$ (iii) $(2\vec{a} + 3\vec{b}) \cdot (\vec{a} - 2\vec{b})$

2. If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that:

(i) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (ii) Vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.

3. Check whether the following vectors are orthogonal or not;

(i) $\vec{a} = 2\hat{i} - 3\hat{j} + 2\hat{k}$, $\vec{b} = 5\hat{i} + 8\hat{j} + 7\hat{k}$

(ii) $\vec{a} = \hat{i} + 3\hat{j} - 4\hat{k}$, $\vec{b} = -2\hat{i} + \hat{j} + 3\hat{k}$

4. Verify the distributive laws for dot product for the following vectors.

(i) $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{c} = 5\hat{i} - \hat{j} + 2\hat{k}$

(ii) $\vec{a} = 3\hat{i} - 5\hat{j} + 7\hat{k}$, $\vec{b} = -4\hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 6\hat{i} + \hat{j} - 2\hat{k}$

5. Find the angle between the given vectors;

(i) $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$

(ii) $3\hat{i}+5\hat{j}-3\hat{k}$ and $\hat{i}-4\hat{j}+\hat{k}$

6. Find the direction cosines and direction angles for the following vectors.

(i) $2\hat{i}+\hat{j}-3\hat{k}$ (ii) $2\hat{i}-4\hat{j}+5\hat{k}$

(iii) $-3\hat{i}+4\hat{j}+5\hat{k}$

7. If α,β,γ are the direction angles of a vector then show that;
$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

8. If measures of two of the direction angles of a vector are 45° and 60°. Find measure of third angle.

9. Find the work done by the force $\vec{F}=7\hat{i}+9\hat{j}-11\hat{k}$ in moving an object along a straight line from (4,2,7) to (6,4,9).

10. Calculate the projection of $\bar{a}$ along $\bar{b}$ and projection of $\bar{b}$ along $\bar{a}$ when;

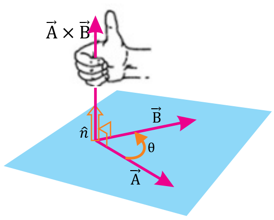
(i) $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}-5\hat{k}$

(ii) $\vec{a}=2\hat{i}+3\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$

11. Calculate the work done by a Force $\vec{F}=3\hat{i}+4\hat{j}+5\hat{k}$ in displacing a body from position B to position A along a straight path. The position vectors of A and B are respectively given as $\vec{r_A}=2i+5j-2k$ and $\vec{r_B}=7i+3j-5k$.

3.6 Cross or Vector Product

3.6.1 Define cross or vector product of two vectors and give its geometrical interpretation



(Fig. 3.33)

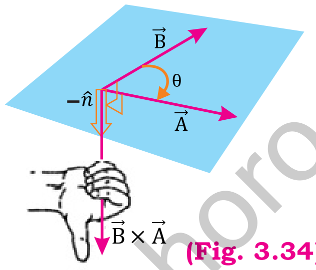
Let $\vec{A}$ and $\vec{B}$ are two non-zero vectors in space. Vectors $\vec{A}$ and $\vec{B}$ are not parallel as such these vectors determine a plane and $\hat{n}$ is a unit vector perpendicular to the plane as per the right-hand rule. The angle θ between $\vec{A}$ and $\vec{B}$ is taken positive as it is measured in anticlockwise direction as shown in Fig.3.33. So, the vector or cross product is defined as

$$\vec{A}\times\vec{B}=|\vec{A}||\vec{B}|\sin\theta\,\hat{n}, \quad (0\le\theta\le\pi)$$

Now, if we reverse the order of the vectors, it reverses the direction of the product. The angle is now from vector $\vec{B}$ to $\vec{A}$ and the unit normal vector is $-\hat{n}$ whereas $\vec{B}\times\vec{A}$ will be negative of $\vec{A}\times\vec{B}$ as shown in Fig. 3.34. Therefore the vector product $\vec{B} \times \vec{A}$ is:

$$\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B}) \quad \ldots(i)$$

Geometrically, $\vec{A} \times \vec{B}$ is a vector whose length is numerically equal to the area of the parallelogram determined by $\vec{A}$ and $\vec{B}$. We will prove it in section 3.6.4.



(Fig. 3.34)

3.6.2 Prove that:

i. $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$,

ii. $\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath} = \hat{k}$,

iii. $\hat{\jmath} \times \hat{k} = -\hat{k} \times \hat{\jmath} = \hat{\imath}$,

iv. $\hat{k} \times \hat{\imath} = -\hat{\imath} \times \hat{k} = \hat{\jmath}$.

When we apply the definition of cross or vector product of two vectors to calculate the pair wise cross products of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, we find

i. $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$

ii. $\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath} = \hat{k}$

iii. $\hat{\jmath} \times \hat{k} = -\hat{k} \times \hat{\jmath} = \hat{\imath}$

iv. $\hat{k} \times \hat{\imath} = -\hat{\imath} \times \hat{k} = \hat{\jmath}$

Proof:

(i) $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$

As the angle between two same vectors is always zero i.e., $\theta = 0°$, so their cross product is calculated as;

$$\hat{\imath} \times \hat{\imath} = |\hat{\imath}||\hat{\imath}| \sin 0° \, \hat{n}$$

$$\hat{\imath} \times \hat{\imath} = \vec{0} \qquad (\because |\hat{\imath}| = 1 \text{ and } \sin 0° = 0)$$

Similarly, $\hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$

(ii) $\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath} = \hat{k}$

Since $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are unit vectors along co-ordinate axes, so these are mutually orthogonal vectors.

By definition of cross product of two vectors;

$$\hat{\imath} \times \hat{\jmath} = |\hat{\imath}||\hat{\jmath}| \sin 90° \, \hat{n}$$

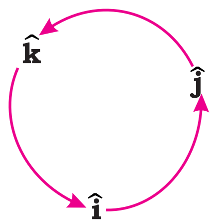
$$= \hat{n} \qquad [\because |\hat{\imath}| = |\hat{\jmath}| = 1 \text{ and } \sin 90° = 1]$$

$$= \hat{k} \qquad (\because \hat{n} = \hat{k} \text{ in this case})$$

Also, $\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath} = \hat{k}$ (Using (i) of section 3.6.1)

Similarly,

(iii) $\hat{\jmath} \times \hat{k} = -\hat{k} \times \hat{\jmath} = \hat{\imath}$



(Fig. 3.35)

(iv) $\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$

Fig. 3.35 helps to remember these results. In anticlockwise direction

$$\hat{i} \times \hat{j} = \hat{k}\,, \hat{j} \times \hat{k} = \hat{i} \text{ and } \hat{k} \times \hat{i} = \hat{j}$$

Whereas, in clockwise direction

$$\hat{j} \times \hat{i} = -\hat{k}\,, \hat{i} \times \hat{k} = -\hat{j} \text{ and } \hat{k} \times \hat{j} = -\hat{i}.$$

3.6.3 Express cross product in terms of components

If $\quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $\quad \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then

$$\begin{aligned}
\vec{a} \times \vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\
&= a_1b_2(\hat{i} \times \hat{j}) + a_1b_3(\hat{i} \times \hat{k}) + a_2b_1(\hat{j} \times \hat{i}) + a_2b_3(\hat{j} \times \hat{k}) + a_3b_1(\hat{k} \times \hat{i}) + a_3b_2(\hat{k} \times \hat{j}) \\
&= a_1b_2\hat{k} - a_1b_3\hat{j} - a_2b_1\hat{k} + a_2b_3\hat{i} + a_3b_1\hat{j} - a_3b_2\hat{i} \\
&= a_2b_3\hat{i} - a_3b_2\hat{i} - a_1b_3\hat{j} + a_3b_1\hat{j} + a_1b_2\hat{k} - a_2b_1\hat{k} \\
&= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}
\end{aligned}$$

This is the cross product of two vectors in terms of components and it can be written in determinant form as:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: If $\vec{a} = -2\hat{i} + 5\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} + \hat{k}$ then find $\bar{a} \times \bar{b}$.

Solution: Here, $\vec{a} = -2\hat{i} + 5\hat{j} - 3\hat{k}$, and $\vec{b} = 3\hat{i} + 2\hat{j} + \hat{k}$

We know that;

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 5 & -3 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \hat{i}(5+6) - \hat{j}(-2+9) + \hat{k}(-4-15)$$

$$= 11\hat{i} - 7\hat{j} - 19\hat{k}$$

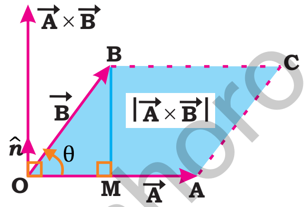
3.6.4 Prove that the magnitude of $\vec{A} \times \vec{B}$ represents the area of a parallelogram with adjacent sides $\vec{A}$ and $\vec{B}$

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector in the direction given by the right-hand rule, however, its magnitude is equal to the area of the parallelogram that is determined by $\vec{A}$ and $\vec{B}$ as shown in Fig 3.36.

i.e., $|\vec{A} \times \vec{B}|$ = Area of the parallelogram determined by $\vec{A}$ and $\vec{B}$.

Proof:

Let $\vec{A}$ and $\vec{B}$ are two non-zero vectors in space as the adjacent sides of the parallelogram OACB and θ is the angle between them measured from $\vec{A}$ to $\vec{B}$ as shown in Fig 3.36. The cross product of the vectors is defined by the formula



(Fig. 3.36)

$$\vec{A} \times \vec{B} = (|\vec{A}||\vec{B}|\sin\theta)\hat{n}$$

where, $\hat{n}$ is the unit vector in the direction of $\vec{A} \times \vec{B}$.

Magnitude of vector $\vec{A}$ is the base of the parallelogram. Draw a perpendicular from head of vector $\vec{B}$ to the base of $\vec{A}$. Now, $|\overline{BM}| = |\vec{B}|\sin\theta$, is the height of the parallelogram.

From elementary geometry, we have

Area of the parallelogram = base x height

$$= |\vec{A}||\vec{B}|\sin\theta$$

Hence, $\quad |\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin\theta$ = Area of the parallelogram

Area of Triangle

If $\vec{A}$ and $\vec{B}$ are two vectors as sides of triangle, also as adjacent sides of parallelogram and $|\vec{A}|, |\vec{B}|$ represent the length of adjacent sides of a parallelogram.

From elementary geometry

$$\text{Area of triangle} = \frac{1}{2}\ (\text{area of parallelogram})$$

$$= \frac{1}{2}|\vec{A} \times \vec{B}|$$

Example 1. Find the area of parallelogram with two adjacent sides represented by vectors.

$\hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $2\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$

Solution: Let $\vec{A} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{B} = 2\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$

Here, $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -5 & 2 \end{vmatrix} = \hat{\imath}(-2+15) - \hat{\jmath}(2-6) + \hat{k}(-5+2) = 13\hat{\imath} + 4\hat{\jmath} - 3\hat{k}$

Now $|\vec{A} \times \vec{B}| = \sqrt{(13)^2 + (4)^2 + (-3)^2} = \sqrt{194} = 13.9$ square units.

Hence, area of the parallelogram is 13.9 square units.

Example 2. Find the area of triangle with vertices A(3,5,7), B(3,7,9) and C(5,3,2).

Solution: Here, $\overrightarrow{AB} = (3-3)\hat{\imath} + (7-5)\hat{\jmath} + (9-7)\hat{k} = 2\hat{\jmath} + 2\hat{k}$

$$\overrightarrow{AC} = (5-3)\hat{i} + (3-5)\hat{j} + (2-7)\hat{k} = 2\hat{i} - 2\hat{j} - 5\hat{k}$$

Now, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 2 \\ 2 & -2 & -5 \end{vmatrix} = \hat{i}(-10+4) - \hat{j}(0-4) + \hat{k}(0-4) = -6\hat{i} + 4\hat{j} - 4\hat{k}$

So, $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-6)^2 + (4)^2 + (-4)^2} = \sqrt{68} = 2\sqrt{17}$

Area of triangle $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{2\sqrt{17}}{2} = \sqrt{17}$ square units.

3.6.5 Find the condition for parallelism of two non-zero vectors

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors which are parallel vectors then the angle θ between them is zero or π.

We know that the cross product $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta° \hat{n}$, which is zero at θ = 0 or π.

i.e., Two non-zero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

This is the condition of parallelism of two vectors $\vec{a}$ and $\vec{b}$.

Example: Let $\vec{A} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{B} = 9\hat{i} - 3\hat{j} + 6\hat{k}$ are two vectors. Show that they are parallel.

Solution:

As non-zero vectors $\vec{A}$ and $\vec{B}$ are parallel if and only if $\vec{A} \times \vec{B} = \vec{0}$

Now, $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 9 & -3 & 6 \end{vmatrix} = \hat{i}(-6+6) - \hat{j}(18-18) + \hat{k}(-9+9) = \vec{0}$

Hence, $\vec{A}$ and $\vec{B}$ are parallel.

3.6.6 Prove that $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$

Let $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then, from the definition of cross product

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

By using the property of determinant

$$\vec{A} \times \vec{B} = -\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \; ; \; R_{23}$$

Thus, $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$

Example: If $\vec{A} = 3\hat{i} - \hat{j} + \hat{k}$ and $\vec{B} = 3\hat{i} + 5\hat{j} - \hat{k}$ then verify that

$$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$$

Solution: We have $\vec{A} = 3\hat{i} - \hat{j} + \hat{k}$ and $\vec{B} = 3\hat{i} + 5\hat{j} - \hat{k}$

Now, $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 3 & 5 & -1 \end{vmatrix} = \hat{i}(1-5) - \hat{j}(-3-3) + \hat{k}(15+3) = -4\hat{i} + 6\hat{j} + 18\hat{k}$

and $\vec{B} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & -1 \\ 3 & -1 & 1 \end{vmatrix} = \hat{i}(5-1) - \hat{j}(3+3) + \hat{k}(-3-15)$

$$= -\left(-4\hat{i} + 6\hat{j} + 18\hat{k}\right) = -(\vec{A} \times \vec{B})$$

Thus, $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$ Hence verified.

3.6.7 Prove the distributive laws for cross product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors, then distributive law is defined as:

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

Proof:

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

then, $\vec{b} + \vec{c} = \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k} + c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\right)$

$$= (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$$

and $\vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, using property of determinant

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Hence, the distributive property holds for the cross product.

Example: If $\vec{a} = \hat{i} - 2\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} - 2\hat{j}$

then verify that; $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Verification:

Here $\vec{a} = \hat{i} - 2\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} - 2\hat{j}$

$$\vec{b} + \vec{c} = \left(2\hat{i} + \hat{j} - 3\hat{k}\right) + (3\hat{i} - 2\hat{j}) = 5\hat{i} - \hat{j} - 3\hat{k}$$

and $\vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 5 & -1 & -3 \end{vmatrix} = \hat{i}(0-2) - \hat{j}(-3+10) + \hat{k}(-1-0)$

$$= -2\hat{i} - 7\hat{j} - \hat{k}$$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 2 & 1 & -3 \end{vmatrix} = \hat{i}(0+2) - \hat{j}(-3+4) + \hat{k}(1-0)$

$$= 2\hat{i} - \hat{j} + \hat{k}$$

and $\quad \vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 3 & -2 & 0 \end{vmatrix} = \hat{i}(0-4) - \hat{j}(0+6) + \hat{k}(-2-0)$

$$= -4\hat{i} - 6\hat{j} - 2\hat{k}$$

$$\bar{a} \times \bar{b} + \bar{a} \times \bar{c} = 2\hat{i} - \hat{j} + \hat{k} - 4\hat{i} - 6\hat{j} - 2\hat{k} = -2\hat{i} - 7\hat{j} - \hat{k}$$

Thus, $\quad \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

3.6.8 Use cross product to find the angle between two vectors

Let $\vec{a}$ and $\vec{b}$ are two non-zero vectors then by definition of cross product;

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta^\circ \hat{n}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta^\circ$$

We can find the angle θ from: $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right)$$

Example: If $\vec{a} = 3\hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + 7\hat{j} - 7\hat{k}$ then find angle between $\vec{a}$ and $\vec{b}$.

Solution:

Here, $\vec{a} = 3\hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + 7\hat{j} - 7\hat{k}$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 0 \\ 1 & 7 & -7 \end{vmatrix} = \hat{i}(7-0) - \hat{j}(-21-0) + \hat{k}(21+1) = 7\hat{i} + 21\hat{j} + 22\hat{k}$

So, $\quad |\vec{a} \times \vec{b}| = \sqrt{(7)^2 + (21)^2 + (22)^2} = = \sqrt{49 + 441 + 484} = \sqrt{974}$

$$|\vec{a}| = \sqrt{(3)^2 + (-1)^2 + (0)^2} = \sqrt{10}$$

and $\quad |\vec{b}| = \sqrt{(1)^2 + (7)^2 + (-7)^2} = \sqrt{99}$

Now, $\quad \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{974}}{\sqrt{10}\sqrt{99}} = 0.99$

$$\theta = \sin^{-1}(0.99) = 82^\circ \text{ approximately.}$$

3.6.9 Find the vector moment of a given force about a given point

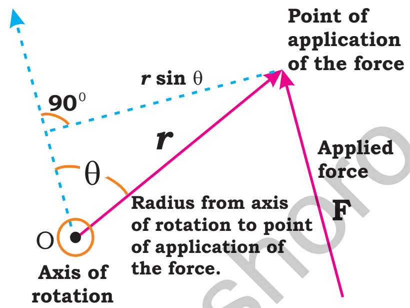
The moment M_o or torque of a force $\vec{F}$ about a point O or axis of rotation, is defined as the vector product of $\vec{r}$ and $\vec{F}$.

where, $\vec{r}$ is the radius vector or position vector, from point O of the axis of rotation to the point of application of the force $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ as shown in Fig. 3.37.

Mathematically, it can be written as

$$M_o = \vec{r} \times \vec{F}$$

Here, $|\vec{r}|\sin\theta$ is the perpendicular distance from the axis of rotation to the line of action of $\vec{F}$ called moment arm of the force, whereas θ is the angle between Axis of rotation and radius vector. The vector M_o is called vector moment of the given force $\vec{F}$ about point O.



(Fig. 3.37)

3.6.10 Application in daily life based on Cross or Vector Product

Example 1.

A force of 22N is applied to the end of 0.15 meter wrench at an angle of 75 degrees with the axis of rotation. Calculate the magnitude of the moment $\vec{M}_o$ produced by applied force.

Solution:

$$\vec{M}_o = \vec{r} \times \vec{F}$$

$$|\vec{M}_o| = |\vec{r}|\,|\vec{F}|\,Sin\,\theta$$

$$= (0.15)(22)Sin\,75° = 3.18Nm$$

Example 2. Find the moment about a point O $(1,-1,0)$ of the force $\vec{F} = \hat{i} - \hat{j} + 3\hat{k}$ applied at $A(-3,1,2)$.

Solution:

Here $\vec{r}$ is the vector $\overrightarrow{OA}$, joining points O and A, point O is on the axis of rotation and A is the point of application of force. So moment $\vec{M}_o$ about O is given by

$$\vec{M}_o = \vec{r} \times \vec{F}$$

Here, $\vec{F} = \hat{i} - \hat{j} + 3\hat{k}$

and, $\vec{r} = \overrightarrow{OA} = (-3-1)\hat{i} + (1+1)\hat{j} + (2-0)\hat{k} = -4\hat{i} + 2\hat{j} + 2\hat{k}$

So, $\vec{M}_o = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 2 & 2 \\ 1 & -1 & 3 \end{vmatrix} = \hat{i}(6+2) - \hat{j}(-12-2) + \hat{k}(4-2)$

Thus, $\vec{M}_o = 8\hat{i} + 14\hat{j} + 2\hat{k}$

Exercise 3.5

1. Find $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ if:

(i) $\vec{a}=4\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

(ii) $\vec{a}=-2\hat{i}+5\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}+2\hat{j}+\hat{k}$

2. Verify that $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$ for $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$.

3. Find a unit vector which is orthogonal to both the vectors.
$\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$

4. Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where; $\vec{a}=3\hat{i}+5\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-5\hat{k}$.

5. Find the area of the parallelogram whose adjacent sides are represented by
$\vec{a}=5\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=\hat{i}+\hat{k}$

6. Find the area of triangle with vertices $A(1,1,2)$, $B(2,3,5)$ and $C(1,5,5)$.

7. If $\vec{a}=2\hat{i}+\hat{j}+7\hat{k}$, $\vec{b}=\hat{i}-3\hat{j}+5\hat{k}$ and $\vec{c}=-\hat{i}+5\hat{j}+\hat{k}$ then verify that $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.

8. Using cross product, find the angle between $\hat{i}-7\hat{j}+7\hat{k}$ and $3\hat{i}-2\hat{j}+2\hat{k}$.

9. Find the moment about a point $A(1,3,5)$ of the force $\vec{F}=\hat{i}+2\hat{j}-\hat{k}$ applied at $B(3,-2,5)$.

10. If you apply 40N force to the end of 0.3 meter wrench at an angle of 110 degrees. Calculate the magnitude of the moment $\vec{M}_o$ produced by applied force.

3.7 Scalar Triple Product

3.7.1 Define scalar triple product of vectors

The scalar triple product of three vectors is defined as a dot product of a vector with cross product of other two vectors. The scalar triple product of three vectors $\vec{a},\vec{b}$ and $\vec{c}$ is $\vec{a}\cdot(\vec{b}\times\vec{c})$. It can be written as $[\vec{a}\ \ \vec{b}\ \ \vec{c}]$ or $[\vec{a},\ \vec{b},\ \vec{c}]$.

3.7.2 Express scalar triple product of vectors in terms of components (determinant form)

We know that if $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ are three vectors then their scalar triple product is; $\vec{a}\cdot(\vec{b}\times\vec{c})$

By definition of cross product.

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=\hat{i}(b_2c_3-b_3c_2)-\hat{j}(b_1c_3-b_3c_1)+\hat{k}(b_1c_2-b_2c_1)$$

and

$$\vec{a}\cdot(\vec{b}\times\vec{c})=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot[\hat{i}(b_2c_3-b_3c_2)-\hat{j}(b_1c_3-b_3c_1)+\hat{k}(b_1c_2-b_2c_1)]$$
$$=a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$$

This is scalar triple product of vectors $\vec{a},\vec{b}$ and $\vec{c}$ in components form which is written in determinant form as:

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix},$$

Example: Find the scalar triple product of the vectors $3\hat{i}-2\hat{j}$; $7\hat{i}+\hat{j}-3\hat{k}$ and $3\hat{i}-\hat{j}+5\hat{k}$.

Solution: Let $\vec{a}=3\hat{i}-2\hat{j}$, $\vec{b}=7\hat{i}+\hat{j}-3\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+5\hat{k}$ then;

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=\begin{vmatrix}3 & -2 & 0\\ 7 & 1 & -3\\ 3 & -1 & 5\end{vmatrix}$$
$$=3(5-3)+2(35+9)+0$$
$$=6+88=94$$

3.7.3 Prove that: (i) $\hat{i}\cdot\hat{j}\times\hat{k}=\hat{j}\cdot\hat{k}\times\hat{i}=\hat{k}\cdot\hat{i}\times\hat{j}=1$ and (ii) $\hat{i}\cdot\hat{k}\times\hat{j}=\hat{j}\cdot\hat{i}\times\hat{k}=\hat{k}\cdot\hat{j}\times\hat{i}=-1$

Proof: (i) $\hat{i}\cdot\hat{j}\times\hat{k}=\hat{j}\cdot\hat{k}\times\hat{i}=\hat{k}\cdot\hat{i}\times\hat{j}=1$

Here, $\hat{i}\cdot\hat{j}\times\hat{k}=\hat{i}\cdot\hat{i}=1$ $\quad[\because \hat{j}\times\hat{k}=\hat{i}$ and $\hat{i}\cdot\hat{i}=1]$

Similarly, $\hat{j}\cdot\hat{k}\times\hat{i}=\hat{j}\cdot\hat{j}=1$ $\quad[\because \hat{k}\times\hat{i}=\hat{j}$ and $\hat{j}\cdot\hat{j}=1]$

and $\hat{k}\cdot\hat{i}\times\hat{j}=\hat{k}\cdot\hat{k}=1$ $\quad[\because \hat{i}\times\hat{j}=\hat{k}$ and $\hat{k}\cdot\hat{k}=1]$

so, $\hat{i}\cdot\hat{j}\times\hat{k}=\hat{j}\cdot\hat{k}\times\hat{i}=\hat{k}\cdot\hat{i}\times\hat{j}=1$

or $[\hat{i},\hat{j},\hat{k}]=[\hat{j},\hat{k},\hat{i}]=[\hat{k},\hat{i},\hat{j}]=1$

Proof: (ii) $\hat{i}\cdot\hat{k}\times\hat{j}=\hat{j}\cdot\hat{i}\times\hat{k}=\hat{k}\cdot\hat{j}\times\hat{i}=-1$

Here, $\hat{i}\cdot\hat{k}\times\hat{j}=\hat{i}\cdot(-\hat{i})=-1$ $\quad[\because \hat{k}\times\hat{j}=-\hat{i}]$

$\hat{j}\cdot\hat{i}\times\hat{k}=\hat{j}\cdot(-\hat{j})=-1$ $\quad[\because \hat{i}\times\hat{k}=-\hat{j}]$

$\hat{k}\cdot\hat{j}\times\hat{i}=\hat{k}\cdot(-\hat{k})=-1$ $\quad[\because \hat{j}\times\hat{i}=-\hat{k}]$

So, $\hat{i}\cdot\hat{k}\times\hat{j}=\hat{j}\cdot\hat{i}\times\hat{k}=\hat{k}\cdot\hat{j}\times\hat{i}=-1$

i.e., $[\hat{i},\hat{k},\hat{j}]=[\hat{j},\hat{i},\hat{k}]=[\hat{k},\hat{j},\hat{i}]=-1$

3.7.4 Prove that dot and cross are inter-changeable in scalar triple product

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be the three vectors then by definition of scalar triple product.

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \text{[interchanging } R_1 \text{ and } R_2]$$

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \qquad \text{[interchanging } R_2 \text{ and } R_3]$$

$$= \vec{b}\cdot(\vec{c}\times\vec{a})$$

And $\quad \vec{b}\cdot(\vec{c}\times\vec{a}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$

$$= -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \qquad \text{[interchanging } R_1 \text{ and } R_2]$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{[interchanging } R_2 \text{ and } R_2]$$

$$= \vec{c}\cdot(\vec{a}\times\vec{b})$$

Hence, $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$ or $[\vec{a},\vec{b},\vec{c}] = [\vec{b},\vec{c},\vec{a}] = [\vec{c},\vec{a},\vec{b}]$

This shows that operation of dot and cross are interchangeable in scalar triple product.

3.7.5 Find the volume of :

- **a parallelepiped,**
- **a tetrahedron determined by three given vectors**

(i) Parallelepiped

Consider the parallelepiped as shown in Fig. 3.38, with three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ as its co-terminal edges. The volume of the parallelepiped is the area of the base times the height. From the geometric definition of the cross product, we know that the magnitude, $|\vec{b}\times\vec{c}|$, is the area of the parallelogram base, and that the direction of the vector $\vec{b}\times\vec{c}$ is perpendicular to the base.

The height h of the parallelepiped is the component of vector $\vec{a}$,
i.e., height $= h = |\vec{a}|\cos\theta$ in the direction normal to the base, i.e., in the direction of vector $\vec{b}\times\vec{c}$., where θ is the angle between $\vec{a}$ and $\vec{b}\times\vec{c}$.

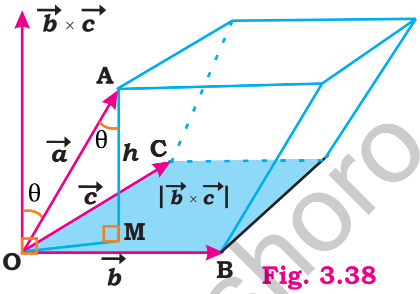


Fig. 3.38

The volume V of the parallelepiped is:
V = (base area of parallelepiped) .(height of parallelepiped)

$$= |\vec{b}\times\vec{c}|\,|\vec{a}|\cos\theta$$
$$= (\vec{b}\times\vec{c})\cdot\vec{a}$$
$$= \vec{a}\cdot(\vec{b}\times\vec{c})$$

So, we have volume of parallelepiped $= \vec{a}\cdot(\vec{b}\times\vec{c})$

Thus, $V = \vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Example: Calculate the volume of the parallelepiped determined by the three vectors $-2\hat{i}-\hat{j}+3\hat{k}$, $\hat{i}+7\hat{j}-2\hat{k}$ and $-\hat{j}+\hat{k}$

Solution: Let $\vec{a} = -2\hat{i}-\hat{j}+3\hat{k}$, $\vec{b} = \hat{i}+7\hat{j}-2\hat{k}$ and $\vec{c} = -\hat{j}+\hat{k}$
We know that

Volume of parallelepiped $= \vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= \begin{vmatrix} -2 & -1 & 3 \\ 1 & 7 & -2 \\ 0 & -1 & 1 \end{vmatrix}$ (by using components of vectors)

$= -2(7-2)+1(1-0)+3(-1-0)$
$= -10+1-3 = -12$

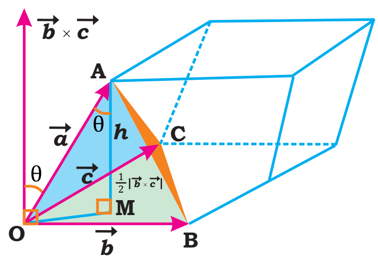
$\because$ volume is always positive,

$\therefore$ Required volume $= |\vec{a}\cdot(\vec{b}\times\vec{c})| = |-12|$
$= 12$ cubic units

(ii) Tetrahedron

Consider a tetrahedron having vertices O, A, B and C such that

$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ with respect to origin as shown in Fig. 3.39.



(Fig. 3.39)

Area of base ΔOBC of tetrahedron $= \frac{1}{2}|\vec{b}\times\vec{c}|$

Let $\hat{n}$ be the unit vector perpendicular to the plane of ΔOBC.

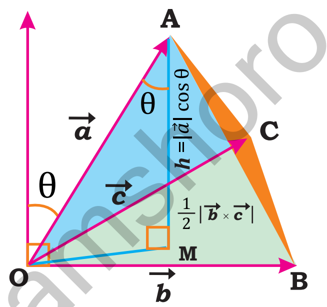
Let $\overline{AM}$ be perpendicular to $\overline{OM}$, is the height of the tetrahedron and $\angle OAM = \theta$ $|\vec{a}| \cos\theta$

So, $h = |\vec{a}| \cos\theta$, from Fig. 3.40

From elementary geometry,

Volume of tetrahedron

$$= \frac{1}{3}(\text{Base Area}) \times (\text{height of tetrahedron})$$



(Fig. 3.40)

Hence,

$$\text{Volume of tetrahedron} = \frac{1}{3}\left(\frac{1}{2}|\vec{b} \times \vec{c}|\right)(|\vec{a}| \cos\theta)$$

$$= \frac{1}{6}(\vec{b} \times \vec{c}) \cdot \vec{a} = \frac{1}{6}\vec{a} \cdot (\vec{b} \times \vec{c})$$

Hence, volume of tetrahedron $= \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$.

Example 1. Compute the volume of tetrahedron with $\vec{a} = (7, -1, 5)$, $\vec{b} = (-3, 1, 0)$ and $\vec{c} = (3, -1, 2)$ are its coterminal edges.

Solution: We know that

$$\text{Volume of tetrahedron} = \frac{1}{6}[\vec{a}\quad \vec{b}\quad \vec{c}]$$

$$= \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \frac{1}{6}\begin{vmatrix} 7 & -1 & 5 \\ -3 & 1 & 0 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= \frac{1}{6}\{7(2-0) + 1(-6-0) + 5(3-3)\}$$

$$= \frac{1}{6}(14 - 6 + 0)$$

$$= \frac{1}{6} \times 8 = 1.33 \text{ cubic units}$$

Example 2. Find the volume of tetrahedron whose vertices are $A(2,1,7), B(5,-1,3), C(4,3,5)$ and $D(0,2,3)$

Solution:

Here,

$$\overrightarrow{AB} = (5-2, -1-1, 3-7) = (3, -2, -4)$$
$$\overrightarrow{AC} = (4-2, 3-1, 5-7) = (2, 2, -2)$$
$$\overrightarrow{AD} = (0-2, 2-1, 3-7) = (-2, 1, -4)$$

we have, volume of tetrahedron $= \frac{1}{6}[\overrightarrow{AB} \quad \overrightarrow{AC} \quad \overrightarrow{AD}]$

$$= \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \ldots\text{(i)}$$

$$= \frac{1}{6}\begin{vmatrix} 3 & -2 & -4 \\ 2 & 2 & -2 \\ -2 & 1 & -4 \end{vmatrix}$$

$$= \frac{1}{6}[3(-8+2)+2(-8-4)-4(2+4)]$$

$$= \frac{1}{6}[-18-24-24] = \frac{1}{6}|-66|$$

So, volume of tetrahedron $= 11$ cubic units

3.7.6 Define Coplanar vectors and find the condition for coplanarity of three vectors

Coplanar Vectors

Vectors lying on the same plane are called coplanar vectors.

Condition for co-planarity of three vectors

If $\vec{a}, \vec{b}, \vec{c}$ are three coplanar vectors, then $\vec{a} \times \vec{b}$, being perpendicular to the plane containing vectors $\vec{a}$ and $\vec{b}$ is also perpendicular to $\vec{c}$ as shown in Fig. 3.41. Since dot product of two perpendicular vectors is zero,

Therefore, $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$, i.e., $[\vec{a} \quad \vec{b} \quad \vec{c}] = 0$

Hence, three vectors are coplanar if their scalar triple product is zero. This is called the condition for co-planarity of three vectors.

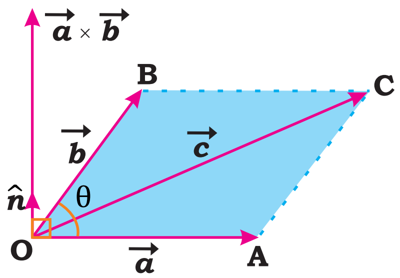


Fig. 3.41

Example 1. Show that $\vec{a} = \hat{i} + 3\hat{j} + 5\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 6\hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} - 11\hat{k}$ are coplanar.

Solution:

Vectors $\vec{a}, \vec{b}, \vec{c}$ will be coplanar if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Now, $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 3 & 5 \\ -2 & 4 & -6 \\ -3 & 1 & -11 \end{vmatrix}$

$$= 1(-44+6) - 3(22-18) + 5(-2+12)$$

$$= -38 - 12 + 50 = 0$$

Hence, vectors are coplanar.

Example 2. For what value of λ, the vectors $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}-3\hat{j}+4\hat{k}$ and $3\hat{i}+\hat{j}+\lambda\hat{k}$ are coplanar.

Solution:

Here, vectors are $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}-3\hat{j}+4\hat{k}$ and $3\hat{i}+\hat{j}+\lambda\hat{k}$

If the given vectors are coplanar, then $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \\ 3 & 1 & \lambda \end{vmatrix} = 0$

$\Rightarrow \quad 1(-3\lambda-4)-2(2\lambda-12)+3(2+9)=0$ [Expanding by R_1]

$\Rightarrow \quad -3\lambda-4-4\lambda+24+33=0$

$\Rightarrow \quad \lambda = \dfrac{53}{7}$

Exercise 3.6

1. Find the scalar triple product of vectors $\hat{i}+2\hat{j}+3\hat{k}, -\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$.
2. Compute $[\hat{i}+\hat{j},\ \hat{i},\ \hat{i}-\hat{j}+\hat{k}]$.
3. Find the volume of parallelepiped whose edges are represented by the vectors:
   (i) $3\hat{i}+2\hat{j}-\hat{k}$; $\hat{i}-2\hat{j}+\hat{k}$; $\hat{i}+2\hat{j}-4\hat{k}$
   (ii) $\hat{i}-2\hat{j}+3\hat{k}$; $2\hat{i}-\hat{j}-\hat{k}$; $\hat{i}+2\hat{j}-4\hat{k}$
4. Find the volume of the tetrahedron whose vertices are; $A(2,1,8), B(3,2,9), C(2,1,4)$ and $D(3,3,10)$
5. Show that the vectors $4\hat{i}-\hat{j}+\hat{k}, 3\hat{i}-2\hat{j}-\hat{k}$ and $\hat{i}+\hat{j}+2\hat{k}$ are coplanar.
6. Find λ if the vectors $\hat{i}+\hat{j}+2\hat{k}, \lambda\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}-2\hat{j}-\hat{k}$ are coplanar.
7. Show that $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-2\hat{j}+3\hat{k}$ and $2\hat{i}-\hat{j}+3\hat{k}$ are not coplanar vectors.

Review Exercise 3

1. Select true option.

i. The unit vector in the direction of $\hat{r}=2\hat{i}+\hat{j}-\hat{k}$ is:

(a) $\frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}-\hat{k})$ (b) $\sqrt{6}(2\hat{i}+\hat{j}-\hat{k})$

(c) $6(2\hat{i}+\hat{j}-\hat{k})$ (d) $\frac{1}{6}(2\hat{i}+\hat{j}-\hat{k})$

ii. The magnitude of $\vec{a}\times\vec{b}$ represents the _________ of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$.

(a) opposite sides (b) diagonal (c) area (d) volume

iii. The volume of tetrahedron determined by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is:

(a) $[\vec{a}, \vec{b}, \vec{c}]$ (b) $\frac{1}{3}[\vec{a}, \vec{b}, \vec{c}]$ (c) $\frac{1}{6}[\vec{a}, \vec{b}, \vec{c}]$ (d) None

iv. $\vec{a} \cdot (\vec{b} \times \vec{c}) =$ ___________.

(a) $\vec{a}.\vec{b}.\vec{c}$ (b) $\vec{a} \times \vec{b} \times \vec{c}$ (c) $\vec{b} \cdot (\vec{a} \times \vec{c})$ (d) $(\vec{a} \times \vec{b}).\vec{c}$

v. If $\vec{a}$ and $\vec{b}$ are parallel then $\vec{a} \cdot \vec{b} =$ ___________

(a) 1 (b) -1 (c) 0 (d) ab

vi. If three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}] =$ ___________

(a) 1 (b) 0 (c) $\vec{c}$ (d) $\vec{a}$

vii. Direction cosines of the vector $\hat{i} + \hat{j} - \hat{k}$ are:

(a) $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ (b) $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

(c) $1, 1, -1$ (d) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

viii. $\vec{a} \times \vec{b}$ is _____________ to plane of $\vec{a}$ and $\vec{b}$:

(a) Parallel (b) Perpendicular (c) Opposite (d) None of these

ix. If $\vec{a} = \overrightarrow{P_1P_2}$, where $P_1(0, 0, 1)$ and $P_2(-3, 1, 2)$, then $|\vec{a}| =$

(a) $\sqrt{12}$ (b) $\sqrt{10}$ (c) $\sqrt{13}$ (d) $\sqrt{11}$

x. If $\vec{p} = 4\hat{i} + 6\hat{k}$ and $\vec{q} = 6\hat{i} - 4\hat{j}$ then $|\vec{p} - \vec{q}|$ is

(a) $\sqrt{8}$ (b) $2\sqrt{14}$ (c) $2\sqrt{26}$ (d) $2\sqrt{3}$

xi. For non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \times \vec{b}$ is a unit vector and $|\vec{a}| = |\vec{b}| = \sqrt{2}$, then angle θ between vectors $\vec{a}$ and $\vec{b}$ is:

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $-\frac{\pi}{2}$

xii. Magnitude of a vector $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$ is:

(a) 13 (b) $\sqrt{12}$ (c) $\sqrt{14}$ (d) $\sqrt{11}$

xiii. The position vector of the point (1, 0, 2) is:

(a) $\hat{i} + \hat{j} + 2\hat{k}$ (b) $\hat{i} + 2\hat{j}$ (c) $2\hat{i} + 3\hat{k}$ (d) $\hat{i} + 2\hat{k}$

xiv. If O be the origin and $\overline{OP} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\overline{OQ} = 5\hat{i} + 4\hat{j} - 3\hat{k}$, then $\overline{PQ}$ is equal to:

(a) $7\hat{i} + 7\hat{j} - 7\hat{k}$ (b) $-3\hat{i} + \hat{j} - \hat{k}$ (c) $3\hat{i} + \hat{j} + \hat{k}$ (d) $-7\hat{i} - 7\hat{j} + 7\hat{k}$

xv. $\hat{k} \times \hat{j} =$ __________.

(a) $\hat{i}$ (b) $\hat{k}$ (c) $\hat{j}$ (d) $-\hat{i}$

xvi. If $|\vec{a} \times \vec{b}| = |\vec{a} . \vec{b}|$ then the angle between $\vec{a}$ and $\vec{b}$:

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) π

xvii. The distance of the point $(-3,4,5)$ from the origin is:

(a) 50 (b) $5\sqrt{2}$ (c) 6 (d) None of these

xviii. The vector having, initial and terminal points as (2, 5, 0) and $(-3,7,4)$ respectively is:

(a) $-\hat{i}+12\hat{j}+4\hat{k}$ (b) $-5\hat{i}+2\hat{j}-4\hat{k}$ (c) $-5\hat{i}+2\hat{j}+4\hat{k}$ (d) $-\hat{i}+\hat{j}+\hat{k}$

xix. If $|\vec{a}|=10$, $|\vec{b}|=2$ and $\vec{a}\cdot\vec{b}=0$, then the value of $|\vec{a}\times\vec{b}|$ is:

(a) 10 (b) 16 (c) 14 (d) 20

xx. The number of vectors of unit length perpendicular to the plane of the vectors

$\vec{a}=2\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{j}+\hat{k}$ is:

(a) One (b) Two (c) Three (d) Infinite

2. Write $\overrightarrow{PQ}$ in the form $x\hat{i}+y\hat{j}$

(i) P (4,6), Q(−4,5) (ii) $P\left(-\frac{5}{2},9\right), Q(10,8)$

3. Find the vector PQ joining the points P(3,4,2) and Q(−2,3,4) and also find direction cosines of $\overrightarrow{PQ}$.

4. Show that P(2,3,6), Q(3,7,4) and R(4,11,−2) are collinear.

5. Show that the vectors $\frac{1}{7}\ \hat{i}+\hat{j}+\hat{k}$, $2\hat{i}-3\hat{j}+3\hat{k}$ and $4\hat{i}+\hat{j}-5\hat{k}$ are mutually perpendicular.

6. Find the area of parallelogram whose adjacent sides are $3\hat{i}-5\hat{j}+6\hat{k}$ and $\hat{i}+3\hat{j}-4\hat{k}$. Also find unit vector parallel to its diagonal through common initial point.

7. Prove that $[\vec{p}+\vec{q},\vec{q}+\vec{r},\vec{r}+\vec{p}] = 2\,[\vec{p}\ \vec{q}\ \vec{r}]$.

8. For the vectors $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=3\hat{i}-\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}+8\hat{j}+\hat{k}$, verify the distributive property of cross product.