Sequences and Series

Unit 4

4.1 Sequence

An ordered set of numbers, formed according to a definite rule, is called a sequence (or progression) and the individual members are called terms (or elements) of the sequence.

4.1.1 Define a sequence (progression) and its terms

We may also define a sequence of numbers as follows:

A function $a: \mathbb{N} \longrightarrow \mathbb{R}$ or $\mathbb{C}$ is called a sequence, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ are the set of natural, real and complex numbers respectively.

For any $n \in \mathbb{N}, a(n) \in \mathbb{R}$ $(or\ \mathbb{C})$ and is called the nth term or general term of the sequence.

We usually write $a(n)$ as a_n and the sequence as $\{a_n\}$ where $n \in \mathbb{N}$. A common representation of a finite sequence, one that has finite number of terms n is

$$a_1, a_2, a_3, \ldots, a_n$$

If the sequence has an unlimited number of terms, then it is called an infinite sequence, we may write

$$a_1, a_2, a_3, \ldots$$

or $\{a_k\}$, where $k = 1,2,3, \ldots$

The nth term of a sequence is also represented by T_n (or t_n). The terms T_1, T_2, T_3 (or t_1, t_2, t_3) will denote first, second and third terms of the sequence respectively. In T_n (or t_n), n indicates the position or rank of the term in the sequence.

Some examples of sequences along with their general terms are:

(i) $1, 2, 3, 4, \ldots;$ where $a_n = n$

(ii) $5, 9, 13 \ldots;$ where $a_n = 4n + 1$

(iii) $-1, 1, -1, 1, -1, \ldots;$ where $a_n = (-1)^n$

(iv) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots;$ where $a_n = \frac{n}{n+1}$

4.1.2 Know that a sequence can be constructed from a formula or an inductive definition

If we are able to find a pattern from the given initial terms of a sequence, then we can deduce a rule or formula for the terms of the sequence as explained in the following example.

Example: Find n^{th} term of the sequence:

$$6, 11, 16, 21, \dots$$

Solution: Here, $a_1 = 6 = 5 \times 1 + 1$

$a_2 = 11 = 5 \times 2 + 1$

$a_3 = 16 = 5 \times 3 + 1$

$a_4 = 21 = 5 \times 4 + 1$

From the above pattern, we deduce that $a_n = 5n + 1 \quad (\forall n \in \mathbb{N})$

We can find any term of the sequence by giving corresponding value to n in the nth or general term a_n of a sequence. In this way, we can construct a sequence from a formula or an inductive definition as explained in the following examples.

Example 1. Find the sequence if $a_n = 2n - 1$

Solution: $a_n = 2n - 1$

For n =1, $a_1 = 2(1) - 1 = 1$

For n =2, $a_2 = 2(2) - 1 = 3$

For n =3, $a_3 = 2(3) - 1 = 5$

For n =4, $a_4 = 2(4) - 1 = 7$

Thus, the required sequence is 1, 3, 5, 7, …

Example 2. Find the sequence if $a_n - a_{n-1} = n + 1$ and $a_4 = 14$

Solution: Putting $n = 2, 3, 4, 5$ in $a_n - a_{n-1} = n + 1$, we have,

$a_2 - a_1 = 3$ …(i)

$a_3 - a_2 = 4$ …(ii)

$a_4 - a_3 = 5$ …(iii)

$a_5 - a_4 = 6$ …(iv)

From (iv), $a_5 = a_4 + 6$

$= 14 + 6 = 20 \quad (\because a_4 = 14)$

From (iii), $a_3 = a_4 - 5$

$= 14 - 5 = 9 \quad (\because a_4 = 14)$

From (ii), $a_2 = a_3 - 4$

$= 9 - 4 = 5 \quad (\because a_3 = 9)$

and from (i), $a_1 = a_2 - 3$

$= 5 - 3 = 2 \quad (\because a_2 = 5)$

Thus, the sequence is $2, 5, 9, 14, 20, \dots$

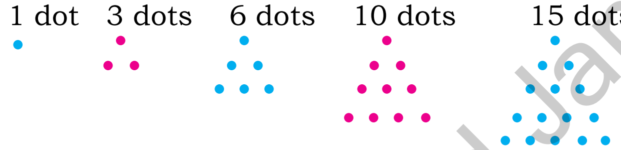
4.1.3 Recognize triangle, factorial and Pascal sequences as fractional form

(a) Triangular Sequence

Consider the following sequence;

$$1, 3, 6, 10, 15, 21, 28, 36, 45, \ldots$$

which simply represents the number of dots in each triangular pattern as shown in Fig. 4.1, so this is called triangular sequence:

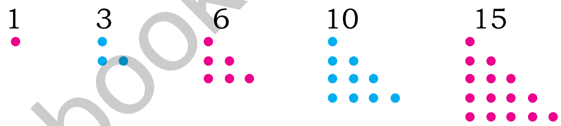


(Fig. 4.1)

By adding a row of dots in preceding pattern and counting all the dots, we can find the next term of the sequence. The first pattern has just one dot. The second pattern has another row with 2 extra dots, making $1 + 2 = 3$

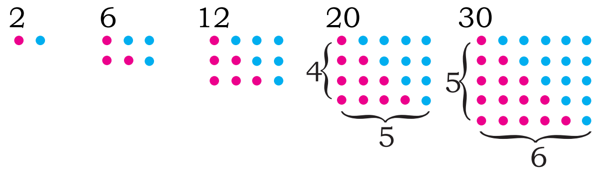
The third pattern has another row with 3 extra dots, making $1 + 2 + 3 = 6$. The fourth has $1 + 2 + 3 + 4 = 10$ etc.

We can make a "Rule" to calculate any term or triangular number. First, rearrange the dots as shown is Fig 4.2:



(Fig. 4.2)

Then double the number of dots, and arrange them into a rectangular pattern as shown in Fig 4.3



(Fig. 4.3)

Now number of dots in each rectangular pattern is:

$$\text{Number of dots} = n(n+1)$$

Where n shows the position of term of the sequence.
But remember we doubled the number of dots, so number of dots in each term of triangular sequence is: $\frac{n(n+1)}{2}$.

Hence, $a_n = \frac{n(n+1)}{2}$ is the general term of the triangular sequence.

Example: Find the number of dots in triangular sequence for $n = 6, 10, 13$ and 16.

Solution: Using formula: $t_n = \frac{n(n+1)}{2}$

(i) When n = 6, then $t_6 = \frac{6(6+1)}{2} = 21$

(ii) When n = 10, then $t_{10} = \frac{10(10+1)}{2} = 55$

(iii) When n = 13, then $t_{13} = \frac{13(13+1)}{2} = 91$

(iv) When n = 16, then $t_{16} = \frac{16(16+1)}{2} = 136$

(b) Factorial Sequence

In mathematics, the factorial of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n. For example, $3! = 3 \times 2 \times 1 = 6$ and $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

We define $0! = 1$, a sequence involving factorial is called factorial sequence, for example, a sequence with $a_n = \frac{1}{n!}$

i.e., $\frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots$

Example: Find the first four terms of the sequence with $a_n = \frac{2^n}{n!}$

Solution: Here, $a_n = \frac{2^n}{n!}$, so first four terms of the sequence are a_1, a_2, a_3 and a_4

Now,

$$a_1 = \frac{2^1}{1!} = 2$$

$$a_2 = \frac{2^2}{2!} = \frac{4}{2 \cdot 1} = 2$$

$$a_3 = \frac{2^3}{3!} = \frac{8}{3 \cdot 2 \cdot 1} = \frac{8}{6} = \frac{4}{3}$$

$$a_4 = \frac{2^4}{4!} = \frac{16}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{16}{24} = \frac{2}{3}$$

(c) Pascal Sequence

Pascal's Triangle

Pascal's triangle is a triangular arrangement of numbers which represent the coefficients of the expansion of power of binomial like $(a+b)^n$ as shown in Fig 4.4.

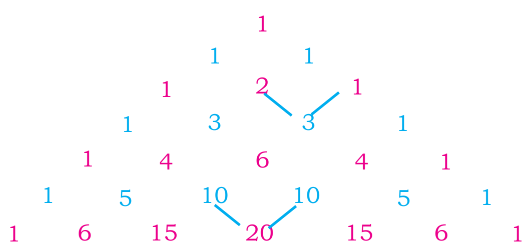


Fig. 4.4

Pascal Sequence

The number '1' is at the tip of Pascal's Triangle, which makes up the zeroth row. The first row (1 and 1) contains two 1's, both formed by adding the two numbers above them to the left and the right, in this case 1 and 0 (all numbers outside the triangle are 0's). By the same process we get the 2nd row (1, 2, 1): $0+1=1$; $1+1=2$; $1+0=1$. And the third row (1, 3, 3, 1): $0+1=1$; $1+2=3$; $2+1=3$; $1+0=1$. In this way, the rows of the triangle go on infinitely. And every row is called Pascal sequence. For finding terms of Pascal sequence, we use the formula: $\frac{n!}{r!(n-r)!}$; $r \le n$

Where n denotes the row of Pascal triangle and r denotes its define column.

Example: Find the Pascal sequence when $n = 4$.

Solution: By using formula of Pascal sequence, when $n = 4$ and $r = 0,1,2,3,4$

$$r = 0;\ \frac{4!}{0!(4-0)!} = \frac{4!}{1!(4-0)!} = \frac{4!}{4!} = 1 \qquad [\because\ 0! = 1]$$

$$r = 1;\ ;\ \frac{4!}{1!(4-1)!} = \frac{4!}{1!(3!)} = \frac{4\cdot 3!}{3!} = 4 \qquad [\because\ 1! = 1]$$

$$r = 2;\ \frac{4!}{2!(4-2)!} = \frac{4!}{2(2!)} = \frac{4\cdot 3\cdot 2!}{2\cdot 2!} = 6$$

$$r = 3;\ \frac{4!}{3!(4-3)!} = \frac{4\cdot 3!}{3!(1!)} = \frac{4}{1} = 4$$

$$r = 4;\ \frac{4!}{4!(4-4)!} = \frac{4!}{4!(0!)} = 1$$

Hence the required Pascal sequence for $n = 4$ is (1, 4, 6, 4, 1).

Exercise 4.1

1. Find the nth term (rule of formation) of each of the following sequences:

(i) $2, 4, 6, \ldots$ (ii) $1^2, 2^2, 3^2, \ldots$ (iii) $3, 9, 27, \ldots$

(iv) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ (v) $1, \frac{-1}{3}, \frac{1}{9}, \frac{-1}{27}, \ldots$ (vi) $1, 8, 27, 64, \ldots$

2. Find the first five terms of sequences with the given general terms.

(i) $\frac{2n}{3}(n+1)$ (ii) $(-1)^{n+1}\cdot 3^{n-1}$ (iii) $\frac{1}{4}n^2(n+1)^2$

(iv) $\frac{n}{3n+1}$ (v) $\frac{1}{6}n(n+1)(n+2)$ (vi) $\frac{1}{6}n(n+1)(2n+7)$

(vii) $a_{n+1} = 6 + a_n$, $a_1 = 2$, (viii) $a_n = \frac{n}{2}a_{n+1}$, $a_1 = 1$

3. Find the sequence by using

$T_{n+1} = (n+1)T_n$, where $T_1 = 2$,

4. Find the values of n^{th} term of triangular sequence when n = 7, 9, 12 and 16.

5. Find the first five terms of the sequence with general term: $a_n = \frac{(n+1)!}{2!}$.

6. Find the Pascal sequence when $n = 5, 6, 7$ and 8.

4.2 Arithmetic Sequence

4.2.1 Define an arithmetic sequence

A sequence in which each term is formed by adding a fixed number to the one preceding it, is called an arithmetic sequence or an arithmetic progression (A.P.).

In A.P, the difference between the two consecutive terms is same. Here difference means that second term minus the first term or third term minus second term and so on. The difference is called the common difference denoted by 'd'. For instance, the sequence 5, 7, 9, 11, 13, 15...; is an arithmetic progression with common difference of 2.

4.2.2 Find the *n*th or general term of an arithmetic sequence

If the first term of an A.P. is a and the common difference d then by definition

a_1 = the first term $= a \quad = \quad a + (1-1)d$
a_2 = the second term $= a + d \quad = \quad a + (2-1)d$
a_3 = the third term $= a + 2d \quad = \quad a + (3-1)d$
a_4 = the fourth term $= a + 3d \quad = \quad a + (4-1)d$
a_5 = the fifth term $= a + 4d \quad = \quad a + (5-1)d$ and so on.

Hence, we conclude that

a_n = the nth term $= a + (n-1)d$

or $\quad a_n = a + (n-1)d$

This is the formula for finding the nth or general term of an arithmetic sequence whose first term is a and the common difference d, whereas

$$a, a + d, a + 2d, \ldots, a + (n-1)d \ldots$$

is known as the standard form of A.P.

4.2.3 Solve problems involving arithmetic sequence

The formula to find nth term of an A.P. is given by

$$a_n = a + (n-1)d$$

The formula involves four elements, namely; a_n, a, n, and d. We shall now consider various types of problems based on this formula.

Example 1. Find the 10^{th} term of the A.P: $2, 5, 8, 11, \ldots$

Solution:

Here, $n = 10,\ a = 2,\ \ d = 5 - 2 = 8 - 5 = 3$

By using $a_n = a + (n-1)d$

We get, $a_{10} = 2 + (10-1)3 = 29$

Thus the 10^{th} term of the given A.P. is 29.

Example 2. Find the number of terms in the A.P., if $a = 3, d = 7$ and $a_n = 59$.

Solution: Using $a_n = a + (n-1)d$,

We get, $59 = 3 + (n-1)7$ $\quad(\because a_n = 59, a = 3 \text{ and } d = 7)$

$\Rightarrow \quad \frac{56}{7} = n - 1$

$\Rightarrow \quad n = 9$

Thus the number of terms in the A.P. is 9.

Example 3. Find the thirteenth term of the A.P. whose first term and the common difference are 3 and -4 respectively. Also write its first four terms.

Solution: Here, $a = 3$ and $d = -4$

We know that $a_n = a + (n-1)d$.

So, $a_n = 3 + (n-1)(-4) = 3 - 4n + 4$

or $a_n = 7 - 4n$ $\quad\ldots$(i)

Thus, the general term of the A.P. is $7 - 4n$.

Taking $n = 13$ in (i), we have

$$a_{13} = 7 - 4(13) = 7 - 52 = -45$$

We can find a_2, a_3, a_4 by using $n = 2,3,4$ in (i), that is,

$$a_2 = 7 - 4(2) = -1$$
$$a_3 = 7 - 4(3) = -5$$
$$a_4 = 7 - 4(4) = -9$$

Hence, the first four terms of the sequence are $3, -1, -5, -9$.

Example 4. If $a_{n-2} = 4n - 13$, then find the nth term of the sequence.

Solution: For the first term we take

$$n - 2 = 1$$
$$\text{or} \quad n = 3$$

So,

For $n = 3$, $\quad a_1 = 4(3) - 13 = -1 = a$

For $n = 4$, $\quad a_2 = 4(4) - 13 = \ \ 3$

For $n = 5$, $\quad a_3 = 4(5) - 13 = 7$

So, we get an A.P $-1, 3, 7, \ldots$

Thus $\ a_n = a + (n-1)d = -1 + (n-1)4$ $\quad(\because a = -1 \text{ and } d = 4)$

$$a_n = 4n - 5$$

Hence this is the nth term of the sequence.

Example 5. If the 5th term of an A.P.is 13 and 17th term is 49.
Find a_n and a_{13}.

Solution: Given that $a_5 = 13$ and $a_{17} = 49$.
Putting $n = 5$ in $a_n = a + (n-1)d$, we have

$$a_5 = a + (5-1)d = a + 4d$$

$$\Rightarrow \quad 13 = a + 4d \quad \ldots(i)$$

Also $$a_{17} = a + (17-1)d$$

$$\Rightarrow \quad 49 = a + 16d \quad \ldots(ii)$$

Subtracting (i) from (ii), we get

$$12d = 36 \Rightarrow d = 3$$

By using $d = 3$ in (i), we get $13 = a + 4(3) \Rightarrow a = 13 - 12 = 1$

Thus, $a_{13} = 1 + (13-1)3 = 37$ and $a_n = 1 + (n-1)3 = 3n - 2$

Hence, $a_{13} = 37$ and $a_n = 3n - 2$

Example 6. An object falling from rest, falls 16 meters during the first second, 32 meters during the next second, 48 meters during the third second and so on. How much will it fall during the 9th second?

Solution: Fall of object for 1st second $= a_1 = 16m$
Fall of object for 2nd second $= a_2 = 32m$
Fall of object for 3rd second $= a_3 = 48m$
Fall of object for 9th second $= a_9 = ?$

So, $16, 32, 48, \ldots$ is an A.P.

Here, $a = 16, d = 32 - 16 = 16$ and $n = 9$

We know that $a_n = a + (n-1)d$

So, $a_9 = 16 + (9-1)(16)$

$$a_9 = 16 + (8)(16)$$

$$a_9 = 16 + 128 \Rightarrow a_9 = 144m$$

Hence, the object falls $144m$ during the 9th second.

Exercise 4.2

1. Find the indicated term in each of the following A.P.

(i) $1, 5, 9, \ldots; a_{12}$ (ii) $-15, -9, -3, \ldots; a_{10}$

(iii) $2, 6, 10, 14, \ldots; a_7$ (iv) $-5, 4, 13, \ldots; a_{30}$

(v) $23, 26, 29, \ldots; a_{14}$ (vi) $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots;\ a_6$

2. Find the first five terms of the following arithmetic sequences, if

(i) $a_n = 2n - 3$ (ii) $a_{n-2} = 3n - 5$

(iii) $a_n = \frac{n}{2n+1}$ (iv) $a_n = (n+1)a_{n-1}, a_1 = 1$

(v) $a_n - a_{n-1} = n + 2, a_1 = 2$

(vi) $a = 5$ and other three consecutive terms are 23, 26, 29.

3. If $a_{n-3} = 2n - 5$, find nth term of the sequence.

4. Find 20th term of the sequence $1, 2 - x, 3 - 2x, \ldots$

5. Find the 21st term of the A.P if its 6th term is 11 and the 15th term is 47.

6. (a) Which term of the sequence $-15, -9, -3, \ldots$ is 75?
   (b) Which term of the sequence $5, 2, -1, \ldots$ is -85 ?
   (c) Which term of the sequence $-2, 4, 10, \ldots$ is 148?

7. Find the nth term of the sequence $\left(\frac{6}{7}\right)^2, \left(\frac{11}{7}\right)^2, \left(\frac{16}{7}\right)^2, \ldots$

8. If a, b, c are the lth, mth and nth terms of an A.P., show that:
   (i) $a(m-n) + b(n-l) + c(l-m) = 0$
   (ii) $l(b-c) + m(c-a) + n(a-b) = 0$

9. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., show that $b = \frac{2ac}{a+c}$.

10. If $\frac{1}{a-b}, \frac{1}{b-c}, \frac{1}{c-a}$ are in A.P., then show that $\frac{a+c}{a-b} = \frac{a+b}{c-a}$.

11. An object falling from rest, falls 12 metres during the first second, 24 metres during the next second, 36 metres during the third second and so on. How much will it fall during the 8th second?

12. A man deposits Rs.13,000 in a bank in the first month; Rs. 14,500 in the second month; Rs. 16,000 in third month and so on. Find how much he has to deposit in the bank at the end of a year.

13. A boy saves Rs. 200 at the end of the first week and goes on increasing his saving for Rs. 25 weekly. After how many weeks, his weekly saving will be Rs. 2000.

4.3 Arithmetic Mean

4.3.1 Know arithmetic mean between two numbers

When three numbers are in arithmetic progression (A.P)., then the middle term is called their arithmetic mean i.e., if a, A, b are in A.P., then A is called the arithmetic mean (A.M.) of a and b, where a and b are called extremes. When more than three numbers are in A.P., then all the numbers between the extreme numbers (i.e. all the terms between the first and the last terms) are called arithmetic means. That is, if $a, A_1, A_2, A_3, \ldots A_n, b$ are in A.P, then there are $n+2$ terms in A.P., and $A_1, A_2, A_3, \ldots A_n$ are termed as the n arithmetic means between a and b. For example,

(i) As 2,5,8 are in A.P. So, 5 is the A.M. of 2 and 8

(ii) As 3, 5, 7, 9, 11, 13, 15, 17, 19, 21 are in A.P. So, 5, 7, 9, 11, 13, 15, 17 and 19 are the eight A.Ms between 3 and 21.

4.3.2 Insert an arithmetic mean between two numbers

Let A be the A.M. between a and b. Then a, A, b are in arithmetic progression, so that the common difference is the same,

i.e., $A - a$ = Common difference = $b - A$

or $A - a = b - A$ or $A + A = a + b$ or $2A = a + b$

$$\Rightarrow \boxed{A = \frac{a+b}{2}}$$

Example: Find the A.M., between $\sqrt{2}$ and $3\sqrt{2}$.

Solution: Let, A be the A.M between $\sqrt{2}$ and $3\sqrt{2}$.

Here, $a = \sqrt{2}$ and $b = 3\sqrt{2}$

Now, $A = \frac{a+b}{2} = \frac{\sqrt{2}+3\sqrt{2}}{2} = 2\sqrt{2}$

4.3.3 Insert n arithmetic means between two numbers

Let $A_1, A_2, \ldots, A_n$ be the n arithmetic means between any two given numbers a and b. Then we have an A.P: $a, A_1, A_2, \ldots, A_n, b$ with a as its first term and b its $(n+2)th$ term. Suppose the common difference of this A.P. is d. Then, $a_{n+2} = b = a + \{(n+2) - 1\}d$,

or $b = a + (n+1)d$

or $b - a = (n+1)d$

or $d = \frac{b-a}{n+1}$

So, $A_1 =$ the second term $= a + d = a + \frac{b-a}{n+1} = \frac{na+b}{n+1}$,

$A_2 =$ the third term $= a + 2d = a + \frac{2(b-a)}{n+1} = \frac{(n-1)a+2b}{n+1}$

… … … … … … … … … … … … … …

and $A_n =$ the $(n+1)th$ term $= a + nd = a + \frac{n(b-a)}{n+1} = \frac{a+nb}{n+1}$,

Hence, $\boxed{\frac{na+b}{n+1}, \frac{(n-1)a+2b}{n+1}, \ldots, \frac{a+nb}{n+1}}$

are the n arithmetic means between a and b.

Example: Find three arithmetic means between $\sqrt{3}$ and $3\sqrt{3}$.

Solution: Let, A_1, A_2, A_3 be three arithmetic means between $\sqrt{3}$ and $3\sqrt{3}$.

Then $\sqrt{3}, A_1, A_2, A_3, 3\sqrt{3}$ are in A.P.

Here, $a = \sqrt{3}$ and $a_5 = 3\sqrt{3}$

Using $a_n = a + (n-1)d$

We get, $a_5 = a + (5-1)d$ or $3\sqrt{3} = \sqrt{3} + 4d$ or $4d = 2\sqrt{3}$

Therefore, $d = \frac{\sqrt{3}}{2}$

Now, $A_1 = a + d = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$,

Similarly, $A_2 = A_1 + d = \frac{3\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$,

$A_3 = A_2 + d = 2\sqrt{3} + \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$,

Hence, the three arithmetic means between $\sqrt{3}$ and $3\sqrt{3}$ are $\frac{3\sqrt{3}}{2}$, $2\sqrt{3}$, $\frac{5\sqrt{3}}{2}$

Exercise 4.3

1. Find the A.M between.

(i) 18 and 26

(ii) $10 + 5\sqrt{3}$ and $4 - 5\sqrt{3}$

(iii) $3a + 2b$ and $5a - 6b$

(iv) $5 + 7\sqrt{5}i$ and $8 - 7\sqrt{5}i$

(v) $3a^2 - 5a + 6$ and $-a^2 + 7a - 4$

(vi) $3a - 5$ and $5a + 3$

2. Insert three A.Ms. between 3 and 11.

3. Insert four A.Ms. between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$.

4. Insert 5 A.Ms. between $\frac{\sqrt{5}}{2}$ and $4\sqrt{5}$.

5. Insert 6 A.Ms. between −30 and 30.

6. If 5 and 8 are two A.Ms. between c and d, find c and d.

7. Find n so that $\frac{a^{n+5}+b^{n+5}}{a^{n+4}+b^{n+4}}$ may be the A.M. between a and b.

4.4 Arithmetic Series

4.4.1 Define an arithmetic series

If $t_1, t_2, t_3, \dots, t_n$ is an arithmetic sequence, then the expression

$$t_1 + t_2 + t_3 + \cdots + t_n$$

is called an arithmetic series. If an arithmetic series consists of a finite number of terms, it is called a finite arithmetic series; otherwise, it is called an infinite arithmetic series.

In general, an arithmetic series with a as its first term and d as common difference is

$$a + (a + d) + (a + 2d) + \cdots$$

The following are some examples of arithmetic series:

(i) $3 + 5 + 7 + \cdots + (2n + 1)$;

(ii) $-1 - 8 - 15 - \cdots$

4.4.2 Establish the formula to find the sum to n term of an arithmetic series

In general, an arithmetic series of n terms with a as its first term and d its common difference is:

$$a + (a + d) + (a + 2d) + \cdots + \{a + (n - 1)d\}$$

If S_n denotes the sum of the series to n terms, then

$$S_n = a + (a + d) + (a + 2d) + \cdots + \{a + (n - 2)d\} + \{a + (n - 1)d\} \quad \ldots \text{(i)}$$

By writing the sum of the terms of the series in the reverse order, we have

$$S_n = \{a + (n - 1)d\} + \{a + (n - 2)d\} + \cdots + (a + 2d) + (a + d) + a \quad \ldots \text{(ii)}$$

Adding the corresponding terms of (i) and (ii), we get

$$2S_n = \{2a + (n - 1)d\} + \{2a + (n - 1)d\} + \cdots + \{2a + (n - 1)d\}; \text{ (up to } n \text{ terms)}$$

$$\Rightarrow \quad \boxed{2S_n = n\{2a + (n - 1)d\}}$$

Hence, $$S_n = \frac{n}{2}\{2a + (n - 1)d\}$$

Let l denotes the last term, i.e., $l = T_n = a + (n - 1)d$; then we have

$$S_n = \frac{n}{2}\{2a + (n - 1)d\} = \frac{n}{2}[a + \{a + (n - 1)d\}]$$

or $$\boxed{S_n = \frac{n}{2}(a + l)}$$

Thus, the sum to n terms of a series of A.P. is equal to n times the average of the first and the last term.

In each of the above two formulae for the sum of an arithmetic sequence, there are four elements, viz. S_n, n, a and d or l. Any three being known, the fourth one may be evaluated as illustrated by means of the examples given below.

Example 1. Find the sum of the first n terms of the arithmetic series $3 + 8 + 13 + \cdots$; also find the sum of the first 13 terms.

Solution: Given arithmetic series is: $3 + 8 + 13 + \cdots$

Here $a = 3, d = 8 - 3 = 5$

We have to find S_n and S_{13}

We know that $S_n = \frac{n}{2}[2a + (n - 1)d]$ ... (i)

B using $a = 3, d = 5$ in equation (i)

We get, $$S_n = \frac{n}{2}[2(3) + (n - 1)(5)] = \frac{n}{2}[6 + 5n - 5]$$

$$\Rightarrow \quad S_n = \frac{n}{2}[5n + 1] \quad \ldots \text{(ii)}$$

By using $n = 13$ in equation (ii)

We get, $$S_{13} = \frac{13}{2}[5(13) + 1] = 429$$

Thus, the sum of first 13 terms is 429.

Example 2. Find the arithmetic series if $S_5 = 30$ and $S_9 = a_5 - 32$.

Solution: We know that $S_n = \frac{n}{2}[2a + (n-1)d]$ ... (i)

By using $n = 5$ in equation (i)

we get, $S_5 = \frac{5}{2}[2a + (5-1)d] = \frac{5}{2}(2a + 4d)$

$\Rightarrow \quad 30 = 5(a + 2d) \qquad (\because S_5 = 30)$

$\Rightarrow \quad a + 2d = 6$ ... (ii)

Now, using $n = 9$ in equation (i)

$$S_9 = \frac{9}{2}[2a + (9-1)d] = \frac{9}{2}(2a + 8d)$$

$\Rightarrow \quad S_9 = 9(a + 4d)$

$\Rightarrow \quad S_9 = 9a + 36d$

We know that $a_n = a + (n-1)d$

and $a_5 = a + (5-1)d$

or $a_5 = a + 4d$

Now, we have, $S_9 = a_5 - 32$

i.e., $9a + 36d = a + 4d - 32$

$8(a + 4d) = -32$

$a + 4d = -4$ ... (iii)

Subtracting equation (iii) from (ii), we have

$(a + 2d) - (a + 4d) = 6 - (-4)$

$\Rightarrow \quad d = -5$

By using $d = -5$ in equation (ii) weget, $a + 2(-5) = 6 \Rightarrow \quad a = 16$

Now, $a_2 = a_1 + d = 16 + (-5) = 11$

$a_3 = a_2 + d = 11 + (-5) = 6$

$a_4 = a_3 + d = 6 + (-5) = 1$

Hence, the required arithmetic series is $16 + 11 + 6 + 1 + \cdots$

Example 3. The sum of three numbers in A.P. is 21 and their product is 231. Find the numbers.

Solution: Let the three numbers in A.P. are $a - d,\ a, a + d$

$\because$ Sum of three numbers = 21

$\therefore \quad (a - d) + a + (a + d) = 21$

$3a = 21$ or $a = 7$.

Now, product of three numbers = 231

i.e., $(a - d)(a)(a + d) = 231$ ....(1)

Put $a = 7$ in equation (1)

$(7 - d)(7)(7 + d) = 231$

$$(7-d)(7+d)=\frac{231}{7}$$
$$49-d^2=33$$
$$49-33=d^2$$
$$d^2=16 \text{ or } d=\pm 4.$$

If $d=4$, then first number $=a-d=7-4=3$, second number $=a=7$
and third number $=a+d=7+4=11$

Hence, three numbers are 3, 7 and 11.

If $d=-4$, then first number $=a-d=7+4=11$, second number $=a=7$
and third number $=a+d=7-4=3$

Hence, three numbers are 11, 7 and 3.

Thus, required numbers are 3, 7, 11 or 11, 7, 3.

Example 4. Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.

Solution: Let four numbers in A.P. are $a-3d,\ a-d, a+d, a+3d$

Sum of four numbers = 20

i.e., $a-3d+a-d+a+d+a+3d=20$

$\Rightarrow 4a=20$ or $a=5$.

Now, sum of squares of the numbers = 120

i.e., $(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2=120$ ...(1)

Using $a=5$ in equation (1)

We get $(5-3d)^2+(5-d)^2+(5+d)^2+(5+3d)^2=120$

$\Rightarrow 25-30d+9d^2+25-10d+d^2+25+10d+d^2+25+30d+9d^2=120$

$\Rightarrow 20d^2+100=120$

$\Rightarrow 20d^2=20$

$\Rightarrow d^2=1$

$\Rightarrow d=\pm 1$

If $a=5$ and $d=1$, then

First number $=a-3d=5-3=2$

Second number $=a-d=5-1=4$

Third number $=a+d=5+1=6$

and Fourth number $=a+3d=5+3=8$

Hence, four numbers are 2, 4, 6 and 8.

If $a=5$ and $d=-1$, then

First number $=a-3d=5-(-3)=8$

Second number $=a-d=5-(-1)=6$

Third number $=a+d=5-1=4$

Fourth number $=a+3d=5+(-3)=2$

Hence, four numbers are 8, 6, 4, 2.

Thus required numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

Note: For the sake of convenience, we take
two numbers in A.P as: $a-d, a+d$
four numbers in A.P as: $a-3d, a-d, a+d, a+3d$ and so on

Similarly,
three numbers in A.P as: $a-d,\ a,\ a+d$
five numbers in A.P as: $a-2d, a-d, a, a+d, a+2d$ and so on.

Example 5. The sums of the first n terms of two A.P.'s are in the ratio $5n-3:3n+31$. Show that their 9th terms are equal. Find also $a_n:a'_n$ and $a_9:a'_{11}$.

Solution: Let a, a' be the first terms and d, d' be the common differences of the two A.P.'s. If S_n and S'_n denotes the sum of these A.P.'s, then we are given

$$\frac{S_n}{S'_n}=\frac{\frac{n}{2}\{2a+(n-1)d\}}{\frac{n}{2}\{2a'+(n-1)d'\}}=\frac{5n-3}{3n+31} \qquad \ldots \text{(i)}$$

Express R.H.S of equation (i) in the form of L.H.S

$$\therefore \quad \frac{S_n}{S'_n}=\frac{\frac{n}{2}\{2a+(n-1)d\}}{\frac{n}{2}\{2a'+(n-1)d'\}}=\frac{\frac{n}{2}\{2(1)+(n-1)(5)\}}{\frac{n}{2}\{2(17)+(n-1)(3)\}}$$

By comparing we have

$$a=1,\ d=5 \text{ and } a'=17, d'=3$$

Now, $a_9=a+8d=1+8(5)=1+40=41$

and $a'_9=a'+8d'=17+8(3)=17+24=41$

Thus, $a_9=a'_9=41$ showed.

$$\Rightarrow \qquad a_n:a'_n=\frac{a_n}{a'_n}=\frac{1+(n-1)(5)}{17+(n-1)(3)}=\frac{5n-4}{3n+14}$$

and $a_9:a'_{11}=\frac{5(9)-4}{3(11)+14}=\frac{45-4}{33+14}=\frac{41}{47}=41:47.$

4.4.3 Show that sum of n arithmetic means between two numbers is equal to n times their arithmetic mean

Using the results from the article (4.3.3), we can find the sum of n arithmetic means as:

$$\text{Sum}=\{a+d\}+\{a+2d\}+\cdots+\{a+nd\}$$

$$=\left\{a+1\frac{(b-a)}{(n+1)}\right\}+\left\{a+2\frac{(b-a)}{(n+1)}\right\}+\cdots+\left\{a+\frac{n(b-a)}{(n+1)}\right\}$$

$$=(a+a+a\ldots\text{upto } n \text{ terms})+\left\{\frac{1(b-a)}{n+1}+\frac{2(b-a)}{n+1}+\cdots+\frac{n(b-a)}{n+1}\right\}$$

$$=na+\frac{(b-a)}{n+1}(1+2+\cdots+n)$$

$$= na + \frac{(b-a)}{(n+1)} \times \frac{n(n+1)}{2}$$

$$= na + \frac{n(b-a)}{2} = \frac{2na + nb - na}{2}$$

$$= n\left(\frac{a+b}{2}\right) = n \text{ times the A.M. of between } a \text{ and } b.$$

Hence, the sum of n A.M's between a and b is n times the single mean between them. Therefore, the sum of p A.M's $= S_p = p\left(\frac{a+b}{2}\right)$

Similarly, the sum of q A.M's $= S_q = q\left(\frac{a+b}{2}\right)$

Note: $\frac{S_p}{S_q} = \frac{p}{q}$ or $S_p : S_q = p : q$

4.4.4 Solve real life problems involving arithmetic series

Example: A grocery store, displays cans in such a way that 27 cans are in the bottom row, 24 cans in the next row and forming an arithmetic sequence. The top row has 3 cans. Find the total number of cans in the display.

Solution: Since the display of cans are in arithmetic sequence with

$$a = 27, a_n = 3 \text{ and } d = -3$$

Now, $a_n = a + (n-1)d \Rightarrow 3 = 27 - 3n + 3$

$\Rightarrow n = 9$

Now the total number of cans is given by

$$S_n = \frac{n}{2}(a + l) = \frac{9}{2}(27 + 3)$$

$$= 9(15) = 135 \text{ cans}$$

Exercise 4.4

1. Sum the series.

(i) $3+8+13+\cdots$ to 16 terms.

(ii) $(-3)+(-1)+1+3+5+\cdots$ to 18 terms.

(iii) $4.57+4.87+5.17+\cdots$ to 22 terms.

(iv) $\frac{3}{\sqrt{2}} + 2\sqrt{2} + \frac{5}{\sqrt{2}} + \cdots$ to n terms.

(v) $\frac{3}{\sqrt{2}} + 2\sqrt{2} + \frac{5}{\sqrt{2}} + \cdots$ to 15 terms.

2. Find the number of terms of the following.

(i) $(-7)+(-5)+(-3)+\cdots$; if sum = 65

(ii) $-7+(-4)+(-1)+\cdots$; if sum = 114

(iii) $-9+(-6)+(-3)+\cdots$; if sum = 3225

(iv) $-9-6-3+0+\cdots$; if sum = 66

3. Sum the series.

(i) $3+5-7+9+11-13+15+17-19+\cdots$ to $3n$ terms.

(ii) $1+4-7+10+13-16+19+22-25+\cdots$ to $3n$ terms.

(iii) $9+12-15+18+21-24+27+30-33+\cdots$ to $3n$ terms.

4. If $S_n = n(2n-1)$, then find the series.

5. Find the sum of first 100 natural numbers which are neither exactly divisible by 3 nor by 7.

6. If S_2, S_3, S_5 are the sum of $2n, 3n$ and $5n$ terms of a_n A.P. show that $S_5 = 5(S_3 - S_2)$.

7. The sum of n terms of two arithmetic series are in the ratio of $3n+2:n+1$. Find the ratio of their 8^{th} terms.

8. The sum of three numbers in A.P. is 27 and their product is 405. Find the numbers.

9. Find five numbers in A.P. whose sum is 25 and the sum of whose squares is 135.

10. The sum of Rs. 42,000 is distributed among five persons so that each person after the first receives Rs. 80 less than the preceding person. How much does each person receive?

11. A well digging company charges Rs. 1,250 for the first meter, Rs.1500 for the second meter and Rs. 1,750 for the third meter and so on. What is the depth of a well that costs Rs. 50,000.

12. A man borrows Rs. 25,000 and agrees to repay with a total profit of Rs. 10,000 in 10 installments, each installment being less than the preceding by Rs. 200. What should be his first installment?

4.5 Geometric Sequence

4.5.1 Define a geometric sequence

A geometric sequence, also known as a geometric progression (G.P) is a sequence of numbers where each term after the first non-zero term is found by multiplying the previous one by a fixed, non-zero number called the common ratio denoted by 'r'. For example, the sequence 3, 6, 12, ...; is a geometric progression with common ratio 2.

Also, the sequence

$$2,\ 6,\ 18,\ 54,\ \ldots$$

is a G.P. because its common ratio is 3

i.e., $6 \div 2 = 3,\ 18 \div 6 = 3,$ and $54 \div 18 = 3$

4.5.2 Find the *n*th or general term of geometric sequence

If the first term of a G.P. is a and the common ratio is r, then by definition,

$$a = \text{the first term} \quad = a \quad = \quad ar^{1-1} \; ;$$
$$a_2 = \text{the second term} = ar \quad = \quad ar^{2-1} \; ;$$
$$a_3 = \text{the third term} \quad = ar^2 \quad = \quad ar^{3-1} \; ;$$
$$a_4 = \text{the fourth term} \quad = ar^3 \quad = \quad ar^{4-1} \; ;$$

and so on.

Hence, we deduce that

$$\boxed{a_n = \text{the } n\text{th term} \quad = ar^{n-1}} \qquad \ldots\text{(i)}$$

which is the formula for finding the nth term of a geometric sequence whose first term is a and the common ratio is r.

The sequence

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$$

is known as the standard form of a G.P.

Example 1. Find 9th term of the G.P: 6, 12, 24, …

Solution: Here, first term $= a = 6$

$$\text{Common ratio} = r = \frac{12}{6} = 2$$

$$9^{\text{th}} \text{ term} \quad = a_9 = ?$$

We know that

$$a_n = ar^{n-1}$$

So, $\quad a_9 = (6)(2)^{9-1} \quad = (6)(2)^8 \Rightarrow a_9 = 1536$

Thus, the required 9th term of the given G.P is 1536.

Example 2. In a G.P., $a_3 = 12$ and $a_8 = 384$. Find the nth term.

Solution: We know that $\quad a_n = ar^{n-1} \qquad \ldots$(i)

Using $n = 3$ in equation (i), $\quad a_3 = ar^{3-1} \Rightarrow ar^2 = 12 \qquad \ldots$(ii)

Now, using $n = 8$ in equation (i), $\quad a_8 = ar^{8-1} \Rightarrow ar^7 = 384 \qquad \ldots$(iii)

From equation (ii) and (iii), by division

$$\frac{ar^7}{ar^2} = \frac{384}{12} \quad \Rightarrow \quad r^5 = 32 \quad \Rightarrow r^5 = 2^5 \quad \Rightarrow \quad r = 2$$

Using $\quad r = 2$ in equation (ii)

We get $\quad a(2)^2 = 12$

$\Rightarrow \quad 4a = 12 \quad$ or $\quad a = 3$

Now, $\quad a_n = ar^{n-1}$

Using $a = 3$ and $r = 2$

We get $\quad a_n = 3(2)^{n-1}$

Hence, the required nth term $\quad a_n = 3(2)^{n-1}$

Example 3. Suppose that the fourth term of a geometric sequence is 81 and the sixth term is 729. Find the first term and common ratio of the sequence.

Solution: Here, fourth term $= a_4 = 81$ and 6th term $= a_6 = 729$.

We have to find r and a.

We know that $a_n = ar^{n-1}$

$a_4 = ar^{4-1} \Rightarrow 81 = ar^3$ ...(i)

and $a_6 = ar^{6-1} \Rightarrow 729 = ar^5$ ...(ii)

From equation (ii) and (i), by division we have

$$\frac{729}{81} = \frac{ar^5}{ar^3} \Rightarrow r = \pm 3$$

Taking $r = 3$ in eq. (i), we get

$$81 = a(3)^3 \Rightarrow a = 3$$

Here $a = 3$ and $r = 3$ the required geometric sequence will be $3, 9, 27, 81, \ldots$

Taking $r = -3$ in eq. (i), we get

$$81 = a(-3)^3 \Rightarrow a = -3$$

Here $a = -3$ and $r = -3$, the required geometric sequence will be

$$-3, 9, -27, 81, \ldots$$

Example 4. Find three consecutive numbers in G.P. whose sum is 14 and their product is 64.

Solution: Let the three numbers in G.P. are: a, ar and ar^2

As, sum of three numbers = 14

so, $a + ar + ar^2 = 14$

$a(1 + r + r^2) = 14$ ... (1)

Now, product of three numbers = 64

$(a) \times (ar) \times (ar^2) = 64$

$\Rightarrow (ar)^3 = (4)^3$

$\Rightarrow ar = 4$ ...(2)

From equation (1) by (2), by division

$$\frac{a(1 + r + r^2)}{ar} = \frac{14}{4}$$

$$\Rightarrow \frac{(1 + r + r^2)}{r} = \frac{7}{2}$$

$$\Rightarrow 2(1 + r + r^2) = 7(r)$$

$$\Rightarrow 2 + 2r + 2r^2 = 7r$$

$$\Rightarrow 2r^2 - 5r + 2 = 0$$

$$\Rightarrow (r - 2)(2r - 1) = 0$$

$$\Rightarrow r = 2 \quad \text{or} \quad r = \frac{1}{2}$$

By using $r = 2$ in equation (2), we get: $a(2) = 4$ or $a = 2$.

When $a = 2$ and $r = 2$ then

First number $= a = 2$
Second number $= ar = 2(2) = 4$
Third number $= ar^2 = 2(2)^2 = 8$

Hence the three numbers are: 2, 4 and 8.

By using $r = \frac{1}{2}$ in equation (2), we get: $a\left(\frac{1}{2}\right) = 4$ or $a = 8$.

When $a = 8$ and $r = \frac{1}{2}$, then

First number $= a = 8$
Second number $= ar = 8\left(\frac{1}{2}\right) = 4$
Third number $= ar^2 = 8\left(\frac{1}{2}\right)^2 = 2$

Hence the three numbers are: 8, 4 and 2.
Thus the required numbers are 2, 4, 8 or 8, 4, 2.

Note: For the sake of convenience, we take

two numbers in G.P as: $\frac{a}{r}$, ar

four numbers in G.P as: $\frac{a}{r^3}, \frac{a}{r}, ar, ar^2$ and so on.

Similarly,

three numbers in G.P as: $\frac{a}{r}$, a, ar

five numbers in G.P as: $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$ and so on.

4.5.3 Solve problems involving geometric sequence

Example 1. If the population of a town increases geometrically at the rate of 7% annually and the present population is 70,000. What will be the population after 6 years from now?

Solution: Present population $= a = 70{,}000$

$\because$ Number of years $= n = 6$
$\because$ Number of terms $= n = 7$

Rate of increase of population $= r = 1 + 7\,\% = 1.07$
Population after 6 years $= a_7 = ?$

We know that $a_n = ar^{n-1}$

$a_7 = 70{,}000(1.07)^{7-1} = 70{,}000\ (1.07)^6$

$a_7 = 70{,}000(1.500730) = 105051$ approx.

Hence population after 7 years will be 105051.

Example 2. The number of bacteria in a culture increased in G.P from 1250 to 10,000 in 4 days. Find the daily rate of increase, assuming the rate of increase to be constant.

Solution: Here $a = 1250, a_n = 10,000, n = 4$ and $r = ?$
We know that

$$a_n = ar^{n-1}$$
$$10,000 = 1250r^{4-1}$$
$$\frac{10,000}{1250} = r^3$$
$$8 = r^3 \quad \text{or} \quad 2^3 = r^3 \quad \text{or} \quad r = 2$$

Hence the rate of increase of bacteria is 200%.

Exercise 4.5

1. Find the indicated terms of each of the following G.P.
(i) $3, 6, 12, \ldots; 5th$ term (ii) $32, 16, 8, \ldots;$ 9^{th} term
(iii) $4, 2\sqrt{2}, 2 \ldots,$ 12^{th} term (iv) $i, 1, -i, \ldots;$ 16^{th} term
(v) $b^2 - c^2, b + c, \frac{b+c}{b-c}, \ldots,$ 9^{th} term

2. In a G.P, $a_1 = \frac{5}{9}, a_6 = \frac{15625}{9}$. Find its nth term.

3. Find the nth term of the geometric sequence if:
$\frac{a_5}{a_3} = \frac{4}{9}$ and $a_2 = \frac{4}{9}$

4. Show that the reciprocals of the terms of the G.P. $a_1, a_1r^2, a_1r^4, \ldots$ form another G.P.

5. How many terms are in G.P. $\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \ldots, \frac{1}{196608}$.

6. Find three consecutive numbers in G.P. whose sum is 39 and their product is 729.

7. The number of bacteria in a culture increased in G.P from 515,000 to 15,45,000 in 7 days. Find the daily rate of increase, assuming the rate of increase to be constant.

8. Find the profit on Rs. 1000 for 5 years at 4% per annum compound profit.

4.6 Geometric Mean

4.6.1 Know geometric mean between two numbers

When three numbers are in G.P., the middle one is called their geometric mean, i.e., if numbers a, G, b are in G.P. then G is called a geometric mean of a and b where a and b are called extremes.

Similarly, if $a, G_1, G_2, G_3 \ldots, G_n, b$ are in G.P, then $G_1, G_2, G_3 \ldots, G_n$ are n geometric means between a and b.

Example: (i) As, 2, 6, 18 are in G.P, so 6 is a G.M between 2 and 18.
(ii) As, 1, 3, 9, 27, 81 are in G.P, so 3, 9, 27 are three geometric means between 1 and 81.

4.6.2 Insert a geometric mean between two numbers

Let G be a G.M. between a and b, then a, G, b are in G.P.

So that the common ratio is the same, i.e. $\frac{G}{a}$ = common ratio = $\frac{b}{G}$

Now, $G^2 = ab$ or $G = \pm\sqrt{ab}$

According to definition, $+\sqrt{ab}$ and $-\sqrt{ab}$ are two possible geometric means between a and b.

Thus, a geometric mean between two numbers is equal to the square root of their product.

Example: Find G.M between 3 and 6.

Solution: Here, $a = 3$ and $b = 6$

We know that $G = \pm\sqrt{ab} = \pm\sqrt{(3)(6)} = \pm 3\sqrt{2}$

Hence the required G.M between 3 and 6 is $\pm 3\sqrt{2}$.

4.6.3 Insert *n* geometric means between two numbers

Let $G_1, G_2, G_3, \ldots, G_n$ be n geometric means between any two given numbers a and b. Then $a, G_1, G_2, G_3, \ldots, G_n, b$ is a geometric progression with a as its first term and b its $(n+2)$th term.

Suppose the common ratio of this G.P. is r.

So, $a_{n+2} = b = ar^{(n+2)-1} \Rightarrow b = ar^{n+1} \Rightarrow \frac{b}{a} = r^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

So, G_1 = the second term $= ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

G_2 = the third term $= ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$

… … … … … … … … … … … …

and G_n = the $(n+1)$th term $= ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

Hence, $$a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \ldots, a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

are the required n G.M's between a and b.

Now, the product of all geometric means = $G_1 . G_2 . G_3 \ldots G_n$

$$= a\left(\frac{b}{a}\right)^{\frac{1}{n+1}} \cdot a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} \cdot \ldots \cdot a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

$$= a^n\left(\frac{b}{a}\right)^{\frac{(1+2+3+\cdots+n)}{n+1}}$$

$$= a^n\left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}} = a^n\left(\frac{b}{a}\right)^{\frac{n}{2}}$$

$$= (ab)^{\frac{n}{2}} = \left(\sqrt{ab}\right)^n$$

Hence the product of n G.Ms between two numbers a and b is the n^{th} power of G.M of a and b.

Example: Insert three G.Ms between 3 and 243

Solution: Let, G_1, G_2, G_3 be three geometric means between 3 and 243 Then $3, G_1, G_2, G_3, 243$ is a G.P.

Here, $a = 3$ and $a_5 = 243$

Using $a_n = ar^{n-1}$,

$a_5 = ar^{5-1}$ or $243 = 3r^4$ or $r^4 = 81$

Therefore, $r = \pm 3$

If $r = 3$

then $G_1 = ar = 3(3) = 9$,

Similarly, $G_2 = G_1 r = 9(3) = 27$,

and $G_3 = G_2 r = 27(3) = 81$

If $r = -3$

$G_1 = ar = 3(-3) = -9$,

Similarly, $G_2 = G_1 r = -9(-3) = 27$,

$G_3 = G_2 r = 27(-3) = -81$

Hence, the three geometric means between 3 and 243 are 9, 27, 81 or −9, 27, −81.

Exercise 4.6

1. Find the G.M. between:

(i)	-2 and -8	(ii)	3 and $\frac{1}{3}$
(iii)	$8\sqrt{2}$ and $9\sqrt{2}$	(iv)	$-2i$ and $8i$
(v)	$7+i$ and $7-i$	(vi)	$\frac{16}{9}$ and $\frac{9}{25}$

2. Insert:
   (i) Two G.Ms between 3 and 81.
   (ii) Three G.Ms between 2 and $\frac{1}{2}$.
   (iii) Four G.Ms between 3 and 96.
   (iv) Five G.Ms between 16 and 1024.

3. The A.M. between two numbers is 29 and the geometric mean is 21, find the numbers.
4. For what value of n, is $\frac{a^{n-2}+b^{n-2}}{a^{n-3}+b^{n-3}}$ the G.M. between a and b?
5. Show that the nth root of the product of n geometric means between x and y is the geometric mean between x and y.
6. The A.M. of two positive integral numbers exceeds their (positive) G.M by 2 and their sum is 20. Find the numbers.
7. The A.M. between two numbers is 5 and their (positive) G.M. is 4. Find the numbers.

4.7 Geometric Series

4.7.1 Define a geometric series

If $a_1, a_2, a_3, \ldots, a_n$ is a geometric sequence, then the expression $a_1 + a_2 + a_3 + \cdots + a_n$ is called a geometric series. If the series consists of a finite number of terms, it is called a finite geometric series; otherwise it is called an infinite geometric series.

In general, a geometric series with a as its first term and r as its common ratio is: $a + ar + ar^2 + \cdots$

The following are some examples of geometric series:

(i) $2 + 4 + 8 + 16 + 32 + \cdots + 2^n$

(ii) $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots + \frac{2}{3^n}$

(iii) $\frac{1}{y} - \frac{1}{y^2} + \frac{1}{y^3} - \cdots + \frac{(-1)^{n+1}}{y^n}$

4.7.2 Find the sum of *n* terms of a geometric series

In general, a geometric series of n terms with a as its first term and r as its common ratio is: $a + ar + ar^2 + \cdots + ar^{n-1}$

If S_n denotes the sum to n terms of the series, we have

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad \ldots. \text{(i)}$$

Multiplying each side by r, we have

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad \text{.... (ii)}$$

Hence, from (i) and (ii), by subtraction:

$$S_n - rS_n = a - ar^n \quad \text{or} \quad S_n(1-r) = a(1-r^n).$$

If $r \neq 1$, dividing by $(1-r)$, we get $S_n = \frac{a(1-r^n)}{1-r}$ when $r < 1$ .... (iii)

or $S_n = \frac{a(r^n-1)}{r-1}$ when $r > 1$ ...(iv)

If l denotes the last term, i.e., if $l = a_n = ar^{n-1}$; then we have

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a - ar^n}{1-r} = \frac{\{a - r(ar^{n-1})\}}{1-r}$$

i.e., $S_n = \frac{a-rl}{1-r}$ when $r < 1$ .... (v)

or $S_n = \frac{rl-a}{r-1}$ when $r > 1$ .... (vi)

Thus, we can use the formula (v) and (vi) to find the sum of geometric series when its last term l is given.

Note: If $r = 1$, every term of the series is equal to a : hence the sum of the first n terms $= S_n = a + a + a + \cdots$ to n terms, i.e. $S_n = na$.

Example 1.

Find the sum of the first 7 terms of the G.P: $-4, 12, -36, \ldots$

Solution: Here $a = -4$ and $r = -3$. Using the formula $S_n = \frac{a(1-r^n)}{1-r}$,

$$S_7 = \frac{(-4)\{1-(-3)^7\}}{1-(-3)} = \frac{(-4)\{1-(-2187)\}}{4} = -2188$$

Thus the required sum of given G.P is -2188.

Example 2. Find the sum of n terms of the geometric series if $a_n = (-3)\left(\frac{2}{5}\right)^n$.

Solution: As we know

$$a_n = ar^{n-1} \quad \text{...(i)}$$

Given general term is: $a_n = (-3)\left(\frac{2}{5}\right)^n$

or $a_n = -3\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)^{n-1} = \left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$, that is $a_n = \left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$

Comparing it with eq. (i), we get $a = -\frac{6}{5}$ and $r = \frac{2}{5} < 1$

Thus, $S_n = \frac{a(1-r^n)}{1-r} = \frac{-\frac{6}{5}\left[1-\left(\frac{2}{5}\right)^n\right]}{1-\frac{2}{5}} = \left(-\frac{6}{5}\right)\left(\frac{5}{3}\right)\left[1-\left(\frac{2}{5}\right)^n\right]$

Thus, the sum of n terms of the geometric series is: $S_n = (-2)\left[1-\left(\frac{2}{5}\right)^n\right]$

4.7.3 Find the sum of an infinite geometric series

Consider the infinite geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \qquad \ldots\text{(i)}$$

We know that the sum S_n, of the first n terms of the infinite geometric series (i), is given by

$$S_n = a + ar + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

i.e., $$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r} \qquad \ldots\text{(ii)}$$

The following three cases arise for an infinite geometric series.

Case I: when $r = 1$

If $r = 1$, then sum of the first n terms of infinite geometric series is $S_n = a + a + a + \cdots$ to n terms $= na$.

Since a is constant, as the number of terms tends to infinity, the sum also tends to infinity and we say that series is divergent. Thus, the sum of an infinite number of terms, S, of a G.P. is infinite, if $r = 1$.

For example, the sum of the infinite geometric series

$$2 + 2 + 2 + \cdots$$

with $r = 1$ is infinite or that the series is divergent.

Case II: when $|r| > 1$

In this case the sum of the series also tends to infinity as the number of terms tends to infinity. This result is otherwise also obvious, for if, the absolute value of r, $|r| > 1$, then r^n and consequently ar^n tends to infinity as n tends to infinity. Thus, the sum to an infinite number of terms, of a G.P. is infinite, if $|r| > 1$.

For example, the sum of the infinite geometric series $2 - 6 + 18 - 54 + \cdots$ with $|r| = |-3| > 1$, is infinite or that the series is divergent.

Case III: when $|r| < 1$

If $|r| < 1$, then the value of r^n and consequently of $\frac{ar^n}{1-r}$ tends to zero as n tends to infinity. But as n tends to infinity the first part of (ii); viz. $\frac{a}{1-r}$ remains unaffected.

Thus, the sum to an infinite number of terms of a G.P. is $S = \frac{a}{1-r}$, if $|r| < 1$

For example, the sum of the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$

with $a = \frac{1}{2}$ and $|r| = \frac{1}{2} < 1$ is $\dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ and we say that the series is convergent.

Note: (i) If the sum of a series is a definite finite value it, the series is said to be convergent and if its sum is infinity, it is said to be divergent.
(ii) Infinite geometric series is convergent if $|r| < 1$ and it is divergent if $|r| > 1$ or $r = 1$.

Example 1. Find the sum of the infinite series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

Solution:

Here, $a = 1$ and $|r| = \left|-\frac{1}{2}\right| < 1$

So, $S = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$

Example 2. If $a = 1 - x + x^2 - x^3 + \cdots$ where $|x| < 1$
$b = 1 + x + x^2 + x^3 + \cdots$ where $|x| < 1$

show that $2ab = a + b$

Solution: Here $a = \frac{1}{1-(-x)}$ $\because$ $(r = -x)$

$\Rightarrow$ $a = \frac{1}{1+x} \Rightarrow 1 + x = \frac{1}{a}$ ...(i)

Similarly, $b = \frac{1}{1-x}$ $(\because r = x)$

$\Rightarrow$ $1 - x = \frac{1}{b}$ ...(ii)

Adding (i) and (ii), we obtain $2 = \frac{1}{a} + \frac{1}{b}$ $\Rightarrow$ $2ab = a + b$

4.7.4 Convert recurring decimal fraction into an equivalent common fraction

A repeating or recurring decimal is the decimal representation of a number whose digits are periodic, (repeating its values at regular intervals). We convert recurring decimal into equivalent common fraction with the help of following example:

Example: Convert the recurring decimal $1.\dot{3}$ into an equivalent common fraction (Vulgar fraction).

Solution: $1.\dot{3} = 1.33333\ldots = 1 + 0.33333\ldots = 1 + 0.3 + 0.03 + 0.003 + \cdots$ ...(i)

Now, consider the series $0.3 + 0.03 + 0.003 + \cdots$

Here, $a = 0.3$ and $r = \frac{0.03}{0.3} = 0.1$

We know that $S = \frac{a}{1-r} = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}$

Hence, $0.3 + 0.03 + 0.003 + \cdots = \frac{1}{3}$

Thus the given recurring decimal $1.\dot{3}$ is converted into an equivalent common fraction using equation (i)

i.e., $1.\dot{3} = 1 + \frac{1}{3} = \frac{3+1}{3} = \frac{4}{3}$

4.7.5 Solve real life problems involving geometric series

Example 1. A business man earned Rs. 10,000 in the first month, Rs. 15,000 in the second month, Rs. 22,500 in the third month and so on. Find the total amount he will earn in 12 months.

Solution:

Amount earned in first month $= a = Rs.\,10{,}000$

Amount earned in second month $= a_2 = Rs.\,15{,}000$

Amount earned in third month $= a_3 = Rs.\,22{,}500$

Amount earned by 12 months $= S_{12} = ?$

We have geometric series:

$$10000 + 15000 + 22500 + \cdots \text{ to 12 terms.}$$

Here, $a = 10000$ and $r = \frac{15000}{10000} = \frac{3}{2} = 1.5 > 1$

We know that

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{10000\left[\left(\frac{3}{2}\right)^{12} - 1\right]}{\frac{1}{2}} = 20000\left(\frac{531441 - 4096}{4096}\right)$$

$$= 2574926.75$$

$$= 2{,}575{,}000 \text{ rupees (approximately)}$$

Hence, the total amount in 12 months is 2,575,000 rupees.

Example 2. The starting salary of a peon was Rs. 8000 and after each subsequent year his salary was increased by 15%. What total amount of salary he got for the first twelve years?

Solution: Initial salary of the peon $= a = \text{Rs.}\,8000$

Rate of increase in salary $= r = 1 + 15\% = 1 + 0.15 = 1.15$

Number of years $= 12$

We know that

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{12} = \frac{8000(1.15^{12} - 1)}{1.15 - 1}$$

$$S_{12} = \frac{8000(5.35025 - 1)}{0.15}$$

$$S_{12} = \frac{34802.00084}{0.15} = 232013.339$$

Total amount of salary of the peon will be Rs. 232013 (approximately).

Exercise 4.7

1. Find the sum:

(i) $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{512}$ (ii) $2^6+2^7+2^8+\cdots+2^{13}$

(iii) $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots+\frac{1}{x^8}$

(iv) $0.9+0.99+0.999+\cdots$ to n terms

(v) $7+77+777+\cdots$ to n terms.

(vi) $0.2+0.22+0.222+\cdots$ to n terms

(vii) $3+33+333+\cdots$ to n terms

2. Find the sum to n terms of the following series:

(i) $1+(a+b)+(a^2+ab+b^2)+(a^3+a^2b+ab^2+b^3)+\cdots;$ where $a>1, b>1$ and $a>b$

(ii) $r+(1+k)r^2+(1+k+k^2)r^2+\cdots$, where k and r are proper fractions.

3. If $a_n=\left(\frac{1}{4}\right)^n$ then find its sum up to n terms.

4. Find the sum of the following infinite geometric series.

(i) $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots$ (ii) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$

(iii) $\frac{9}{4}+\frac{3}{2}+1+\frac{2}{3}+\cdots$ (iv) $2+\sqrt{2}+1+\cdots$

(v) $2+1+0.5+\cdots$ (vi) $4+2\sqrt{2}+2+\sqrt{2}+1+\cdots$

(vii) $0.1+0.05+0.025+\cdots$

5. Find the Vulgar fraction equivalent to the following recurring decimals.

(i) $0.\dot{7}$ (ii) $1.\dot{7}\dot{4}$ (iii) $0.\dot{2}\dot{5}\dot{9}$

(iv) $1.1\dot{4}\dot{8}$ (v) $2.\dot{2}\dot{3}$

6. If $a=\frac{b}{2}+\frac{b^2}{4}+\frac{b^3}{8}+\cdots$ if $0<b<2$, then prove that $b=\frac{2a}{1+a}$.

7. The sum of an infinite geometric series is half the sum of the squares of its terms. If the sum of its first two terms is $4\frac{1}{2}$, find the series.

8. Joining the midpoints of the sides of an equilateral triangle, an equilateral triangle having half the perimeter of the original triangle is obtained. We form a sequence of nested equilateral triangles in this manner with the original triangle having perimeter $\frac{5}{2}$. What will be the total perimeter of all the triangles formed in this way?

4.8 Harmonic Sequence

4.8.1 Recognize a harmonic sequence

A sequence is said to be a harmonic sequence or a harmonic progression (H.P.) if the reciprocals of its terms are in arithmetic progression.

The harmonic progression derives its name from the fact that musical strings of equal thickness and tension will produce harmony if their lengths are to one another as the reciprocals of the natural numbers.

Examples of harmonic sequence

(i) $\frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \ldots;$ is an H.P. Its corresponding A.P is: 5, 10, 15,…

(ii) $\frac{4}{5}, \frac{2}{5}, \frac{4}{15}, \ldots;$ is an H.P. Its corresponding A.P is: $\frac{5}{4}, \frac{5}{2}, \frac{15}{4}, \ldots$

Since the general form of an A.P. is: $a, a+d, a+2d, \ldots, a+(n-1)d.$

Therefore H.P. is a sequence of the form: $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots, \frac{1}{a+(n-1)d}$ provided that none of the denominators is zero.

Hence to every H.P, there is a corresponding A.P, the terms of which are the reciprocals of the corresponding terms of the H.P.

Most of the examples on an H.P. can be worked out by converting the given H.P into A.P and making use of the corresponding A.P.

Example: Find the nth term and 8th term of H.P: $\frac{1}{8}, \frac{1}{26}, \frac{1}{44}, \ldots$

Solution: Corresponding A.P is 8,26,44, …

Here, $a = 8$ and $d = 26 - 8 = 18$

Using these values in, $a_n = a + (n-1)d,$

we have $a_n = 8 + (n-1)18$

$\Rightarrow$ $a_n = 18n - 10$

Thus the nth term of the given H.P is $T_n = \frac{1}{a_n} = \frac{1}{18n-10}$

so, $T_8 = \frac{1}{18 \times 8 - 10} = \frac{1}{134}.$

4.8.2 Find *n*th term of harmonic sequence

Since the reciprocals of the terms of H.P form A.P. Therefore, the nth term of the harmonic progression is equal to the reciprocal of the nth term of the corresponding A.P. Thus, the formula to find the nth term of the harmonic progression sequence is given as:

The nth term of the Harmonic Progression (H.P) is: $T_n = \frac{1}{[a+(n-1)d]}$, where

"a" is the first term of corresponding A.P

"d" is the common difference of corresponding A.P

"n" is the number of terms of corresponding A.P

We will also find the general or nth term when the first two terms of an H.P. are given.

Let a and b respectively be the first and the second terms of an H.P. Then $\frac{1}{a}$ and $\frac{1}{b}$ are, respectively, the first and the second terms of the corresponding A.P. If d be the common difference of this A.P., then the common difference can be calculated as: $d = \frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab}$.

Thus, the nth term of this A.P.

$$a_n = \frac{1}{a} + (n-1)\left\{\frac{a-b}{ab}\right\} = \frac{b + (n-1)\ (a-b)}{ab}$$

Hence the corresponding nth term of the H.P is:

$$T_n = \frac{ab}{b + (n-1)(a-b)}$$

Example 1. Find the 5th term of H.P: $\frac{1}{3}, \frac{1}{10}, \frac{1}{17}, \ldots$

Solution: The given harmonic sequence is: $\frac{1}{3}, \frac{1}{10}, \frac{1}{17}, \ldots$

Here , $a = 3, \quad n = 5, \quad d = 7$

We know that, $a_n = \frac{1}{a+(n-1)d}$

$\Rightarrow a_5 = \frac{1}{3+(5-1)7} = \frac{1}{31}$

Thus, the fifth term of the given H.P. is $\frac{1}{31}$.

Example 2. Find the 8th term of the H.P. $\frac{4}{5}, \frac{2}{5}, \frac{4}{15}, \ldots$

Solution:

Here H.P. is $\frac{4}{5}, \frac{2}{5}, \frac{4}{15}, \ldots$

We know that $T_n = \frac{ab}{b+(n-1)\ (a-b)}$ where, $a = \frac{4}{5}, \quad b = \frac{2}{5}$ and $n = 8$

So, $$T_8 = \frac{\left(\frac{4}{5}\right)\left(\frac{2}{5}\right)}{\frac{2}{5}+(8-1)\left(\frac{4}{5}-\frac{2}{5}\right)} = \frac{\frac{8}{25}}{\frac{2}{5}+\frac{14}{5}} = \frac{\frac{8}{25}}{\frac{16}{5}}$$

$$= \frac{8}{25} \times \frac{5}{16} = \frac{1}{10}$$

Example 3. If the 3rd term and 7th term of an H.P. are $\frac{2}{9}$ and $\frac{2}{25}$ respectively, find the sequence.

Solution: Since the 3rd term of the H.P. is $\frac{2}{9}$ and its 7th term is $\frac{2}{25}$, therefore the 3rd and 7th terms of the corresponding A.P. are $\frac{9}{2}$ and $\frac{25}{2}$ respectively.

Now taking a, the first term and d, the common difference of the corresponding A.P., we have

$$a + 2d = \frac{9}{2} \qquad \text{(3rd term)} \qquad \ldots\text{(i)}$$

and $$a + 6d = \frac{25}{2} \qquad \text{(7th term)} \qquad \ldots\text{(ii)}$$

Subtracting (i) from (ii), gives: $4d = \frac{25}{2} - \frac{9}{2} \Rightarrow d = 2$

From (i), we get $a = \frac{9}{2} - 2d = \frac{9}{2} - 4 = \frac{1}{2}$

Thus a_2 of the A.P. $= a + d = \frac{1}{2} + 2 = \frac{5}{2}$

and a_4 of the A.P. $= a + 3d = \frac{1}{2} + 3(2) = \frac{13}{2}$

Hence the required H.P. is $\frac{2}{1}, \frac{2}{5}, \frac{2}{9}, \frac{2}{13}, \ldots$

Exercise 4.8

1. Find the indicated terms in the following harmonic progressions.

(i)	$\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \ldots$; 9th term	(ii)	$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$; 12th term
(iii)	$\frac{9}{5}, \frac{9}{13}, \frac{9}{21}, \ldots$; 8th term	(iv)	$-\frac{1}{5}, -\frac{1}{3}, -1, \ldots$; 15th term
(v)	$\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$; 8th term	(vi)	$\frac{1}{3}, \frac{1}{8}, \frac{1}{13}, \ldots$; 11th term

2. If the 7th and 10th terms of an H.P are $\frac{1}{3}$ and $\frac{5}{21}$ respectively, find its 14th and 20th terms.

3. If the sum of first and sixth terms of H.P. is $\frac{31}{116}$. Find the harmonic sequence if the first term is $\frac{1}{4}$.

4. The first term of H.P is $-\frac{1}{3}$ and fifth term is $\frac{1}{5}$. Find its 9th term.

5. If the pth term of an H.P. is q, the qth term is p: prove that the $(p + q)$th term is $\frac{pq}{(p+q)}$.

4.9 Harmonic Mean

4.9.1 Define a harmonic mean

When three numbers are in Harmonic Progression (H.P), then the middle one is called their harmonic mean i.e., if a,H,b are in H.P., then H is called the harmonic mean (H.M.) of a and b. When more than three numbers are in H.P., all the numbers between the extreme numbers are called Harmonic Means. That is, if

$$a,\ H_1, H_2,\ H_3, \ldots, H_n, b \text{ are in H.P.}$$

Then there are $n+2$ numbers in H.P., and $H_1, H_2, H_3, \ldots, H_n$ are termed as the n harmonic means between a and b.

For example,

(i) $\frac{2}{5}$ is the H.M. of 1 and $\frac{1}{4}$ because $1, \frac{2}{5}, \frac{1}{4}$ form an H.P.

(ii) $\frac{3}{5}, \frac{3}{7}, \frac{3}{9}, \frac{3}{11}, \frac{3}{13}, \frac{3}{15}, \frac{3}{17}$ and $\frac{3}{19}$ are the eight H.Ms between 1 and $\frac{3}{21}$ because

$$1, \frac{3}{5}, \frac{3}{7}, \frac{3}{9}, \frac{3}{11}, \frac{3}{13}, \frac{3}{15}, \frac{3}{17}, \frac{3}{19}, \frac{3}{21} \text{ form an H.P}$$

4.9.2 Insert a harmonic mean between two numbers

Let H be the H.M. between a and b. Then a, H, b are in H.P.

So, $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in corresponding A.P. and $\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$

or $$\frac{2}{H} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$$

Therefore, $$H = \frac{2ab}{a+b}$$

Thus, the harmonic mean of any two numbers is equal to twice their product divided by their sum.

Example: Find the harmonic mean of 2 and 5.

Solution: Here $a = 2$ and $b = 5$, using the formula of harmonic mean:

$$H = \frac{2ab}{a+b}$$

We get, $$H = \frac{2(2)(5)}{2+5} = \frac{20}{7}$$

Thus, the harmonic mean between 2 and 5 is $\frac{20}{7}$.

4.9.3 Insert n harmonic means between two numbers

Let, $H_1, H_2, \ldots, H_n$ be the n harmonic means between any two given numbers a and b. Then the corresponding A.P. will be:

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \ldots, \frac{1}{H_n}, \frac{1}{b}$$

Let d be the common difference of this A.P.,

So, $\qquad \frac{1}{b} = a_{n+2} = \frac{1}{a} + \{(n+2) - 1\}d$

i.e. $\qquad \frac{1}{b} = \frac{1}{a} + (n+1)d$

or $\qquad d = \frac{a-b}{(n+1)ab}$

Now, $\qquad \frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{(a-b)}{(n+1)ab} = \frac{a+nb}{(n+1)ab}$

Therefore, $\qquad H_1 = \frac{(n+1)ab}{a+nb}$

Now, $\qquad \frac{1}{H_2} = \frac{1}{a} + 2d = \frac{1}{a} + 2\frac{(a-b)}{(n+1)ab} = \frac{2a+(n-1)b}{(n+1)ab}$

So, $\qquad H_2 = \frac{(n+1)ab}{2a+(n-1)b}$

Similarly, $\qquad H_3 = \frac{(n+1)ab}{3a+(n-2)b}$

and $\qquad H_n = H_{n-1} + d = \frac{(n+1)ab}{na+b}$

Hence, $\qquad \frac{(n+1)ab}{a+nb}, \frac{(n+1)ab}{2a+(n-1)b}, \frac{(n+1)ab}{3a+(n-2)b}, \ldots, \frac{(n+1)ab}{na+b}$

are n harmonic means between a and b.

Relation Between Arithmetic, Geometric and Harmonic Means

Following are two important relations involving the arithmetic, the geometric and the harmonic means between two given numbers.

(i) If A, G and H are respectively the arithmetic, the geometric and the harmonic means between any two given numbers, then A, G, H are in G.P.

$$\text{i.e.,} \quad \frac{G}{A} = \frac{H}{G} \quad \text{or} \quad G^2 = AH$$

(ii) If A, G and H are respectively the arithmetic, the positive geometric and the harmonic means between any two real, positive and unequal numbers then A > G > H.

Example 1. If A.M and H.M between two numbers are 5 and $\frac{21}{5}$ respectively. Find the numbers.

Solution: Let the numbers are a and b

$$\because \quad A.M = 5$$

$$\therefore \quad \frac{a+b}{2} = 5$$

$$a + b = 10 \qquad \ldots \text{(i)}$$

and
$$H.M = \frac{21}{5}$$

i.e.,
$$\frac{2ab}{a+b} = \frac{21}{5} \qquad \ldots \text{(ii)}$$

From equation (i), using $a + b = 10$ in equation (ii), we get

$$\frac{2ab}{10} = \frac{21}{5}$$

$$\Rightarrow 2ab = \frac{21 \times 10}{5}$$

$$\Rightarrow ab = 21 \qquad \ldots \text{(iii)}$$

From equation (i), we get $b = 10 - a$ ... (iv)

Using $b = 10 - a$ in equation (iii),

we get
$$a(10 - a) = 21$$

$$a^2 - 10a + 21 = 0$$

$$\Rightarrow (a - 7)(a - 3) = 0$$

$$\Rightarrow a = 7 \text{ and } a = 3$$

Using $a = 7$ and $a = 3$ in equation (iv), we get

$b = 10 - 7 = 3$ and $b = 10 - 3 = 7$

Hence the required numbers are 3, 7 or 7, 3.

Example 2. If G.M and H.M between two numbers are 15 and $\frac{75}{13}$ respectively. Find the numbers.

Solution: Let the numbers be a and b

$$\because \quad G.M = 15$$

$$\therefore \quad \pm\sqrt{ab} = 15$$

Squaring both sides

$$\left(\pm\sqrt{ab}\right)^2 = (15)^2$$

$$ab = 225 \qquad \ldots \text{(i)}$$

and
$$H.M = \frac{75}{13}$$

$$\frac{2ab}{a+b} = \frac{75}{13} \quad \ldots \text{(ii)}$$

From equation (i), using $ab = 225$ in equation (ii), we get

$$\frac{2(225)}{a+b} = \frac{75}{13}$$

$$a + b = \frac{26 \times 225}{75}$$

$$a + b = 78 \quad \ldots\text{(iii)}$$

From equation (iii), we get $b = 78 - a$ ...(iv)

Using $b = 78 - a$ in equation (i),

$$a(78 - a) = 225$$

$$\Rightarrow \quad a^2 - 78a + 225 = 0$$

$$\Rightarrow \quad (a - 75)(a - 3) = 0$$

$$\Rightarrow \quad a = 75 \text{ and } a = 3$$

Using $a = 75$ and $a = 3$ in equation (iv), we get

$$b = 78 - 75 = 3 \text{ and } b = 78 - 3 = 75$$

Hence the required numbers are 3, 75 or 75, 3.

Exercise 4.9

1. Insert the H.M between:

(i) $\frac{1}{64}$ and $\frac{1}{81}$ (ii) $\frac{1}{5}$ and $\frac{1}{17}$ (iii) $3 + \sqrt{2}$ and $3 - \sqrt{2}$

(iv) $2 + 3i$ and $2 - 3i$ (v) $\frac{1}{7}$ and $\frac{1}{11}$ (vi) 3 and 7

2. Insert four H.Ms between:

(i) 4 and 20 (ii) $\frac{1}{9}$ and $\frac{1}{81}$

3. Insert five harmonic means between $\frac{1}{4}$ and $\frac{1}{24}$

4. Prove that the square of the geometric mean of two numbers equals the product of the A.M and the H.M of the two numbers.

5. Find n so that $\frac{x^{n-5}+y^{n-5}}{x^{n-6}+y^{n-6}}$ may be H.M. between x and y.

6. If 5 is the harmonic mean between 2 and b, find b?

7. Find the arithmetic, harmonic and geometric means between 1 and 9. Also verify that $AH = G^2$.

8. The A.M of two numbers is 8 and H.M is 6. Find the numbers.

9. The H.M of two numbers is $\frac{24}{5}$ and G.M is 6. Find the numbers.

Review Exercise 4

1. Select correct option.

i. A sequence is a function whose domain is set of:

(a) integers (b) rational numbers

(c) natural numbers (d) real numbers

ii. If H is the harmonic mean between x and y then H is:

(a) $\frac{2(x+y)}{xy}$ (b) $\frac{x+y}{2xy}$ (c) $\frac{2xy}{x+y}$ (d) $\frac{xy}{x+y}$

iii. If $a_n - a_{n-1} = n + 1$ and $a_4 = 14$ then $a_5 =$ -------------:

(a) 3 (b) 5 (c) 14 (d) 20

iv. A sequence $\{a_n\}$ in which $a_{n+1} - a_n$ is the same number for all $n \in \mathbb{N}$ is called:

(a) A.P (b) G.P (c) H.P (d) None of these

v. If a_{n-1}, a_n, a_{n+1} are in A.P, then a_n is called:

(a) A.M (b) G.M (c) H.M (d) Mid-point

vi. Arithmetic mean between c and d is:

(a) $\frac{c+d}{2}$ (b) $\frac{c+d}{2cd}$ (c) $\frac{2cd}{c+d}$ (d) $\frac{2}{c+d}$

vii. The harmonic mean between $\sqrt{2}$ and $3\sqrt{2}$ is:

(a) $4\sqrt{2}$ (b) $\frac{4}{\sqrt{2}}$ (c) $\frac{3}{2}\sqrt{2}$ (d) None of these

viii. For any G.P the common ratio r is equal to:

(a) $\frac{a_n}{a_{n+1}}$ (b) $\frac{a_{n-1}}{a_n}$ (c) $\frac{a_n}{a_{n-1}}$ (d) $a_{n+1} - a_n$ for $n \in \mathbb{N}$, $n > 1$

ix. No term of a G.P is:

(a) 0 (b) 1 (c) Negative (d) Imaginary number

x. The sum of infinite geometric series is a finite number if:

(a) $|r| > 1$ (b) $|r| = 1$ (c) $|r| \geq 1$ (d) $|r| < 1$

xi. If the reciprocals of the terms of a sequence form an A.P, then it is:

(a) Harmonic sequence (b) Arithmetic sequence

(c) Reciprocal sequence (d) Series

xii. If $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is A.M between a & b, then n is equal to:

(a) 0 (b) -1 (c) 1 (d) $\frac{1}{2}$

xiii. If $1, x-1, 3$ are in A.P., then x = ------------

(a) 0 (b) 1 (c) 2 (d) 3

xiv. G.M between −2 and 8 is:

(a) $4i$ or $-4i$ (b) 4 or −4 (c) 16 or −16 (d) 3 or −5

xv. General term of a sequence is $(-1)^n n^2$. Its 4th term is:

(a) −4 (b) −16 (c) 16 (d) 4

xvi. If $|r| > 1$ then infinite geometric series is --------------:

(a) Convergent (b) Divergent (c) Undefined (d) both a and b

xvii. The harmonic mean of $\frac{1}{3}$ and $\frac{2}{5}$ is:

(a) $\frac{4}{11}$ (b) $\frac{3}{4}$ (c) $\frac{5}{11}$ (d) $\frac{11}{4}$

xviii. If $\frac{a-b}{b-c} = \frac{a}{b}$, then a, b and c are in:

(a) A.P (b) G.P (c) H.P (d) None of these

2. The 5th term of an arithmetic sequence is 60 and 8th term is 90. Find 12th term.

3. There are n A.Ms. between 8 and 32 such that the ratio of the third and 7th means is 3:5, find the value of n.

4. Find the sum of all the integers between 280 and 350 which are exactly divisible by 9.

5. How many terms are there in a G.P if $a = 8, a_n = \frac{1}{512}$ and $r = \frac{1}{2}$.

6. The yearly depreciation of a certain machine is 20% of the value at the beginning of the year. If the original cost of the machine is Rs. 50,000, find the value after 5 years.

7. Find r, if $S_{10} = 244S_5$ in a G.P.

8. If A.M and H.M between two numbers are 5 and $4\frac{1}{5}$ respectively. Find the numbers.

9. If G.M. and H.M. between two numbers are 15 and $5\frac{10}{13}$ respectively. Find the numbers.

10. The second term of an H.P is $\frac{1}{2}$ and the fifth term is $\frac{-1}{4}$. Find the 12th term.