Miscellaneous Series

Unit 5

5.1 Evaluation of Σn, Σn^2 and Σn^3

5.1.1 Recognize sigma (Σ) notation

Let $x_1 + x_2 + x_3 + \cdots + x_n$ be a series of first n terms, the sum of this series is denoted by

$$\sum_{i=1}^{n} x_i$$

where Σ is a Greek letter "Sigma" which is used for summation.

i.e., $$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n$$

In summation notation $\sum_{i=1}^{n} x_i$, i is called index, x_i is the ith element of the series, whereas 1 and n are called lower and upper limits respectively.

For example, $\sum_{i=1}^{10} x_i$ means the sum of the values of x, starting with x_1 and ending at x_{10}.

i.e., $$\sum_{i=1}^{10} x_i = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$$

Similarly, the expression $\sum_{i=3}^{10} x_i$ means sum of the values of x, starting with x_3 and ending at x_{10}.

i.e., $$\sum_{i=3}^{10} x_i = x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$$

5.1.2 Find sum of

- **the first n natural numbers (Σn)**
- **the squares of the first n natural numbers (Σn^2)**
- **the cubes of the first n natural numbers (Σn^3)**

Let us find the sum of positive integral powers of natural numbers by using $\sum_{k=1}^{n}[k^m-(k-1)^m]$, where m, n and k are natural numbers.

Now, $\sum_{k=1}^{n}[k^m-(k-1)^m]=(1^m-0^m)+(2^m-1^m)+(3^m-2^m)+(4^m-3^m)+\cdots$

$$+[(n-1)^m-(n-2)^m]+[n^m-(n-1)^m]=n^m$$

i.e $$\boxed{\sum_{k=1}^{n}[k^m-(k-1)^m]=n^m} \quad \ldots \text{(i)}$$

• Sum of the first n natural numbers (Σn)

Taking $m=1$ in equation (i), we get

$$\sum_{k=1}^{n}[k-(k-1)]=n$$

i.e., $$\sum_{k=1}^{n}1=1+1+1+\cdots \text{ to } n \text{ terms}=n \quad \ldots \text{(ii)}$$

Taking $m=2$ in equation (i), we get

$$\sum_{k=1}^{n}[k^2-(k-1)^2]=n^2$$

$$\Rightarrow \sum_{k=1}^{n}[k^2-(k^2-2k+1)]=n^2$$

$$\Rightarrow \sum_{k=1}^{n}(2k-1)=n^2$$

$$\Rightarrow 2\sum_{k=1}^{n}k-\sum_{k=1}^{n}1=n^2$$

$$\Rightarrow 2\sum_{k=1}^{n}k-n=n^2 \quad \text{[using (ii)]}$$

$$\Rightarrow \sum_{k=1}^{n} k = \frac{n^2+n}{2}$$

$$\text{or} \quad \boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}} \quad \text{or} \quad \boxed{\sum n = \frac{n(n+1)}{2}} \qquad \ldots\text{(iii)}$$

This is the formula for getting the sum of first n natural numbers.

Example 1. Find the sum of

(i) first fifteen natural numbers (ii) first fifty-one natural numbers

Solution:

We know that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(i) By using $n = 15$

$$\sum_{k=1}^{15} k = \frac{15(15+1)}{2}$$

$$= 120$$

Hence $1 + 2 + 3 + \cdots + 15 = 120$.

(ii) By using $n = 51$

$$\sum_{k=1}^{51} k = \frac{51(51+1)}{2}$$

$$= 51(26) = 1326$$

Hence $1 + 2 + 3 + \cdots + 51 = 1326$.

Example 2. (i) Find the sum of first fifteen natural numbers starting from 5.

(ii) Find the sum of first fifty one natural numbers starting from 19.

Solution:

By using the formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(i) The sum of first fifteen terms of natural numbers starting from 5 is:

$$\sum_{k=5}^{19} k = \sum_{k=1}^{19} k - \sum_{k=1}^{4} k$$

$$= \frac{19(19+1)}{2} - \frac{4(4+1)}{2}$$

$$= 190 - 10 = 180$$

So, $\sum_{k=5}^{19} k = 180$

Hence, $5 + 6 + 7 + \cdots + 19 = 180$.

(ii) The sum of first fifty one natural numbers starting from 19 is:

$$\sum_{k=19}^{69} k = \sum_{k=1}^{69} k - \sum_{k=1}^{18} k$$

$$= \frac{69(69+1)}{2} - \frac{18(18+1)}{2}$$

$$= 69(35) - 9(19) = 2244$$

So, $$\sum_{k=19}^{69} k = 2244$$

Hence, $19 + 20 + 21 + \cdots + 69 = 2244$.

- **Sum of the squares of the first n natural numbers (Σn^2)**

Taking $m = 3$ in equation (i), we get

$$\sum_{k=1}^{n} [k^3 - (k-1)^3] = n^3$$

$$\Rightarrow \sum_{k=1}^{n} [k^3 - (k^3 - 3k^2 + 3k - 1)] = n^3$$

$$\Rightarrow \sum_{k=1}^{n} (k^3 - k^3 + 3k^2 - 3k + 1) = n^3$$

$$\Rightarrow \sum_{k=1}^{n} (3k^2 - 3k + 1) = n^3$$

$$\Rightarrow 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = n^3$$

$$\Rightarrow 3\sum_{k=1}^{n} k^2 = n^3 + \frac{3n(n+1)}{2} - n \quad \text{[using (ii) and (iii)]}$$

$$3\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + 3n - 2n}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(2n^2 + 3n + 1)}{(2)(3)}$$

$$\sum_{k=1}^{n} k^2 = \frac{n[(2n(n+1) + 1(n+1)]}{6}$$

Thus, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ or $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$ ...(iv)

This is the formula to find the sum of squares of first n natural numbers.

Example:

(i) Find the sum of squares of first six natural numbers.

(ii) Find the sum of squares of first thirty five natural numbers.

Solution:

By using formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(i) For $n = 6$

$$\sum_{k=1}^{6} k^2 = \frac{6(6+1)[2(6)+1]}{6} = \frac{6(7)(13)}{6} = 91$$

Hence, $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91$

(ii) For $n = 35$

$$\sum_{k=1}^{35} k^2 = \frac{35(35+1)[2(35)+1]}{6} = \frac{35(36)(71)}{6} = 14910$$

Hence, $1^2 + 2^2 + 3^2 + \cdots + 35^2 = 14910$

- **Sum of the cubes of the first n natural numbers (Σn^3)**

Taking $m = 4$ in equation (i), we get

$$\sum_{k=1}^{n} [k^4 - (k-1)^4] = n^4$$

$$\Rightarrow \sum_{k=1}^{n} [k^4 - (k^4 - 4k^3 + 6k^2 - 4k + 1)] = n^4$$

$$\Rightarrow \sum_{k=1}^{n} (4k^3 - 6k^2 + 4k - 1) = n^4$$

$$\Rightarrow 4\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = n^4$$

$$4\sum_{k=1}^{n} k^3 - \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} - n = n^4$$ [using (ii), (iii) and (iv)]

$$\Rightarrow 4\sum_{k=1}^{n} k^3 - n(n+1)(2n+1) + 2n(n+1) - n = n^4$$

$$\Rightarrow 4\sum_{k=1}^{n} k^3 = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$

$$\Rightarrow 4\sum_{k=1}^{n} k^3 = n[n^3 + (n+1)(2n+1) - 2(n+1) + 1]$$

$$\Rightarrow 4\sum_{k=1}^{n} k^3 = n(n^3 + 2n^2 + n + 2n + 1 - 2n - 2 + 1) = n(n^3 + 2n^2 + n)$$

$$\Rightarrow \sum_{k=1}^{n} k^3 = \frac{n^2(n^2+2n+1)}{4} = \left[\frac{n(n+1)}{2}\right]^2$$

Thus,

$$\boxed{\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2} \qquad \ldots(\text{v})$$

This is the formula to find the sum of cubes of first n natural numbers.

Example: Find the sum of cubes of first seven natural numbers.

Solution: By using formula

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

For $n = 7$, $$\sum_{k=1}^{7} k^3 = \left[\frac{7(7+1)}{2}\right]^2 = \left[\frac{7(8)}{2}\right]^2 = (28)^2 = 784$$

Hence, $1^3 + 2^3 + 3^3 + \cdots + 7^3 = 784$

Exercise 5.1

Sum the following series up to indicated terms.

1. $1^2 + 2^2 + 3^2 + \cdots + 10^2$

2. $1^3 + 2^3 + 3^3 + \cdots + 15^3$

3. $1^2 + 3^2 + 5^2 + \cdots$ up to n terms

4. $2^2 + 4^2 + 6^2 + 8^2 + \cdots$ up to n terms

5. $1^3 + 3^3 + 5^3 + \cdots$ up to n terms

6. $2 \times 1^2 + 5 \times 2^2 + 8 \times 3^2 + \cdots$ up to n terms

7. $1 + (1+2) + (1+2+3) + \cdots$ up to n terms

8. $2 + (2+5) + (2+5+8) + \cdots$ up to n terms

9. Find the sum of n terms of the series whose nth term is:

(i) $3n^2 + n + 1$ (ii) $n^2 + 4n + 1$

(iii) $n^3 + 3n^2 + 2n + 1$

10. (i) Find the sum of first forty six natural numbers starting from 10.
(ii) Find the sum of last 12 terms of the series: $1 + 2 + 3 + \cdots + 39$.

11. Find the sum of 9th to 21st terms of the series of squares of first n natural numbers.

12. Find the sum of last 12 terms of the series of cubes of first 30 natural numbers.

5.2 Arithmetico-Geometric Series

5.2.1 Define arithmetico-geometric series

A series obtained by multiplying the corresponding terms of an A.P. and a G.P. is called an artithmetico-geometric (A.G.) series. Such a series will be of the form

$$a + (a + d)r + (a + 2d)r^2 + \cdots$$

This is the series which is obtained by multiplying the arithmetic series $a + (a + d) + (a + 2d) + \cdots$

and the geometric series $1 + r + r^2 + \cdots$

The n^{th} term of A.G series is, $T_n = \{a + (n-1)d\}r^{n-1}$

5.2.2 Find sum to n terms of the arithmetico-geometric series

Sum of n terms S_n of the arithmetico-geometric series is obtained as under.

$$\text{Let } S_n = a + (a + d)r + (a + 2d)r^2 + \cdots + [a + (n-1)d]r^{n-1} \quad \ldots(i)$$

Multiplying both sides by r

$$rS_n = ra + (a + d)r^2 + (a + 2d)r^3 + \cdots + [a + (n-1)d]r^n \quad \ldots(ii)$$

Subtracting equation (ii) from (i), we get

$$S_n - rS_n = a + (dr + dr^2 + dr^3 + \cdots + dr^{n-1}) - [a + (n-1)d]r^n$$

$$(1 - r)S_n = a + \left[\frac{dr(1 - r^{n-1})}{1 - r}\right] - [a + (n-1)d]r^n$$

$$(1 - r)S_n = a + \frac{dr}{1 - r} - \frac{dr^n}{1 - r} - [a + (n-1)d]r^n$$

Dividing both sides by $1 - r$

$$S_n = \frac{a}{1 - r} + \frac{dr}{(1 - r)^2} - \frac{dr^n}{(1 - r)^2} - \frac{[a + (n-1)d]r^n}{1 - r} \quad \ldots(iii)$$

Example 1.

Find the sum of the series: $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + 100 \cdot 2^{100}$.

Solution:

Let $S = 1\cdot 2 + 2\cdot 2^2 + 3\cdot 2^3 + \cdots + 100\cdot 2^{100}$ ...(i)

Multiplying both sides by 2

$2S = 1\cdot 2^2 + 2\cdot 2^3 + \cdots + 99\cdot 2^{100} + 100\cdot 2^{101}$ ...(ii)

Subtracting eq. (ii) from eq. (i), we get

$-S = 1\cdot 2 + 1\cdot 2^2 + 1\cdot 2^3 \ldots + 1\cdot 2^{100} - 100\cdot 2^{101}$

$\Rightarrow \quad -S = 2 + 2^2 + 2^3 + \cdots + 2^{100} - 100\cdot 2^{101}$

$\Rightarrow \quad -S = 2(1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{99}) - 100\cdot 2^{101}$

$\Rightarrow \quad -S = \dfrac{2(2^{99} - 1)}{2 - 1} - 100\cdot 2^{101}$

$\Rightarrow \quad -S = 2^{100} - 2 - 100\cdot 2^{101}$

$\Rightarrow \quad -S = 2^{100}(1 - 100\cdot 2^1) - 2$

$\Rightarrow \quad -S = 2^{100}(-199) - 2$

$\Rightarrow \quad S = 2^{100}(199) + 2$

or $\quad S = 199\cdot 2^{100} + 2$

Example 2. Evaluate the sum of the series: $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots$ to infinite terms.

Solution:

Let, $S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots$ ...(i)

Multiplying both sides by $\frac{1}{5}$

$\frac{1}{5}S = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \cdots$ ...(ii)

Subtracting eq. (ii) from eq. (i), we get

$$\left(1 - \frac{1}{5}\right)S = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \cdots$$

$$\Rightarrow \quad \frac{4}{5}S = 1 + 3\left(\frac{\frac{1}{5}}{1 - \frac{1}{5}}\right) \qquad \left[\because S = \frac{a}{1-r} \quad \text{where } |r| < 1\right]$$

$$\Rightarrow \quad \frac{4}{5}S = 1 + 3\left[\frac{1}{5}\cdot\frac{5}{4}\right] = 1 + \frac{3}{4}$$

$$\Rightarrow \quad S = \frac{7}{4}\cdot\frac{5}{4}$$

Hence $\quad S = \dfrac{35}{16}$

Thus, the required sum of the given infinite series

$$1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots \quad \text{is } \frac{35}{16}$$

Example 3. Find the sum of the infinite series: $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+\frac{7}{81}+\cdots$

Solution: Let S denote the sum of the given AG series. Then

$$S = 1.\frac{1}{3}+3.\frac{1}{9}+5.\frac{1}{27}+7.\frac{1}{81}+\cdots \qquad \text{... (i)}$$

Multiplying both sides by $\frac{1}{3}$

$$\frac{1}{3}S = 1.\frac{1}{9}+3.\frac{1}{27}+5.\frac{1}{81}+\cdots \qquad \text{... (ii)}$$

Subtracting equation(ii) from equation (i)

$$S-\frac{1}{3}S = 1.\frac{1}{3}+2.\frac{1}{9}+2.\frac{1}{27}+2.\frac{1}{81}+\cdots$$

$$\Rightarrow \quad \frac{2}{3}S = \frac{1}{3}+2\left(\frac{\frac{1}{9}}{1-\frac{1}{3}}\right)$$

$$= \frac{1}{3}+2\left(\frac{1}{9}\times\frac{3}{2}\right)$$

$$\Rightarrow \quad \frac{2}{3}S = \frac{1}{3}+\frac{1}{3}=\frac{2}{3}$$

$$\Rightarrow \quad S = 1$$

Thus, the required sum of the given infinite series is 1,

i.e., $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+\frac{7}{81}+\cdots = 1$

Exercise 5.2

Sum the following series up to the indicated terms.

1. $1\cdot 2+2\cdot 2^2+3\cdot 2^3+4\cdot 2^4+\cdots$ to 50 terms

2. $1+2.5+3.5^2+4.5^3+\cdots$ to 30 terms

3. $1+\frac{3}{2}+\frac{5}{2^2}+\frac{7}{2^3}+\cdots$ to 25 terms

4. $1+\frac{8}{5}+\frac{15}{5^2}+\frac{22}{5^3}+\cdots$ to n terms

Find the sum to infinity of the following series:

5. $1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\cdots$

6. $1+9a+25a^2+49a^3+\cdots$ where $|a|<1$,

7. $1+4b+7b^2+10b^3+\cdots$ where $|b| < |$

8. $1+\frac{4}{4}+\frac{7}{4^2}+\frac{10}{4^3}+\cdots$

9. $1+\frac{5}{3}+\frac{12}{3^2}+\frac{22}{3^3}+\frac{35}{3^4}+\cdots$

10. $1+5a+9a^2+13a^3+\cdots$ where $|a| < |$

5.3 Method of Differences

5.3.1 Define method of differences

The Method of Differences is the method in which differences of successive terms of the given series are used to find the sum of the series particularly when the general term of the series is that of an "Algebraic fraction".

Theorem:

If a series can be expressed in the form of $f(r+1)-f(r)$ or $f(r)-f(r+1)$ where $f(r)$ is the known function of r, then

$$\sum_{r=1}^{n}[f(r+1)-f(r)] = f(n+1)-f(1)$$

or

$$\sum_{r=1}^{n}[f(r)-f(r+1)] = f(1)-f(n+1)$$

Proof: Consider a series that is expressed in the form $f(r)-f(r+1)$

Now, $\sum_{r=1}^{n}[f(r)-f(r+1)] = \{f(1)-f(2)\}+\{f(2)-f(3)\}+\{f(3)-f(4)\}+\cdots+$
$\{f(n-1)-f(n)\}+\{f(n)-f(n+1)\}$

$= f(1)-f(2)+f(2)-f(3)+f(3)+\cdots+f(n-1)-f(n)+f(n)-f(n+1)$

$= f(1)-f(n+1)$

i.e, $$\sum_{r=1}^{n}[f(r)-f(r+1)] = f(1)-f(n+1) \qquad \ldots \text{(i)}$$

Multiplying both sides by -1

$$(-1)\sum_{r=1}^{n}[f(r)-f(r+1)] = (-1)[f(1)-f(n+1)]$$

$$\Rightarrow \sum_{r=1}^{n}[f(r+1)-f(r)] = f(n+1)-f(1) \qquad \ldots\text{(ii)}$$

Hence, both relations (i) and (ii) are proved.

Example: Sum the series $\sum_{1}^{50} \frac{1}{(r+1)(r+2)}$ by the method of differences.

Solution:

Method 1:

The function can be written in the following form

$$\frac{1}{(r+1)(r+2)}=\frac{1}{r+1}-\frac{1}{r+2} \qquad \text{(by partial fractions)}$$

Taking summation on both sides up to fifty terms, we get

$$\sum_{r=1}^{50}\frac{1}{(r+1)(r+2)}=\sum_{r=1}^{50}\frac{1}{r+1}-\sum_{r=1}^{50}\frac{1}{r+2} \qquad \text{... (i)}$$

Now,

$$\sum_{r=1}^{50}\frac{1}{r+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{50}+\frac{1}{51}$$

and

$$\sum_{r=1}^{50}\frac{1}{r+2}=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{50}+\frac{1}{51}+\frac{1}{52}$$

Substituting in equation (i), we get

$$\sum_{r=1}^{50}\frac{1}{(r+1)(r+2)}=\left[\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{50}+\frac{1}{51}\right]-\left[\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{50}+\frac{1}{51}+\frac{1}{52}\right]$$

After cancelling the values, we have

$$\sum_{r=1}^{50}\frac{1}{(r+1)(r+2)}=\frac{1}{2}-\frac{1}{52}=\frac{25}{52}$$

Method 2: We know that

$$\sum_{r=1}^{n}[f(r)-f(r+1)]=f(1)-f(n+1) \qquad \text{... (i)}$$

Here $\frac{1}{(r+1)(r+2)}=\frac{1}{r+1}-\frac{1}{r+2}$ with $f(r)=\frac{1}{r+1}$ and $f(r+1)=\frac{1}{r+2}$

Now, $$\sum_{r=1}^{50}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)=\frac{1}{1+1}-\frac{1}{51+1} \qquad [\text{using (i)}]$$

$$=\frac{1}{2}-\frac{1}{52}=\frac{25}{52}$$

5.3.2 Apply this method to find the sum of n terms of the series whose differences of the consecutive terms are either in arithmetic or in geometric sequence

We apply the method of differences to find the sum of a series which is not in any standard progression. However, differences of the consecutive terms of that series are either in arithmetic progression or in geometric progression. The following examples show the method of finding the sum of the series whose differences of the consecutive terms are either in A.P or in G.P respectively.

Example 1: Find sum of the series: $5 + 10 + 19 + 32 + 49 + \cdots$ to n terms

Solution: Here, differences of consecutive terms are in A.P.

i.e., $5, 9, 13, 17, \ldots$; is A.P.

So, we use method of differences

Let $\quad S_n = 5 + 10 + 19 + 32 + 49 + \cdots$ to n terms ... (i)

or $\quad \underline{S_n = \pm 5 \pm 10 \pm 19 \pm 32 \pm \cdots \pm a_n}$ ... (ii)

$0 = 5 + \{5 + 9 + 13 + 17 + \cdots \text{ to } (n-1)\text{terms}\} - a_n$

[by taking difference of equations (i) & (ii)]

$$\Rightarrow \quad a_n = 5 + \frac{(n-1)}{2}\{2(5) + (n-2)4\} \qquad \because S_n = \frac{n}{2}[2a + (n-1)d]$$

$$= 5 + \frac{(n-1)}{2}(4n+2)$$

$$= 5 + (n-1)(2n+1)$$

$$a_n = 2n^2 - n + 4$$

For sum of the series,

$$S_n = 2\sum_{1}^{n} n^2 - \sum_{1}^{n} n + \sum_{1}^{n} 4$$

$$S_n = 2\,\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n$$

$$= \frac{n\{2(n+1)(2n+1) - 3(n+1) + 24\}}{6}$$

$$= \frac{n\{2(2n^2+3n+1) - 3n - 3 + 24\}}{6}$$

$$= \frac{n(4n^2 + 6n + 2 - 3n + 21)}{6}$$

$$S_n = \frac{n(4n^2+3n+23)}{6}$$

Hence, sum of the n terms of the given series is $S_n = \frac{n(4n^2+3n+23)}{6}$

Example 2: Find sum of the series: $3 + 7 + 15 + 31 + 63 + \cdots$ to n terms.

Solution: Here, differences of consecutive terms are in G.P.

i.e., $4, 8, 16, 32, \ldots$; is G.P.

So, we use method of differences

Let, $S_n = 3 + 7 + 15 + 31 + 63 + \cdots$ to n terms ... (i)

or $S_n = \pm 3 \pm 7 \pm 15 \pm 31 \pm \cdots \pm a_n$ ... (ii)

$$0 = 3 + \{4 + 8 + 16 + 32 + \cdots \text{ to } (n-1) \text{ terms}\} - a_n$$

[by subtracting equation (ii) from (i)]

$$a_n = 3 + \frac{4(2^{n-1}-1)}{2-1} \qquad \left[\because S_n = \frac{a(r^n-1)}{r-1}\right]$$

$$= 3 + 4(2^{n-1} - 1)$$

$$= 3 + 2 \cdot 2^n - 4$$

$$a_n = 2 \cdot 2^n - 1$$

For sum of the series,

$$S_n = 2\sum_{1}^{n} 2^n - \sum_{1}^{n} 1$$

$$= 2\{2^1 + 2^2 + 2^3 + \cdots + 2^n\} - n$$

$$= 2\left[\frac{2(2^n - 1)}{2 - 1}\right] - n$$

$$S_n = 4(2^n - 1) - n$$

Hence, sum of the n terms of the given series is: $S_n = 4(2^n - 1) - n$

Exercise 5.3

Find the sum of the following series.

1. $3 + 6 + 12 + 21 + 33 + \cdots$ to n terms.
2. $5 + 10 + 17 + 26 + 37 + \cdots$ to n terms.
3. $7 + 14 + 26 + 43 + 65 + \cdots$ to n terms.
4. $3 + 6 + 15 + 42 + 123 + \cdots$ to n terms.
5. $7 + 14 + 45 + 100 + 179 + \cdots$ to n terms.
6. $3 + 6 + 21 + 96 + 471 + \cdots$ to n terms·

5.4 Summation of Series using Partial Fractions

Partial fractions also help in obtaining the sum of the series whose general terms are rational expressions which can be expressed as a sum of two or more fractions.

5.4.1 Use partial fractions to find the sum to n terms and to infinity the series of the type $\frac{1}{a(a+d)}+\frac{1}{(a+d)(a+2d)}+\cdots$

Let us use partial fractions to find the sum to n terms and to infinity the series of the type

$$\frac{1}{a(a+d)}+\frac{1}{(a+d)(a+2d)}+\frac{1}{(a+2d)(a+3d)}+\cdots$$

with general term $\quad a_n=\frac{1}{[a+(n-1)d]\cdot[a+nd]}$

Example 1. By using partial fractions, evaluate the sum of the following series to n terms.

$$\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\frac{1}{8\cdot 11}+\cdots$$

Solution: We have

$$\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\frac{1}{8\cdot 11}+\cdots\text{to } n \text{ terms}=\sum_{1}^{n}\frac{1}{\{2+(n-1)3\}(2+3n)}=\sum_{1}^{n}\frac{1}{(3n-1)(2+3n)}$$

Now we find partial fractions of general term

Let, $$\frac{1}{(3n-1)(3n+2)}=\frac{A}{3n-1}+\frac{B}{3n+2}\qquad\ldots(i)$$

Multiplying both sides by $(3n-1)(3n+2)$

We get $$1=A(3n+2)+B(3n-1)\qquad\ldots(ii)$$

By using $n=\frac{1}{3}$ in (ii)

We get $1=A(1+2)\quad\Rightarrow\quad 3A=1\quad$ or $\quad A=\frac{1}{3}$

By using $n=-\frac{2}{3}$ in equation (ii)

We get $1=B(-2-1)\quad\Rightarrow\quad -3B=1\quad$ or $\quad B=-\frac{1}{3}$

By using values of A and B in equation (i)

We get $$\frac{1}{(3n-1)(2+3n)}=\frac{1}{3(3n-1)}-\frac{1}{3(3n+2)}$$

Now, $$\sum_{1}^{n}\frac{1}{(3n-1)(3n+2)}=\sum_{1}^{n}\left\{\frac{1}{3(3n-1)}-\frac{1}{3(3n+2)}\right\}$$

$$=f(1)-f(n+1)\qquad\left[\because f(r)=\frac{1}{3(3r-1)}\right]$$

$$=\frac{1}{3(3\times 1-1)}-\frac{1}{3\{3(n+1)-1\}}$$

$$=\frac{1}{6}-\frac{1}{3(3n+2)}$$

$$= \frac{3n+2-2}{6(3n+2)}$$

$$= \frac{n}{2(3n+2)}$$

So, $\frac{1}{2\cdot5} + \frac{1}{5\cdot8} + \frac{1}{8\cdot11} + \cdots$ to n terms $= \frac{n}{2(3n+2)}$.

Example 2. By using partial fractions, evaluate the sum of the following infinite series.

$$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \cdots$$

Solution: We have

$$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

Now we find partial fractions of general term

Let $\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$ ... (i)

Multiplying both sides by $n(n+1)(n+2)$

We get $1 = A(n+1)(n+2) + Bn(n+2) + Cn(n+1)$ ... (ii)

For $n = 0$, (ii) becomes, $1 = 2A \implies A = \frac{1}{2}$

For $n = -1$, (ii) becomes, $1 = B(-1)(-1+2) \implies B = -1$

For $n = -2$, (ii) becomes, $1 = C(-2)(-2+1) \implies C = \frac{1}{2}$

By using values of A, B and C in (i)

We get, $\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} + \frac{-1}{n+1} + \frac{1}{2(n+2)}$

Now,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{2n} + \frac{-1}{n+1} + \frac{1}{2(n+2)}\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2n} + \sum_{n=1}^{\infty} \frac{-1}{n+1} + \sum_{n=1}^{\infty} \frac{1}{2(n+2)}$$

$$= \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots\right] + \left[\frac{-1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \cdots\right] + \left[\frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots\right]$$

$$= \frac{1}{4} \qquad \text{(By cancellation of terms)}$$

Hence, $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots = \frac{1}{4}$.

Exercise 5.4

Find the sum of the following series:

1. $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots$ to n terms.

2. $\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + \cdots$ to n terms.

3. $\frac{1}{1 \cdot 5 \cdot 9} + \frac{1}{5 \cdot 9 \cdot 13} + \frac{1}{9 \cdot 13 \cdot 17} + \cdots$ to infinity.

4. $\frac{1}{1 \cdot 4 \cdot 7} + \frac{1}{4 \cdot 7 \cdot 10} + \frac{1}{7 \cdot 10 \cdot 13} + \cdots$ to infinity.

Find sum of the series:

5. $\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}$

6. $\sum_{k=1}^{n} \frac{1}{9k^2 + 3k - 2}$

7. $\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$

8. $\sum_{k=2}^{n} \frac{1}{(k^2 - k)}$

Review Exercise 5

1. **Select correct option.**

i. If $a_n = 5 - 3n + 2n^2$, then $a_{2n} =$ --------------

(a) $5 - 6n + 2n^2$ (b) $5 - 6n + 4n^2$

(c) $5 + 6n + 4n^2$ (d) $5 - 6n + 8n^2$

ii. If $a_{n-2} = 3n - 11$, then $a_n =$ --------------

(a) $3n + 5$ (b) $3n - 5$ (c) $3n - 9$ (d) $3n - 13$

iii. $\sum_{1}^{50} n =$ -----------

(a) 1274 (b) 1275 (c) 1280 (d) 1285

iv. $\frac{1}{n(n+1)} =$ -------------

(a) $\frac{1}{n}+\frac{1}{n+1}$ (b) $\frac{1}{n}-\frac{1}{n+1}$ (c) $\frac{2}{n}+\frac{1}{n+1}$ (d) $\frac{1}{n}-\frac{3}{n+1}$

v. The n^{th} term of $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}$ ... is:

(a) $\frac{1}{3n-1}$ (b) $3n-1$ (c) $2n+1$ (d) $\frac{1}{3n+1}$

vi. $\sum n$ is equal to:

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{6}$ (c) $\frac{n^2(n+1)^2}{2}$ (d) n^2

vii. $\sum n^2$ is equal to:

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{6}$ (c) $\frac{n^2(n+1)^2}{2}$ (d) n^2

viii. $\sum n^3$ is equal to:

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{6}$ (c) $\frac{n^2(n+1)^2}{4}$ (d) $\frac{n(n+1)^2}{4}$

ix. If $S_n = (n+1)^2$, then S_{2n} is equal to:

(a) $2n+1$ (b) $4n^2+4n+1$

(c) $(2n-1)^2$ (d) Cannot be determined

x. $\sum_{n=3}^{20} n^0 =$:-----------

(a) 1 (b) 19 (c) 20 (d) 18

2. Sum the following series up to n terms

$2\times 1^2 + 4\times 2^2 + 6\times 3^2 + \cdots$

3. Sum the series $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots$ to infinity.