Permutation, Combination and Probability

Unit 6

6.1 Factorial of a Natural Number

6.1.1 Know Kramp's factorial notation to express the product of first *n* natural numbers by n!

In mathematics, the factorial of a positive integer n, is the product of all positive integers (i.e., natural numbers) less than or equal to n. It may be noted that the factorial values for negative integers are not defined.

The product of first n natural numbers or factorial of n is denoted by n! and read as 'n factorial'. The factorial notation n! was introduced in 1808 AD by a French mathematician named Christian Kramp. The factorial notation is frequently used to write continued products in simplified form.

Now, we find some factorials by definition.

$$1! = 1$$
$$2! = 2 \cdot 1 = 2$$
$$3! = 3 \cdot 2 \cdot 1 = 6$$
$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Thus, $n! = n(n-1)(n-2) \ldots .3 \cdot 2 \cdot 1$

or $n! = n(n-1)(n-2)!$ or $n! = n(n-1)!$

Also we define $0! = 1$.

Note: The symbol $\lfloor n$ is also used for n!

Example 1. Evaluate: $\dfrac{15!}{2!10!}$

Solution: $\dfrac{15!}{2!10!} = \dfrac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10!}{2!\ 10!} = \dfrac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{2} = 180180$

Example 2. Write $\dfrac{n-4}{(n-2)(n-1)}$ in factorial form.

Solution: $\dfrac{n-4}{(n-2)(n-1)} = \dfrac{(n-4)}{(n-2)(n-1)} \times \dfrac{(n-1)!}{(n-1)!} \times \dfrac{(n-4)!}{(n-4)!}$

$$= \frac{(n-4)}{(n-2)(n-1)} \times \frac{(n-1)(n-2)(n-3)!}{(n-1)!} \times \frac{(n-4)!}{(n-4)(n-5)!}$$
$$= \frac{(n-3)!\,(n-4)!}{(n-1)!\,(n-5)!}$$

Example 3. Express $\frac{(n-1)!}{r!} - \frac{(n+1)!}{(r-1)!}$ as a single fraction:

Solution: $\frac{(n-1)!}{r!} - \frac{(n+1)!}{(r-1)!}$

$$= \frac{(n-1)!}{r(r-1)!} - \frac{(n+1)n(n-1)!}{(r-1)!}$$
$$= \frac{(n-1)!}{(r-1)!}\left[\frac{1}{r} - n(n+1)\right]$$
$$= \frac{(n-1)!}{r(r-1)!}[1 - nr(n+1)]$$
$$= \frac{(1-n^2r-nr)(n-1)!}{r!}$$

Exercise 6.1

1. Evaluate the following:

(i) $4!$ (ii) $6!$ (iii) $\frac{8!}{5!}$ (iv) $\frac{10!}{7!}$

(v) $5! \times 7! \times 3!$ (vi) $10! \div 8! + 5!$ (vii) $\frac{16!}{8!12!}$ (viii) $\frac{(9!)^2}{(5!)^3 \times 7!}$

(ix) $\frac{9!}{2! \times (9-2)!}$

2. Write the following in factorial form:

(i) $6 \cdot 5 \cdot 4$ (ii) $12 \cdot 11 \cdot 10$

(iii) $\frac{n(n+1)(n+2)}{6 \cdot 5}$ (iv) $20 \cdot 19 \cdot 18 \cdot 17$

3. Express the following as a single fraction:

(i) $\frac{(n+1)!}{(r+1)!} + \frac{n!}{r!}$ (ii) $\frac{(n+1)!}{r!} + \frac{n!}{(r+1)!}$

6.2 Permutation

6.2.1 Recognize the fundamental principle of counting and illustrate this principle using tree diagram

A process of determining the number of elements contained in a set is called counting. The symbol O(A) or n(A) or |A| is used to denote the number of elements in the set A.

For example, If $A = \{a, b, c\}$, then $O(A) = 3$.

Sum Principle of counting

If A and B are two sets, then

$$O(A \cup B) = O(A) + O(B) - O(A \cap B) \qquad \text{... (i)}$$

This is known as the Sum Principle.

If A and B are two disjoint sets, then

$$O(A \cup B) = O(A) + O(B) \qquad \text{.... (ii)}$$

This is known as the Sum Principle for disjoint sets.

This principle can be extended to any finite number of disjoint sets:

i.e., $O(A_1 \cup A_2 \cup A_3 \cup ... \cup A_n) = O(A_1) + O(A_2) + O(A_3) + \cdots + O(A_n)$ ... (iii)

Fundamental Principle or Product Principle of counting

If A and B are any two sets and A×B is their Cartesian product, then

$$O(A \times B) = O(A) \cdot O(B) \qquad \text{.... (iv)}$$

This is known as the fundamental principle or Product Principle.

The Product Principle can be extended to any finite number of finite sets:

i.e., $O(A_1 \times A_2 \times A_3 \times ... \times A_n) = O(A_1) \cdot O(A_2) \cdot O(A_3) \cdot ... \cdot O(A_n)$ ... (v)

Product principle can also be defined as:

If an operation performed in r distinct ways, and corresponding to each of these ways there are s distinct ways of performing a second operation, then the two consecutive operations can be performed together in rs distinct ways in that order.

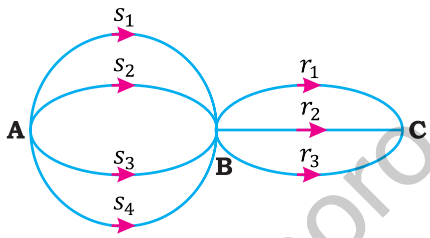
Example 1. There are 4 roads from village A to village B and 3 roads from village B to village C. By how many different routes may one can travel from A to C by way of B? Also enlist the various routes.

Solution: One can go from village A to village B by any one of the four paths s_1, s_2, s_3, s_4 and then he can go from village B to village C by any one of the three different paths r_1, r_2, r_3 (Fig. 6.1).

Note that by going from A to B by the path s_1, he can choose any of the three paths r_1, r_2, r_3 to go from B to C. Therefore, from A to C, he can go by any of the following three different routes:

$$s_1r_1 \quad s_1r_2 \quad s_1r_3$$

Thus, path s_1 from A to B is yielding three different paths to reach C. Similarly, each of the remaining three paths s_2, s_3 and s_4 from A to B, will also yield three different paths to reach at C. Hence, by the Fundamental Principle, he can go from A to C by 4 times 3, or 12, different routes.



(Fig. 6.1)

A list of various routes is as follows:

s_1r_1	s_1r_2	s_1r_3
s_2r_1	s_2r_2	s_2r_3
s_3r_1	s_3r_2	s_3r_3
s_4r_1	s_4r_2	s_4r_3

Example 2. How many positive odd numbers, each having three digits, can be formed from the digits 1,2,3,4,5,6 if no digit is to be repeated in a given number.

Solution: Since the required numbers are odd, the unit's place may be filled in one of three ways: by the 1, 3 or 5. In turn, the digit, in the tenth's place may be filled in any one of the five ways, and the digit in the hundredth's place may be filled in any one of the four ways. Hence, by the Fundamental Principle of counting, the three places can be filled in

$$(4)(5)(3) = 60 \text{ ways}$$

Hence, 60 positive odd numbers of different digits can be formed.

Example 3. How many two-digit whole numbers can be formed from the digits 3,4,5?

Solution: Any one of the three digits can be used in the tenth's place. Since the digits used need not to be different, so, the unit's place may also be filled in any one of the three different ways. Hence, by the Fundamental Principle, the two places can be filled in

$$3 \cdot 3 = 9 \text{ ways}$$

Hence, 9 whole numbers of two digits can be formed.

Tree Diagram

A tree diagram is a special type of diagram used in strategic decision making, valuation or probability calculations. The diagram starts at a single node, with branches emanating to additional nodes, which represent mutually exclusive decisions or events.

Tree diagrams are useful for organizing and visualizing the different possible outcomes of a sequence of events. For each possible outcome of the first event, we draw a branch where we write down the possibility of that outcome.

Then, for each possible outcome of the second event we do the same thing.

Below is an example of a simple tree diagram.

Example: Farhan has 3 books and Hashir has 2 books. In how many ways can they exchange a book? Also illustrate using tree diagram.

Solution: Farhan can give a book to Hashir in 3 ways (any one of his books F_1, F_2, or F_3 may be given); and then Hashir can give one of his 2 books H_1, H_2, to Farhan, so Hashir can give a book in exchange by 2 ways. Hence by the Fundamental Principle they can exchange a book in

$$(3)(2) = 6 \text{ ways.}$$

The various possibilities can be shown with the help of the following tree diagram (Fig. 6.2)

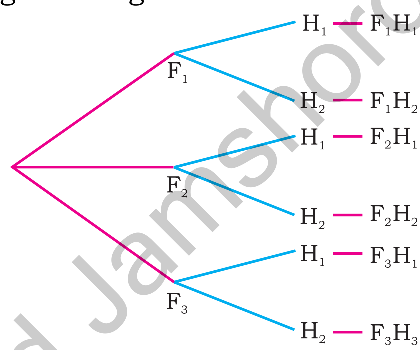


Fig. 6.2

6.2.2 Explain the meaning of permutation of 'n' different objects taken r at a time and know the notation ^nP_r

Permutation is one of the methods of counting, very useful in the study of certain fields of mathematics.

We know that an ordered r-tuple (finite ordered list) whose first, second, ... , r^{th} elements are respectively $t_1, t_2, \ldots, t_r$ is written as $(t_1, t_2, \ldots, t_r)$. When $r = 2$, we have the ordinary ordered pair; viz. (t_1, t_2).

An ordered r-tuple all of whose elements belong to a set S of n elements $(n \geq r)$, is called a permutation of r elements selected from the set S.

Permutation of r elements from a set of n elements $(n \geq r)$ are traditionally called "Permutations of n things taken r at a time" and denoted as ^nP_r.

The concept of a permutation is nothing more than an ordered set, with each ordering of the elements of the set being a different permutation.

From the above discussion, it is obvious that a permutation is an arrangement of all or part of a set of objects, with regard to the order of the arrangement. For example, suppose we have a set of three letters: A, B and C. If we arrange 2 letters from that set then each possible arrangement would be an example of a permutation. The complete list of possible permutations would be: AB, AC, BA, BC, CA, and CB. Thus, in this case $^3P_2 = 6$.

6.2.3 Prove that $^nP_r = n(n-1)(n-2)\cdots(n-r+1)$ and hence deduce that

(i) $^nP_r = \frac{n!}{(n-r)!}$ **(ii)** $^nP_n = n!$ **(iii)** $0! = 1$

We have already studied that if n distinct objects are given and we have to arrange r $(n \geq r)$ out of them at a time in which order is important, such an arrangement is called a permutation of n objects taken r at a time. The number of permutations is determined by the following theorem.

Theorem: The number of permutations of n elements of a set taken r at a time is given by $^nP_r = n(n-1)(n-2)\ldots(n-r+1)$, up to r factors

Proof:

Let $S = \{t_1, t_2, \ldots, t_n\}$ be a set of n elements. The number of permutations of n elements taken r at a time from this set are the ordered r-tuples which can be formed by using its elements. The number of permutations of r elements out of n is the same as the number of ways of filling up r blank places in an ordered r-tuple.

The first element of the ordered r-tuple can be chosen in n ways because there are n element of S for us to choose from. After the first element of the r-tuple is chosen in any one of these n ways, we are left with (n–1) elements of the set S. Thus the second element of the r-tuple can be chosen in (n–1) ways, so both 1st and 2nd elements can be chosen in $n(n-1)$ ways. After the first two elements of the r-tuple are chosen, we are left with (n–2) elements of the set S. Thus the third element of r-tuple can be chosen in (n–2) ways. Again, by the Fundamental Principle, the first three elements of the r-tuple can be chosen in n(n–1)(n–2) ways and so on. Continuing in this way, the number of ways in which the 'r' elements of the ordered r-tuple can be chosen is

$$n(n-1)(n-2)\ldots \text{ up to r factors.}$$

Therefore, the number of possible ordered r-tuples is:

$$^nP_r = n(n-1)(n-2)\ldots[n-(r-1)]$$
$$= n(n-1)(n-2)\ldots(n-r+1)$$

Hence, proved.

Deductions:

(i) $^nP_r = \frac{n!}{(n-r)!}$

Using the relation $^nP_r = n(n-1)(n-2)\ldots(n-r+1)$

Multiplying and dividing by $(n-r)!$ we get:

$$= \frac{[n(n-1)(n-2)\ldots(n-r+1)](n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}$$

Hence $^nP_r = \frac{n!}{(n-r)!}$

(ii) $^nP_n = n!$

If r = n, then

$$^nP_n = \frac{n!}{(n-n)!}$$
$$= \frac{n!}{0!} = \frac{n!}{1}$$
$$^nP_n = n!$$

(iii) $0! = 1$

We have, $^nP_n = 1$

By using $n = 1$

We get, $^1P_1 = 1!$

$$\Rightarrow \frac{1!}{(1-1)!} = 1$$
$$\Rightarrow \frac{1!}{0!} = 1$$
$$\Rightarrow 1 = 0!$$

6.2.4 Apply ^nP_r to solve problems of finding the number of arrangements of *n* objects taken *r* at a time (when all *n* objects are different and when some of them are alike).

Example 1. How many different arrangements can be made by using all the letters of the word “DAUGHTER”?

Solution: There are 8 different letters in the word “DAUGHTER” and all the 8 letters are to be used. This is a case of permutations of 8 different things taking all of them at a time. Hence the total number of arrangements is:

$$^8P_8 = 8! = 40320.$$

Example 2. In how many ways can 6 books of Mathematics and 5 books of Biology be placed on a shelf so that the books on the same subject always remain together?

Solution: The 6 books of Mathematics can be arranged in $^6P_6$, i.e., 6! ways among themselves.

The 5 books of Biology can be arranged in $^5P_5$, i.e., 5! ways among themselves.

The 2 groupings of the books on the two subjects can be permutated in $^2P_2$, i.e., 2! ways. Hence by fundamental principle.

The total number of arrangements:

$$= 6!\ .5!\ .2!$$
$$= (720)(120)(2)$$
$$=1,72,800 \text{ arrangements.}$$

Example 3. How many different three digit whole numbers can be formed with the numbers 9,8,7,5,3,2. Assuming that no digit can be repeated?

Solution: This is the case of permutations of 6 digits taking 3 at a time. Hence the total of three-digit numbers that can be formed is:

$$^6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6.5.4.(3!)}{3!} = 120$$

Example 4. How many signals can be made with six flags of different colours, by hoisting 1 or 2 or 3 or 4 or 5 or all of them?

Solution: When only 1 flag is to be displayed at a time,

the number of signals = $^6P_1 = \frac{6!}{5!} = 6.$

When 2 flags are to be displayed at a time,

the number of signals = $^6P_2 = \frac{6!}{4!} = 30.$

When 3,4,5 and 6 flags are to be displayed at a time, the number of signals are $^6P_3$, $^6P_4$, $^6P_5$ and $^6P_6$ respectively,

Hence the total number of signals:

$$= {}^6P_1 + {}^6P_2 + {}^6P_3 + {}^6P_4 + {}^6P_5 + {}^6P_6$$
$$= 6 + 30 + 120 + 360 + 720 + 720$$
$$= 1956 \text{ signals}$$

Example 5. If $^nP_3 = 12 . {}^{\frac{n}{2}}P_3$, find n.

Solution: We have, $\quad {}^nP_3 = 12 . {}^{\frac{n}{2}}P_3$

$$\Rightarrow \quad n(n-1)(n-2) = 12.\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)$$
$$\Rightarrow \quad 2n(n^2-3n+2) = 3n\,(n-2)(n-4)$$
$$\Rightarrow \quad 2(n^2-3n+2) = 3(n^2-6n+8)$$
$$\Rightarrow \quad n^2-12n+20 = 0$$
$$\Rightarrow \quad (n-10)(n-2) = 0$$
$$\Rightarrow \quad n = 10 \text{ or } 2.$$

But $\quad n = 2$ is inadmissible.

Hence, $\quad n = 10.$

The number of permutations can be determined by the following theorem if some objects are alike.

Theorem

The number of distinct permutations of n things taken all at a time, when r of them are alike of one kind, s of them alike of another kind, t of them are alike of a third kind and the rest, if any, all different is

$$\frac{n!}{r!\,s!\,t!}$$

Proof:

Let P be the required number of permutations. Take any one of these permutations and replace the r like things by r unlike things, different from each other and from all the rest, and permute them in all possible ways among themselves, leaving the others unchanged. We thus get $r!$ permutations out of the original one. Hence out of the P original permutations, we shall get $(r! \cdot P)$ permutations.

Similarly, if we now replace the s like things by s unlike things, different from each other and from all the rest, and permute them in all possible ways among themselves, we obtain $(r!\cdot s!\cdot \text{P})$ permutations out of the $(r!\cdot \text{P})$ permutations previously obtained.

Repeat the process with the 't' like things of the third kind, replacing them by 't' unlike things different from each other and from all the rest, and permuting them in all possible ways among themselves. The number of permutations is thus increased $t!$ times. Hence we have $(r!\cdot s!\cdot t!\cdot \text{P})$ permutations in the end. But the 'n' things are now all different, and the total number of their permutations must be $n!$. Therefore, $r!\cdot s!\cdot t!\cdot \text{P} = n!$

or $$\text{P} = \frac{n\,!}{r!\ s!\ t!}$$

This number of permutations is symbolically written as $\binom{n}{r,\ s,\ t}$

The above theorem can be easily extended to cases where there are more than three kinds of like things.

Note: If 'r', 's' and 't' are such that none of the things are left over, then $r + s + t = n$.

Example 1. Find the number of permutations of 9 fruits, taken all at a time when 3 are apples, 2 are oranges and 2 are mangoes.

Solution: Here 3 apples are like things of the first kind, 2 oranges are like things of the second kind and 2 mangoes of the third kind and rest are the different. The total number of fruits is 9.

Hence the required number of permutations is:

$$\binom{9}{3\,,2\,,2} = \frac{9!}{3!\,2!\,2!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4(3!)}{(3!)2\cdot 1\cdot 2\cdot 1} = 15120$$

Example 2. How many whole numbers can be formed with the digits $1, 2, 3, 4, 3, 2, 1$ so that the even digits occupy the even places?

Solution: The arrangement $1, 2, 3, 4, 3, 2, 1$ is one of the type required. The even digits 2, 4, 2 can be permuted among themselves in their three places in:

$$\frac{3!}{2!} = 3 \text{ ways.}$$

The odd digits 1, 3, 3, 1 can be permuted among themselves in their four places in:

$$\frac{4!}{2!\ 2!} = 6 \text{ ways.}$$

By fundamental principle

The required number = (3)(6) = 18.

Example 3. A coin is tossed nine times repeatedly. In how many possible ways can we get 5 heads and 4 tails.

Solution: Out of 9 tosses, we are to have 5 heads and 4 tails. This can be done in:

$$\binom{9}{5,4} = \frac{9!}{5!\ \ 4!}$$
$$= \frac{9\cdot 8\cdot 7\cdot 6\cdot (5!)}{(5!).4\cdot 3\cdot 2\cdot 1}$$
$$= 126 \text{ ways.}$$

6.2.5 Find the arrangement of different objects around a circular permutation

Circular Permutation:

In an ordinary permutation of r things, the elements are arranged in a definite order and their places can be marked as the first, the second, the third... etc., so that every arrangement has a beginning and an end. In a circular permutation, where the elements are arranged round the circumference of a circle, there is neither a beginning nor an end, and the positions cannot be marked out absolutely as the first, the second, the third... etc. In a circular permutation therefore, it is the relative positions of elements that matter.

Example 1. Suppose a man *a* , a lady *b*, a boy *c*, and a girl *d*, are sitting around a round table as shown in Fig. 6.3.

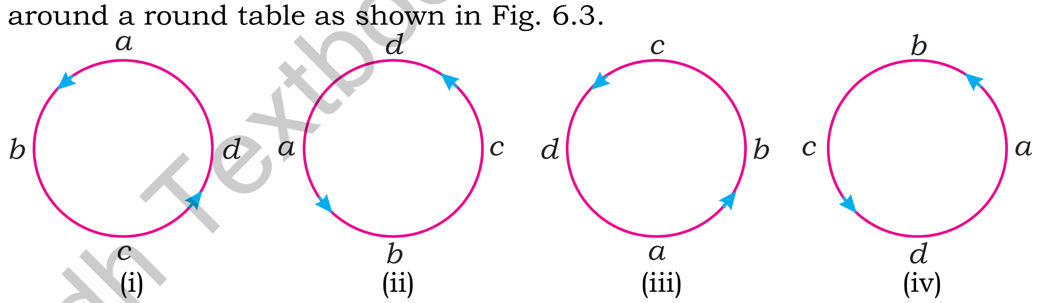


Fig. 6.3

Moving round the circumference in the anti-clockwise sense as indicated by the arrows, the four letters *a*, *b*, *c* and *d* occur in the same order and their relative positions are unchanged. Hence the various arrangements thus obtained are to be regarded as identical. The four figures show the four different forms in which the same arrangement appears to four different persons standing outside the circle opposite *a*, *b*, *c* and *d* respectively.

These identical arrangements round a circle correspond to four distinct ordinary permutations:

abcd , *dabc* , *cdab* , *bcda*

obtained by taking in succession as first each of the four letters in the circular permutation. Hence the total number of circular permutations of four different things is one-fourth of the number of ordinary permutations. It is therefore equal to $\frac{1}{4}$. (4!) = 3!.

Since these identical arrangements arise because all the things are moved around without changing their relative positions, it follows that if we keep one of them fixed in position and permute all the rest among themselves, we shall get the required number of circular permutations.

In the above situation clockwise and anti-clockwise arrangements are distinct.

In the case of a necklace of beads (or a garland of flowers) the same necklace gives one order of beads if looked at from one side and the reverse order of looked at from the opposite side. If one of these arrangements is turned over, it becomes identical with the other.

Hence the same necklace is the result of both the orders of beads. In such cases, therefore, the clockwise and anti-clockwise arrangements give only one distinct permutation.

Theorem

The number of circular permutations of n elements of a set taken all at a time, is $(n-1)!$, if clockwise and anti-clockwise arrangements are regarded as distinct; if they are regarded as identical, the number is $\frac{1}{2}[(n-1)!]$

Proof:

The number of circular permutations of n elements can be obtained by fixing the position of any one of these elements wherever it may be and then by permuting the remaining $(n-1)$ elements.

This can be done in $(n-1)!$ ways.

Hence the required number of circular permutations is $(n-1)!$

If the clockwise and anti-clockwise arrangements are not regarded as distinct, the number is reduced by one-half.

Hence in such cases the required number is: $\frac{1}{2}[(n-1)!]$.

Example 1. In how many ways can 5 persons be seated at round table conference?

Solution: The number of circular permutations of n distinct objects is $(n-1)!$. So the number of ways in which 5 person be seated at round table conference is $(5-1)! = 4! = 24$ ways

Example 2. In how many ways, can 10 keys be arranged in a ring?

Solution: Since the clockwise and anticlockwise arrangements are identical.

So, the number of permutations $= \frac{(n-1)!}{2} = \frac{(10-1)!}{2} = \frac{9!}{2} = 181440$

6.2.6 Solve daily life problems involving permutation

Example 1. There are 9 competitors in a race for 3 prizes. In how many ways, can the prizes be given?

Solution:

Here $n = 9$ and $r = 3$

Now $\quad {}^9P_3 = \frac{9!}{(9-3)!}$

$= \frac{9!}{6!}$

$= \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6!}$

$= (9)(8)(7) = 504$

Hence in 504 ways 3 prizes can be given to 9 competitors.

Example 2. In how many ways, can Fiza take one Red ball, one Yellow ball and one Green ball from a box containing 4 Red, 3 Yellow and 5 Green balls?

Solution:

Fiza can take one red ball, one yellow ball and one green ball in

$P_1^4 \cdot P_1^3 \cdot P_1^5$ ways

$= 4 \cdot 3 \cdot 5$ ways.

$= 60$ ways.

Example 3. There are 9 different posts vacant of which 5 must be held by postgraduates and 3 by graduates, while the remaining 1 may be given to either postgraduates or graduates if 8 postgraduates and 6 graduates apply for these posts, in how many ways can the posts be filled?

Solution: There are 8 postgraduates and 5 different posts for postgraduates alone. These can, therefore, be filled in ${}^8P_5$ ways.

Similarly, the 6 graduates can fill the 3 posts for them alone in ${}^6P_3$ ways. When this has been done, 1 post still remains to be filled up and there remain 6 candidates for the same. Thus the remaining 1 post can be filled up by the remaining 6 candidates in ${}^6P_1$ ways.

Hence the total number of ways required to fill all the posts are:

$$= {}^8P_5 \,.\, {}^6P_3 \,.\, {}^6P_1 = \frac{8!}{3!} \cdot \frac{6!}{3!} \cdot \frac{6!}{5!}$$

$$= (8)(7)(6)(5)(4)(6)(5)(4)(6)$$

$$= 4838400 \text{ ways.}$$

Exercise 6.2

1. Evaluate the following:

(i) $^{20}P_3$ (ii) $^{16}P_4$ (iii) $^{12}P_5$ (iv) $\binom{8}{3,2}$ (v) $\binom{9}{2,3,4}$

2. Find the value of n in each of the following:

(i) $^{n}P_2 = 30$ (ii) $^{n}P_3 = 60$ (iii) $^{8}P_n = 8.7.6$

(iv) $^{11}P_n = 11.10.9$ (v) $^{n+2}P_4 : {}^{n+1}P_3 = 5:1$ (vi) $^{n}P_4 : {}^{n-1}P_3 = 9:1$

3. In how many ways different batting orders are possible for a cricket team consisting of 11 players.

4. In how many ways can two English books, three Chemistry books and four Physics books be arranged on a shelf so that all the books of same subject are together?

5. How many different arrangement can be formed of the word "MATRIX" using all letters? Also find the number of arrangements if each of them start with "R".

6. How many signals can be given by 8 flags of different colours when 5 of them are used at a time?

7. A fair coin is tossed three times. How many outcomes are possible? Find by fundamental principle and illustrate using tree diagram.

8. How many 5-digit numbers can be formed, without repeating any digit:

(i) 5,6,7,8,9 (ii) 1,2,3,5,7,9 (iii) 2,4,6,8,0

9. Prove that:

(i) $^{n}p_r = {}^{n-1}p_r + r \cdot {}^{n-1}p_{r-1}$ (ii) $^{n}p_r = n \cdot {}^{n-1}p_{r-1}$

10. How many arrangements can be formed using all the letters of the following words.

(i)	STATISTICS	(ii)	PLANE
(iii)	OBJECT	(iv)	FASTENS
(v)	MATHEMATICS	(vi)	ASSISTANCE

11. How many 8-digit numbers can be formed from the digits 2,2,2,3,3,3,5,6? How many among these are greater than 60,000,000?

12. How many distinct permutations of letters of the word "ESSENTIAL" are possible? How many will have the two S's (i) together (ii) separate.

13. In how many ways, can 8 persons be seated around a round table?

14. In how many ways, 7 keys be arranged in a circular key ring?

15. How many 3-digit numbers can be formed by using each one of the digits 2,3,5,7,9 only once?

16. How many numbers greater than 23,000 can be formed from the digits 1,2,3,5,6 if digits may repeat?

17. Find the number of 5-digit numbers that can be formed from the digits 1,2,4,6,8 (when no digit is repeated), but
(i) the digits 2 and 8 are next to each other.
(ii) the digits 2 and 8 are not next to each other.

18. How many 6-digit numbers can be formed without repeating any digit from the digits 0,1,2,3,4,5? In how many of them will 0 be at the tens place?

19. How many 5-digit multiplies of 5 can be formed from the digits 2,3,5,7,9 when digits may repreat?

6.3 Combination

Combination is another method of counting, like permutation. If we form subsets of a given set S of n elements, each one of which has a specified number of elements, say r $(r \leq n)$ without any regard to the order of the elements in any of these subsets, then each of these subsets is called a combination.

6.3.1 Define combination of n different objects taken r at a time

Combinations of r elements from a set of n elements $(n \geq r)$, are traditionally called "Combinations of n things taken r at a time".

The number of ways of selecting r things out of n different objects is denoted by ${}^{n}C_{r}$ or $\binom{n}{r}$. In other words, the number of combinations of n objects taken r at a time $(r \leq n)$ is denoted by ${}^{n}C_{r}$ or $\binom{n}{r}$. In combination order does not matter.

6.3.2 Prove the formula ${}^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ and deduce that

- $\binom{n}{n} = \binom{n}{0} = 1$
- $\binom{n}{r} = \binom{n}{n-r}$, $\binom{n}{1} = \binom{n}{n-1} = n$,
- $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

The general formula for finding the number of combinations of n elements of a set taken r at a time is given by the following theorem.

Theorem: The number of combinations of n elements of a set taken r at a time is

$$\binom{n}{r} = \frac{^nP_r}{r!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!\,(n-r)!} = \frac{n!}{r!\,(n-r)!}$$

Proof: We know that the number of permutations of the n elements of a set taken r at a time is given by

$$^nP_r = n(n-1)(n-2)\ldots(n-r+1). \qquad \ldots\text{(i)}$$

Each one of the combinations of r elements that can be formed provides $r!$ of these permutations. Thus by the Product Principle we have: $\binom{n}{r} r! = {^nP_r}$

Dividing each member by $r!$, we get:

$$\binom{n}{r} = \frac{^nP_r}{r!} \qquad \ldots\text{(ii)}$$

Substituting the value of ^nP_r from (i) in (ii), we get:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

Multiplying the numerator and the denominator of the right member by $(n-r)!$, we have if $n \geq r$.

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{r!\,(n-r)!}$$

or $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Deductions:

(i) $\binom{n}{n} = \binom{n}{0} = 1$

We have, $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

If $r = n$, then

We get, $$\binom{n}{n} = \frac{n!}{n!\,(n-n)!} = 1$$

i.e. the number of combinations of n things taken n (all) at a time is 1.

Now, if $r = 0$, then

We get, $$\binom{n}{0} = \frac{n!}{0\,!\,(n-0)!} = 1$$

(ii) $\binom{n}{r} = \binom{n}{n-r}$

Here we have $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

If $r = n-r$, then $\binom{n}{n-r} = \dfrac{n!}{(n-r)![n-(n-r)]!}$

$$= \frac{n!}{(n-r)!\,r!}$$

$$= \binom{n}{r}$$

When we are selecting r elements out of n, in fact, we are leaving $(n-r)$ elements. Sometimes it is simpler to select $(n-r)$ elements which are left out. For example, if we have to select 90 students out of 100, it would be simpler to select 10 students whom we want to reject. Such combinations of r elements or $(n-r)$ elements, out of n elements are called complementary combinations.

(iii) $\binom{n}{r} = \binom{n}{n-1} = n$

We have, $\binom{n}{n-r} = \binom{n}{r}$

Taking $r = 1$

We get, $\binom{n}{n-1} = \binom{n}{1} = \dfrac{n!}{1!(n-1)!} = \dfrac{n!}{(n-1)!} = n$

Hence, $\binom{n}{1} = \binom{n}{n-1} = n.$

(iv) $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

Proof: By using $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$,

we have, $\binom{n}{r-1} + \binom{n}{r} = \dfrac{n!}{(n-r+1)!(r-1)!} + \dfrac{n!}{(n-r)!r!}$

$$= \frac{n!\,r + n!\cdot(n-r+1)}{(n-r+1)!\,r!}$$

$$= \frac{n!\,(r+n-r+1)}{(n-r+1)!\,r!}$$

$$= \frac{(n+1)n!}{(n-r+1)!\,r!}$$

$$= \frac{(n+1)!}{(n+1-r)!\,r!}$$

$$= \binom{n+1}{r}$$

Hence, $\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$

This rule is called Pascal's rule.

Example 1. Find the value of $\binom{11}{5}$.

Solution:

$$\binom{11}{5}=\frac{11!}{(11-5)!\cdot 5!}$$
$$=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6!}{6!\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$
$$=462$$

Example 2. If ${}^{20}C_r={}^{20}C_{r+2}$, find ${}^{r}C_5$

Solution: Here $\quad {}^{20}C_r={}^{20}C_{r+2}$

By using $\quad \binom{n}{r}=\frac{n!}{r!(n-r)!}$

We have, $\quad \frac{20!}{(20-r)!\cdot r!}=\frac{20!}{(20-r-2)!\cdot(r+2)!}$

$$\Rightarrow \quad \frac{1}{(20-r)!\cdot r!}=\frac{1}{(20-r-2)!\cdot(r+2)!}$$

$$\Rightarrow \quad \frac{1}{(20-r)!\cdot r!}=\frac{1}{(20-r-2)!\cdot(r+2)(r+1)!}$$

$$\Rightarrow \quad \frac{1}{(20-r)(20-r-1)(20-r-2)!\cdot r!}=\frac{1}{(20-r-2)!\cdot(r+2)(r+1)\cdot r!}$$

Cancelling out $[(20-r-2)!]\cdot r!$ from the denominators

We get, $\quad \frac{1}{(20-r)(20-r-1)}=\frac{1}{(r+2)(r+1)}$

$$\Rightarrow \quad (r+2)(r+1)=(20-r)(20-r-1)$$
$$\Rightarrow \quad (r+2)(r+1)=(20-r)(19-r)$$
$$\Rightarrow \quad r^2+3r+2=380-39r+r^2$$
$$\Rightarrow \quad 42r=378$$
$$\Rightarrow \quad r=9.$$

Hence, $\quad {}^{r}C_5={}^{9}C_5$

$$=\frac{9\cdot 8\cdot 7\cdot 6\cdot(5!)}{(9-5)!\cdot(5!)}=\frac{9\cdot 8\cdot 7\cdot 6}{4!}=\frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}=126.$$

Alternatively,

we know that $\quad {}^{n}C_r={}^{n}C_{n-r}$

So, $\quad {}^{20}C_r={}^{20}C_{20-r}$

But we are given that $\quad {}^{20}C_r={}^{20}C_{r+2}$

Therefore by comparison,

We get, $20 - r = r + 2$

$\Rightarrow$ $r = 9$

Thus ${}^{r}C_5 = {}^{9}C_5 = 126.$

Example 3. Verify Pascal's rule when $n = 8$ and $r = 4$.

Solution: If $n = 8$ and $r = 4$ then, according to Pascal's rule $\binom{8}{3} + \binom{8}{4} = \binom{9}{4}$.

$$\text{L.H.S} = \binom{8}{3} + \binom{8}{4} = \frac{8!}{(8-3)!\cdot 3!} + \frac{8!}{(8-4)!\cdot 4!}$$

$$= \frac{8!}{5!\cdot 3!} + \frac{8!}{4!\cdot 4!} = \frac{8!}{5\cdot 4!\cdot 3!} + \frac{8!}{4!\cdot 4\cdot 3!}$$

$$= \frac{8!}{4!\cdot 3!}\left(\frac{1}{5} + \frac{1}{4}\right)$$

$$= \frac{9\cdot 8!}{5\cdot 4! \times 4\cdot 3!} = \frac{9!}{5!\cdot 4!} = \frac{9!}{(9-4)!\,4!} = \binom{9}{4} = \text{R.H.S}$$

Hence, $\binom{8}{3} + \binom{8}{4} = \binom{9}{4}$

i.e., Pascal rule is verified.

6.3.3 Solve daily life problems involving combination

Example 1. How many matches can be played among five cricket teams, if each team has to play once against other.

Solution: This is the case of combination in which two teams are to be selected out of five teams.

So, required number of matches $= \binom{5}{2} = \frac{5!}{2!3!} = \frac{5\times4\times3!}{2\times3!} = 10.$

Example 2. A department in a college consists of 3 professors and 6 students. A study tour is to be arranged. In how many ways can a party of five tourist be chosen so as to include at least one professor?

Solution: The party may consist of

(i) 1 professor and 4 students, or

(ii) 2 professors and 3 students, or

(iii) 3 professor and 2 students

These can be chosen in:

(i) $\binom{3}{1}\cdot\binom{6}{4}$ (ii) $\binom{3}{2}\cdot\binom{6}{3}$ (iii) $\binom{3}{3}\cdot\binom{6}{2}$ ways respectively.

All these choices are independent. Hence by the sum Principle, the required number of ways is:

$$\binom{3}{1}\cdot\binom{6}{4} + \binom{3}{2}\cdot\binom{6}{3} + \binom{3}{3}\cdot\binom{6}{2}$$

$$= (3)(15) + (3)(20) + 15 = 120.$$

Example 3. In how many ways can a party of 5 students and 4 teachers be formed out of 18 students and 6 teachers?

Solution: The students can be chosen in $\binom{18}{5}$ ways and the teachers in $\binom{6}{4}$ ways.

Hence, by the Product Principle, the required number of ways is:

$$\binom{18}{5}\cdot\binom{6}{4}$$

$$=\frac{18!}{(18-5)!\cdot 5!}\times\frac{6!}{(6-4)!\cdot 4!}$$

$$=\frac{18\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13!}{13!\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}\times\frac{6\cdot 5\cdot 4\cdot 3\cdot 2!}{2!\cdot 4!}=(8568)(15)$$

$$=128520 \text{ ways.}$$

Exercise 6.3

1. Evaluate the following:

(i) $^{12}C_{10}$ (ii) $^{12}C_3$ (iii) $\binom{20}{17}$

(iv) $\binom{7}{4}$

2. Find the value of n when:

(i) $^nC_2 = 21$ (ii) $^nC_3 = 4$ (iii) $^nC_{12} = {}^nC_6$

(iv) $^nC_{15} = \frac{17\times 16}{2!}$ (v) $^nC_5 = {}^nC_4$ (vi) $^nC_{10} = \frac{12\times 11}{2!}$

3. Find the values of n and r, when:

(i) $^nP_r = 210$ and $^nC_r = 35$ (ii) $^{n-1}C_{r-1}: {}^nC_r: {}^{n+1}C_{r+1} = 3:6:11$

4. If (i) $^{12}C_r = {}^{12}C_{r+4}$ then find ^6C_r.

(ii) $^nC_{12} = {}^nC_{10}$ then find $^{25}C_n$.

5. The members of a club are 12 boys and 8 girls. In how many ways,
(i) a committee of 5 members be formed?
(ii) a committee of 4 boys and 3 girls be formed?

6. A box contains 7 red balls and 6 black balls. In how many ways can 5 balls be selected such that exactly 3 are red?

7. How many (a) diagonals and (b) triangles can be drawn in a plane by joining the vertices of the polygon having:

(i) 6 sides (ii) 8 sides (iii) 10 sides

8. Show that: $^9C_6 + {}^9C_5 = {}^{10}C_6$.

9. There are 11 men and 9 women members of a club. How many committees of 8 members can be formed, having:

(i) exactly five men (ii) at most five women

(iii) at least five women.

10. Find the number of combinations of the letters of the word "Question" taken 5 at a time.

11. How many words can be formed by 3 vowels and 4 consonants out of 5 vowels and 7 consonants.

6.4 Probability

The word "Probability" is commonly used in everyday speech. We say: "It will probably rain tomorrow"; "He will probably pass the examination"; or "The black horse will probably win the race".

Probability has become a science that predicts the chances of success or failure of an untold number of occurrences for man's benefit. The numerical measure of uncertain statements is, infact, called probability.

The theory of probability is of great importance in a number of modern fields. It is particularly useful in solving problems related to mortality and insurance, biological, physical, medical, engineering and social phenomena. It is also useful in our daily life.

6.4.1 Define the following:

- **statistical experiment**
- **sample space and an event**
- **mutually exclusive events**
- **equally likely events**
- **dependent and independent events**
- **exhaustive events**
- **impossible event**
- **simple and compound events**

(i) Statistical Experiment

The process by which an observation is made is called an statistical experiment or a trial.

Examples:

(i) Rolling a dice.

(ii) Tossing a coin.

- **Outcome**

The results of an experiment are called outcomes or logical possibilities.

Examples:

(i) Outcomes of an experiment of tossing a coin are head and tail.

(ii) A possible outcome of an experiment of rolling a die is 1,2,3,4,5 or 6.

- **Sample Point**

Every possible outcome, no two of which may be outcomes at the same time, is called sample point or an element.

Example: Head is the sample point for the experiment of rolling a die.

(ii) Sample Space and an event

(a) Sample Space

A set of all sample points or outcomes of an experiment is called the sample space or an outcome set and is usually denoted by S.

The sample space for rolling a die is $S = \{1,2,3,4,5,6\}$.

(b) Event

Any subset of a sample space is called an event.

A subset of a sample space having no element at all is called a null space or empty space and is denoted by $\varnothing$.

(iii) Mutually Exclusive events

Two events are said to be mutually exclusive or incompatible if they cannot occur simultaneously in a single outcome. In other words, if one of the events excludes the occurrence of the other event in an outcome, then they are said to be mutually exclusive. Thus, two events A and B are said to be mutually exclusive if $A \cap B = \emptyset$, i.e. if A and B are disjoint sets. For example, in a single toss of a coin, the events A={head} and B={tail} are mutually exclusive, for if one occurs the other cannot happen.

(iv) Equally likely events

Outcomes of an experiment or a trial are said to be equally likely if taking into consideration all the relevant evidences, there is no reason to expect one in preference to the other.

For example, in the tossing of an unbiased coin, both the faces are equally likely to come up; or in the rolling of a fair die, all the six faces are equally likely to occur. If each member of a set has an equal chance of being selected, we say that there is a random choice or a random selection.

(v) Dependent and Independent Events

(a) Dependent Events

Two events are dependent if the outcome or occurrence of the first affects the outcome or occurrence of the second.

Example: If we draw two colored balls from a bag, and the first ball is not replaced before you draw the second ball then the occurrence of the second drawn ball will be affected, hence both events are dependent events.

(b) Independent Events

Two events are independent if the outcome or occurrence of the first event does not affect the occurrence of the second event.

Some examples of independent events are:

(i) Getting a head in tossing a coin and getting a 5 in rolling a die.

(ii) Choosing a marble from a jar end getting tail in tossing a coin.

(vi) Exhaustive events

Let A and B be two events of a sample space S. Then A and B are said to be exhaustive, if $A \cup B = S$.

In the case of rolling a die, the two events,

$E = \{4\}$ and $F = \{3\}$ are not exhaustive for $E \cup F \neq S$ while the events $A = \{head\}$ and $B = \{tail\}$ in the tossing of a coin, are exhaustive as $A \cup B = S$ in this case.

(vii) Impossible event

An impossible event is an event that cannot happen.

Example: In flipping a coin once, an impossible event would be getting both a head and a tail.

(viii) Simple and Compound events

(a) Simple Event

An event containing only one element of the sample space, is called a simple or an elementary event.

For example, in throwing a dice, the event A of getting 4 is simple event. In this case sample space $= \{1, 2, 3, 4, 5, 6\}$ and event $A = \{4\}$.

(b) Compound Event

A compound event is one that can be expressed as the union of simple events.

The event B = A set of getting head or tail in tossing a coin. It is a compound event, since $B = \{head\} \cup \{tail\} = \{head, tail\}$

It may be noted that the union of simple events produces a compound event that is still a subset of the sample space.

Complementary Events

Two events A and B of sample space S are called complementary events if $A \cup B = S$ and $A \cap B = \emptyset$.

Note: The compelement event of the event A is denoted by A' or A^c

For example, the events A = {2, 4, 6} and B = {1, 3, 5} in the rolling of a die with S = {1, 2, 3, 4, 5, 6} are complementary events, because $A \cup B = S$ and $A \cap B = \emptyset$.

6.4.2 Recognize the formula for probability of occurrence of an event *E*, that is

$$P(E) = \frac{n(E)}{n(S)}, \qquad 0 \le P(E) \le 1$$

If S is a sample space and E is an event such that $E \subseteq S$, then the probability that the event E occurs is defined as

$$P(E) = \frac{n(E)}{n(S)}$$

Since $E \subseteq S$

$\Rightarrow$ $n(E) \le n(S)$

$\Rightarrow$ $P(E) \le P(S)$

i.e. $P(E) \le 1$ ...(i) ($\because P(S) = 1$)

$\because$ E is an event

$\therefore n(E) \ge 0$

$\Rightarrow$ $P(E) \ge 0$ ...(ii)

Combining (i) and (ii), we get, $0 \le p(E) \le 1$.

Thus this result guarantees that no probability may be less than zero or greater than one. P(E) = 0 means that E cannot occur, i.e., when $E = \emptyset$ and P(E) = 1 means that E must occur, i.e. when E = S.

6.4.3 Apply the formula for finding probability in simple cases

Example 1. A bag contains a red, a yellow and a blue marble. What is the probability of picking a red marble?

Solution: There are three possible outcomes, picking a red marble (r), a yellow marble (y) and a blue marble (b).

Therefore, S = {r, y, b}

Let, E be the event of picking a red marble, i.e, A = {r}.

Hence the probability of picking the red marble is:

$$P(E) = \frac{O(E)}{O(S)} = \frac{1}{3}$$

Example 2. A basket contains two white balls and two black balls. What is the probability of drawing two black balls?

Solution:

Here, the sample space $S = \{(w, w), (w, b), (b, w), (b, b)\}$.

There is only one favorable case for the event E.

E = An event set of drawing two black balls = $\{(b, b)\}$.

Hence the probability of drawing two black balls is: P(A) $= \frac{O(E)}{O(S)} = \frac{1}{4}$.

Example 3. Two coins are tossed together once. Find the probability of getting at least one head.

Solution: Here, sample space, $S = \{HH, HT, TH, TT\}$, where H denotes head and T denotes tail.

Let, E be the event of getting at least one head.

i.e., $= \{HH, HT, TH\}$

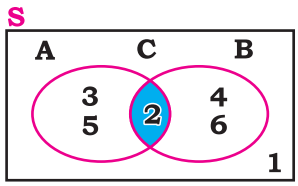
Therefore $O(S) = 4$ and $O(E) = 3$

Hence $P(E) = \frac{3}{4}$.

6.4.4 Use Venn diagrams and tree diagrams to find the probability for the occurrence of an event

Venn Diagram:

A Venn diagram can be used to represent the outcomes of an experiment and is very helpful to find probability. It generally consists of a rectangle that represents the sample space S together with circles or ovals. The circles or ovals represent events. For example, in throwing a dice, the events A, B and C along with the sample spaces are shown through Venn diagram (Fig 6.4) where A be the event of getting a prime number, B be the event of getting an even number and C be the event of getting both prime and even.



(Fig. 6.4)

The use of Venn diagram in finding probability is explained with the help of the following examples.

Example 1.

Let A be the event of getting tail on the first coin. and B be the event of getting tail on the second coin when two coins are tossed.

Find P(A∩B) and P(AUB) with the help of Venn diagrams

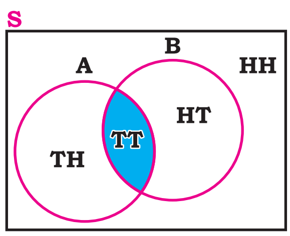
Solution:

Here, sample space is

S = {HH, HT, TH, TT}.

Event A ={TT, TH} and event B = {TT, HT}.

Therefore, A∩B = {TT} as shown in Fig 6.5.

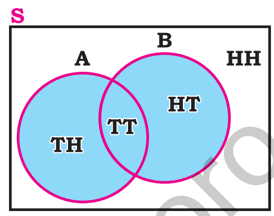


A ∩ B
(Fig. 6.5)

Using Fig 6.5, $P(A \cap B) = \frac{1}{4} = 0.25$

and $A \cup B$ ={TH, TT, HT}. as shown in Fig 6.6.

Using Fig 6.6, $P(A \cup B) = \frac{3}{4} = 0.75$

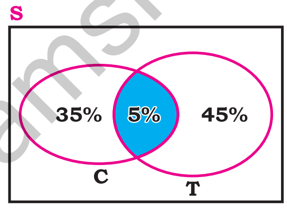


A U B

(Fig. 6.6)

Example 2. 40% of the students at a local college belong to a club and 50% work part time. 5% of the students work part time and belong to the club. Find the probability with the help of Venn diagram (i) that a student belongs to club and works part time.

(ii) that a student belongs to club or works part time.



C ∩ T

(Fig. 6.7)

Solution:

Let C be the event that student belongs to a club

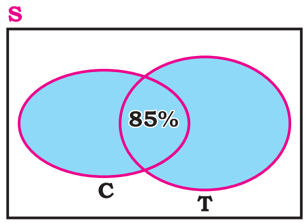
and T be the event that student works part time.

By using Fig 6.7

- The probability that the student belongs to a club is: $P(C) = 0.40$
- The probability that the student works part time is: $P(T) = 0.50$

Using Fig 6.7

- the probability that the student belongs to a club and works part time is: $P(C \cap T) = 0.05$



C U T

(Fig. 6.8)

Using Fig. 6.8,

- the probability that the student belongs to a club or works part time is:

$$P(C \cup T) = 0.85$$

Tree Diagram

We have already studied that a tree diagram is a special type of diagram used to determine the number of outcomes of an experiment. It consists of "branches" that are labelled with numbers of outcomes or probabilities. Tree diagrams can make some probability problems easier to visualize and solve. The following example illustrates how to use a tree diagram.

Example 1. There are 11 balls in a basket in which three balls are red (R) and eight balls are blue (B). Draw two balls, one at a time, with replacement. With the help of tree diagram, find the number of BR outcomes and

a. Calculate P(RR).

b. Calculate P(RB U BR).

c. Calculate P(R on 1st draw ∩ B on 2nd draw).

d. Calculate P(BB).

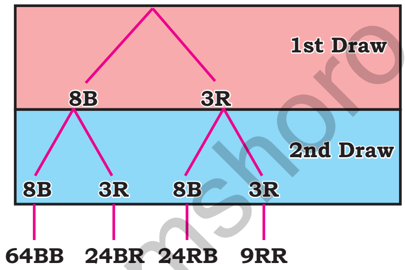
Solution:

"With replacement" means that we put the first ball back in the basket before we select the second ball. The tree diagram shows all the possible outcomes (Fig. 6.9)

We have, Total numbers of outcomes

= 64 + 24 + 24 + 9 = 121

Tree diagram showing outcomes



(Fig. 6.9)

The first set of branches represents the first draw. The second set of branches represents the second draw. Each of the outcomes is distinct. We can list each draw containing red balls as R1, R2, and R3 and blue balls as B1, B2, B3, B4, B5, B6, B7, and B8. All BR outcomes can be listed as under.

B1R1 B1R2 B1R3	B2R1 B2R2 B2R3	B3R1 B3R2 B3R3
B4R1 B4R2 B4R3	B5R1 B5R2 B5R3	B6R1 B6R2 B6R3
B7R1 B7R2 B7R3	B8R1 B8R2 B8R3	(24 outcomes)

Also, with the help of tree diagram

Number of BR outcomes = 24

Using tree diagram (Fig. 6.9)

a: $P(RR) = \frac{9}{121}$

b: $P(RB \text{ U } BR) = \frac{48}{121}$

c: $P(R \text{ on 1st draw } \cap \text{ B on 2nd draw}) = P(RB) = \frac{24}{121}$

d: $P(BB) = \frac{64}{121}$

Example 2. In a standard deck of 52 playing cards, twelve cards are face cards (F) and 40 cards are not face cards (N). Two cards are drawn one at a time, without replacement. Using tree diagram

a. Find P(FN or NF).

b. Find P(at least on face card).

Tree diagram showing probabilities

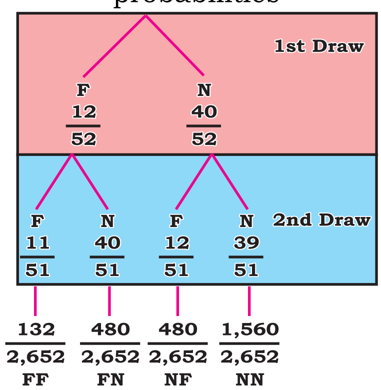


Fig (6.10)

Solution: The tree diagram is labeled with all possible probabilities. (Fig. 6.10)

a. $P(FN \text{ or } NF) = \frac{480}{2,652} + \frac{480}{2,652}$

$= \frac{960}{2,652} = 0.362$

b. P(at least one face card)

$$= \frac{(132 + 480 + 480)}{2,652} = 0.4118$$

6.4.5 Define the conditional probability

The conditional probability of an event B in relationship to an event A is the probability when event A has already occurred. The notation for conditional probability is P(B/A) [read as probability of event B given A] and is determined by:

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Example 1. A single six sided die is rolled once. Determine the probability that a 2 is rolled. Given that an even number has already been rolled.

Solution: The sample space for this experiment is S = {1,2,3,4,5,6}

A is an event that 2 is rolled i.e., A = {2}, B is an event that even number is rolled i.e., B = {2,4,6}. Now probability of A on the occurrence of B is:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3}$$

Example 2. A family has two children. Determine the probability that the family has; (i) one boy and one girl given that the first child is a boy.
(ii) two girls given that at least one is a girl.

Solution: Here, S = {BB, BG, GB, GG}

(i) Let A is an event of having one boy and one girl
i.e., A = {BG, GB} and B is an event of having first child is a boy
i.e., B = {BB, BG}

We have, $A \cap B = \{B\,G\}$

Now, $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

(ii) Let A is an event of having both girls i.e., A = {G G} and B is an event of having at least one a girl i.e., B = {BG, GB, GG}

Now, $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3}$$

6.4.6 Recognize the addition theorem (or law) of probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where, A and B are two events. Deduce that $P(A \cup B) = P(A) + P(B)$ where A and B are mutually exclusive events

To write down the elements of a sample space S and to count the number of favourable cases often proves tedious in practical problems. To facilitate the computations of probabilities in such cases. We have the following theorems in which we shall denote

the probability of A or B by $P(A \cup B)$,

and the probability of A and B by $P(A \cap B)$.

Theorem 1: (Addition law of probability)

If A and B are two events of sample space S, then

$$P(A \cup B) + P(A \cap B) = P(A) + P(B).$$

This law is called addition law of probability.

This can also be written as:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

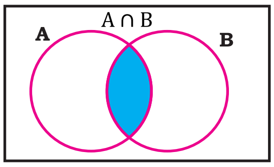
Proof: From Venn diagram (Fig. 6.11), it is clear that

$$O(A \cup B) + O(A \cap B) = O(A) + O(B)$$

Dividing both sides by O(S)

We get, $$\frac{O(A \cup B)}{O(S)} + \frac{O(A \cap B)}{O(S)} = \frac{O(A)}{O(S)} + \frac{O(B)}{O(S)}$$

i.e., $$P(A \cup B) + P(A \cap B) = P(A) + P(B)$$



(Fig. 6.11)

Corollary: If A and B be mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$

This result can be extended to any finite number of mutually exclusive events $A_1, A_2, \ldots, A_n$. That is, if $A_1, A_2, \ldots, A_n$ are disjoint subsets of S, then the probability of at least one of them is given by

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Theorem 2:

If A and A′ are complementary events in a sample space S, then

$$P(A) + P(A') = 1$$

Proof: From Venn diagram (Fig 6.12)

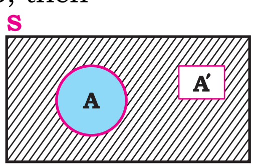
$$A \cup A' = S$$

$\Rightarrow$ $$P(A \cup A') = P(S)$$

$\Rightarrow$ $P(A) + P(A') = 1$, (By using addition law of probability)

$$P(A) = 1 - P(A')$$

and $$P(A') = 1 - P(A).$$



(Fig. 6.12)

Theorem 3:

$$P(\emptyset) = 0.$$

Proof: Since $\emptyset \cup \emptyset' = S$

Therefore, $P(S) = P(\emptyset \cup \emptyset')$

$\Rightarrow \quad 1 = P(\emptyset) + P(\emptyset')$, (By using addition law of probability)

$1 = P(\emptyset) + P(S)$, [Since $\emptyset' = S - \emptyset = S$]

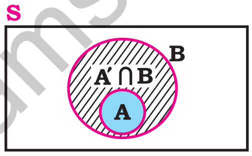
$\Rightarrow \quad 1 = P(\emptyset) + 1$, [Since $p(S)=1$]

$\Rightarrow \quad p(\emptyset) = 0.$

Theorem 4:

If $A \subseteq B \subseteq S$, then

$P(A) \leq P(B).$



(Fig. 6.13)

Proof: According to Venn diagram (Fig. 6.13)

$$B = A \cup (A' \cap B).$$

Also, $A \cup (A' \cap B) = (A \cup A') \cap (A \cup B)$ (distributive law)

$= S \cap (A \cup B)$

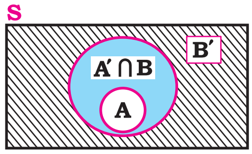
$= A \cup B$

$= B$, $(\because A \subseteq B)$

From Venn diagram (Fig. 6.14)

$A \cap (A' \cap B) = \emptyset$

i.e., A and $(A' \cap B)$ are disjoint sets.



(Fig. 6.14)

Therefore $P(B) = P[A \cup (A' \cap B)]$

$= P(A) + P(A' \cap B)$

But $P(A' \cap B) \geq 0.$

Therefore $P(B) \geq P(A)$

or $P(A) \leq P(B).$

Hence proved.

6.4.7 Recognize the multiplication theorem (or law) of probability: $P(A \cap B) = P(A) \cdot P(B|A)$ or $P(A \cap B) = P(B) \cdot P(A|B)$ where $P(B|A)$ and $P(A|B)$ are conditional probabilities. Deduce that $P(A \cap B) = P(A) \cdot P(B)$ where A and B are independent events

Multiplication theorem of Probability

If A and B are events of sample space, then the probability that both A and B occur is equal to the probability of the event A times the probability of B given that A has occurred.

i.e., $P(A \cap B) = P(A) \cdot P(B/A)$

Also, $P(A \cap B) = P(B) \cdot P(A/B)$

where P(B/A) and P(A/B) are conditional probabilities. This is multiplication rule for two dependent events.

In the case where A and B are independent (where A has no effect on the probability of event B), the conditional Probability of event B given by event A, is simply the probability of event B, that is P(B).

So, $P(A \cap B) = P(A) \cdot P(B)$

Example 1. A box contain 5 black and 7 red balls. Two balls are drawn from the box one after the other without replacement, what is the probability that both balls are black balls.

Solution:

The total number of balls in the box is $5 + 7 = 12$. Let B_1 is the event of getting first black ball and B_2 is the event of getting second black ball.

Here, both events are dependent events.

So, $P(B_1 \cap B_2) = P(B_1)P(B_2/B_1)$

$$= \frac{5}{12} \times \frac{4}{11} = \frac{20}{132} = \frac{5}{33}$$

Example 2. A coin is tossed and a single 6-sided die is rolled. Find the probability of getting head of the coin and rolling a 3 on the die.

Solution:

Let A be the event of getting a head and B be the event of getting a 3 on the dice.

∵ Both events are independent

∴ $P(A \cap B) = P(A).P(B)$

$$= \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

6.4.8 Use theorems of addition and multiplication of probability to solve related problems from daily life

(i) Problems of theorems of addition of probability.

Example 1. An integer is chosen at random from the first 200 positive integers. What is the probability that the chosen integer is divisible by 6 or 8?

Solution: Here $S = \{1, 2, 3, \dots, 200\}$

Now the number of integers divisible by 6 in the first 200 positive integers = 33.

Again, the number of integers divisible by 8 in the first 200 positive integers = 25.

Also, the number of integers divisible by 24 (L.C.M. of 6 and 8) in the first 200 positive integers is 8.

Let A be the event that the chosen integer is divisible by 6.

i.e., $A = \{6,\ 12,\ 18, \ldots, 198\}$.

Let B be the event that the chosen integer is divisible by 8.

i.e., $B = \{8,\ 16,\ 24, \ldots, 200\}$.

$$A \cap B = \{24, 48, 72, 96, 120, 144, 168, 192\}.$$

Now, $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{33}{200} + \frac{25}{200} - \frac{8}{200}$$
$$= \frac{50}{200} = \frac{1}{4}.$$

Example 2. If the probability of solving a problem by two students Ahsan and Umar are $\frac{1}{2}$ and $\frac{1}{3}$ respectively then what is the probability of the problem to be solved.

Solution: Let A and B be the events of solving the problem by Ahsan and Umar respectively. We have $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$.

The problem will be solved if it is solved by at least one of them.

So, we need to find $P(A \cup B)$.

$\therefore$ A and B are independent events

$\therefore$ $P(A \cap B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{3}$

By addition theorem of probability, we have
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

So, $$P(A \cup B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{2} \times \frac{1}{3} = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{3+2-1}{6} = \frac{2}{3}$$

(ii) Problems of theorems of multiplication of probability

Example 1. Three cards are chosen at random from a deck of 52 cards without replacement. What is the probability of choosing 3 aces?

Solution: $$P(3\text{ aces}) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50}$$
$$= \frac{24}{1{,}32{,}600}$$
$$= \frac{1}{5{,}525}$$

Example 2. In a shipment of 20 computers, 3 are defective. Three computers are randomly selected and tested. What is the probability that all three are defective if the first and second ones are not replaced after being tested?

Solution: P(3 defectives) $= \frac{3}{20} \cdot \frac{2}{19} \cdot \frac{1}{18} = \frac{6}{6840} = \frac{1}{1140}$

Example 3. Suppose an individual applying to a college determines that he has an 80% chance of being accepted, and he knows that dormitory housing will only be provided for 60% of all of the accepted students. What is the chance of the student being accepted and receiving dormitory housing?

Solution: P (Accepted and Dormitory Housing)

$= P(\text{Dormitory Housing}|\text{Accepted}) \cdot P(\text{Accepted})$

$= (0.60) \cdot (0.80) = 0.48.$

Example 4. A jar contains 3 red, 5 green, 2 blue and 6 yellow marbles. A marble is chosen at random from the jar. After replacing it, a second marble is chosen. What is the probability of choosing a green and then a yellow marble?

Solution: P (green) $= \frac{5}{16}$

P (yellow) $= \frac{6}{16}$

∵ This is the case of independent events.

∴ P (green and yellow) = P (green) · P (yellow)

$= \frac{5}{16} \cdot \frac{6}{16} = \frac{30}{256}$

$= \frac{15}{128}$

Exercise 6.4

1. Two fair coins are tossed. Find the probability of getting:

(i) Same faces (ii) All heads (iii) At most one head

(iv) At least two tail (v) At least one tail

2. Two dice, one red and the other green, are rolled simultaneously. The numbers of dots on the tops are added. Find the probability of getting a sum of:

(i) 8 (ii) 10 (iii) 12

3. A bag contains 40 balls out of which 5 are green, 15 are red and the remaining are black. A ball is drawn out of the bag. Find the probability of getting:

(i) The ball is green (ii) The ball is black (iii) The ball is not green

4. A card drawn from a well shuffled deck of cards, find the probability of:

(i) getting a King (ii) getting a club (iii) getting a face card

5. A die is rolled. Find the probability using Venn diagram of getting:
(i) an even number (ii) a number greater than 4
(iii) a number which is even and greater than 4.

6. In a three child family, by using tree diagram. Find the probability of having:
(i) three boys (ii) exactly two boys (iii) at least one girl

7. In a single throw of two fair dice, find the probability that the product of the numbers on the dice is:
(i) between 2 and 10 (both inclusive) (ii) divisible by 5

8. A marble is drawn at random from a box containing 20 red, 10 white, 25 orange and 15 blue marbles. Find the probability that it is:
(i) orange or red (ii) not blue or red (iii) red, white or blue

9. The king, queen and jack of clubs are removed from a deck of 52 playing cards and then shuffled. A card is drawn from the remaining cards. Find the probability of getting:
(i) a heart (ii) a queen (iii) a club (iv) '9' of red color

10. A pair of fair dice is thrown. If the two numbers appearing are different, find the probability that (i) the sum is 10, (ii) the sum is six or less.

11. Given that $P(A) = 0.3$, $P(B) = 0.7$, $P(A \cap B) = 0.21$ then find:
(i) $P(A/B)$ (ii) $P(B/A)$

12. If one card is selected at random from a deck of 52 playing cards, what is the probability that the card is a club or a face or both?

13. From two events A and B, $P(A) = 0.5$, $P(B) = 0.2$, $P(A \cup B) = 0.4$ then find $P(A \cap B)$?

14. A natural number is chosen out of first 35 natural numbers. What is the probability that the chosen number is divisible by 8 or 9?

15. A bag contains 15 black, 25 red and 10 white balls. A ball is drawn at random. Find the probability that it is either red or white?

16. Two events A and B are such that $P(A) = \frac{1}{4}$, $P(A/B) = \frac{1}{2}$, $P(B/A) = \frac{2}{3}$. Find $P(A \cap B)$ and $P(B)$?

17. A fair die is thrown twice. Find the probability that an even number of dots appear in first and the number of dots in the second throw is less than 4?

18. Three missiles are fired at a target. If the probability of hitting the target are 0.5, 0.3 and 0.6 respectively, and if the missiles are fired independently, what is the probability that all the missiles hit the target?

Review Exercise 6

1. Select correct answer.

i. If $n = 0$, then $\frac{(n+1)!}{n!} =$ ______________

(a) 0 (b) 1 (c) n (d) ∞

ii. Probability of getting 7 in throwing a dice is:

(a) 0 (b) 1 (c) -1 (d) Not defined

iii. The factorial form of $12 \cdot 11 \cdot 10 \cdot$ is:

(a) $\frac{12!}{9!}$ (b) 12! (c) $\binom{12}{9}!$ (d) $(12!) \cdot (9!)$

iv. If two independent events A and B occur in p and q ways respectively, then number of ways that both events can occur is:

(a) $p + q$ ways (b) $p.q$ ways (c) $(pq)^r$ ways (d) $rp + qr$ ways

v. An arrangement of n objects according to some definite order is called:

(a) Combination (b) Permutation
(c) Factorial (d) none of these

vi. An arrangement of n objects without any order is called:

(a) Combination (b) Permutation
(c) Factorial (d) none of these

vii. The number of permutation of the letters of the word COMMITTEE is:

(a) $\binom{9}{2,\ 2,\ 2}$ (b) $\binom{6}{1,\ 2,\ 2}$ (c) $\binom{9}{2,\ 2,\ 1}$ (d) $\binom{9}{2,\ 3,\ 1}$

viii. 8.7.6 is equal to:

(a) ${}^{8}P_{3}$ (b) ${}^{8}C_{3}$ (c) ${}^{8}P_{5}$ (d) ${}^{8}C_{5}$

ix. If $r = n$, then ${}^{n}P_{r}$ is equal to:

(a) $r!$ (b) $(n - r)!$ (c) 1 (d) 0

x. The number of ways that a necklace of n beads of different colours be made is:

(a) $(n - 1)!$ (b) $\frac{n!}{2}$ (c) $\frac{n!-1}{2}$ (d) $\frac{(n-1)!}{2}$

xi. Any subset of a sample space is called:

(a) Sample space (b) an event
(c) a Trial (d) Random variable

xii. For two events A and B if $A \cap B = \emptyset$, then events A and B are called:

(a) Mutually exclusive (b) Not mutually exclusive
(c) Overlapping (d) Dependent events

xiii. When a dice is rolled and coin is tossed, all possible outcome are:

(a) 6 (b) 12 (c)18 (d)24

xiv. If two events A and B have equal chance of occurrence, then the events are:

(a) Equally likely (b) Not equally likely
(b) Dependent (d) Not mutually exclusive

xv. If E be an event of a sample space S, then:

(a) $P(E) = \frac{n(S)}{n(E)}$ (b) $0 \leq P(E) \leq 1$
(c) $0 < P(E) < 1$ (d) all of these

xvi. The probability of getting the tail in a single toss of a coin is:

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

xvii. Three dice are rolled simultaneously, then $n(S)$ is equal to:

(a) 36 (b) 18 (c) 216 (d) 6

xviii. Two teams A and B are playing a match, the probability that team A does not lose is:

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) 0

xix. If ${}^{n}C_6 = {}^{n}C_{12}$, then n equals:

(a) 18 (b) 12 (c) 6 (d) 20

2. Evaluate the following:

(i) $\frac{{}^{6}C_3 \times {}^{4}C_3}{{}^{10}C_4}$ (ii) $\frac{{}^{6}P_3 \times {}^{4}P_3}{{}^{10}P_4}$ (iii) $\sum_{r=1}^{3} {}^{3}C_r$

3. There are seven seats available in a compartment. In how many ways can seven persons be seated?

4. A room has 3 lamps. From a collection of 10 light bulbs of which 6 are no good, a person selects 3 at random and puts them in the sockets. What is the probability that he will have light?

5. If two dice are thrown simultaneously, what is the probability of obtaining a sum of 7 or a sum of 11?

6. A bag contains 8 white, 10 black and 12 red balls. 3 balls are drawn from the bag. What is the probability that the first ball is white, the second ball is black and the third ball is red, when every time the ball is replaced?

7. In how many ways can a football team of 11 players be selected out of 15 players? How many of them will include a particular player?