Unit 7

Mathematical Induction and Binomial Theorem

7.1 Mathematical Induction

A powerful method of proof, frequently used in mathematics is mathematical induction. This method is not to be confused with the method of inductive logic used in the experimental science in which generalization is formulated by observing many specific cases. In contrast, mathematical induction is a form of deductive reasoning in which conclusions are established beyond any doubt.

7.1.1 Describe principle of mathematical induction

Principle of Mathematical Induction

If P(n) is a proposition about a positive integer (natural number) n such that

- P (n) is true for $n = 1$, and
- P (n) is true for $n = k$ implies that it is also true for $n = k + 1$.

Then P (n) is true for all positive integers (natural numbers) n.

Principle of Mathematical Induction when Proposition P (n) is not true for the first few values of n

Sometimes a proposition P(n) is not true for the first few values of n and is true for all successive values after a certain stage.

For an application of the method of mathematical induction to such cases, the principle of mathematical induction is modified and restated as under.

If P(n) is a proposition about a positive integer n such that

- P(n) is true for $n = i$, where i is a positive integer, and
- P (n) is true for $n = k + 1$, whenever P(n) is true for any positive integer $n = k$.

7.1.2 Apply the principle to prove the statements, identities or formulae

Example 1. By using principle of mathematical induction, prove that the following formula is true for all positive integral values of n.

$$2 + 4 + 6 + \cdots + 2n = n\,(n+1).$$

Proof:

(i) For $n = 1$, we have $2 = 1(1+1)$ or $2 = 2$ i.e., P(n) is true for $n = 1$

(ii) Assume that the formula or P(n) is true for some positive integer $n = k$.

i.e., $2 + 4 + 6 + \cdots + 2k = k\,(k+1)$ ...(i)

Now we shall prove that the proposition is true for $n = k+1$, that is, we shall prove that: $2 + 4 + 6 + \cdots + 2k + 2(k+1) = (k+1)\{(k+1)+1)\}$.

By our hypothesis, we have

$$2 + 4 + 6 + \cdots + 2k = k(k+1)$$

Adding $2(k+1)$ to both the sides of equation (i), we get

$$\begin{aligned} 2 + 4 + 6 + \cdots + 2k + 2(k+1) &= k(k+1) + 2(k+1) \\ &= (k+1)(k+2) \\ &= (k+1)\{(k+1)+1\}. \end{aligned}$$

i.e., P(n) is true for $n = k+1$ whenever it is true for $n = k$. Hence by principle of mathematical induction the proposition is true for all positive integral values of n.

Example 2. Prove that $2^n > (2n+1)$ for all integral values of $n \geq 3$.

Solution:

It can well be seen that for $n = 1$ and 2, the proposition gives $2 > 3$ and $4 > 5$ which are false.

We therefore, apply the modified form of mathematical induction.

(i) Here, $i = 3$ and so P(n) for $n = 3$ gives

$2^3 > (2 \cdot 3 + 1)$

or $8 > 7$ which is true. So, P(n) is true for $i = 3$.

(ii) Assume P(n) to be true for $n = k$, i.e.

$$2^k > (2k+1)$$

Now, multiplying both sides of the above inequality by 2, we get

$2.2^k > 2(2k+1)$,

$\Rightarrow$ $2^{k+1} > 4k + 2$,

$\Rightarrow$ $2^{k+1} > 2k + 2k + 2$.

Deducting $2k$ from right side and keeping 1 instead, the inequality statement holds and we have,

$2^{k+1} > 2k + 1 + 2$

or $\quad 2^{k+1} > 2k + 2 + 1.$

i.e., $\quad 2^{k+1} > 2\,(k+1) + 1.$ Therefore, P (n) is also true for $n = k + 1$.

Hence, by the principle of mathematical induction P(n) is true for all integral values of $n \geq 3$.

Example 3. Using the principle of mathematical induction prove that:

$$1 + 2 + 3 + \cdots n = \frac{n(n+1)}{2}, \text{ for all natural numbers } n.$$

Proof:

(i) For $n = 1$, P(n) becomes $1 = \frac{1(1+1)}{2}$

or $\quad 1 = 1$

So, P(n) is true for $n = 1$.

(ii) Assuming the result to be true for $n = k$, we get the hypothesis:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

Now we have to prove that the result is also true for $n = k + 1$. To do so, we add $(k + 1)$ to both sides of the above hypothesis and get

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= (k+1)\ (\frac{k}{2} + 1) = (k+1) \cdot \frac{(k+2)}{2}$$

$$= \frac{(k+1)\{\,(k+1) + 1\,\}}{2}$$

i.e., P(n) is true for $n = k + 1$.

Thus, by the principle of mathematical induction P$(n)$is true for all natural numbers n.

Example 4. Prove that:

$$1^2 + 2^2 + 3^2 + \cdots +\ n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{for all natural numbers } n.$$

Proof:

If P (n) represents the above proposition, then

(i) for $n = 1$, P(n) becomes $1^2 = \frac{1(1+1)(2\cdot1+1)}{6}$

$$\Rightarrow \quad 1 = \frac{1(2)(3)}{6}$$

or $\quad 1 = 1$ which is true.

i.e., P(n) is true for $n = 1$.

(ii) Suppose that P(n) is true for $n = k$, i.e.,

$$1^2 + 2^2 + 3^2 + \cdots +\ k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \ldots\text{(i)}$$

To prove P(n) to be true for $n = k + 1$ we add $(k+1)^2$ to both sides of the above equation (i).

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)}{6}\{k(2k+1) + 6(k+1)\}$$

$$= \frac{(k+1)}{6}\{2k^2 + 7k + 6\}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)\{(k+1)+1\}\{2(k+1)+1\}}{6}$$

This, in fact, is the same as the given proposition P(n) for $n = k + 1$.
Thus, the truth of P(n) for $n = k$ implies its truth for $n = k + 1$.
Hence, by the principle of mathematical induction, P(n) is true for all natural numbers n.

Example 5. Prove that: $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$, for all natural numbers n.

Proof:

i. For $n = 1$, P(n) becomes

$$1^3 = \left[\frac{1.(1+1)}{2}\right]^2$$

$$1 = 1, \text{ which is true.}$$

i.e., P(n) is true for $n = 1$

ii. Assume that P(n) is true for $n = k$.

i.e., $$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

Adding $(k+1)^3$ to both the sides, we have

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

$$= \frac{(k+1)^2}{4}\{k^2 + 4(k+1)\}$$

$$= \frac{(k+1)^2}{4}\{k^2 + 4k + 4\}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left[\frac{(k+1)\{(k+1)+1\}}{2}\right]^2,$$

Thus, it is true for $n = k + 1$.

Hence, by mathematical induction P(n) is true for all natural numbers.

Example 6. Prove that $2^{3n} - 7n - 1$, is divisible by 49 where n is any positive integer.

Proof:

(i) For $n = 1$, the given expression becomes

$$2^{3n} - 7n - 1 = 2^{3\cdot1} - 7\cdot1 - 1$$
$$= 8 - 7 - 1 = 0$$

Since zero is divisible by 49, the given statement is true for $n = 1$.

(ii) Assume that the statement is true for $n = k$, i.e.,

$$2^{3k} - 7k - 1 \text{ is divisible by } 49$$

Now for, $n = k + 1$, we have

$$2^{3(k+1)} - 7(k+1) - 1 = 2^{3k+3} - 7k - 7 - 1$$
$$= 8.2^{3k} - 7k - 8$$
$$= 8.2^{3k} - 8(7k) - 8 + 7(7k)$$
$$= 8(2^{3k} - 7k - 1) + 49k$$

By our hypothesis $(2^{3k} - 7k - 1)$ is divisible by 49 and $49k$ is obviously divisible by 49. Therefore, the given expression is also divisible by 49 for $n = k + 1$.

Hence by mathematical induction the given statement is true for all positive integral values of n.

Exercise 7.1

1. Prove the following propositions by mathematical induction for every positive integer n.

(i) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

(ii) $3 + 6 + 9 + \cdots + 3n = \frac{3}{2}n(n + 1)$

(iii) $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$

(iv) $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 2(1 - \frac{1}{2^n})$

(v) $2 + 6 + 18 + \cdots + 2\cdot3^{n-1} = 3^n - 1$

(vi) $2 + 6 + 12 + \cdots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$.

(vii) $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2}{3}n(n + 1)\ (2n + 1)$.

(viii) $1\cdot3 + 2\cdot4 + 3\cdot5 + \cdots + n(n + 2) = \frac{1}{6}n(n + 1)(2n + 7)$.

(ix) $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

(x) $1.2+2.2^2+3.2^2+4.2^2+\cdots+n.2^n=(n-1).2^{n+1}+2$.

(xi) $1!\cdot 1+2!\cdot 2+3!\cdot 3+\cdots+n!\cdot n=(n+1)!-1$.

(xii) $\frac{1}{a(a+1)}+\frac{1}{(a+1)(a+2)}+\cdots+\frac{1}{(a+n-1)(a+n)}=\frac{n}{a(a+n)}$

(xiii) $\frac{1^2}{1.3}+\frac{2^2}{3.5}+\cdots+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}$.

(xiv) (i) $a+ar+ar^2+\cdots+a\,r^{n-1}=\frac{a(a-r^n)}{1-r}, (r\neq 1)$.

(ii) $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n$.

(iii) $\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r};\ n\geq r$.

2. Prove the following statement by mathematical induction.

(i) $2^{3n+2}-28n-4$ is divisible by $49, \forall\ n\in\mathbb{N}$.

(ii) $3^{2n+2}-8n-9$ is divisible by $64, \forall\ n\in\mathbb{N}$.

(iii) 7^n-4^n is divisible by 3

3. Prove that:

(i) $2^{n+1}>(2n+3)$, for all integral values of $n\geq 2$,

(ii) $3^{n-1}>2^n$, for all integral values of $n\geq 3$,

(iii) $n!>3^{n-1}$, for all integral values of $n\geq 5$.

7.2 Binomial Theorem

7.2.1 Use Pascal's triangle to find the expansion of $(x+y)^n$ where n is a small positive integer

An expression consisting of two terms connected by +ve or –ve sign is called a binomial expression or simply a binomial.
For example, $a+b$, $2x-3y$, z^3-2z, a^2+b^2 are all binomial expressions.
For $(x+y)^n$ where $(x+y)$ is a binomial and natural number n is its exponent or index. The following products can be verified by actual multiplication.

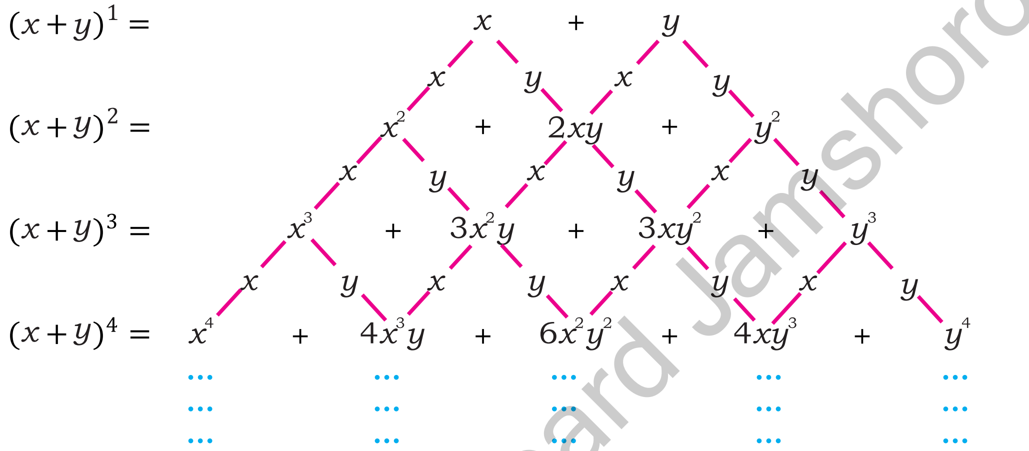
$$(x+y)^1=x+y$$

$$(x+y)^2=x^2+2xy+y^2$$

$$(x+y)^3=x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

In the binomial theorem we wish to state a general law of formation that lies behind these expansions; i.e., we wish to state a general law for the expansion of $(x+y)^n$, where n is a natural number. The following diagram illustrates the process.



(Fig. 7.1)

The coefficients in the above expansions may be shown in the following scheme, which is called Pascal's triangle.

			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1
⋯		⋯		⋯		⋯		⋯
⋯		⋯		⋯		⋯		⋯
⋯		⋯		⋯		⋯		⋯

(Fig. 7.2)

A very definite pattern is evident in this array. The first and the last number in each row is 1. Each of the other elements is the sum of the two elements to its left and right in the row immediately preceding. Thus we would expect the coefficients of the expansion of $(x+y)^5$ to be

1 5 10 10 5 1

Direct multiplication will verify this result. Similarly, for $(x+y)^6$, we have 1 6 15 20 15 6 1

It may also be noted that there is a connection between the coefficients in these expansions and the possible number of combinations of x and y. Thus, Pascal's triangle can also be given in the following form.

$$
\begin{array}{ccccccccc}
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

(Fig. 7.3)

Each element in a row, other than the 1's is the sum of the two elements to its left and right in the row immediately preceding it, that is, in general

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

This is called Pascal's rule. Apart from coefficients, a definite pattern about the terms of the expansion of $(x+y)^n$ is also evident from (Fig. 7.1). Each term contains product of powers of x and y. From first term to the last term the products are:

$$x^n y^0, x^{n-1}y, x^{n-2}y^2, \ldots, x^0 y^n$$

Example: Using Pascal's triangle, expand $(x+y)^5$.

Solution: According to Pascal's triangle, the row related to $n = 5$, is: $1, 5, 10, 10, 5, 1$

So, $\quad (x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

7.2.2 State and prove binomial theorem for positive integral index

It is difficult to find the co-efficients in the expansion of $(x+y)^n$ from the scheme given above when the exponent n is large. In this section we state and prove a theorem, known as the Binomial Theorem, due to UMER KHYAM (1074 A.D).

Statement:

If n is a positive integer, then

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

Proof: We prove that

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r-1}a^{n-r+1}b^{r-1} + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$$

with the help of the Principle of Mathematical Induction.

(i) For $n = 1$, we have

$$(a+b)^1 = a^1 + \binom{1}{1} a^{1-1} b^1 = a + b$$

Thus, the theorem is true for $n = 1$

(ii) Assume the theorem to be true for $n = k$, i.e., let the hypothesis be

$$(a+b)^k = a^k + \binom{k}{1} a^{k-1} b + \binom{k}{2} a^{k-2} b^2 + \cdots + \binom{k}{r-1} a^{k-r+1} b^{r-1} + \binom{k}{r} a^{k-r} b^r + \cdots + b^k$$

Multiplying each term of the formula in the above hypothesis by $a + b$, we have

$$(a+b)^{k+1} = a\left\{a^k + \binom{k}{1} a^{k-1} b + \binom{k}{2} a^{k-2} b^2 + \cdots + \binom{k}{r-1} a^{k-r+1} b^{r-1} + \binom{k}{r} a^{k-r} b^r + \cdots + b^k\right\}$$

$$+b\left\{a^k + \binom{k}{1} a^{k-1} b + \binom{k}{2} a^{k-2} b^2 + \cdots + \binom{k}{r-1} a^{k-r+1} b^{r-1} + \binom{k}{r} a^{k-r} b^r + \cdots + b^k\right\}$$

$$= \left\{a^{k+1} + \binom{k}{1} a^k b + \binom{k}{2} a^{k-1} b^2 + \cdots + \binom{k}{r-1} a^{k-r+2} b^{r-1} + \binom{k}{r} a^{k-r+1} b^r + \cdots + ab^k\right\}$$

$$+\left\{a^k b + \binom{k}{1} a^{k-1} b^2 + \binom{k}{2} a^{k-2} b^3 + \cdots + \binom{k}{r-1} a^{k-r+1} b^r + \binom{k}{r} a^{k-r} b^{r+1} + \cdots + b^{k+1}\right\}$$

$$= a^{k+1} + \left\{\binom{k}{1} + \binom{k}{0}\right\} a^k b + \left\{\binom{k}{2} + \binom{k}{1}\right\} a^{k-1} b^2 + \cdots + \left\{\binom{k}{r} + \binom{k}{r-1}\right\} a^{k-r+1} b^r + \cdots + b^{k+1}$$

[By grouping the like terms]

But by Pascal's rule

$$\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$$

Therefore, $\binom{k}{1} + \binom{k}{0} = \binom{k+1}{1}$; $\binom{k}{2} + \binom{k}{1} = \binom{k+1}{2}$; ... and so on

Hence,

$$(a+b)^{k+1} = a^{k+1} + \binom{k+1}{1} a^k b + \binom{k+1}{2} a^{k-1} b^2 + \cdots + \binom{k+1}{r} a^{k+1-r} b^r + \cdots + b^{k+1}$$

Therefore, the theorem is also true for $n = k + 1$. Hence by the Principle of Mathematical Induction the theorem is true for all positive integral exponents.

Characteristics of Binomial Theorem

We may notice the following points in connection with the binomial formula for the positive integral index.

(i) Using the other notation for combinations the formula can also be written in the form as:

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2}.b^2 + \cdots + {}^nC_n.b^n.$$

(ii) The binomial formula is frequently written as:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!} a^{n-r} b^r + \cdots + b^n$$

(iii) The number of terms in the expansion of $(a+b)^n$ is $n + 1$.

(iv) In the successive terms index of "a" decreases by one and index of "b"

increases by one so that the sum of two indices is always n.

(v) The coefficients of successive terms are 1, ${}^{n}C_1$, ${}^{n}C_2, \ldots,$ ${}^{n}C_{n-1}$, 1. These are known as binomial coefficients.

(vi) Since the expansion of $(a+b)^n$ is symmetrical w.r.t. a and b, it follows that the coefficients of $a^{n-r}b^r$ and $a^r b^{n-r}$ are equal, i.e., the coefficients of terms equidistant from the beginning and the end are equal.

(vii) since $\binom{n}{0} = 1 = \binom{n}{n}$, we may, for the sake of uniformity write $\binom{n}{0}$ as the coefficient of a^n and $\binom{n}{n}$ as the coefficient of b^n in the expansion.

(viii) If we put $a = b = 1$ in the binomial formula, then

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{r} + \cdots + \binom{n}{n}$$

This formula also determines the total number of subsets of a set S consisting of n elements. Whereas ${}^{n}C_r$ is the total number of subsets of S each consisting of r elements.

(ix) Since $(a-b) = \{a + (-b)\}$, we have

$$\{a+(-b)\}^n = a^n + \binom{n}{1}a^{n-1}(-b) + \binom{n}{2}a^{n-2}(-b)^2 + \cdots + \binom{n}{r}a^{n-r}(-b)^r + \cdots + (-b)^n;$$

i.e., $(a-b)^n = a^n - \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 - \cdots + (-1)^r\binom{n}{r}a^{n-r}(b)^r + \cdots + (-1)^n b^n$

Thus, the terms in the expansion of $(a-b)^n$ are alternatively positive and negative, the last term being $+b^n$ or $-b^n$ according as n is even and odd respectively.

(x) Putting $a = 1$ and $b = x$ in the expansion of $(a-b)^n$, we have

$$(1-x)^n = 1 - \binom{n}{1}x + \binom{n}{2}x^2 - \cdots + (-1)^r\binom{n}{r}x^r + \cdots + (-1)^n x^n$$

Similarly, $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + x^n$.

Example 1. Expand $(x+y)^7$ by the binomial theorem.

Solution:

$$(x+y)^7 = x^7 + 7x^6y + \frac{7.6}{2!}x^5y^2 + \frac{7.6.5}{3!}x^4y^3 + \frac{7.6.5.4}{4!}x^3y^4 + \frac{7.6.5.4.3}{5!}x^2y^5 + \frac{7.6.5.4.3.2}{6!}xy^6 + y^7$$

$$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

Example 2. Expand $(ax - \frac{b}{x})^6$ by the binomial theorem.

$$(ax - \frac{b}{x})^6 = (ax)^6 - 6(ax)^5\left(\frac{b}{x}\right) + \frac{6.5}{2!}(ax)^4\left(\frac{b}{x}\right)^2 - \frac{6.5.4}{3!}(ax)^3\left(\frac{b}{x}\right)^3 + \frac{6.5.4.3}{4!}(ax)^2\left(\frac{b}{x}\right)^4$$

$$-\frac{6.5.4.3.2}{5!}(ax)\left(\frac{b}{x}\right)^5 + \left(\frac{b}{x}\right)^6$$

$$= a^6x^6 - 6a^5bx^4 + 15a^4b^2x^2 - 20a^3b^3 + \frac{15a^2b^4}{x^2} - \frac{6ab^5}{x^4} + \frac{b^6}{x^6}$$

Example 3. Compute $(1.01)^9$ by means of the binomial theorem correct to three decimal places.

Solution: $(1.01)^9 = (1 + 0.01)^9$

$$= 1 + \frac{9}{1} \cdot 1^9 \cdot (0.01) + \frac{9.8}{1.2} \cdot 1^8 \cdot (0.01)^2 + \frac{9.8.7}{1.2.3} \cdot 1^7 \cdot (0.01)^3 + \text{up to last term}$$

$$= 1 + 0.09 + 0.0036 + 0.000084 + \cdots \text{ up to the last term}$$

$$= 1.093684 \text{ approx: } = 1.094 \text{ correct to three decimal places.}$$

7.2.3 Expand $(x + y)^n$ using binomial theorem and find its general term

If n is a positive integer, then

$$(x + y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n-1}xy^{n-1} + y^n$$

which is called the binomial expansion.

The term $T_{r+1} = \binom{n}{r}x^{n-r} \cdot y^r$ is called the general term in the expansion of $(x + y)^n$ where n is a +ve integer, we observe that

$$T_1 = \binom{n}{0}x^n = x^n$$

$$T_2 = \binom{n}{1}x^{n-1}y = nx^{n-1}y,$$

$$T_3 = \binom{n}{2}x^{n-2}y^2 = \frac{n(n-1)}{2!}x^{n-2}y^2,$$ and so on.

Example 1. Find the sixth term in the expansion of

$$\left(\frac{2x}{3} - \frac{3}{2x}\right)^{10}$$

Solution: $T_6 = T_{5+1}$

$$= \binom{10}{5} \cdot \left(\frac{2x}{3}\right)^{10-5} \cdot \left(-\frac{3}{2x}\right)^5$$

$$= \frac{10!}{5!\,(10-5)!} \cdot \frac{2^5x^5}{3^5} \cdot \left(-\frac{3^5}{2^5x^5}\right)$$

$$= -\frac{10.9.8.7.6(5!)}{5.4.3.2.1(5!)} = -252$$

Example 2. Find the coefficient of x^6 in $(a^3 + 3bx^2)^6$

Solution: Here, $T_{r+1} = \binom{6}{r}(a^3)^{6-r}(3bx^2)^r$

$$= \binom{6}{r}(a^3)^{6-r}(3b)^r x^{2r}$$

Now, T_{r+1} will contain x^6, if $2r = 6$ or $r = 3$.

So, $\quad T_4$ contains x^6.

i.e., $$T_4 = \binom{6}{3}(a^3)^{6-3}(3bx^2)^3$$
$$= \frac{6!}{3!\,(6-3)!}a^9(27b^3x^6)$$
$$= 540a^9b^3x^6$$

So, the coefficient of x^6 is $540a^9b^3$.

7.2.4 Find the specified term in the expansion of $(x+y)^n$

In the expansion of $(x+y)^n$, where n being a positive integral index. We can find specified terms like middle terms, the term involving particular power of x, the term independent of x etc.

The term involving particular power of x and independent of x

Example 1. Write in the simplified form the term involving x^{-17} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

Solution: Suppose x^{-17} occurs in T_{r+1}.

Now, $$T_{r+1} = \binom{15}{r}\cdot(x^4)^{15-r}\left(-\frac{1}{x^3}\right)^r$$
$$= (-1)^r\binom{15}{r}x^{60-4r}\cdot x^{-3r}$$
$$= (-1)^r\cdot\binom{15}{r}\cdot x^{60-7r}$$

Thus $\quad 60-7r = -17 \quad$ or $\quad r = 11$.

So, $$T_{r+1} = T_{11+1}$$
$$= (-1)^{11}\cdot\binom{15}{11}\cdot x^{60-7(11)} = -\frac{15.14.13.12.(11\,!)}{(11)!\,.4.3.2.1}x^{-17} = -1365x^{-17}.$$

Hence the term involving x^{-17} is $-1365\ x^{-17}$.

Example 2. Find the term independent of x in $\left(2x + \frac{1}{3x^2}\right)^9$

Solution: Let $(r+1)$th term be independent of x.

Now, $$T_{r+1} = \binom{9}{r}\cdot(2x)^{9-r}\cdot\left(\frac{1}{3x^2}\right)^r = \binom{9}{r}\cdot\frac{2^{9-r}}{3^r}\cdot\frac{x^{9-r}}{x^{2r}} = \binom{9}{r}\cdot\frac{2^{9-r}}{3^r}\cdot x^{9-3r}$$

Since this term is supposed to be independent of x, we must have $9-3r=0$. or $r=3$.

Thus, the required term, $T_{r+1} = T_{3+1}$.

Now, $$T_{3+1} = \binom{9}{3}\cdot\frac{2^{9-3}}{3^3}x^{9-3(3)} = \frac{9.8.7.6!}{3.2.1.6\,!}\cdot\frac{2^6}{3^3} = 3.4.7\cdot\frac{2^6}{3^3} = \frac{1792}{9}$$

Hence, the term independent of x is $\frac{1792}{9}$.

Example 3. Does the expansion of $\left(x+\frac{1}{x}\right)^7$ contain a term

(i) independent of x, (ii) involving x^6

Solution:

(i) Suppose the term independent of x (i.e. the constant term) is $(r+1)$th term.

Now, $T_{r+1} = \binom{7}{r} x^{7-r}.\left(\frac{1}{x}\right)^r = \binom{7}{r}\ x^{7-2r}$

Since, this term is supposed to be independent of x, we must have

$7-2r = 0$ or $r = \frac{7}{2}$ Then $r+1 = \frac{7}{2}+1 = \frac{9}{2}$

Since the position of a term cannot be fractional.

Therefore, there does not exist a term independent of x in the given expansion.

(ii) Let x^6 occurs in $(r+1)$th term of the expansion.

Now $T_{r+1} = {}^7C_r\ x^{7-r}\frac{1}{x^r} = {}^7C_r\ x^{7-2r}$,

According to the supposition $7-2r = 6$

$\Rightarrow$ $r = \frac{1}{2}$, which is a fraction.

So there does not exist a term involving x^6 in the given expansion.

Middle Term:

We shall find the middle term or terms in the expansion of $(x+y)^n$, n being a positive integral index.

The number of terms in the expansion of $(x+y)^n$ is $n+1$. If n is even then there is one middle term and if n is odd, then there are two middle terms.

If n is even, say $n = 2k$, then the number of terms is $(2k+1)$. Hence only one term, i.e., $(k+1)th = \left(\frac{n+2}{2}\right)$th term is the middle term. On the other hand. If n is odd, say $n = 2k+1$, then in the expansion there are $n+1 = 2k+2$ terms, i.e., the number of terms is even. In this case there are two, $(k+1)$th and $(k+2)$th, middle terms. Thus the required middle terms are $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms.

Example 1. Find the middle term in the expansion of $\left(x-\frac{2y}{3}\right)^{10}$.

Solution: Here, $n = 10$ is an even. So, there is only one middle term that is $\left(\frac{10+2}{2}\right)$th term = 6th term.

So, $T_6 = T_{5+1} = \binom{10}{5}(x)^{10-5}\left(\frac{-2y}{3}\right)^5$

$$= (-1)^5 \frac{10!}{5! \times 5!} \times x^5 \times \frac{2^5 \cdot y^5}{3^5}$$

$$= \frac{-252x^5 \cdot 32 \cdot y^5}{243}$$

$$= \frac{-896x^5y^5}{27}$$

So, the required middle term is $\frac{-896x^5y^5}{27}$.

Example 2. Find the middle terms of $\left(x^3 + \frac{1}{x^2}\right)^7$.

Solution: Here, $n = 7$ is an odd. So, there are two middle terms, that is $\left(\frac{7+1}{2}\right)$th and $\left(\frac{7+3}{2}\right)$th the terms. Hence T_4 and T_5 are two middle terms.

Now, $T_4 = \binom{7}{3}(x^3)^{7-3}\left(\frac{1}{x^2}\right)^3$

$$= 35\, x^{12} \times \frac{1}{x^6}$$

$$= 35x^6$$

and $T_5 = \binom{7}{4}(x^3)^{7-4}\left(\frac{1}{x^2}\right)^4$

$$= 35\, x^9 \times \frac{1}{x^8}$$

$$= 35x$$

Hence, $35x^6$ and $35x$ are the two required middle terms.

Exercise 7.2

1. Expand by means of the binomial theorem:

(i) $(a+b)^8$ (ii) $(2x-3y)^4$ (iii) $\left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^4$

(iv) $\left(\frac{x^2}{2} - \frac{2}{x}\right)^6$ (v) $(3x^2 - 2y^3)^5$

2. (a) Use the binomial theorem, to find the value of:

(i) $(11)^5$ (ii) $(19)^6$ (iii) $(99)^4$

(b) Use the binomial theorem to compute the values of the following correct to four places of decimal.

(i) $(1.01)^6$ (ii) $(1.02)^7$ (iii) $(2.03)^5$

3. Evaluate:

(i) $(1+2\sqrt{a})^4-(1-2\sqrt{a})^4$ (ii) $(2-\sqrt{1-a})^5+(2+\sqrt{1-a})^5$

(iii) $(2+\sqrt{3})^5-(2-\sqrt{3})^5$

4. Find the indicated term in each of the following expansions.

(i) $\left(\frac{2}{3}x-\frac{3}{2}\right)^{10}$; the 8^{th} term (ii) $\left(\frac{a}{b}-\frac{b}{a}\right)^{8}$; the 7^{th} term

(iii) $\left(2x-\frac{1}{y}\right)^{10}$ the last term

5. Find the middle term in the expansion of:

(i) $\left(b-\frac{2c}{3}\right)^{10}$ (ii) $\left(1-\frac{1}{2}a^2\right)^{14}$ (iii) $\left(\frac{a}{b}-\frac{b}{a}\right)^{18}$ (iv) $\left(a-\frac{1}{a}\right)^{2n}$

6. Find the two middle terms of:

(i) $\left(x^3+\frac{1}{x^2}\right)^7$ (ii) $\left(2b-\frac{b^2}{4}\right)^9$

7. Write, in the simplified form, the term independent of x in

(i) $\left(2x+\frac{1}{x^2}\right)^9$ (ii) $\left(2x+\frac{5}{x}\right)^6$

8. Obtain in the simplified form:

(i) the term involving x^6 in the expansion of $\left(2x^3-\frac{1}{x^2}\right)^7$.

(ii) the term involving a^8 in the expansion of $\left(a^2-\frac{1}{a^2}\right)^{12}$.

(iii) the coefficient of x in the expansion of $\left(x^2+\frac{a^2}{x}\right)^5$.

9. Does the expansion of $\left(\frac{3a}{2}-\frac{1}{3a^3}\right)^9$ contain a term.

(i) independent of a, (ii) involving a^{10}

10. The coefficients of the fifth, sixth and seventh terms of the expansion of $(1+x)^n$ form an A.P. Find n.

7.3 Binomial Series

7.3.1 Expand $(1+x)^n$ where n is a positive integer and extend this result for all rational values of n

We know that

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n \quad \ldots\text{(i)}$$

By putting $x=1$ and $y=x$ in equation (i)

We get $$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n \quad \ldots\text{(ii)}$$

When n is a positive integer then in the expansion of $(x+y)^n$ or $(1+x)^n$, there are $(n+1)$ terms or finite number of terms and is valid for any real value of x.

But if n is negative or rational (fraction), then the above expansion never ends or have infinite number of terms and is valid only for $-1 < x < 1$ or $|x| < 1$ thus in such a case the expansion is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)(n-2)\ldots(n-r+1)x^r}{r!} + \cdots$$

This series is called binomial series and its general term is

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

Example 1. Expand $(2-x)^{-3}$ to four terms, where $|x| < 1$.

Solution:

$(2-x)^{-3} = 2^{-3}(1-\frac{x}{2})^{-3}$, making the first term of the binomial as 1.

So, $\quad (2-x)^{-3} = 2^{-3}\left(1-\frac{x}{2}\right)^{-3} = 2^{-3}\{1+\left(-\frac{x}{2}\right)\}^{-3}$

$$= 2^{-3}\left\{1 + (-3)\left(-\frac{x}{2}\right) + \frac{(-3)(-3-1)}{2\,!}\cdot\left(-\frac{x}{2}\right)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}\cdot\left(-\frac{x}{2}\right)^3 + \cdots\right\};\ |x| < 1$$

$$= \frac{1}{8}\left(1 + \frac{3x}{2} + \frac{3x^2}{2} + \frac{5x^3}{4} + \cdots\right)$$

Example 2. Expand $\left(1+\frac{2x}{3}\right)^{-\frac{3}{2}}$ to four terms, when $|x| < 1$

Solution:

$$\left(1+\frac{2x}{3}\right)^{-\frac{3}{2}} = 1 + \left(-\frac{3}{2}\right)\left(\frac{2x}{3}\right) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2\,!}\cdot\left(\frac{2x}{3}\right)^2 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)}{3\,!}\cdot\left(\frac{2x}{3}\right)^3 + \cdots;\ |x| < 1$$

$$= 1 - x + \frac{15}{18}x^2 - \frac{105}{162}x^3 + \cdots$$

$$= 1 - x + \frac{5}{6}x^2 - \frac{35}{54}x^3 + \cdots$$

Example 3. Find the first negative term in the expansion of $(1+x)^{\frac{7}{2}}$

Solution: By using the general term fromula

$$T_{r+1} = \frac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\ldots\left(\frac{7}{2}-r+1\right)}{r\,!}x^r$$

This will be the first negative term when $\frac{7}{2} - r + 1$ is negative,

i.e., $\quad \frac{9}{2} - r < 0$ or $r > \frac{9}{2}$,

i.e., when $r > 4\frac{1}{2}$, so $r = 5$.

Hence, when $r = 5$, we get the first negative term,

$$T_6 = \frac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\left(\frac{7}{2}-3\right)\left(\frac{7}{2}-4\right)}{5\,!}x^5 = -\frac{7}{256}x^5.$$

7.3.2 Expand $(1+x)^n$ in ascending power of x and explain its validity/ convergence for $|x|<1$ where n is a rational number

We know that if n is negative or rational (fraction), then the expansion of $(1+x)^n$ has infinite number of terms and is valid only for $-1<x<1$ or $|x|<1$ and the expansion is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \; ; \; |x|<1$$

As we see in the expansion, we get ascending power of x and the terms progressively get smaller and smaller for $-1<x<1$ or $|x|<1$, so the series will be valid or convergent.

Let us explain the validity or convergence of this series

We have, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r\,!}x^r + \cdots,$ ... (i)

Replacing x by $-x$ in (i), we have

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2\,!}x^2 - \frac{n(n-1)(n-2)}{3\,!}x^3 + \cdots + (-1)^r\cdot\frac{n(n-1)\ldots(n-r+1)}{r\,!}x^r + \cdots; |x|<1 \quad \ldots\text{(ii)}$$

Changing the sign of n in (i) and (ii), we get

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2\,!}x^2 - \frac{n(n+1)(n+2)}{3\,!}x^3 + \cdots + (-1)^r\cdot\frac{n(n+1)\ldots(n+r-1)}{r\,!}x^r + \ldots; |x|<1 \quad \ldots\text{(iii)}$$

and

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2\,!}x^2 + \frac{n(n+1)(n+2)}{3\,!}x^3 + \cdots + \cdot\frac{n(n+1)\ldots(n+r-1)}{r\,!}x^r + \cdots; |x|<1 \quad \ldots\text{(iv)}$$

Hence, for $|x|<1$ and $n=1$, we have from (iii) and (iv)

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^r x^r + \cdots \quad \ldots\text{(v)}$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots \quad \ldots\text{(vi)}$$

Similarly for $n=2$, we have

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots + (-1)^r\cdot(r+1)x^r + \cdots \quad \ldots\text{(vii)}$$

and $$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (r+1)x^r + \cdots \quad \ldots\text{(viii)}$$

We see that the expansion (vi) does not hold for $x=2$. In this case the equation (vi) becomes

$(-1)^{-1} = -1 = 1 + 2 + 2^2 + 2^3 + \cdots + 2^r + \cdots$, which is absurd.

Summing the infinite given series on the right-hand side of the expansion (vi), we have

$$1 + x + x^2 + x^3 + \cdots + x^r + \cdots = \frac{1}{1-x} = (1-x)^{-1}$$

which is true only when $|x| < 1$.

Thus expansion (i) is valid or convergent for $|x| < |$ where n is a rational number.

Example 1. If $y = 3x + 6x^2 + 10x^3 + \cdots,$

then prove that, $x = \frac{y}{3} - \frac{1.4}{3^2.2\,!}y^2 + \frac{1.4.7}{3^3.3\,!}y^3 - \cdots$

Solution: Since $y = 3x + 6x^2 + 10x^3 + \cdots$

So, $1 + y = 1 + 3x + 6x^2 + 10x^3 + \cdots$

i.e., $1 + y = (1 - x)^{-3}$

Therefore, $(1 - x) = (1 + y)^{-\frac{1}{3}}$

i.e., $1 - x = 1 + \left(-\frac{1}{3}y\right) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3} - 1\right)}{2\,!}y^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3} - 1\right)\left(-\frac{1}{3} - 2\right)}{3\,!}y^3 + \cdots$

or $1 - x = 1 - \frac{1}{3}y + \frac{1.4}{3^2.2\,!}y^2 - \frac{1.4.7}{3^3.3\,!}y^3 + \cdots$

Hence, $x = \frac{1}{3}y - \frac{1.4}{3^2.2\,!}y^2 + \frac{1.4.7}{3^3.3\,!}y^3 - \cdots$

Example 2. Evaluate $\sqrt[5]{31}$ to five places of decimals.

Solution:

$$\sqrt[5]{31} = (31)^{\frac{1}{5}} = (32 - 1)^{\frac{1}{5}} = \left\{32\left(1 - \frac{1}{32}\right)\right\}^{\frac{1}{5}}$$

$$= (32)^{\frac{1}{5}}\left(1 - \frac{1}{32}\right)^{\frac{1}{5}} = 2\left(1 - \frac{1}{32}\right)^{\frac{1}{5}}$$

$$= 2\left\{1 + \frac{1}{5}\cdot\left(\frac{-1}{32}\right) + \left(\frac{1}{5}\right)\left(-\frac{4}{5}\right)\cdot\frac{1}{2!}\left(-\frac{1}{32}\right)^2 + \frac{1}{5}\cdot\left(-\frac{4}{5}\right)\left(-\frac{9}{5}\right)\cdot\frac{1}{3!}\left(-\frac{1}{32}\right)^3 + \cdots\right\}$$

$$= 2(1 - 0.00625 - 0.000078125 - 0.000001464 + \cdots)$$

$$= 2 \times 0.99367 = 1.98734.$$

7.3.3 Determine the approximate values of the binomial expansions having indices as – ve integers or fractions

We know that

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \frac{n(n-1)(n-2)}{3\,!}x^3 + \cdots \quad \text{when } |x| < 1 .$$

where n is negative integer or fraction.

We see that the terms progressively get smaller and smaller. In such a case, to get an approximate value of the expansion, we may omit the terms containing squares and higher powers of x.

Thus, we can get approximation of this expression.

$(1 + x)^n = 1 + nx$ approximately and this approximation is called 1st approximation.

Similarly, up to a second approximation, we have

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2\,!}x^2$$

and so on.

Example 1. If $|x| < 1$, find the first approximation of

$$\frac{\sqrt[3]{1-7x}}{\sqrt[4]{(1+2x)^3}}$$

Solution: $\dfrac{\sqrt[3]{1-7x}}{\sqrt[4]{(1+2x)^3}}$

$$= (1-7x)^{\frac{1}{3}}\,(1+2x)^{-\frac{3}{4}}$$

$$= \left\{1 - \frac{1}{3}(7x) + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2\,!}(7x)^2 - \cdots\right\}\left\{1 + \left(-\frac{3}{4}\right)(2x) + \frac{\left(-\frac{3}{4}\right)\left(-\frac{3}{4}-1\right)}{2\,!}(2x)^2 + \cdots\right\}$$

$= \left(1 - \frac{7}{3}x\right)\left(1 - \frac{3}{2}x\right)$, neglecting squares and higher powers of x

$$= \left(1 - \frac{7}{3}x - \frac{3}{2}x + \frac{7}{2}x^2\right)$$

$= \left(1 - \frac{23}{6}x\right)$ approximately (neglecting the term containing the square of x)

Example 2. If x be so small that its squares and higher powers can be neglected, prove that

$$\frac{\sqrt{1+2x} + (16+3x)^{\frac{1}{4}}}{(1-x)^2} = 3 + \frac{227}{32}x$$

Solution: The given expression

$$= \frac{(1+2x)^{\frac{1}{2}} + (16)^{\frac{1}{4}}\left(1+\frac{3}{16}x\right)^{\frac{1}{4}}}{(1-x)^2}$$

$= \dfrac{\left(1+\frac{1}{2}2x\right) + 2\left(1+\frac{1}{4}\cdot\frac{3}{16}x\right)}{(1-2x)}$, approximating

$$= \frac{(1+x) + 2\left(1+\frac{3}{64}x\right)}{(1-2x)}$$

$$= \left(3 + \frac{35}{32}x\right)(1-2x)^{-1}$$

$= \left(3 + \frac{35}{32}x\right)(1+2x)$, approximating again

$= 3 + \dfrac{227}{32}x$, approximately.

Example 3. If $(a-b)$ be small as compared with a or b then show that

$$\frac{(n+1)a+(n-1)b}{(n-1)a+(n+1)b}=\left(\frac{a}{b}\right)^{\frac{1}{n}}$$

Solution: Let $a=b+h$ where h is a small quantity.

So, $$\frac{(n+1)a+(n-1)b}{(n-1)a+(n+1)b}=\frac{(n+1)(b+h)+(n-1)b}{(n-1)(b+h)+(n+1)b}$$

$$=\frac{2nb+(n+1)h}{2nb+(n-1)h}$$

$$=\frac{1+\frac{n+1}{2n}\cdot\frac{h}{b}}{1+\frac{n-1}{2n}\cdot\frac{h}{b}}$$

$$=\left(1+\frac{n+1}{2n}\cdot\frac{h}{b}\right)\left(1+\frac{n-1}{2n}\cdot\frac{h}{b}\right)^{-1}$$

$$=\left(1+\frac{n+1}{2n}\cdot\frac{h}{b}\right)\left(1-\frac{n-1}{2n}\cdot\frac{h}{b}\right), \text{ approximating}$$

$$=1+\frac{h}{b}\cdot\frac{2}{2n} \quad \text{(Again approximating)}$$

$$=1+\frac{h}{nb} \qquad \ldots\text{(i)}$$

Also $$\left(\frac{a}{b}\right)^{\frac{1}{n}}=\left(\frac{b+h}{b}\right)^{\frac{1}{n}}=\left(1+\frac{h}{b}\right)^{\frac{1}{n}} \qquad [\text{Since, } a=b+h]$$

$$=1+\frac{h}{nb}, \text{ approximating} \qquad \ldots\text{(ii)}$$

Hence, from (i) and (ii), we have

$$\frac{(n+1)a+(n-1)b}{(n-1)a+(n+1)b}=\left(\frac{a}{b}\right)^{\frac{1}{n}}$$

7.3.4 Application of summation of series

In general, most of the infinite series can be summed up very quickly by identifying them with some binomial expansion, as is shown in the following examples.

Example 1. Sum to infinity the series:

$$1+\frac{1}{3^2}+\frac{1.4}{1.2}\cdot\frac{1}{3^4}+\frac{1.4.7}{1.2.3}\cdot\frac{1}{3^6}+\cdots$$

Solution: Identifying the given series with

$$(1+x)^n=1+nx\frac{n(n-1)}{2\,!}x^2+\cdots$$

By comparing,

we get $\quad nx = \frac{1}{9}$ ...(i)

$\Rightarrow \quad n^2x^2 = \frac{1}{81}$ ... (ii)

and $\quad \frac{n(n-1)x^2}{2!} = \frac{2}{81}$ ...(iii)

From (ii) and (iii), by division, we have $\frac{n-1}{2n} = 2$ or $n - 1 = 4n$

Therefore $\quad 3n = -1$ or $n = -\frac{1}{3}$

By using $\quad n = -\frac{1}{3}$ in equation (i), we get $x = -\frac{1}{3}$

Using $x = -\frac{1}{3}, \; n = -\frac{1}{3}$

we have $\quad (1 + x)^n = \left(1 - \frac{1}{3}\right)^{-\frac{1}{3}}$

$$= \left(\frac{2}{3}\right)^{-\frac{1}{3}} = \sqrt[3]{\frac{3}{2}}.$$

Hence, the sum of the given series is $\sqrt[3]{\frac{3}{2}}$.

Example 2. If $x = \frac{1}{3} + \frac{1.3}{3.6}\frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \cdots$, prove that $x^2 + 2x - 2 = 0$.

Solution: Adding 1 to both the sides of the given series,

We have $\quad x + 1 = 1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \cdots$ ...(i)

Let the series on the right hand side of (i) be identified with

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \cdots$

By comparing

we get, $\quad nx = \frac{1}{3}$ ...(ii)

$\Rightarrow \quad n^2x^2 = \frac{1}{9}$ ...(iii)

and $\quad \frac{1}{2}n(n-1)x^2 = \frac{1.3}{3.6}$ ...(iv)

From (iii) and (iv) by division, we get

$\frac{n-1}{2n} = \frac{3}{2}$ or $2(n-1) = 6n$ or $2n - 2 = 6n$ or $4n = -2$

Therefore, $\quad n = -\frac{1}{2}$.

Hence, from (ii), $x = -\frac{2}{3}$

By substituting $x = -\frac{2}{3}$ and $n = -\frac{1}{2}$ in $(1 + x)^n$ we have

Sum of the given series is

$$= \left(1-\frac{2}{3}\right)^{-\frac{1}{2}} = \left(\frac{1}{3}\right)^{-\frac{1}{2}} = \sqrt{3} \qquad \ldots(iv)$$

Thus, from (i) and (iv) give we get $\quad x+1=\sqrt{3}$

Squaring both sides, we have

$$x^2+2x+1=3 \quad \text{or} \quad x^2+2x-2=0$$

Exercise 7.3

1. Find the first four terms in the following expansions:

(i) $\sqrt[3]{8-16x}$ (ii) $\frac{3}{3+x}$ (iii) $(1+3x)^{\frac{-1}{3}}$ (iv) $\frac{\sqrt{1+2x}}{1-x}$

2. Find first negative term in the expansion of

(i) $(1+y)^{\frac{4}{3}}$ (ii) $(1+2x)^{\frac{5}{2}}$

3. If x is so small that its square and higher powers may be neglected, then show that:

(i) $\sqrt{1+\frac{1}{4}x} \approx 1+\frac{1}{8}x$ (ii) $\sqrt[4]{1+8x} \approx 1+2x$

4. Using binomial series, find the value of the following up to three places of decimals:

(i) $\sqrt{24}$ (ii) $\sqrt[3]{28}$ (iii) $\sqrt[5]{241}$

(iv) $(1280)^{\frac{1}{4}}$ (v) $\frac{1}{\sqrt[5]{252}}$

5. Identify the following series as binomial expansion and find the sum in each case.

(i) $1+\frac{3}{4}+\frac{3\cdot5}{4\cdot8}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+\cdots$ (ii) $1+\frac{1}{2}\cdot\frac{1}{5}-\frac{1}{2!}\cdot\frac{1}{4}\cdot\frac{1}{5^2}+\cdots$

(iii) $1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1\cdot3}{2!4}+\left(\frac{1}{4}\right)^2-\frac{1\cdot3\cdot5}{3!8}\left(\frac{1}{4}\right)^3+\cdots$

(iv) $1-\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1\cdot3}{2\cdot4}\left(\frac{1}{2}\right)^2-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{2}\right)^3+\cdots$

(v) $1-\frac{1}{2}\cdot\frac{1}{3}+\frac{1\cdot3}{2\cdot4}\left(\frac{1}{3}\right)^2-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{3}\right)^3+\cdots$

6. Use binomial theorem to show that:

(i) $1+\frac{1}{4}+\frac{1\cdot3}{4\cdot8}+\frac{1\cdot3\cdot5}{4\cdot8\cdot12}\ldots=\sqrt{2}$ (ii) $1+\frac{1}{4}+\frac{1\cdot3}{4\cdot6}+\frac{1\cdot3\cdot5}{4\cdot6\cdot8}+\cdots=\sqrt[3]{4}$

7. If $y=\frac{1}{2}\cdot\frac{1}{16}-\frac{1}{2!\cdot4}\cdot\frac{1}{16^2}+\frac{1}{3!}\cdot\frac{1\cdot3}{8}\cdot\frac{1}{16^3}\ldots$

then show that $16y(y+2)=1$.

8. If $\frac{1}{x}=\frac{2}{5}+\frac{1\cdot3}{2!}\left(\frac{2}{5}\right)^2+\frac{1\cdot3\cdot5}{3!}\left(\frac{2}{5}\right)^3+\cdots,$

then show that $4x^2 - 2x - 1 = 0$.

9. If $\frac{y}{2} = \frac{1}{2}\left(\frac{4}{9}\right) + \frac{1\cdot 3}{2^2\cdot 2!}\left(\frac{4}{9}\right)^2 + \frac{1\cdot 3\cdot 5}{2^3 3!}\left(\frac{4}{9}\right)^3 + \cdots$,
then show that $5y^2 + 20y - 16 = 0$.

Review Exercise 7

1. Select correct answer.

i. The general term of the binomial expansion $(a+x)^n$ is where $n \in N$
(a) $\binom{n}{r} a^n x$ (b) $\binom{n}{r} a^{n-r} x^r$ (c) $\binom{n}{r} a^r x^{n-r}$ (d) $\binom{n}{r} (ax)^{n-r}$

ii. The number of terms in the expansion of $(a+b)^{2n}$ are:
(a) n (b) $2n+1$ (c) 2^n (d) 2^{n-1}

iii. In the expansion $(a+x)^n$, the exponent of 'x':
(a) decreases from n to 0 (b) increases from 0 to n
(c) remains n every where (d) becomes 0 at the end

iv. Middle term in the expansion of $(a+b)^{2n}$ is:
(a) nth term (b) $(n+1)$th term
(c) $(2n+1)$th term (d) $(2n-1)$th term

v. The term independent of x in the expansion of $(a+2x)^n$ is:
(a) First term (b) Middle term
(c) Last term (d) 2^{nd} last term

vi. The coefficient of the last term in the expansion of $(2-x)^7$ is/are:
(a) 1 (b) -1 (c) 7 (d) -7

vii. In the expansion $\left(a+\frac{1}{2}\right)^7$, the number of middle terms is/are:
(a) one (b) two (c) three (d) four

viii. Sum of odd binomial coefficients in the expansion of $(a+x)^n$ is:
(a) 2^n (b) 2^{n-1} (c) 2^{n+1} (d) $n+1$

ix. $\binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n+1}$ is equal to:
(a) 2^n (b) 2^{n+1} (c) 2^{n-1} (d) Cannot be determined

x. The number of terms in the expansion of $(1+x)^{\frac{1}{3}}$ is __________.
(a) $\frac{4}{3}$ (b) 4 (c) ∞ (d) 2

xi. $1 - x + x^2 - x^3 + \cdots$ is equal to:
(a) $(1+x)^{-1}$ (b) $(1-x)^{-1}$ (c) $(1+x)^{-2}$ (d) $(1-x)^{-2}$

xii. The expansion of $(1-2x)^{-2}$ is valid if:
(a) $|x| < 0$ (b) $|x| < \frac{1}{2}$ (c) $|x| < 2$ (d) $|x| < 1$

xiii. The middle term in the expansion of $(a+b)^n$ is $\left(\frac{n}{2}+1\right)$ th term; then n is:

(a) Odd (b) Even
(c) Prime (d) None of these

xiv. In $\left(a+\frac{1}{a}\right)^8$, the sum of the binomial coefficients is:

(a) 64 (b)128 (c)256 (d) 512

2. Prove by principle of mathematical induction $\forall n \in \mathrm{N}$ that $8 \times 10^n - 2$ is divisible by 6.

3. Prove by mathematical induction

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$$

4. Given that the coefficient of y^2 in the expansion $\frac{1}{(1+my)^2}$ is 96, find m.

5. Given that the terms involving m^4 and higher powers may be neglected and that $\frac{1}{(a+bm)^3} - \frac{1}{(1+3m)^4} = cm^2 + dm^3$, find the values of a, b, c and d.