Functions and Graph

Unit 8

8.1 Function

8.1.1 Recall

- **function as a rule or correspondence**
- **domain, co-domain and range of a function**
- **one to one and onto functions**

a. Function as a rule or correspondence

A function f from a set X to a set Y is a rule or correspondence that assigns each element of X to, one and only one element of Y. The elements of X are called pre-images of the function and the corresponding elements of Y are called the images of the function.

Symbolically, we write it as, a function $f: X \rightarrow Y$, where $y = f(x)$, $\forall x \in X$ and $y \in Y$.

The variable x is called the independent variable and y is called the dependent variable.

Example 1. Let $X = \{a, b, c\}$ and $Y = \{4,5,6\}$. State whether or not the relations indicated by the following figures are functions from X to Y.

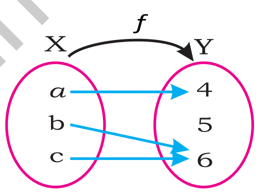


Fig. 8.1

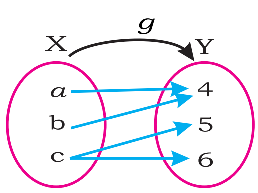


Fig. 8.2

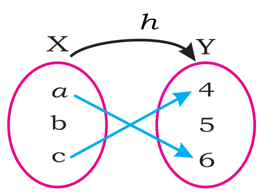


Fig. 8.3

Solution:

f is a function, because each element of X has a unique image in Y. (Fig. 8.1)
g is not a function, because the element c of X has two images in Y. (Fig. 8.2)
h is not a function, because the element b of X has no image in Y. (Fig. 8.3)

Example 2. If function $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x + 5$, then find

(i) $f(2)$ (ii) $f(-3)$ (iii) $f\left(\frac{2}{3}\right)$

Solution: (i) $f(2) = 3(2) + 5 = 11$

(ii) $f(-3) = 3(-3) + 5 = -4$

(iii) $f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right) + 5 = 7$

b. Domain, co-domain and range of a function

Let $f: X \to Y$ be a function from a set X to a set Y, then X is called the domain and Y is called co-domain of the function f. Whereas the range is the set of all images of the function.

Example 1.

If $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, the function f is defined by $f(x) = 2x + 1$, $\forall\, x \in X$, then find range of f.

Solution:

Here $f(x) = 2x + 1$

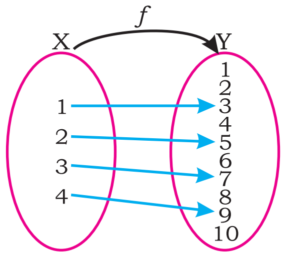
for $x = 1$, we get $f(1) = 3$

for $x = 2$, we get $f(2) = 5$

for $x = 3$, we get $f(3) = 7$

for $x = 4$, we get $f(4) = 9$,

Thus, Range of $f = \{3,5,7,9\}$ as shown in Fig. 8.4.



(Fig. 8.4)

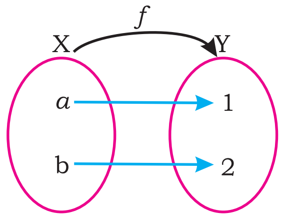
Example 2. If function $f: X \to Y$ is a function as shown in Fig. 8.5, where $X = \{a, b\}$ and $Y = \{1,2\}$, then write domain of f and show that Co-domain of f = Range of f.

Solution: Here, from Fig. 8.5, we have

Domain of $f = \{a, b\}$

Co-domain of $f = \{1,2\}$ and Range of $f = \{1,2\}$

Hence, Co-domain of f = Range of f.



(Fig. 8.5)

Example 3. Find the domain of $f(x) = \frac{(x-2)(x-4)}{(x-1)(x-3)}$

Solution: We have to find those values of x for which $f(x)$ is undefined, so that these values may be excluded from $\mathbb{R}$.

The function is undefined when the denominator is zero.

Let $(x - 1)(x - 3) = 0 \implies x = 1$ or $x = 3$

So, $f(x)$ is undefined for $x = 1$ or $x = 3$

i.e., $f(1) = \frac{(1-2)(1-4)}{(1-1)(1-3)} = \frac{(-1)(-3)}{(0)(-2)} = \frac{6}{0}$ (Undefined),

$$f(3)=\frac{(3-2)(3-4)}{(3-1)(3-3)}=\frac{(1)(-1)}{(2)(0)}=\frac{-1}{0} \text{ (Undefined)}$$

Thus, the domain of $f = \{x|x \in \mathbb{R} \text{ and } x \neq 1 \text{ or } 3\}$

Example 4. If $f(x) = x^2$ then find range of f.

Solution: Let $y = x^2$

$\because \quad x^2 \geq 0 \quad \forall x \in \mathbb{R}$

$\therefore \quad \text{Range} = \{y|y \in \mathbb{R} \wedge y \geq 0\}$

$= \mathbb{R}^+ \cup \{0\}$.

Example 5. Find domain of the function $f(x) = \sqrt{x+9}$

Solution: For real values, we take radicand as greater than or equal to zero

i.e., $x + 9 \geq 0$

$\Rightarrow \quad x \geq -9$

Hence the domain of $f(x) = \{x|x \in \mathbb{R} \text{ and } x \geq -9\} = [-9, \infty)$

Example 6. Find the domain and range of the function $f(x) = \sqrt{x^2 - 4}$

Solution: For real values, we take radicand as greater than or equal to zero

i.e., $x^2 - 4 \geq 0$

$\Rightarrow \quad x^2 \geq 4$

or $\quad x \geq 2 \text{ or } x \leq -2$

Hence domain of the function $f(x) = \{x|x \in \mathbb{R} \text{ and } x \geq 2 \text{ } or \text{ } x \leq -2\}$

Let $\quad y = f(x) = \sqrt{x^2 - 4}$

Here $\quad f(-2) = f(2) = 0$

and $\quad f(x) > 0 \ \forall x < -2 \text{ or } x > 2$

Hence Range $= \{y|y \in \mathbb{R} \wedge y \geq 0\} = \mathbb{R}^+ \cup \{0\}$

Example 7. Find the domain of the function $f(x) = \frac{1}{\sqrt{x-1}}$

Solution:

We have to find those values of x for which $f(x)$ is undefined or non-real so that these values may be excluded from $\mathbb{R}$.

The function is undefined or non-real when $x - 1$ is less than or equal to 0.

i.e., $\quad x - 1 \leq 0$

$\Rightarrow \quad x \leq 1$

So, the domain of $\quad f = \mathbb{R} - \{x|x \in \mathbb{R} \wedge x \leq 1\}$

or domain of $f = \{x|x \in \mathbb{R} \text{ and } x > 1\}$

c. One to one and onto Functions

(i) One – to – one function (Injective function)

A function $f: X \longrightarrow Y$ is one-to-one (injective) if distinct elements of set X have distinct images in set Y.

i.e., if $x_1, x_2 \in X$ and $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$

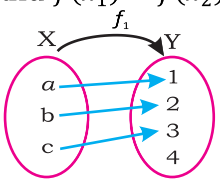


Fig. 8.6

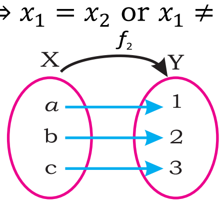


Fig. 8.7

Mapping diagrams of Fig.8.6 and Fig.8.7 represent that f_1 and f_2 are one-to-one functions.

(ii) Onto function (Surjective function)

A function $f: X \longrightarrow Y$ is onto function (Surjective), if the range of f is Y, that is co-domain is equal to range. In other words, if each $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$ then f is onto function.

Let $X = \{a, b, c\}$ and $Y = \{1,2,3\}$ then

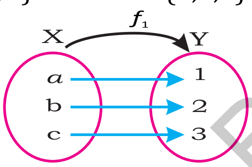


Fig. 8.8

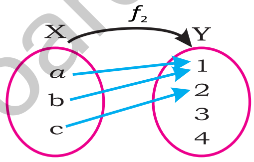
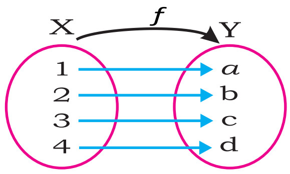


Fig. 8.9

Mapping diagram of Fig.8.8 represents a function f_1 which is onto.
Mapping diagram of Fig.8.9 represents a function f_2 which is not onto.

(iii) One to one and onto (Bijective Function)

A function $f: X \longrightarrow Y$ is called one-to-one and onto or bijective function if each element of Y has one and only one pre-image in X. Functions that are both one-to-one and onto are referred to as Bijective functions.



(Fig. 8.10)

Let $X = \{1,2,3,4\}$ and $Y = \{a, b, c, d\}$, then f is bijective function as shown in mapping diagram of Fig. 8.10.

8.1.2 Know linear, quadratic and square root functions

In this section we are concerned with the definitions of linear, quadratic and square root functions, however, we will first define an important function, which is polynomial function.

(i) Polynomial function

A function $p: \mathbb{R} \longrightarrow \mathbb{R}$ defined as

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

for all $x \in \mathbb{R}$ where the co-efficient $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are all real numbers and n is non-negative integer, is called a polynomial function.

If $a_n \neq 0$, then $p(x)$ is called a polynomial function of degree n and a_n is the leading co-efficient of $p(x)$.

For example, $p(x) = 4x^3 - 3x^2 + 2x + 1$,

$q(x) = x^2 - 4x + 2$,

$r(x) = 2x + 3$

and so on are polynomial functions. The degree of $p(x)$ is 3, $q(x)$ is 2 and $r(x)$ is 1 with leading co-efficients as 4, 1 and 2 respectively.

(ii) Linear Function

A polynomial function of degree 1 is called a linear function. A function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is defined as a linear function of the form $f(x) = ax + b$ where $a, b \in \mathbb{R}$ and $a \neq 0$

For example, $f(x) = 2x + 3$ and $g(x) = -5x + 7$ are linear functions.

(iii) Quadratic Function

A polynomial function of degree 2 is called a quadratic function. A function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is defined as a quadratic function of the form

$f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

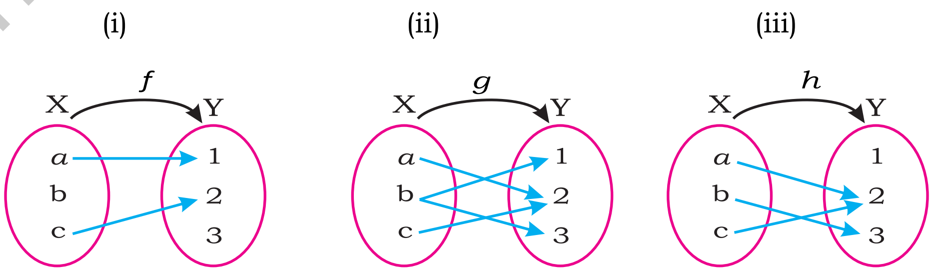
For example, $f(x) = 2x^2 + 5x + 6$, and $g(x) = 3x^2 - 2x - 5$ are quadratic functions.

(iv) Square Root Function

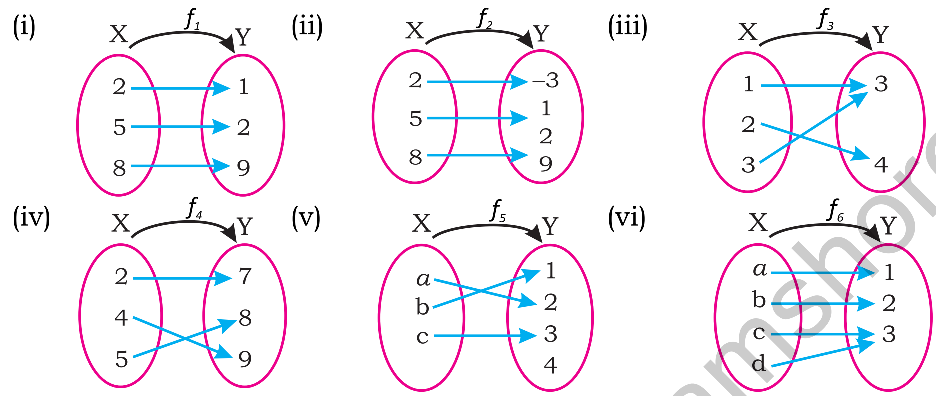
A function $f: \mathbb{R}^{+} \cup \{0\} \longrightarrow \mathbb{R}$ defined as $f(x) = \sqrt{x}$, where $x \geq 0$, is called a square root function.

Exercise 8.1

1. Let $X = \{a, b, c\}$ and $Y = \{1,2,3\}$. State whether or not the relations indicated by the following figures are functions from X to Y.



2. Which of the following functions are injective, surjective and bijective. Give reason.



3. If function $f: Z \longrightarrow \mathbb{R}$ defined by $f(x) = x^3 + 1$, find the following if possible:

(i) $f(0)$ (ii) $f(5)$ (iii) $f\left(\frac{3}{4}\right)$ (iv) $f(-2)$

4. If $f(x) = x^3$, find the values of :

(i) $f(2)$ (ii) $f(-10)$ (iii) $f\left(\frac{1}{2}\right)$ (iv) $f(5a)$ (v) $f\left(\frac{a}{3}\right)$

(vi) $f(a+h)$ (vii) $\frac{f(a+h)-f(a-h)}{2h}, (h \neq 0)$ (viii) $f(a+h) - f(a-h)$

5. Given that $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$, find:

(i) $\frac{f(5+h)-f(5)}{h}, (h \neq 0)$ (ii) $\frac{f(a+h)-f(a)}{h}, (h \neq 0)$

6. The domain of the function $f(x) = 5x + 1$ is $\{0,1,2,3,4,5\}$. Find its range.

7. Find the domain of the following functions:

(i) $f(x) = x^2 + 3$ (ii) $f(x) = \sqrt{x+3}$ (iii) $f(x) = \frac{1}{x}$

(iv) $f(x) = \sqrt{x-5}$ (v) $f(x) = \frac{x+1}{x-2}$ (iv) $f(x) = \frac{x^2}{x^2-9}$

8. The domain of the function $f(x)$ is $\{1,2,3,4,5\}$. Find the range of:

(i) $f(x) = 5x^2 + 3$ (ii) $f(x) = \frac{x}{x+1}$

9. Find the range of the function $f(x) = \sqrt{x^2+1}$.

10. Find the domain and range of the function $f(x) = \frac{1}{1+x^2}$.

11. Find the domain and range of $f(x) = \frac{1}{\sqrt{25-x}}$

12. A function is defined by $f(x) = ax + b$. The images of 1 and 5 are -2 and 10 respectively.

(i) Find the value of a and b.

(ii) The domain of $f = \{1,3,5\}$, find the range of f.

13. If $f(x) = x^3 - ax^2 + bx + 1$. Find the values of a and b, where $f(2) = -3$ and $f(-1) = 0$.

8.2 Inverse Function

8.2.1 Define inverse functions and demonstrate their domain and range with examples

If $f: X \longrightarrow Y$ be a one to one and onto (Bijective) function, then inverse of f, i.e., $f^{-1}: Y \longrightarrow X$ is inverse function of f and is defined as

$$x = f^{-1}(y), \ \forall\, y \in Y$$

if and only if $y = f(x), \forall x \in X$. It is evident that f and f^{-1} are inverses of each other. The Fig 8.11 illustrates the concept of inverse function.

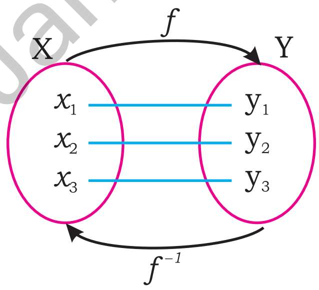


Fig. 8.11

It is clear from the definition of inverse function that

domain of f^{-1} = range of f

and range of f^{-1} = domain of f.

Note: (i) Inverse of f is not always a function.

(ii) Inverse of f is inverse function if f is bijective.

Example: If a function $f: X \longrightarrow Y$, is given by Fig. 8.12. Find f^{-1} and decide whether it is inverse function or not.

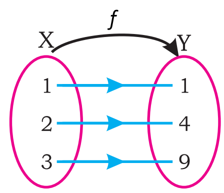


Fig. 8.12

Solution: From Fig 8.12

$f: X \longrightarrow Y$ is defined as $f = \{(1,1),(2,4),(3,9)\}$ or $y = x^2$

where $y = f(x), \forall x \in X$,

with domain of $f = \{1,2,3\}$

and range of $f = \{1,4,9\}$

Now $f^{-1}: Y \longrightarrow X$ is inverse function of f and will be as under

$f^{-1} = \{(1,1),(4,2),(9,3)\}$ as shown in Fig. 8.13 or $x = \sqrt{y}$

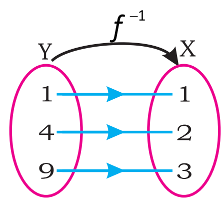


Fig. 8.13

where $x = f^{-1}(y), \ \forall\, y \in Y$

with, domain of $f^{-1} = \{1,4,9\}$

and range of $f^{-1} = \{1,2,3\}$

Since f is both one–to–one and onto, the function is bijective. Therefore f^{-1} is inverse function.

Method for Finding the Inverse of a Function

Inverse of a function $f(x)$ can be found by the following steps:

Step 1. Write $y = f(x)$

Step 2. Express x in terms of y

Step 3. In the resulting equation in step 2, replace x by $f^{-1}(y)$

Step 4. Replace each y by x in the result of step 3 to get $f^{-1}(x)$

Step 5. Verify that $f^{-1}(f(x)) = x$

Example 1. Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = 5x + 7$. Find $f^{-1}(x)$.

Solution: We have $f(x) = 5x + 7$;

Step1. Let $f(x) = 5x + 7 = y$

Step 2. $\Rightarrow \quad x = \frac{y-7}{5}$.

Step 3. Replacing x by $f^{-1}(y)$, we get $f^{-1}(y) = \frac{y-7}{5}$

Step 4. To find $f^{-1}(x)$, replace y by x, we have $f^{-1}(x) = \frac{x-7}{5}$

Step 5. Verification: $f^{-1}(f(x)) = f^{-1}(5x + 7) = \frac{5x+7-7}{5} = x$.

Example 2. Let $f(x) = \frac{x-3}{x-7}, x \neq 7$. Find $f^{-1}$and also find the domain and range of f^{-1}.

Solution: Since, $f(x) = \frac{x-3}{x-7}$ is not defined for $x = 7$,

$\therefore$ Domain of $f = \mathbb{R} - \{7\}$

Let $f(x) = \frac{x-3}{x-7} = y$,

$\Rightarrow \quad (x - 7)y = x - 3$

$\Rightarrow \quad xy - 7y = x - 3$

$\Rightarrow \quad xy - x = 7y - 3$

$\Rightarrow \quad x = \frac{7y-3}{y-1}$

Replacing x by $f^{-1}(y)$, we get $f^{-1}(y) = \frac{7y-3}{y-1}$

To find $f^{-1}(x)$, replace y by x, we have $f^{-1}(x) = \frac{7x-3}{x-1}$

Verification

$$f^{-1}(f(x)) = f^{-1}\left(\frac{x-3}{x-7}\right) = \frac{7\left(\frac{x-3}{x-7}\right) - 3}{\frac{x-3}{x-7} - 1}$$

$$= \frac{\frac{(7x-21)-(3x-21)}{x-7}}{\frac{(x-3)-(x-7)}{x-7}}$$

$$= \frac{(7x-21)-(3x-21)}{(x-3)-(x-7)} = \frac{4x}{4} = x$$

We see that $f^{-1}(x)$ is not defined at $x = 1$, so it is not in the domain of f^{-1}.

So, domain of $f^{-1} = \mathbb{R} - \{1\}$

we know that Range of f^{-1} = Domain of $f = \mathbb{R} - \{7\}$.

Exercise 8.2

1. If $f(x) = -5x + 1$, find the values of :

(i) $f^{-1}(36)$ (ii) $f^{-1}(-14)$ (iii) $f^{-1}(0)$ (iv) $f^{-1}(a)$

2. Given that $g(t) = \frac{1}{t-5}$, $(t \neq 5)$, find the values of:

(i) $g^{-1}\left(\frac{1}{2}\right)$ (ii) $g^{-1}(2)$ (iii) $g^{-1}(-1)$ (iv) $g^{-1}(a)$

3. Find the inverse of the following functions:

(i) $f(x) = 12 - \frac{1}{2}x$ (ii) $f(x) = \frac{1}{2}(x-3)$ (iii) $f(x) = \frac{2x+1}{5}$

(iv) $f(x) = \frac{5}{9}(x-32)$ (v) $g(x) = 180(x-2)$ (vi) $h(x) = 2\pi x$

(vii) $f(t) = t^2 + 5\ (t \geq 0)$ (viii) $f(t) = 5\sqrt{t}, (t \geq 0)$ (ix) $f(t) = (t-5)^3$

(x) $f(t) = \sqrt[3]{t+1}$ (xi) $g(x) = \frac{1}{x-3}, (x \neq 3)$

(xii) $g(x) = \frac{1}{2x+1}, \left(x \neq \frac{1}{2}\right)$

4. Find inverse of f and determine the domain and range of f^{-1} for the real valued function f defined by

i) $f(x) = \frac{x-1}{x-3}$, $x \neq 3$ ii) $f(x) = 5x + 7$

iii) $f(x) = \frac{1}{x+5}$, $x \neq -5$ iv) $f(x) = (x-5)^2, x \geq 5$

v) $f(x) = \sqrt{x+7}, x \geq -7$ vi) $f(x) = \frac{x-1}{x-4}$, $x \neq 4$

8.3 Graphical Representation of Functions

If $y = f(x)$ is a function, then the set of all points (x, y) such that x is in the domain of f and y is in the range of f, is called graph of function $f(x)$.

8.3.1 Sketch graphs of

- **linear function (e.g. $y = ax + b$)**
- **non-linear function (e.g. $y = x^2$)**
- **square root functions (e.g. obtained from $x^2 + y^2 = a^2$)**

- **linear function (e.g. $y = ax + b$)**

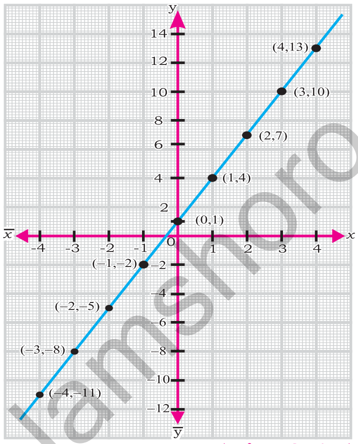
We sketch the graph of a linear function of the form $y = ax + b$, $a, b, x \in \mathbb{R}$, $a \neq 0$ with the help of following example.

Example: Sketch the graph of the function $f(x) = 3x + 1, \forall\, x \in \mathbb{R}$

Solution: By putting some values of x in the given function $y = f(x) = 3x + 1$, we get corresponding values of y, as shown in the following table.

x	-4	-3	-2	-1	0	1	2	3	4
$y = 3x + 1$	-11	-8	-5	-2	1	4	7	10	13

The points are plotted and the graph is obtained which represents a line as shown in Fig. 8.14.



(Fig. 8.14)

- **non-linear functions (e.g., $y = x^2$)**

We sketch the graph of non-linear function with the help of the following example.

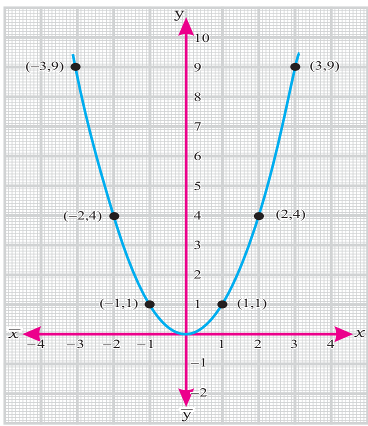
Example: Sketch the graph of the non-linear function $y = f(x) = x^2;\ \forall\, x, y \in \mathbb{R}$.

Solution:

We find some values of y by putting the values of x in the function as shown in the following table.

x	-3	-2	-1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9

The graph of $y = x^2$ represents a parabola as shown in fig. 8.15.



(Fig. 8.15)

- **Square root functions (e.g. obtained from $x^2 + y^2 = a^2$)**

The square root function is the function $f: \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}^+ \cup \{0\}$ defined as $f(x) = \sqrt{x};\ \forall\, x \in \mathbb{R}^+ \cup \{0\}$. We know that the equation $x^2 + y^2 = a^2$ represents a circle of radius a, where $x \le a$.

Now, from $\quad x^2 + y^2 = a^2; \quad \forall x, y \in \mathbb{R}$

We get, $\quad y^2 = a^2 - x^2$

$$\Rightarrow \quad y = f(x) = \pm\sqrt{a^2 - x^2}, \quad \forall\, x \le a$$

Here f is not a function as each element of domain has not unique image

But $y = f(x) = \sqrt{a^2 - x^2}$, $\forall\, x \le a$ and $y = f(x) = -\sqrt{a^2 - x^2}$, $\forall\, x \le a$ represent square root functions.

Example:

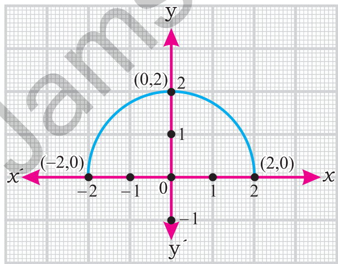
Sketch the graphs of the functions $y = \pm\sqrt{4 - x^2}$ $\forall\, x \le 2$,

Solution:

a) Consider the function $y = \sqrt{4 - x^2}$, $\forall\, x \le 2$

Some values of x and the corresponding values of y are mentioned in the table.

x	-2	-1	0	1	2
$y = \sqrt{4 - x^2}$	0	$\sqrt{3}$	2	$\sqrt{3}$	0



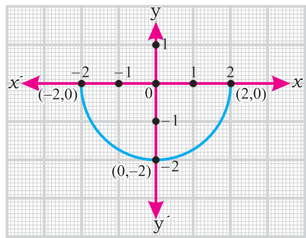
(Fig. 8.16)

Graph of the square root function shows that it is a half circle which opens downward meeting x-axis at (–2,0), (2,0) and y–axis at (0,2). Its center is at origin with radius 2 units as shown in Fig. 8.16.

b) Consider the function $y = -\sqrt{4 - x^2}$, $\forall\, x \le 2$

Some values of x and the corresponding values of y are mentioned in the table.

x	-2	-1	0	1	2
$y = -\sqrt{4 - x^2}$	0	$-\sqrt{3}$	-2	$-\sqrt{3}$	0



(Fig. 8.17)

Graph of the square root function shows that it is a half circle which opens upward meeting x-axis at (–2,0), (2,0) and y–axis at (0,-2). Its center is at origin with radius 2 units as shown in Fig. 8.17.

8.3.2 Sketch the graph of the function $y = x^n$ where

(a) n is a +ve integer

(b) n is a −ve integer ($x \neq 0$),

(c) n is a rational number for $x > 0$

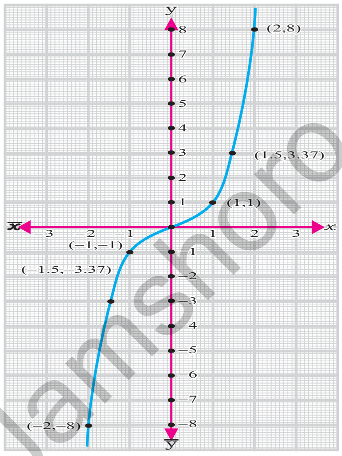
(a) n is a +ve integer

We sketch the graph of the function $y = x^n$, where n is a +ve integer, by taking $n = 3$ and $n = 4$.

(i) For $n = 3$, we have $y = x^3$, $\forall x \in \mathbb{R}$. Some corresponding values of x and y are given in the following table.

x	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5
$y = x^3$	-15.62	-8	-3.37	-1	-0.12	0	0.12	1	3.37	8	15.62

The graph of $y = x^3$ is a curve shown in Fig. 8.18.

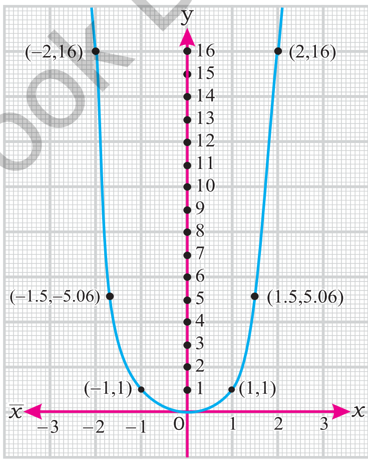


(Fig. 8.18)

(ii) For $n = 4$, we have $y = x^4, \forall x \in \mathbb{R}$. Some corresponding values of x and y are given in the following table.

x	-2	-1.5	-1	0	1	1.5	2
$y = x^4$	16	5.06	1	0	1	5.06	16

The graph of $y = x^4$ is shown in fig. 8.19.



(Fig. 8.19)

(b) *n* is a −ve integer $(x \neq 0)$,

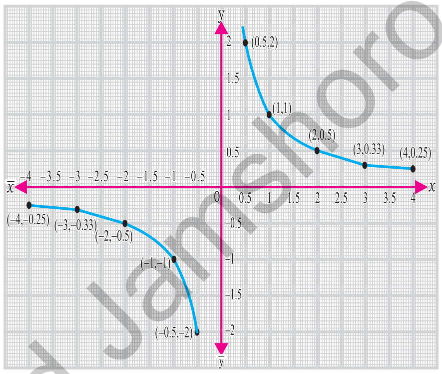
We sketch the graph of $y = x^n$, where n is a negative integer, by taking $-1, -2, -3$ and -4 as values of n.

(i) When $n = -1$, then $y = x^{-1} = \frac{1}{x}$, $x \neq 0$

Some values of x and their corresponding values of y are mentioned in the table.

x	−4	−3	−2	−1	−0.5	0.5	1	2	3	4
$\frac{1}{x}$	−0.25	−0.33	−0.5	−1	−2	2	1	0.5	0.33	0.25

The graph $y = \frac{1}{x}$ is a curve as shown in Fig. 8.20.



(Fig. 8.20)

(ii) When n = −2 , then $y = x^{-2} = \frac{1}{x^2}$, $x \neq 0$

Following table shows some corresponding values of x and y of the function $y = \frac{1}{x^2}$

x	-2.5	−2	−1.5	−1	−0.5	0.5	1	1.5	2	2.5
$\frac{1}{x^2}$	0.16	0.25	0.44	1	4	4	1	0.44	0.25	0.16

The graph $y = \frac{1}{x^2}$ is a curve as shown in Fig.8.21

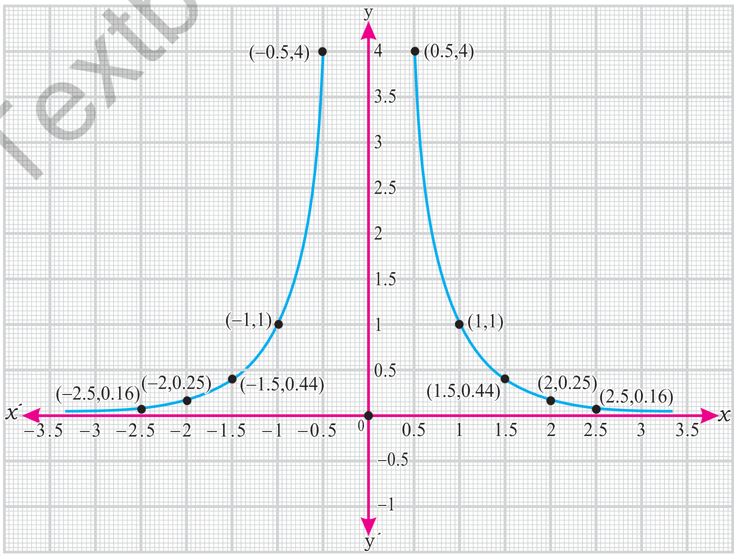


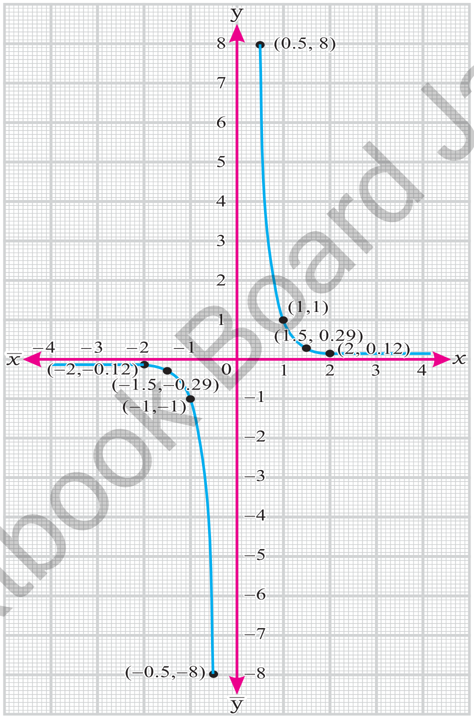
Fig. 8.21

(iii) When $n = -3$, then $y = x^{-3} = \frac{1}{x^3}$, $x \neq 0$

Following table shows some corresponding values of x and y of the function $y = \frac{1}{x^3}$

x	-2	-1.5	-1	-0.5	0.5	1	1.5	2
$y = \frac{1}{x^3}$	-0.12	-0.29	-1	-8	8	1	0.29	0.12

The graph $y = \frac{1}{x^3}$ is a curve as shown in Fig.8.22.



(Fig. 8.22)

(iii) When n= -4 , then $= x^{-4} = \frac{1}{x^4}$, $x \neq 0$

Following table shows some corresponding values of x and y of the function $y = \frac{1}{x^4}$

x	± 0.7	± 0.8	± 0.9	± 1	± 1.5	± 2	± 2.5
$y = \frac{1}{x^4}$	4.16	2.44	1.52	1	0.19	0.06	0.02

The graph $y = \frac{1}{x^4}$ is a curve as shown in Fig.8.23.

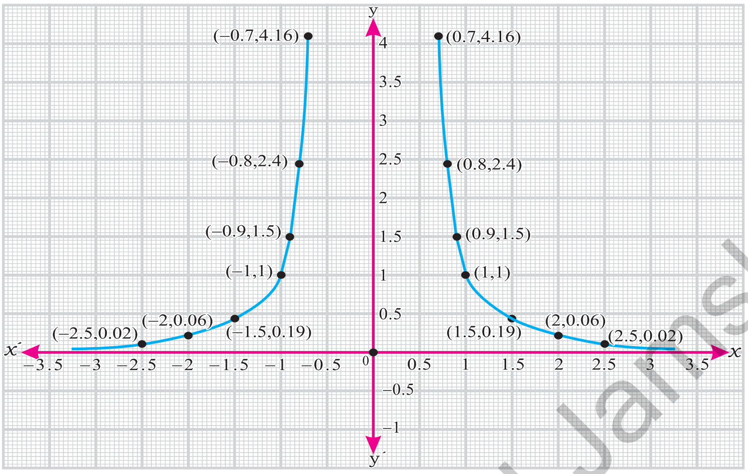


Fig. 8.23

Note: (i) When the value of n is negative odd then the graphs of $y = x^n$ are symmetric about origin. In this case, all the graphs have the same general pattern as of $\frac{1}{x}$.

(ii) When the value of n is negative even then the graphs of $y = x^n$ are symmetric about y axis. In this case, all the graphs have the same general pattern as of $\frac{1}{x^2}$.

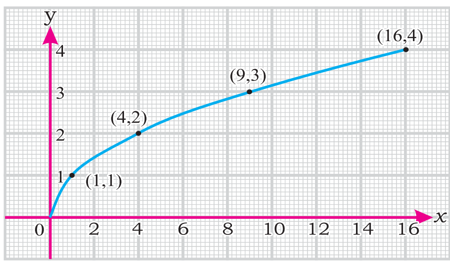
(iii) When the value of n is positive odd then the graphs of $y = x^n$ are symmetric about origin. In this case, all the graphs have the same general pattern as $y = x^3$

(iv) When the value of n is positive even then the graphs of $y = x^n$ are symmetric about y-axis. In this case, all the graphs have the same general pattern as $y = x^2$

(c) n is a rational number for $x > 0$

For n is a rational number, we sketch the graph of the function $y = f(x) = x^n$, $x > 0$ by taking $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ as values of n.

(1) When $n = \frac{1}{2}$ then $y = x^{\frac{1}{2}} = \sqrt{x}$, $x \geq 0$. Following table shows some corresponding values of x and y of the function $y = \sqrt{x}$



(Fig. 8.24)

x	0	1	4	9	16
$y = \sqrt{x}$	0	1	2	3	4

The graph of the function is a curve in xy –plane as shown in Fig. 8.24.

(2) When $n = \frac{1}{3}$ then $y = x^{\frac{1}{3}} = \sqrt[3]{x}$.

Following table shows some corresponding values of x and y of the function $y = \sqrt[3]{x}$

x	0	1	8	27
$y = \sqrt[3]{x}$	0	1	2	3

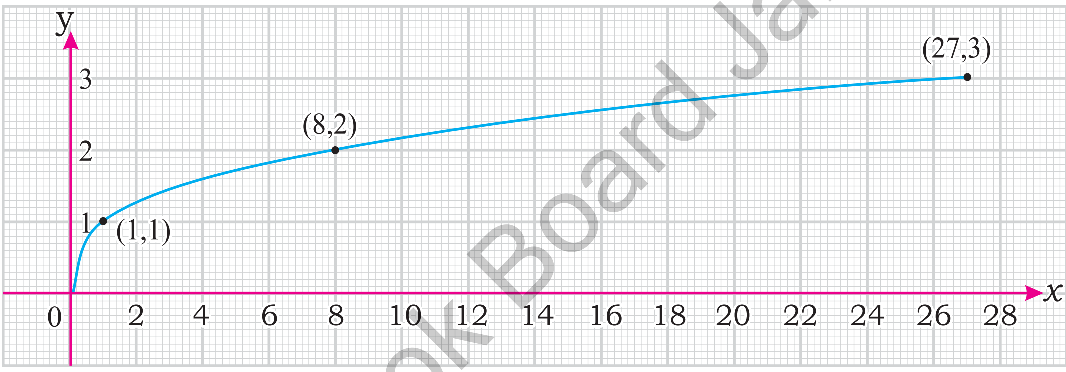


Fig. 8.25

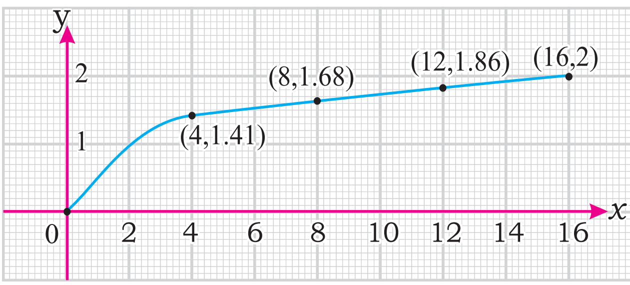
The graph of the function is a curve in xy –plane as shown in Fig. 8.25.

(3) When $n = \frac{1}{4}$ then $y = x^{\frac{1}{4}} = \sqrt[4]{x}$.

Following table shows some corresponding values of x and y of the function $y = \sqrt[4]{x}$

x	0	4	8	12	16
$y = \sqrt[4]{x}$	0	1.41	1.68	1.86	2

The graph of the function is a curve in xy-plane as shown in the Fig. 8.26.



(Fig. 8.26)

8.3.3 Sketch graph of quadratic function of the form $y = ax^2 + bx + c, (a \neq 0), a, b, c$ are integers

A nonlinear function that can be written in the form $y = ax^2 + bx + c$, where $a \neq 0$ and a, b, c are integers, is called a quadratic function. Graph of every quadratic function is a parabola and the parent quadratic function is $y = x^2$.

Example 1. Sketch the graph of the function $y = f(x) = x^2 + 4x + 5$.

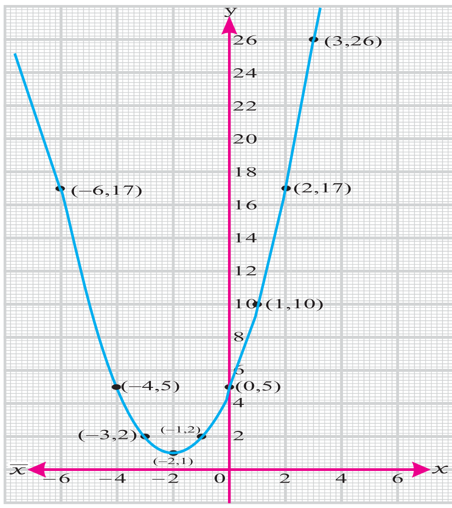
Solution:

$$y = x^2 + 4x + 5$$
$$= x^2 + 4x + 4 + 1$$
$$= (x + 2)^2 + 1$$

Following table shows some corresponding values of x and y of the given function

x	-6	-4	−3	−2	−1	0	1	2	3
y	17	5	2	1	2	5	10	17	26

The graph of the given quadratic function is a parabola. Its vertex is at a point $(-2, 1)$ and it opens upward as shown in Fig. 8.27.



(Fig. 8.27)

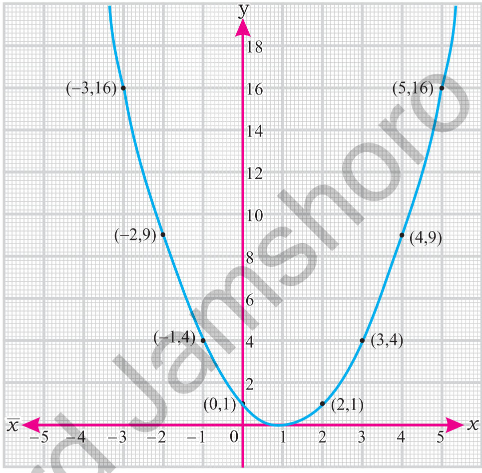
Example 2. Sketch the graph of the function $f(x) = x^2 - 2x + 1$.

Solution: $y = x^2 - 2x + 1 \Rightarrow \quad y = (x - 1)^2$

Some corresponding values x and y of the function $y = x^2 - 2x + 1$ are given in the following table.

x	−3	−2	−1	0	1	2	3	4	5
y	16	9	4	1	0	1	4	9	16

The graph of the given function is a parabola. Its vertex is at point (1,0) and it opens upward as shown in Fig. 8.28.



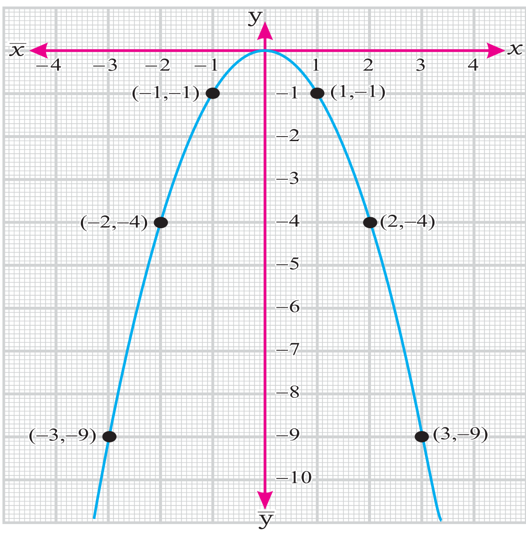
(Fig. 8.28)

Example 3. Sketch the graph of function $y = f(x) = -x^2$.

Solution: Some corresponding values of x and y of the function $y = -x^2$ are given in the following table.

x	−3	−2	−1	0	1	2	3
y	−9	−4	−1	0	−1	−4	−9

The graph of the given function is a parabola. Its vertex is at point (0,0) and it opens downward as shown in Fig. 8.29.



(Fig. 8.29)

Note: (i) When the coefficient of x^2 is positive in quadratic function $y = ax^2 + bx + c, (a \neq 0)$ i.e., $a > 0$, the graph of the function opens upward.

(ii) When the coefficient of x^2 is negative in quadratic function $y = ax^2 + bx + c, (a \neq 0)$ i.e., $a < 0$, the graph of the function opens downward.

8.3.4 Sketch graph using factors

We draw the graph of quadratic function in the form of factors $y = f(x) = a(x - p)(x - q)$ by using the following steps:

1. Identify the points $(p, 0)$ and $(q, 0)$ where the graph of the function cuts x- axis.
2. Take $x = 0$ in the function to identify the point $(0, y)$ where the graph cuts y – axis.
3. The sign of the constant 'a' indicates the shape of the graph opens upward or downward.
4. To draw the graph, we get some additional points of the graph.
5. Graph of each quadratic functions is a parabola. So, we draw parabola through the points.
6. Locate the correct point where the graph is turning.

The method of sketching the graph of a quadratic function using factors is illustrated through the following examples.

Example 1. Sketch the graph of the function $y = f(x) = -2x^2 + 6x$.

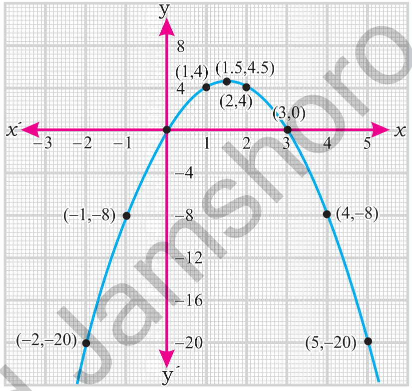
Solution: $y = -2x^2 + 6x \implies y = -2x(x - 3)$

In order to draw the graph, we proceed as follows

i. To identify the points where the graph of function cuts x- axis.
   we compare with $y = f(x) = a(x - p)(x - q)$
   and get $a = -2$; $p = 0$; $q = 3$.
   or we put $y = 0$ in the given equation, we get
   $x = 0$ and $x = 3$
   so the points are (0,0) and (3,0) where the graph cuts the x- axis
ii. To identify the points where the graph cuts y – axis.
   We put $x = 0$ in $y = -2x^2 + 6x$, and get
   $y = 0$. So the point is (0,0) and the graph cuts y-axis at (0,0).
iii. We check the sign of the constant 'a' which is negative in this case, therefore the graph opens downward.
iv. For plotting the graph, some additional points will be obtained from the function $y = -2x^2 + 6x$ as shown in the table.

x	-2	-1	0	1	1.5	2	3	4	5
y	-20	-8	0	4	4.5	4	0	-8	-20

v. We draw the graph of the function which is a parabola that opens downward with vertex at $(1.5, 4.5)$ as shown in Fig. 8.30.



(Fig. 8.30)

Example 2. Sketch the graph of the function $y = f(x) = x^2 - x - 6$.

Solution: $y = x^2 - x - 6 = (x + 2)(x - 3)$

In order to draw the graph, we proceed as follows:

i. To identify the points where the graph of function cuts x- axis.
we compare with $y = f(x) = a(x - p)(x - q)$.
and get $a = +1$; $p = -2$; $q = 3$.
or we put $y = 0$ in the given equation and get $x = -2$ and $x = 3$
So, the points are $(-2,0)$ and $(3,0)$ where the graph cuts the x- axis

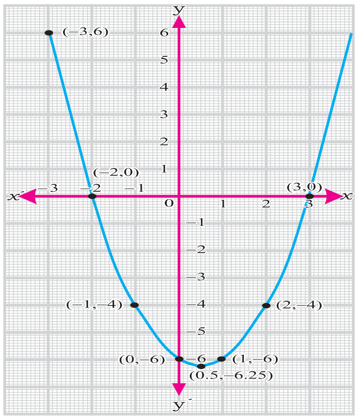
ii. To identify the points where the graph cuts y – axis.
We put $x = 0$ in $y = x^2 - x - 6$ and get $y = -6$
So the point is $(0, -6)$ and the graph cuts y-axis. at $(0, -6)$.

iii. Check the sign of the constant 'a' which is positive in this case, therefore the graph opens upward.

iv. For plotting the graph, some additional points will be obtained from the function $y = x^2 - x - 6$ as under:

x	-3	-2	-1	0	0.5	1	2	3
y	6	0	-4	-6	-6.25	-6	-4	0

v. We draw the graph of the function which is a parabola that opens upward with vertex at $(0.5, -6.25)$ as shown in Fig. 8.31.



(Fig. 8.31)

8.3.5 Predict functions from their graphs (use the factor form to predict the equation of a function of the type $f(x) = ax^2 + bx + c$, if two points where the graph crosses x-axis and third point on the curve' are given)

We use factor form of quadratic equation to predict the equation of a given graph whose function is of the form $f(x) = ax^2 + bx + c$ provided that two points on the graph which cuts the x – axis and third point on the curve are given.

The equation of the curve which passes through $x - axis$ at the points $(p, 0)$ and $(q, 0)$ in the factor form of quadratic equation is

$$y = a(x - p)(x - q) \qquad \ldots\text{(i)}$$

By taking all three points from the graph and substituting in equation (i) we get the value of a. and by the value of p, q and a we can form the desired equation of that curve.

The method is illustrated in the following example.

Example 1: Predict the equation of the function from the given graph of the type $y = ax^2 + bx + c$

Solution: The equation of the curve which passes through $x - axis$ at the points$(p, 0)$ and $(q, 0)$ has the form $y = a(x - p)(x - q)$ ...(i)

Here the curve which passes through the points $(-2,0)$ and $(2,0)$ is shown in figure (8.32). So, $p = -2$ and $q = 2$. By substituting these values in eq.(i)

We get $y = a(x + 2)(x - 2)$ ...(ii)

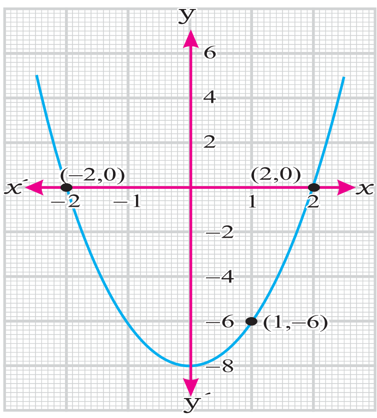
The point $(1, -6)$ lies on the curve, so it must satisfy equation (ii)

so, $-6 = a(1 + 2)(1 - 2) \Rightarrow a = 2$

Therefore equation (ii) becomes

$y = 2(x + 2)(x - 2)$

$y = 2x^2 - 8$, is the required equation of given graph of the parabola.



(Fig. 8.32)

Example 2: Predict equation of the graph of the function of the type $y = ax^2 + bx + c$ which cuts the x – axis at the points (4,0) and (−2,0) and also passes through the point (0,8) as shown in Fig. 8.33.

Solution: The equation of the curve which passes through x – axis at the points$(p, 0)$ and $(q, 0)$ has the form $y = a(x - p)(x - q)$ ...(i)

The curve which passes through the

points (4,0) and (−2,0) is shown in figure (8.33). Here, $p = 4$ and $q = -2$. By substituting these values in equation (i)

We have $\quad y = a(x-4)(x+2) \qquad \ldots$(ii)

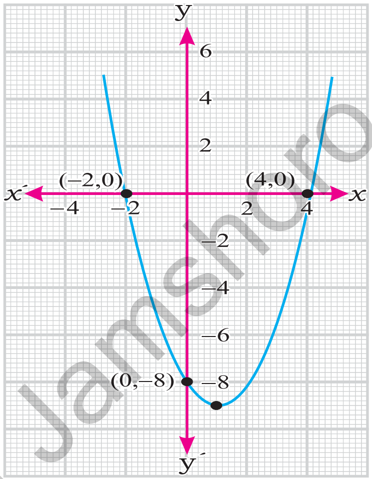
The point (0,8) lies on the curve, so it must satisfy equation (ii)

So, $\quad -8 = a(0-4)(0+2) \quad \Rightarrow \quad a = 1$

Therefore equation (ii) becomes

$$y = (x-4)(x+2)$$

$y = x^2 - 2x - 8$, is the required equation of the graph of parabola.



(Fig. 8.33)

Exercise 8.3

1. Sketch the graph of the following functions:

(i) $y = 5x + 2$ (ii) $y = x^2 + 5$ (iii) $y = x^5$

2. Sketch the graphs of the square root functions obtained from $x^2 + y^2 = 25$.

3. Sketch the graph of the following quadratic functions:

(i) $y = x^2 - 2x + 1$ (ii) $y = -3x^2 + 6x$

4. Sketch the graph of the following functions using factors:

(i) $y = x^2 + 3x + 2$ (ii) $y = -x^2 - 5x$

5. Find the equation of the function of the type $y = f(x) = ax^2 + bx + c$ which cuts the x-axis at the points (−4,0) and (3,0) also passes through the point (2, −4).

6. Find the equation of the graph of the function of the type $y = ax^2 + bx + c$ which crosses the x-axis at the point (−8,0) and (9,0) and also passes through (5,10)

7. Find the equation of the function of the type $y = 3x^2 + bx + c$ of the parabola which cuts x-axis at the points (−3,0) and (5,0).

8. Find the equation of the graph of the function of the type $y = 2x^2 + bx + c$ which crosses the x-axis at the points (−4,0) and (5,0).

8.4 Intersecting Graphs

8.4.1 Find the Intersecting point graphically when the intersection occurs between;

- a linear function and coordinate axes,
- two linear functions,
- a linear and quadratic function.

- **Points of intersection of a linear function and co-ordinate axes**

When we draw the graph of a linear function
$y = f(x) = ax + b$; $\forall$ $a, b, x \in \mathbb{R}$, $a \neq 0$ and $b \neq 0$
we get a straight line that intersects x-axis and y-axis at points P$(a, 0)$ and Q$(0, b)$ respectively as shown in Fig.8.34(i) where a is called x-intercept and b is called y-intercept of the line l on the coordinate axes.

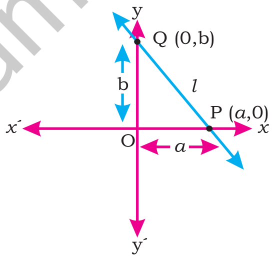


Fig. 8.34(i)

In case, when $b = 0$ in linear function
$y = f(x) = ax + b$; $\forall$ $a, b, x \in \mathbb{R}$, $a \neq 0$
then the line l will pass through origin as shown in Fig. 8.34(ii). So, point of intersection between linear function and coordinate axes is $(0, 0)$.
Method of finding points of intersection between linear function and coordinate axes graphically is explained in the following example.

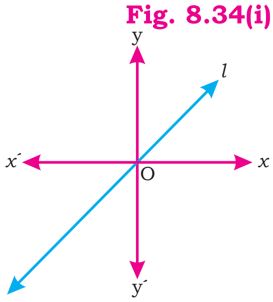


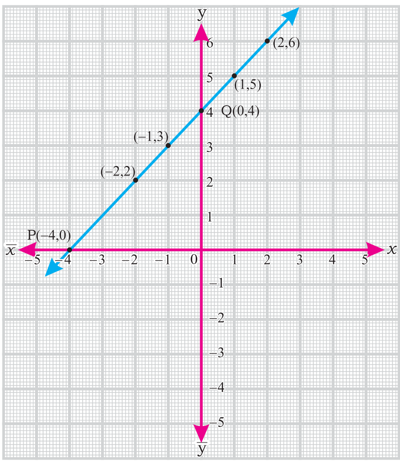
Fig. 8.34(ii)

Example: Find the points of intersection of the linear function $y = f(x) = x + 4$ with co-ordinate axes graphically.

Solution: Here, $y = f(x) = x + 4$
Some corresponding values of x and y of the function are given in the following table.

x	-2	-1	1	2
y	2	3	5	6

From the graph, we see, that x-intercept and y-intercept are -4 and 4 respectively as shown in Fig. 8.35.
Therefore, points of intersections with coordinate axes are P$(-4,0)$ and Q$(0, 4)$.



(Fig. 8.35)

(b) Point of intersection of two linear functions

In order to find the point of intersection of linear functions. we draw the graphs of the functions on the same graph paper and then with the help of graphs we locate the point of intersection of these functions.

If we have two different linear functions described by lines l_1 and l_2

l_1: $y = f_1(x) = a_1x + b_1$; ...(i)

l_2: $y = f_2(x) = a_2x + b_2$; ...(ii) $\forall\ a_1, b_1, a_2, b_2, x \in \mathbb{R}$,

then these lines l_1 and l_2 may or may not intersect. If they intersect, then the point of intersection is unique. In case they do not intersect, the lines will be parallel having no common point.

The method is illustrated in the following example.

Example: Find the point of intersection of the functions
$f(x) = x - 3$ and $g(x) = 2x - 5$ graphically

Solution:

Let $f(x) = y$ then $y = x - 3$.

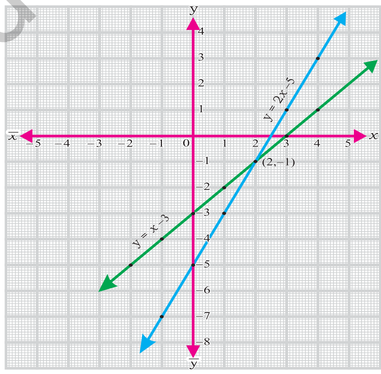
Some corresponding values of x and y are shown in the table.

x	-2	-1	0	1	2	3	4
$y = x - 3$	-5	-4	-3	-2	-1	0	1

Now, let $g(x) = y$ then $y = 2x - 5$.

Some corresponding values of x and y are shown in the table.

x	-1	0	1	2	3	4	5
$y = 2x - 5$	-7	-5	-3	-1	1	3	5



(Fig. 8.36)

From the graph, we find that the two linear functions intersect each other at the point $(2, -1)$ as shown in Fig 8.36.

(c) Points of intersection of a linear function and quadratic function

In order to find the points of intersection, we draw the graph of linear function and quadratic function on the same graph paper. The points of intersection will be located using this graph.

We know that the graphs of linear and quadratic functions are line and parabola respectively. These curves may or may not intersect each other. If they intersect then they will intersect at one or two points.

The method is illustrated by the following example.

Example: Find the points of intersection of functions

$$y = f(x) = 2x + 1 \quad \text{and} \quad y = g(x) = x^2 - 4x + 6,\ \forall\ x \in \mathbb{R}$$

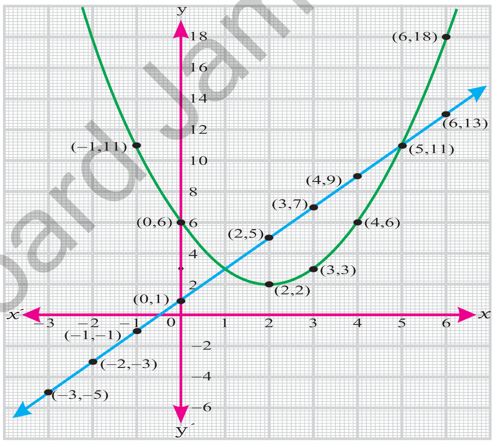
Solution:
Some corresponding values of x and y of the function $y = 2x + 1$ are given in the following table.

x	-3	-2	-1	0	1	2	3	4	5	6
$y = 2x + 1$	-5	-3	-1	1	3	5	7	9	11	13

Some corresponding values of x and y of function $y = x^2 - 4x + 6$ are given in the following table.

x	-1	0	1	2	3	4	5	6
$y = x^2 - 4x + 6$	11	6	3	2	3	6	11	18

Graphs of the above two functions are shown in figure (8.37)
From graph, the points of intersection of linear function and quadratic function are (1,3) and (5,11).



(Fig. 8.37)

8.4.2 Solve, graphically, appropriate problems from daily life

Example 1. A group of 45 school children visited a zoo on special occasion and at discount, paid Rs 60 altogether for entry tickets. The entry ticket of class 1 was Rs. 2 per child where as that of class KG Rs.1 per child, how many children were in the group from each class.

Solution:
Let x be the number of children from class 1
y be the number of children from class KG
According to the given condition
Total number of children are:

$$x + y = 45 \quad \ldots(i)$$

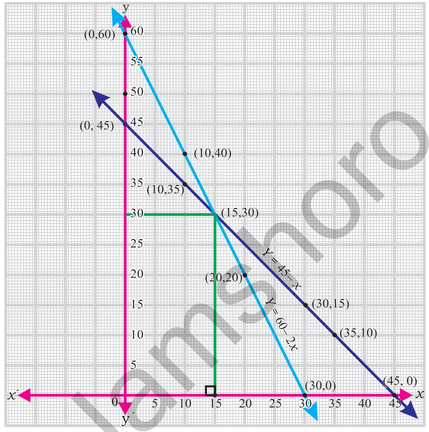
Amount paid for the entry tickets of class 1 and KG children, at the rate of Rs. 2 and Rs.1 respectively is:

$$2x + y = 60 \ldots(ii)$$

Some corresponding values of x and y of eq. (i) and (ii) are given in the following tables.

x	0	15	35	45
$y = 45 - x$	45	30	10	0

x	0	10	20	30
$y = 60 - 2x$	60	40	20	0



(Fig. 8.38)

In order to find children in each group, we plot the graphs of eq. (i) and (ii) as shown in figure 8.38.

From the graph, the intersection point is (15,30)

Hence,

x = number of children from class 1 = 15

and y = number of children from class KG=30

Example 2. The difference of the ages of Waqar and Saqib is 9 years while the age of Waqar is 3 years more than the square of the age of Saqib. Find their ages.

Solution:

Let the age of Saqib be x years.

and age of Waqar be y years.

According to the given condition

Difference of the ages of Waqar and Saqib = 9 years

i.e., $y - x = 9$

$\Rightarrow$ $y = x + 9$ ...(i)

Now, square of the age of Saqib $= x^2$

$\because$ Age of Waqar is 3 years more than the square of the age of Saqib

$\therefore$, $y = x^2 + 3$ ...(ii)

From equation (i)

$y = x + 9$

Some corresponding values of x and y of function $y = x + 9$ are given in the following table.

From equation (i)

x	−3	−2	−1	0	1	2	3	4
$y = x + 9$	6	7	8	9	10	11	12	13

From equation (ii)

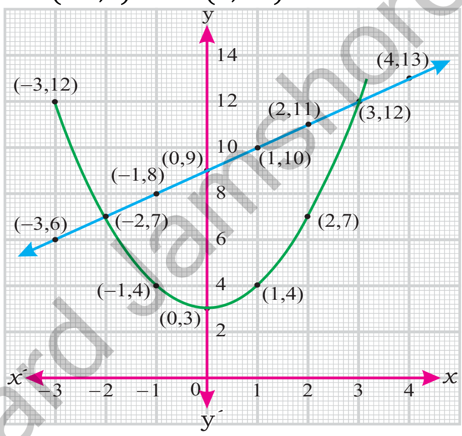
x	-3	-2	-1	0	1	2	3
$y = x^2 + 3$	12	7	4	3	4	7	12

From the graph, we get the intersection points $(-2,7)$ and $(3,12)$ as shown in Fig 8.39.

Since, age cannot be negative, so the required point of intersection is (3, 12).

Thus, age of Saqib = 3 years

and age of Waqar = 12 years



(Fig. 8.39)

Exercise 8.4

1. Graphically find the point of intersection of the following linear functions with co-ordinate axes.

(i) $f(x) = x + 7$ (ii) $f(x) = -x + 5$

(iii) $f(x) = 3x + 2$ (iv) $f(x) = -2x + 4$

2. Find the point of intersection of the following functions graphically where y is function of x.

(i) $2x + y = 5$ and $x + y = 4$ (ii) $3x - 2y = 4$ and $x + 4y = 6$

3. Find the point of intersection of the following functions graphically:

(i) $f(x) = x + 2$ and $g(x) = x^2 - 4x + 6$

(ii) $f(x) = x + 4$ and $g(x) = x^2 - 6x + 10$

4. The paths of two aeroplanes A and B in the plane are determined by the straight lines $2x - y = 6$ and $3x + y = 4$ respectively. Graphically find the point where the two paths cross each other.

5. If the sum of two numbers is 6 and the square of the first number is greater than 6 by the second number. Solve graphically and find the numbers.

Review Exercise 8

1. Select correct answer.

i. If $f: A \to B$ be a function, then it is an onto function if:
(a) Range = B (b) Range $\subset$ B
(c) Image is not repeated (d) Domain $\neq$ A

ii. The graph of $y = x^6$ is symmetric to ____________.
(a) $x -$ axis (b) $y -$ axis (c) origin (d) none

iii. An one to one function is also called __________ function:
(a) Injective (b) Surjective (c) Bijective (d) Inverse

iv. Inverse of a function exists only if it is:
(a) Injective (b) Bijective (c) Surjective (d) all of these

v. The function $f = \{(x, y) \mid y = mx + c\}, m$ and c are real numbers is:
(a) Linear (b) Quadratic (c) A circle (d) A point

vi. The function $f = \{(x, y) \mid y = ax^2 + bx + c, a \neq 0\}$ is:
(a) Linear (b) Quadratic (c) A circle (d) A point

vii. The range of $f(x) = x^2 + 3 \ \forall x \in \mathbb{R}$ is __________.
(a) $\mathbb{R}$ (b) $\mathbb{R}^-$ (c) $(3, \infty)$ (d) $[3, \infty]$

viii. Graph of $y = x^n$ is symmetric to _________ if n is an odd integer.
(a) $x -$ axis (b) $y -$ axis (c) origin (d) none

ix. The graph of linear function is:
(a) Circle (b) Straight line
(c) Parabola (d) Triangle

x. If f is function from A to B. Domain of f is equal to:
(a) Any subset of A (b) A $\times$ B (c) A (d) B

xi. Every function is a:
(a) Relation (b) Inverse function
(c) One to one (d) None of these

xii. If f and g are equal where $f(x) = 7x - 4$ and $g(x) = x$ then $x =$:
(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) 1 (d) $\frac{4}{3}$

xiii. Domain of the function $f(x) = \frac{3x+1}{x-1}$ is set of all:
(a) real numbers (b) rational numbers
(c) complex numbers (d) real numbers except 1

xiv. A function $f(x) = |x| - x^2$ is:
(a) odd (b) Linear
(c) Even (d) Neither even nor odd

2. Find domain of the following functions.

(i) $f(x) = \frac{2x-1}{x+4}$ (ii) $f(x) = \sqrt{x+3}$

3. Graphically, find the point of intersection of the function $f(x) = x$ and $g(x) = x^2 - x$.

4. Find the equation, in the form $y = ax^2 + bx + c$ of the parabola which cuts x-axis at (1,0) and (5,0) and cuts y-axis at (0,15).

5. The function f is defined by $f(x) = ax^2 + bx + c$. Given that $f(0) = 5$, $f(-1) = 15$ and $f(1) = 1$, find the values of a, b and c.

6. The function f is defined by $f(x) = x^2 - 3x + 5$. Find the values of x for which $f(x) = f(3)$.