Unit 9

Linear Programming (LP)

9.1 Introduction

Linear programming was developed as a discipline in 1940s motivated initially by the need to solve complex planning problems in war times operations. Its development accelerated rapidly in the postwar period as many industries found valuable uses for linear programming. During World War-II, linear programming was used extensively to deal with transportation, scheduling, and allocation of resources subject to certain restrictions such as costs and operations.

9.1.1 Define linear programming (LP) as planning of allocation of limited resources to obtain an optimal result

Linear Programming (LP) is a mathematical technique for allocating limited resources in optimum manner.

If we have limited resources at our disposal then we intend to seek optimal utilization of those resources. The resources may be time, money, space etc. For example, we have 50 square feet of office space to use for storage and we have a budget of Rs 20,000 and there are variety of cabinet types and sizes from which to choose. So how best the available space, we should utilize and stay within the allocated budget. In another example a company manufactures three products using the basic raw material, some are more expensive to produce than others and few of them are perishable, and need to be used quickly. How much of each product should the company manufacture to minimize the cost and which combination produces the least waste.

In above examples the situations are complex, as so many variables and constraints are involved for consideration. In order to handle such type of problems, we take the help of linear programming. Linear Programming is a mathematical technique that determines the best way to use available resources. Managers use the process to help make decisions about the most efficient use of limited resources.

9.2 Linear Inequalities

In mathematics, a linear inequality involves a linear expression in two or more variables by using any of the relational symbols such as $<, >, \leq$ or $\geq$.

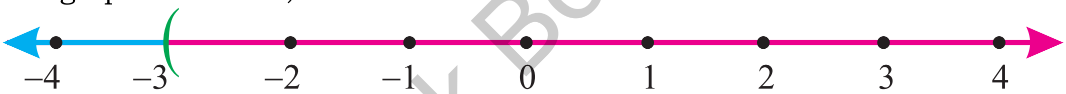
9.2.1 Find algebraic solutions of linear inequalities in one variable and represent them on number line

Inequalities of the form $ax < b, ax \leq b, ax > b$ or $ax \geq b$, where a and b are constants, known as linear inequalities in one variable.

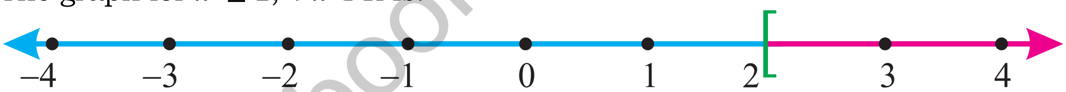
$x < -5$, $5x \leq 10$, $-2x - 6 > 10$ and $-2(x + 2) \geq 4 - x$ are few examples of linear inequalities in one variable. Solutions of a linear inequality in one variable are the values of the variable which satisfy the linear inequality.

The graphic solution of a linear inequality in one variable is represented by a number line. We use the left parenthesis symbol "(" and right parenthesis symbol ")" for ">" and "<" respectively. We also use the left square bracket symbol "[" and right square bracket symbol "]" for "≥" and "≤" respectively for graphic solutions of linear inequalities as shown below.

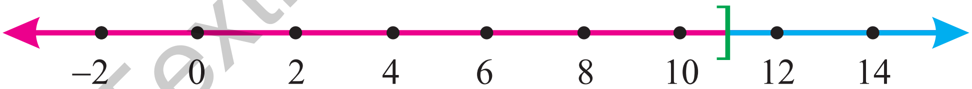
The graph for $x > -3$, $\forall\, x \in \mathbb{R}$ is:



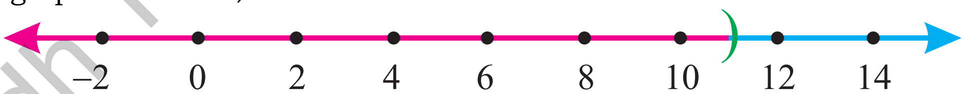
The graph for $x \geq 2$, $\forall\, x \in \mathbb{R}$ is:



The graph for $x \leq 11$, $\forall\, x \in \mathbb{R}$ is:



The graph for $x < 11$, $\forall\, x \in \mathbb{R}$ is:



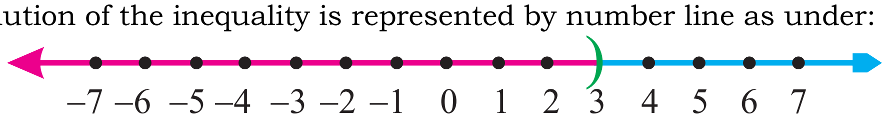
Example 1. Solve the inequality $x - 3 < 0$ and represent the solution on number line, where $x \in \mathbb{R}$.

Solution: We have $\quad x - 3 < 0$

Adding 3 to both sides $\quad x - 3 + 3 < 0 + 3 \Rightarrow x < 3$

Thus, the solution of the inequality is the set of all real values of x that are less than 3, i.e., solution set $= \{x | x \in \mathbb{R} \wedge \; x < 3\}$

The solution of the inequality is represented by number line as under:



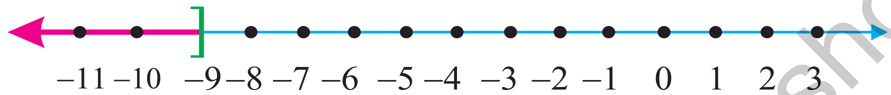
Example 2. Solve the inequality: $2(x+2) \le x-5, \quad \forall x \in \mathbb{R}$ and represent the solution on number line.

Solution: We have $2(x+2) \le x-5$

$\Rightarrow \quad 2x+4 \le x-5 \Rightarrow x \le -9$

The solution set $= \{x | x \in \mathbb{R} \wedge \quad x \le -9\}$

Representation of solution on number line:



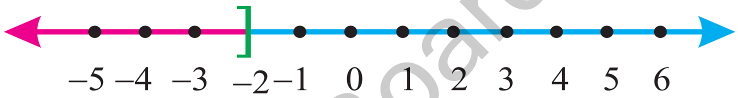
Example 3. Solve the inequality $2(x-3) \ge 3x-4, \forall x \in \mathbb{R}$ and represent the solution on number line.

Solution: We have $2(x-3) \ge 3x-4$

$\Rightarrow \quad 2x-6 \ge 3x-4 \Rightarrow -x \ge 2 \Rightarrow x \le -2$

Thus, solution set $= \{x | x \in \mathbb{R} \wedge \quad x \le -2\}$

Representation of solution on number line:



Note that the graph has an arrow indicating that the line continues without end to the left i.e., $-\infty$.

The solution can be expressed in Interval notation as $(-\infty, -2]$, i.e., all real values of x less than or equal to -2.

> **Note:** Interval notations: $[a, \infty) = \{x \mid x \in \mathbb{R} \wedge x \ge a\}$. Also $(a, \infty) = \{x | x \in \mathbb{R} \wedge x > a\}$
> $(-\infty, a] = \{x \mid x \in \mathbb{R} \wedge x \le a\}$. Also $(-\infty, a) = \{x | x \in \mathbb{R} \wedge x < a\}$

Example 4. Solve the inequality $19 < 3x+7 \le 28$, $\forall x \in \mathbb{R}$ and represent solution on number line.

Solution: We have $19 < 3x+7 \le 28$, $\forall x \in \mathbb{R}$.

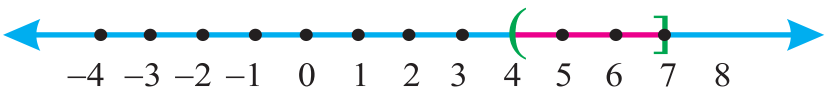
$\Rightarrow \quad 19 < 3x+7$ and $3x+7 \le 28$

$\Rightarrow \quad 12 < 3x \quad \Rightarrow \quad 3x \le 21$

$\Rightarrow \quad 4 < x \quad \Rightarrow \quad x \le 7$

Thus, solution set $= \{x | x \in \mathbb{R} \wedge \quad 4 < x \le 7\}$

Representation on number line:



9.2.2 Interpret graphically the linear inequalities in two variables.

The general form of inequalities: $ax+by < c$, $ax+by > c$, $ax+by \le c$ and $ax+by \ge c$ are known as linear inequalities in two variables, where $a \ne 0$, $b \ne 0$ and c are constants, x and y are variables. The solution of a linear inequality in two variables like $ax + by > c$ is an ordered pair (x, y) that produces a true statement when the values of x and y are substituted into the inequality. The graph of linear inequalities in two variables is a set of all solutions that constitutes a region representing half portion of the plane.

Graphical solution of linear inequalities in two variables

Consider the linear inequality

$$ax + by \le c, \quad a \ne 0,\ b \ne 0 \text{ and } c \ne 0 \qquad \text{.... (1)}$$

Following are the steps to graph the solution region of above inequality.

Step 1. Consider the inequality as an equation

$$ax + by = c, \qquad \text{.... (2)}$$

Step 2. Find x-intercept, and y-intercept

For x-intercept, we put $y = 0$ in (2)

We get $ax + b(0) = c \Rightarrow ax = c \Rightarrow x = \frac{c}{a}$ so, x-intercept is $\frac{c}{a}$ and $\left(\frac{c}{a}, 0\right)$ is the point where line cuts x-axis.

Similarly, for y-intercept, put $x = 0$ in (2)

We get $a(0) + by = c \Rightarrow by = c \Rightarrow y = \frac{c}{b}$ so, y-intercept is $\frac{c}{b}$ and $\left(0, \frac{c}{b}\right)$ is the point where line cuts y-axis.

Step 3. Plot the points in a graph and draw the straight line.

Step 4. Substitute (0, 0) in the inequality (1),

(a) if inequality is true then origin is a part of solution and shade the region involving origin.

(b) if inequality is false then origin is not a part of solution and shade the region which does not involve origin.

Example 1. Graph the solution of the inequality: $3y + 2x \le 6$.

Solution: We have $3y + 2x \le 6$ ...(1)

(i) Consider the inequality as an equation

i.e., $3y + 2x = 6$ ...(2)

(ii) For y-intercept, we put $x = 0$ in (2),

and get $3y + 2(0) = 6 \Rightarrow y = 2$

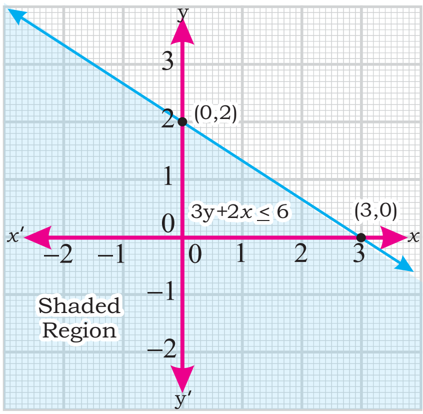
So, intersection point of line and y-axis is (0, 2)

For x-intercept, we put $y = 0$ in (2), and get

$3(0) + 2x = 6 \Rightarrow x = 3$

So, intersection point of line and x-axis is (3, 0)

(iii) We plot the above intersection points (0, 2) and (3, 0) in the graph.



(Fig. 9.1)

(iv) We put $x = 0$ and $y = 0$ in (1) , we get

$3(0) + 2(0) \le 6 \Rightarrow 0 < 6,$

which is true. So, origin is a part of solution of inequality (1).
Now shade the region in the graph as shown in Fig. 9.1 which represents the solution set.

Example 2. Graph, the solution of the inequality $3x+5y \geq 30$

Solution: We have $3x + 5y \geq 30$ ...(1)

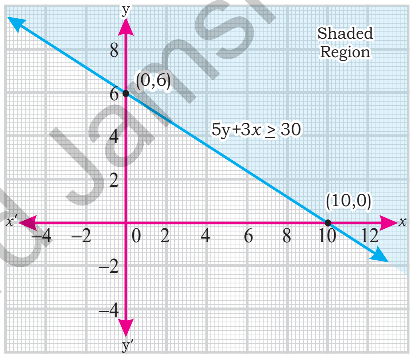
(i) Consider the inequality as an equation

i.e., $3x + 5y = 30$ ...(2)

(ii) For y-intercept, we put $x = 0$ in (2),
and get $3(0) + 5y = 30 \implies y = 6$
Intersection point of line and y-axis is (0,6)
For x-intercept, we put $y = 0$ in (2),
and get $3x + 5(0) = 30 \implies x = 10$
Intersection point of line and x-axis is (10,0)

(iii) We plot the above points (0, 6) and (10, 0) in the graph.

(iv) We put $x = 0$ and $y = 0$ in (1), and get
$3(0) + 2(0) \geq 30 \implies 0 \geq 30$ which is not true.
So, origin is not a part of solution of inequality (1).



(Fig. 9.2)

Now shade the region as shown in the Fig. 9.2 which represents the solution set.

9.2.3 Determine graphically the region bounded by up to 3 simultaneous linear inequalities of non-negative variables and shade the region bounded by them

Two or more linear inequalities form a system of linear inequalities. The solution of the system of linear inequalities in two variables x and y can be obtained by graphing each inequality and then taking intersection of their regions. The common region is the solution region of the system of linear inequalities. In case of '≤', solution region is below the line and in case of '≥', solution region is above the line.

Example 1. Solve graphically

$$5x + y \geq 10 \quad ...(i)$$
$$x + y \geq 6 \quad ...(ii)$$
$$x + 4y \geq 12 \quad ...(iii)$$
$$x, y \geq 0$$

Solution: Let the corresponding equations of above inequalities are

$$5x + y = 10 \quad ...(iv)$$
$$x + y = 6 \quad ...(v)$$
$$x + 4y = 12 \quad ...(vi)$$
$$x, y \geq 0$$

Inequality (i)

For y-intercept, we put $x = 0$ in (iv), and get $5(0) + y = 10 \Rightarrow y = 10$

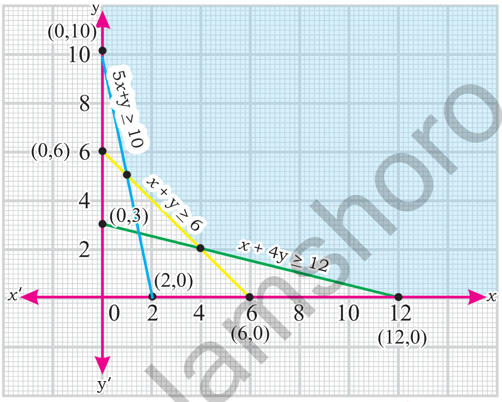
Intersection point of line and y-axis is (0,10)

For x-intercept, we put $y = 0$ in (iv), and get $5x + (0) = 10 \Rightarrow x = 2$

Intersection point of line and x-axis is (2, 0)

For origin, we put $x = 0$ and $y = 0$ in (i) we get , $0 \geq 10$ which is not true

So, origin is not a part of solution of inequality (i)



(Fig. 9.3)

Inequality (ii)

For y-intercept, we put $x = 0$ in (v), and get $0 + y = 6 \Rightarrow y = 6$

Intersection point of line and y-axis is (0,6)

For x-intercept, we put $y = 0$ in (v), and get $x + 0 = 6 \Rightarrow x = 6$

Intersection point of line and x-axis is (6, 0)

For origin, we put $x = 0$ and $y = 0$ in (ii) we get $0 \geq 6$

which is not true. So, origin is a not a part of solution of inequality (ii)

Inequality (iii)

For y-intercept, we put $x = 0$ in (vi), and get $0 + 4y = 12 \Rightarrow y = 3$

Intersection point of line and y-axis is (0,3)

For x-intercept, we put $y = 0$ in (vi), and get $x + 4(0) = 12 \Rightarrow x = 3$

Intersection point of line and x-axis is (12, 0)

For origin, we put $x = 0$ and $y = 0$ in (iii) we get $0 \geq 12$

which is not true. So, origin is a not a part of solution of inequality (iii),

We draw the lines by plotting the above intersection points on the graph. The region bounded by the intersection of three inequalities is shaded (Fig. 9.3) which is the required solution satisfied by all the inequalities.

Note: Since all the inequalities are of type "≥" and coefficients of y are positive. Therefore, the solution region is above all the lines.

Example 2. Solve the following system graphically

$$5x + 10y \leq 50 \quad \ldots(1)$$

$$8x + 2y \geq 16 \quad \ldots(2)$$

$$3x - 2y \leq 6 \quad \ldots(3)$$

$$x, y \geq 0$$

Solution:

Let the corresponding equations of above inequalities are

$$5x + 10y = 50 \quad \ldots(4)$$
$$8x + 2y = 16 \quad \ldots(5)$$
$$3x - 2y = 6 \quad \ldots(6)$$
$$x, y \geq 0$$

Inequality (1)

For y-intercept, we put $x = 0$ in (4), and get $5(0) + 10y = 50 \Rightarrow y = 5$

Intersection point of line and y-axis is (0,5)

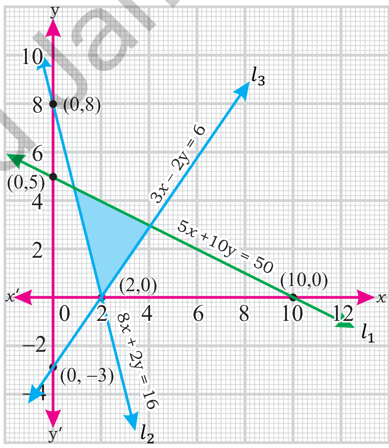
For x-intercept, we put y = 0, and get $5x + 10(0) = 50 \Rightarrow x = 10$

Intersection point of line and x-axis is (10, 0)

For origin, we put $x = 0$ and $y = 0$ in (1)

we get , $0 \leq 50$ which is true

So, origin is a part of solution of inequality (1)



(Fig. 9.4)

Inequality (2)

For y-intercept, put $x = 0$ in (5)

we get $y = 8$

Intersection point of line and y-axis is (0,8)

For x-intercept, we put $y = 0$ in (5)

and get $x = 2$

Intersection point of line and x-axis is (2, 0)

For origin, we put $x = 0$ and $y = 0$ in (2)

We get , $0 \geq 16$ which is not true

So, origin is not a part of solution of inequality (2)

Inequality (3)

For y-intercept, we put $x = 0$ in (6) and get $3(0) - 2y = 6 \Rightarrow y = -3$

Intersection point of line and y-axis is $(0, -3)$

For x-intercept, we put $y = 0$ in (6) and get $3x - 2(0) = 6 \Rightarrow x = 2$

Intersection point of line and x-axis is (2, 0)

For origin, we put $x = 0$ and $y = 0$ in (3)

We get , $0 \leq 6$ which is true.

So, origin is a part of solution of inequality (3)

We draw the lines by plotting the above intersection points on the graph. The region bounded by the intersection of three inequalities is shaded (Fig. 9.4) which is the required solution satisfied by all the inequalities.

Since coefficient y are positive. Therefore

Note: First inequality is of type (≤) so, solution region is below the line l_1.
Second inequality is of type (≥) so, solution region is above the line l_2.
Third inequality is of type (≤) so, solution region is below the line l_3.

Exercise 9.1

1. Solve the following inequalities and represent solution on number line in each case.

(i) $2x + 5 < 9,\ \forall\, x \in \mathbb{R}$ (ii) $3x - 1 > 10, \forall\, x \in \mathbb{R}$

(iii) $4x - 5 \leq 3 + 2x, \forall\, x \in \mathbb{R}$ (iv) $6 < 3(x + 2) < 21, \forall\, x \in \mathbb{R}$

2. Draw graph of the following linear inequalities.

(i) $2x - y \leq 7$ (ii) $3x + 4y \geq 10$

(iii) $x - 2y \geq 5$ (iv) $2x + y \leq 4$

3. Solve the following system of linear inequalities graphically.

(i) $3x - y \geq 8$, $2x + 3y \geq 5$

(ii) $3x - 2y \leq 6$, $x - y \leq 4$

4. Draw graph of the solution region of the following system of linear inequalities. Also check whether the graph is bounded or not.

(i) $2x + y \leq 5$, $3x - 2y \leq 7$, $x \geq 0$

(ii) $x + 2y \leq 6$, $2x - 3y \geq 8$, $x + 2y \leq 4$

9.3 Feasible Region

In linear programming problems, a feasible region is the set of all possible points that satisfy the problem's constraints, including inequalities and equalities.

9.3.1 Define

- linear programming problem,
- objective function,
- problem constraints,
- decision variables.

(i) Linear programming problem

A linear programming problem (LP problem) is one that is concerned with finding the optimal value (maximum or minimum) of a linear function (called objective function) of several variables, subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints).

The term "linear" indicates that all the mathematical relations used in the problem are linear relations while the term "programming" refers to the method of determining a particular programme or plan of action.

(ii) Objective function

An objective function expresses the main aim of the LP problem which is either to be minimized or maximized. The objective function evaluates some quantitative criterion of immediate importance such as cost, profit, utility, or yield. In an LP problem, an objective function $Z = ax +$ by, where a, b are constants, has to be maximized or minimized subject to the given linear inequalities. To find the Maximum Z or Minimum Z, we convert the inequalities into equations.

(iii) Problem Constraints

The linear inequalities or equations or restrictions on the variables of a linear programming problem are called problem constraints. The conditions $x \geq 0$, $y \geq 0$ are called non-negative restrictions.

In LP problem, the constraints are equal or unequal mathematical sentences defining limitations on decisions. Constraints arise from a variety of sources such as limited resources, contractual obligations, or physical laws etc.

(iv) Decision variables

Decision variables describe the quantities that the decision makers would like to determine. They are the unknowns of a mathematical programming problem. In general, decision variables are represented by x, y, z etc.,. These variables are required to be non-negative.

9.3.2 Define and show graphically the feasible region (or solution space) of an LP problem

The common region determined by all the constraints including non-negative constraints $(x \geq 0, y \geq 0)$ of a linear programming problem is called the feasible region (or solution space) for the problem. Points within and on the boundary of the feasible region represent feasible solutions of the constraints. The feasible solution region on the graph is the one which is satisfied by all the constraints.

The method of showing the feasible region of an LP problem is illustrated by the following example.

Example:

Find the feasible region or solution space of the following inequalities

$5x + y \leq 100$; $\quad x + y \leq 60$; $\quad x \geq 0,\ y \geq 0$

Solution:

First, we convert the inequalities into equations as follows:

$$5x + y = 100 \quad \ldots(i)$$

$$x + y = 60 \quad \ldots(ii)$$

$$x \geq 0, y \geq 0$$

We consider equation (i)

For y-intercept, we put $x = 0$ and get $5(0) + y = 100 \Rightarrow y = 100$

Therefore, intersection point of line and y-axis is (0,100)

For x-intercept, we put $y = 0$ and get $5x + 0 = 100 \Rightarrow x = 20$

Therefore, intersection point of line and x-axis is (20, 0)

For origin, we put $x = 0$ and $y = 0$ in $5x + y \leq 100$

We get , $0 \leq 100$ which is true.

So, origin is a part of solution of inequality $5x + y \leq 100$

Now, we consider equation (ii)

For y-intercept, we put $x = 0$ and get

$0 + y = 60 \Rightarrow y = 60$

Therefore, intersection point of line and y-axis is (0, 60)

For x-intercept, we put $y = 0$ and get

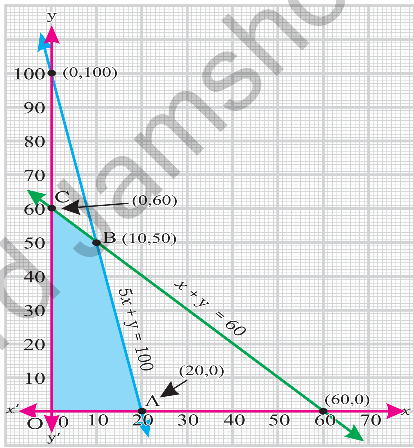
$x + 0 = 60 \Rightarrow x = 60$

Therefore, intersection point of line and x-axis is (60,0)

For origin, we put $x = 0$ and $y = 0$ in $x + y \leq 60$

We get , $0 \leq 60$ which is true.

So, origin is a part of solution of inequality $x + y \leq 60$



(Fig. 9.5)

We draw the lines by plotting the above points in the graph. The shaded region OABC formed by the intersection of these inequalities is the feasible region (or solution space) for the given inequalities as shown in Fig. 9.5.

9.3.3 Identify the feasible region of simple LP problems

The method of identification of feasible region of simple LP problems is explained by the following examples.

Example 1. Find the feasible region of the following LP problem

Maximize $Z = 2x + 3y$

subject to $2x + y \leq 100; \quad x + y \leq 80; \quad x \geq 0, y \geq 0$

Solution:

We first, consider the constraints as equations as follows:

$$2x + y = 100 \quad \ldots(i)$$

$$x + y = 80 \quad \ldots(ii)$$

$$x \geq 0, y \geq 0$$

To get the intersection points with the coordinate axes.

We consider equation (i)

For y-intercept, we put $x = 0$ and get $2(0) + y = 100 \Rightarrow y = 100$

Therefore, intersection point of line and y-axis is (0,100)

For x-intercept, we put $y = 0$ and get
$2x + 0 = 100 \Rightarrow x = 50$
Therefore, intersection point of line and x-axis is (50, 0)

(Fig. 9.6)

For origin, we put $x = 0$ and $y = 0$ in
$2x + y \leq 100$
We get , $0 \leq 100$ which is true.
So, origin is a part of solution of inequality $2x + y \leq 100$
Now, we consider equation (ii)
For y-intercept, we put $x = 0$ and get
$0 + y = 80 \Rightarrow y = 80$
Intersection point of line and y-axis is (0,80)
For x-intercept, we put $y = 0$ and get
$x + 0 = 80 \Rightarrow x = 80$
So, the intersection point of line and x-axis is $(80, 0)$
For origin, we put $x = 0$ and $y = 0$ in $x + y \leq 80$
We get , $0 \leq 80$ which is true.
So, origin is a part of solution of inequality $x + y \leq 80$
By plotting the above points and drawing the lines in the graph, the shaded region OABC formed by the intersection of these inequalities, is the feasible region for the given inequalities as shown in Fig. 9.6.

Example 2. Find the feasible region of the following LP problem

Maximize Z= 7x +3y

subject to $6x + 2y \geq 12$; $2x + 2y \geq 8$; $4x + 12y \geq 24$; $x, y \geq 0$

Solution:

Inequalities can be written in simplified form as:

$$3x + y \geq 6, \quad \ldots(i)$$
$$x + y \geq 4 \quad \ldots(ii)$$
and $$x + 3y \geq 6 \quad \ldots(iii)$$

First, we consider the inequalities (i),(ii) and (iii) as equations

$$3x + y = 6 \quad \ldots(iv)$$
$$x + y = 4 \quad \ldots(v)$$
$$x + 3y = 6 \quad \ldots(vi)$$
$$x \geq 0, y \geq 0$$

To get the intersection points with the coordinate axes:
Now, we consider equation (iv)
For y-intercept, we put $x = 0$ and get $3(0) + y = 6 \Rightarrow y = 6$
Therefore, intersection point of line and y-axis is (0,6)

For x-intercept, we put $y = 0$,

we get $3x + (0) = 6 \quad \Rightarrow \quad x = 2$

Therefore, intersection point of line and x-axis is (2, 0)

For origin, we put $x = 0$ and $y = 0$ in (i), we get $0 \geq 6$ which is not true.

So, origin is not a part of solution of inequality (i)

Now, we consider equation (v)

For y-intercept, we put $x = 0$ and get

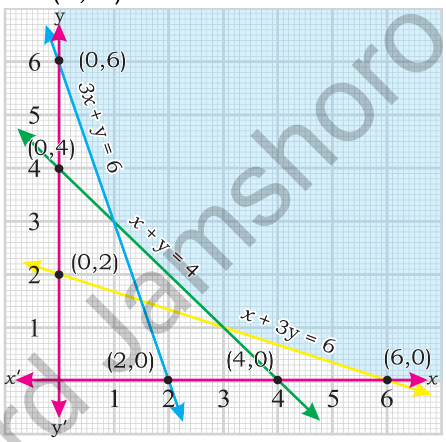
$0 + y = 4 \quad \Rightarrow \quad y = 4$

Therefore, intersection point of line and y-axis is (0,4)

For x-intercept, we put $y = 0$ and get

$x + 0 = 4 \quad \Rightarrow \quad x = 4$

Therefore, intersection point of line and x-axis is (4,0)



(Fig. 9.7)

For origin, we put $x = 0$ and $y = 0$ in (ii), and get $0 \geq 4$ which is not true.

So, origin is not a part of solution of inequality (ii)

Now, we consider equation (vi)

For y-intercept, put $x = 0$, we get $3y = 6 \quad \Rightarrow \quad y = 2$

Therefore, intersection point of line and y-axis is (0,2)

For x-intercept, we put $y = 0$ and get $x + 3(0) = 6 \quad \Rightarrow \quad x = 6$

Therefore, intersection point of line and x-axis is (6,0)

For origin, we put $x = 0$ and $y = 0$ in (iii), we get $0 \geq 6$ which is not true.

So, origin is not a part of solution of inequality (iii)

By plotting the above points and drawing the lines in the graph, the shaded region formed by the intersection of these inequalities is the feasible region (or solution space) for the given inequalities as shown in Fig. 9.7.

Example 3. Find the feasible region and also find its corner points for the following LP problem

Minimize $Z = x - 9y$

subject to $\quad 2x + 3y \leq 48; \quad x \leq 15; \quad y \leq 10; \quad x, y \geq 0$

Solution:

First, we consider the constraints as equations and find intersection points with coordinate axes as under.

Constraint 1: $\quad 2x + 3y = 48$

When $x = 0$, we get $y = 16$

and $\quad y = 0$, we get $x = 24$

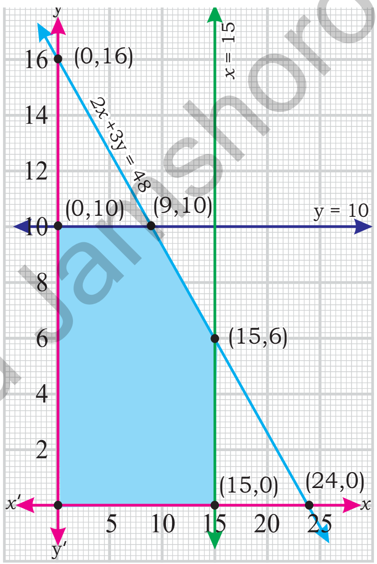
The points of intersection of line with axes are (0, 16) and (24, 0).

For origin, we put $x = 0$ and $y = 0$ in $2x + 3y \leq 48$

We get, $0 \leq 48$ which is true.

So, origin is a part of solution of inequality $2x + 3y \leq 48$

We draw the line through $(0, 16)$ and $(24, 0)$.



(Fig. 9.8)

Constraint 2: $x = 15$

The point of intersection of line with x-axis is (15, 0)

For origin, we put $x = 0$, in $x \leq 15$ we get,

$0 \leq 15$ which is true. So, origin is a part of solution of inequality $x \leq 15$

We draw the line: $x = 15$ through $(15, 0)$.

Constraint 3: $y = 10$

The point of intersection of line with y-axis is (0, 10)

For origin, we put $y = 0$, in $y \leq 10$ we get,

$0 \leq 10$ which is true. So, origin is a part of solution of inequality $y \leq 10$

We draw the line: $y = 10$ through $(0, 10)$

The shaded region as shown in Fig. 9.8 is the feasible region for the given inequalities, with the corner points (0,0), (15,0) (15,6), (9,10) and (0,10).

Exercise 9.2

1. Draw graph of the following system of linear inequalities and identify the feasible region and the corner points.

(i) $2x - 3y \leq 8$, $3x + 2y \leq 10$, $x \geq 0, y \geq 0$

(ii) $-5x + 4y \geq 10$, $2x + y \leq 8$, $x \geq 0, y \geq 0$

(iii) $4x + 3y \leq 12$, $3x - 2y \leq 9$, $x \geq 0, y \geq 0$

2. Draw graph of the following inequalities and also identify the solution space and the corner points.

(i) $5x - y \leq 16$, $3x + 2y \leq 6$, $x + 2y \leq 5$, $x \geq 0, y \geq 0$

(ii) $5x + 3y \leq 12$, $2x - 3y \leq 9$, $x + 2y \leq 4$, $x \geq 0, y \geq 0$

(iii) $6x + 5y \geq 14$, $4x + 2y \leq 10$, $3x - 2y \leq 8$, $x \geq 0, y \geq 0$

9.4 Optimal Solution

9.4.1 Define optimal solution of an LP problem

Feasible solution: A feasible solution to a linear programming problem is a solution that satisfies all constraints.

Optimal solution: An optimal solution to a linear programming problem is the feasible solution with the largest objective function value for a maximization problem and the smallest objective function value for a minimization problem.

9.4.2 Find optimal solution (graphical) through the following systematic procedure:

- establish the mathematical formulation of LP problem,
- construct the graph,
- identify the feasible region,
- locate the solution points,
- evaluate the objective function,
- select the optimal solution,
- verify the optimal solution by actually substituting values of variables from the feasible region.

In order to find the optimal solution of Linear Programming problem the above mentioned steps are to be followed as explained in the following examples.

Example 1. Find an optimal solution of the following LP problem.
Maximize the objective function $z = f(x,y) = 5x + 7y$,
Subject to the constraints $x \leq 6$; $2x + 3y \leq 19$; $x + y \leq 8$; and $x, y \geq 0$

Solution:

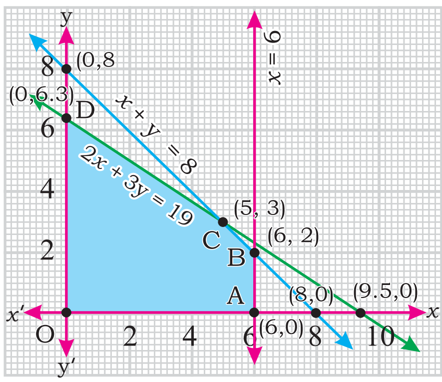
Step 1. Mathematical formulation of LP problem

The problem is already in the form of mathematical formulation

i.e., Maximize $z = 5x + 7y$

Subject to the constraints $x \leq 6$;

$2x + 3y \leq 19$; $x + y \leq 8$; and $x, y \geq 0$



(Fig. 9.9)

Step 2. Construct the graph

For the graph, first we find intersection points of the constraints with axes and draw lines.

Constraint 1: $x \le 6$

Consider the constraint as an equation i.e., $x = 6$, Draw the line for $x = 6$. The line is parallel to y-axis through $(6, 0)$.

Constraint 2: $\mathbf{2x + 3y \le 19}$

Consider the constraint as an equation

$2x + 3y = 19$

When $x = 0$, we get $y = 6.33$. and

when $y = 0$, we get $x = 9.5$.

The intersection points are (0, 6.33) and (9.5, 0). We draw line through these points.

Constraint 3: $x + y \le 8$

Consider the constraint as an equation $x + y = 8$

When $x = 0$, we get $y = 8$.

and for $y = 0$, we get $x = 8$.

The points of intersection are

(0, 8) and (8, 0).

We draw line through these points.

Step 3. Identify the feasible region

The intersection of three linear inequalities is the required feasible region OABCD which is the shaded area in the graph as shown in Fig. 9.9.

Step 4. Locate the solution points

The solution points or corner points of region OABCD are O(0,0), A (6,0), B (6,2), C(5,3) and D (0,6.33) in the graph.

Step 5. Evaluate the objective function

Solution points or Corner points	Objective function $f(x, y) = 5x + 7y$
O (0, 0)	$f(0,0) = 5(0) + 7(0) = 0 + 0 = 0$
A (6, 0)	$f(6,0) = 5(6) + 7(0) = 30 + 0 = 30$
B (6, 2)	$f(6,2) = 5(6) + 7(2) = 30 + 14 = 44$
C (5, 3)	$f(5,3) = 5(5) + 7(3) = 25 + 21 = 46$
D (0, 6.33)	$f(0,6.33) = 5(0) + 7(6.33) = 0 + 44.31 = 44.31$

Step 6. Select the Optimal Solution

From the above table, as the maximum value of the objective function is 46 at the point (5, 3). Therefore, the optimal solution to the given LP problem is: $f_{maximum} = 46$; $x = 5$, $y = 3$

Step 7. Verify the optimal solution

For the optimal solution $(5, 3)$,

1^{st} constraint $x \leq 6$ becomes, $5 \leq 6$ which is true.
2^{nd} constraint $2x + 3y \leq 19$ becomes, $19 \leq 19$ which is true.
3^{rd} constraint $x + y \leq 8$ becomes, $8 \leq 8$ which is true.
$\because$ All the constraints are satisfied by the optimal solution.
$\therefore$ it is verified.

Example 2. Find the optimal solution of the following LP problem:

Maximize $\quad z = 6x - 8y$

Subject to $\quad 30x + 20y \leq 300; \quad 5x + 10y \leq 110;$ and $\quad x, y \geq 0$

Solution:

Step 1. Mathematical formulation of LP problem

The mathematical formulation is already given as

Maximize $\quad z = 6x - 8y$

Subject to $\quad 30x + 20y \leq 300; \quad 5x + 10y \leq 110;$ and $\quad x, y \geq 0$

Step 2. Construct the graph

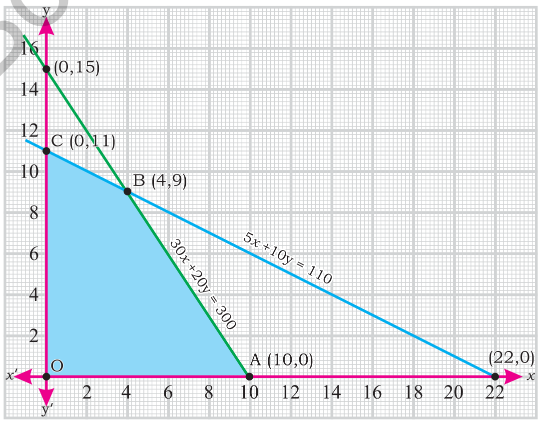
For the graph, first we find intersection points of the constraints with axes.

Constraint 1. $30x + 20y \leq 300$

We consider it as an equation $30x + 20y = 300$
When $y = 0$, we get $x = 10$
The intersection point of line with x-axis is (10, 0)
When $x = 0$, we get $y = 15$
The intersection point of line with y-axis is (0, 15)
We draw line through these points.

Constraint 2. $5x + 10y \leq 110$

We consider it as an equation
$5x + 10y = 110$
When $y = 0$, we get $x = 22$
The intersection point of line with x-axis is (22, 0)
When $x = 0$, we get $y = 11$
The intersection point of line with y-axis is (0, 11)
We draw line through these points.



(Fig. 9.10)

Step 3. Identify the feasible region

The intersection of two linear inequalities is the required feasible region OABC which is the shaded area in the graph as shown in Fig. 9.10.

Step 4. Locate the solution points

The solution points or corner points of the region OABC are O (0,0), A (10,0), B (4,9) and C (0,11) in the graph.

Step 5. Evaluate the objective function

Solution points or Corner points	Objective function $f(x,y) = 6x - 8y$
O (0, 0)	$f(0,0) = 6(0) - 8(0) = 0 - 0 = 0$
A (10, 0)	$f(10,0) = 6(10) - 8(0) = 60 - 0 = 60$
B (4, 9)	$f(4,9) = 6(4) - 8(9) = 24 - 72 = -48$
C (0, 11)	$f(0,11) = 6(0) - 8(11) = 0 - 88 = -88$

Step 6. Select the Optimal Solution

From the above table, as the maximum value of the objective function is 60 at the point (10, 0). Therefore the optimal solution to the given LP problem is: $f_{maximum} = 60$; $x = 10$, $y = 0$

Step 7. Verify the optimal solution

For the optimal solution $(10,0)$,

1st constraint $30x + 20y \leq 300$ becomes, $300 \leq 300$ which is true.

2nd constraint $5x + 10y \leq 110$ becomes, $50 \leq 110$ which is true.

∵ All the constraints are satisfied by the optimal solution.

∴ it is verified.

Example 3. Find the optimal solution of the following LP problem:

Minimize $z = 3x + 4y$

Subject to the constraints

$2x + 3y \geq 6$; $x + y \leq 8$ and $x \geq 0$, $y \geq 0$

Solution:

Step 1. Mathematical formulation of LP problem

The mathematical formulation is as already given

Minimize $Z = 3x + 4y$

Subject to $2x + 3y \geq 6$; $x + y \leq 8$ and $x \geq 0$, $y \geq 0$

Step 2. Construct the graph

For the graph, first we find intersection points of the constraints with axes.

Constraint 1. $2x + 3y \geq 6$

We consider it as an equation $2x + 3y = 6$

When $y = 0$, we get $x = 3$

The intersection point of line with x-axis is (3, 0)

When $x = 0$, we get $y = 2$

The intersection point of line with y-axis is (0, 2)

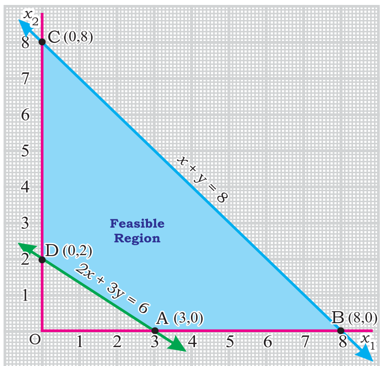
We draw line through these points.

Constraint 2. $x + y \leq 8$

We consider it as an equation $x + y = 8$

When $y = 0$, we get $x = 8$

The intersection point of line with x-axis is (8, 0)



(Fig. 9.11)

When $x = 0$, we get $y = 8$
The intersection point of line with y-axis is (0, 8)
We draw line through these points.

Step 3. Identify the feasible region

The intersection of two linear inequalities is the required feasible region ABCD which is the shaded area in the graph as shown in Fig. 9.11.

Step 4. Locate the solution points

The solution points or corner points of the region ABCD are, A (3,0), B (8,0), C (0,8) and D (0,2) in the graph.

Step 5. Evaluate the objective function

Solution points or Corner points	Objective function $Z = f(x, y) = 3x + 4y$
A (3, 0)	$f(3,0) = 3(3) + 4(0) = 9 + 0 = 9$
B (8, 0)	$f(8,0) = 3(8) + 4(0) = 24 + 0 = 24$
C (0, 8)	$f(0,8) = 3(0) + 4(8) = 0 + 32 = 32$
D (0, 2)	$f(0,2) = 3(0) + 4(2) = 0 + 8 = 8$

Step 6. Select the Optimal Solution

From the above table, as the minimum value of the objective function is 8 at the point (0, 2). Therefore the optimal solution to the given LP problem is: $f_{minimum} = 8$; $x = 0,\ y = 2$

Step 7. Verify the optimal solution

For the optimal solution $(0, 2)$,
1st constraint $2x + 3y \geq 6$ becomes, $6 \geq 6$ which is true.
2nd constraint $x + y \leq 8$ becomes, $2 \leq 8$ which is true.
∵ All the constraints are satisfied with the optimal solution.
∴ it is verified.

9.4.3 Solve real life simple LP problems

Following are some examples of real-life simple LP problems which are solved through graphical method.

Example 1. A company has two flour mills, A and B, which have different capacities for high, medium and low-grade flour. This company has to supply flour to a firm every week 12, 8 and 24 quintals. (1 quintal = 100 kg) of high, medium and low grade respectively. It costs the company Rs.1000 and Rs. 800 per day to run mill A and mill B respectively.

On a day, mill A produces 6, 2 and 4 quintals and mill B produces 2, 2 and 12 quintals of high, medium and low-grade flour respectively.

How many days per week each mill is operated in order to supply the flour to the firm most economically?

Solution:

Step 1. Mathematical formulation of LP problem

The problem can be formulated in the following tabular form:

Flour Production	Capacity		Requirement
	Mill A	Mill B	
High grade	6	2	12
Medium grade	2	2	8
Low grade	4	12	24
Cost (Rs)/day	1000	800	

Let x be the number of days per week mill A operates.

Let y be the number of days per week mill B operates.

The objective is to minimize the total cost of operation and to find the values of x and y.

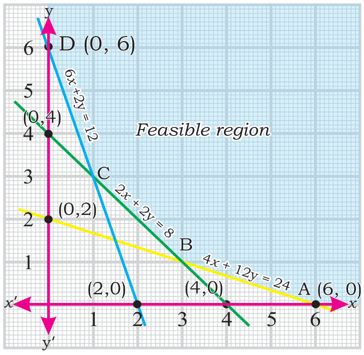
Now, we can write mathematically LP problem as follows:

Minimize $f = 1000\,x + 800\,y$

Subject to $6x + 2y \geq 12$; $2x + 2y \geq 8$; $4x + 12y \geq 24$; $x, y \geq 0$

Step 2. Construct the graph

For the graph, first we find intersection points of the constraints with axes.



(Fig. 9.12)

Constraint 1. $6x + 2y \geq 12$

We consider it as an equation $6x + 2y = 12$

when $x = 0$, we get $y = 6$ and

when $y = 0$, we get $x = 2$

The intersection points of line with axes are (0, 6) and (2, 0).

We draw line through these points.

Constraint 2. $2x + 2y \geq 8$

We consider it as an equation $2x + 2y = 8$

when $x = 0$, we get $y = 4$ and

when $y = 0$, we get $x = 4$

The intersection points of line with axes are (0, 4) and (4, 0).

We draw line through these points.

Constraint 3. $4x + 12y \geq 24$

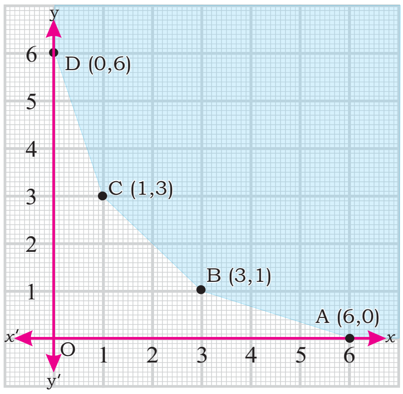
We consider it as an equation $4x + 12y = 24$

when $x = 0$, we get $y = 2$ and

when $y = 0$, we get $x = 6$

The intersection points of line with axes are (0, 2) and (6, 0).

We draw line through these points.



(Fig. 9.13)

Step 3. Identify the feasible region

The intersection of three linear inequalities is the required feasible region ABCD which is the shaded area in the graph as shown in Fig. 9.12 and Fig. 9.13.

Step 4. Locate the solution points

The solution points or corner points ABCD are A (6, 0), B (3,1), C (1,3) and D (0, 6) in the graph.

Step 5. Evaluate the objective function

Solution points or Corner points	Objective function $f(x, y) = 1000x + 800y$
A (6, 0)	$f(6,0) = 1000(6) + 800(0) = 6000 + 0 = 6000$
B (3, 1)	$f(3,1) = 1000(3) + 800(1) = 3000 + 800 = 3800$
C (1, 3)	$f(1,3) = 1000(1) + 800(3) = 1000 + 2400 = 3400$
D (0, 6)	$f(0,6) = 1000(0) + 800(6) = 0 + 4800 = 4800$

Step 6. Select the Optimal Solution

We note that the minimum cost at point C = (1, 3) is Rs. 3400. The company operates mill A for one day per week and mill B for three days per week to supply flour to a firm most economically by paying Rs. 3400.

Therefore, the optimal solution to the given LP problem is

$f_{minimum} = 3400$ Rupees; $x = 1, \; y = 3$

Step 7. Verify the optimal solution

For the optimal solution $(1, 3)$,

1st constraint $6x + 2y \geq 12$ becomes, $12 \geq 12$ which is true.

2nd constraint $2x + 2y \geq 8$ becomes, $8 \geq 8$ which is true.

3rd constraint $4x + 12y \geq 24$ becomes, $40 \geq 24$ which is true.

$\because$ All the constraints are satisfied by the optimal solution.

$\therefore$ it is verified.

Example 2. A workshop has three types of machines A, B and C; it can manufacture two products 1 and 2, and all products have to go to each machine and each one goes in the same order; First to the machine A, then to B and then to C. The following table shows:

- The hours needed at each machine, per product unit
- The total available hours for each machine, per week
- The cost of each product per unit sold

Find maximum profit under given constraints.

Type of Machine	Product 1	Product 2	Available hours per week
A	2	2	16
B	1	2	12
C	4	2	18
Profit per unit	Rs. 1	Rs. 1.5	

Solution:

Step 1. Mathematical formulation of LP problem

Let x be the units produced weekly of the Product 1
and y be the units produced weekly of the Product 2
Now, we can write mathematically LP problem as follows:

Minimize $f = x + 1.5\,y$

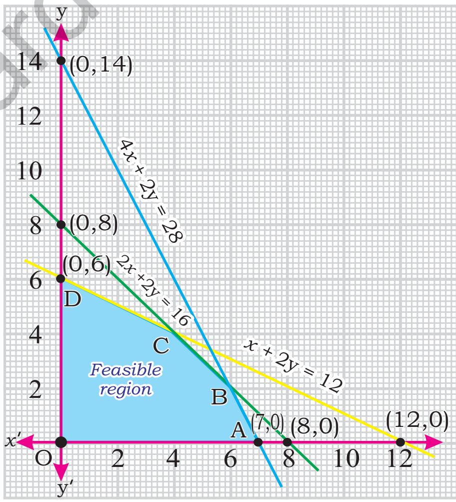
Subject to $2x + 2y \le 16$; $x + 2y \le 12$; $4x + 2y \le 28$; $x, y \ge 0$

Step 2. Construct the graph

For the graph, first we find intersection points of the constraints with axes.

Constraint 1. $2x + 2y \le 16$

We consider it as an equation $2x + 2y = 16$
when $x = 0$, we get $y = 8$ and
when $y = 0$, we get $x = 8$
The points of intersection of line with axes are (0, 8) and (8, 0).
We draw line through these points.



(Fig. 9.14)

Constraint 2. $x + 2y \le 12$

We consider it as an equation $x + 2y = 12$
when $x = 0$, we get $y = 6$ and
when $y = 0$, we get $x = 12$
The intersection points of line with axes are (0, 6) and (12, 0).
We draw line through these points.

Constraint 3. $4x + 2y \le 28$

We consider it as an equation $4x + 2y = 28$
when $x = 0$, we get $y = 14$ and
when $y = 0$, we get $x = 7$
The points of intersection of line with axes are (0,14) and (7,0). We draw line trough these points.

Step 3. Identify the feasible region

The intersection of three linear inequalities is the required feasible region OABCD which is the shaded area in the graph as shown in Fig. 9.14 and Fig. 9.15.

Step 4. Locate the solution points

In the graph, the solution points or corner points of the region OABCD are O(0,0), A (7, 0), B (6, 2), C (4, 4) and D (0, 6).

Step 5. Evaluate the objective function

Solution points or Corner points	Objective function $f(x,y) = 1x + 1.5y$
O (0, 0)	$f(0,0) \;= 1(0) + 1.5(0) = 0 + 0 \;= \;0$
A (7, 0)	$f(7,0) \;= 1(7) + 1.5(0) = 7 + 0 \;= \;7$
B (0, 8)	$f(6,2) \;= 1(6) + 1.5(2) = 6 + 3 = \;9$
C (4, 4)	$f(4,4) \;= 1(4) + 1.5(4) = 4 + 6 = \;10$
D (0, 6)	$f(0,6) \;= 1(0) + 1.5(6) = 0 + 9 = \;9$

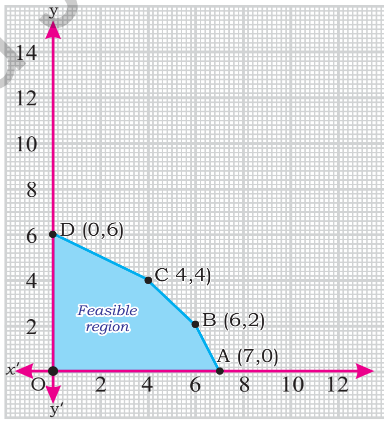
Step 6. Select the Optimal Solution

We note that the maximum profit at the point C = (4,4) is Rs. 10

Therefore, the optimal solution to the given LP problem is

$f_{maximum}$ = 10 rupees $\quad x = 4, \; y = 4$

$x = 4$ units of product 1 and $y = 4$ units of product 2.



(Fig. 9.15)

Step 7. Verify the optimal solution

For the optimal solution (4,4),

1st constraint $2x + 2y \leq 16$ becomes $16 \leq 16$ which is true.

2nd constraint $x + 2y \leq 12$ becomes, $12 \leq 12$ which is true.

3rd constraint $4x + 2y \leq 28$ becomes, $24 \leq 28$ which is true.

$\because$ All the constraints are satisfied by the optimal solution.

$\therefore$ it is verified.

Exercise 9.3

1. Solve the following linear programming problems by graphical method when $x \geq 0, y \geq 0$

(i) Maximize $Z(x,y) = 10x + 11y$

Subject to: $2x + 3y \leq 8$; $\quad 6x + 3y \leq 10$

(ii) Maximize $Z(x,y) = 30x + 36y$

Subject to: $4x + 2y \leq 12$; $\quad 6x + 5y \leq 20$

(iii) Maximize $Z(x,y) = 41x + 38y$

Subject to: $4x + 5y \leq 26$; $\quad 8x + 5y \leq 22$; $\quad 5x + 2y \leq 10$

(iii) Maximize $Z(x, y) = 41x + 38y$

Subject to: $4x + 5y \leq 26$; $8x + 5y \leq 22$; $5x + 2y \leq 10$

(iv) Maximize $Z = 4x + y$

Subject to: $x + y \leq 50$; $3x + y \leq 90$; $x - y \leq 40$

(v) Minimize $Z = 200x + 500y$

Subject to the constraints: $x + 2y \geq 10$; $3x + 4y \leq 24$

(vi) Minimize $Z = 3x + 2y$

Subject to the constraints: $2x + y \geq 6$; $x + y \geq 4$

(vii) Minimize $Z = 0.12x + 0.15y$

Subject to the constraints: $x + y \geq 5$; $2x + y \geq 6$; $x + 3y \geq 9$

2. A landlord has 50 hectares of land to grow two crops X and Y with profits per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide no more than 800 litres, is to be used for crops X and Y at rates of 20 liters and 10 liters per hectare. How much land should be allocated to each crop so as to maximize the total profit.

3. A toy factory manufactures two types of toys A and B and sells for Rs.25 and Rs.20 respectively. The toys A and B respectively, require 20 units and 12 units daily out of 2000 available resource units. Both require a production time of 5 minutes and total working hours are 9 per day. What should be the quantity for each of the toys to maximize the selling amount?

4. At a university, Professor Abdul Sattar wishes to employ two people, Farhan and Sarfaraz, to grade papers for his classes. Farhan is a graduate student and can grade 20 papers per hour; Farhan earns $15 per hour for grading papers. Sarfaraz is a post-doctoral associate and can grade 30 papers per hour; Sarfaraz earns $25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment. If Prof. Abdul Sattar has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?

Review Exercise 9

1. Select correct answer.

i. If $2x - 5 \geq 3$ then its solution set is ___________, where $x \in \mathbb{Z}$.

(a) $\{2, 3, 4, \ldots\}$ (b) $\{4, 5, \ldots\}$ (c) $\{2, 1, 0, \ldots\}$ (d) $\{1, 0, -1, \ldots\}$

ii. If $4x - 3y = 8$ is the corresponding equation of an inequality $4x - 3y \leq 8$ then its x-intercept is ________.

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) 2 (d) $\frac{1}{2}$

iii. $\{x | x \in \mathbb{R} \wedge x > 5\} =$ ___________.

(a) $[5, \infty)$ (b) $(-\infty, 5)$ (c) $(5, \infty)$ (d) $(-\infty, 5]$

iv. Linear Programming (LP) is used to obtain _______solution.

(a) Feasible (b) Trivial (c) Optimal (d) Infeasible

v. Solution space of the linear inequality $2x + 3y \leq 6, \forall x, y \in \mathbb{R}$ includes ____.

(a) All points above the line (b) All points on and below the line

(c) All points below the line (d) All points on and above the line

vi. If $f(x, y) = 5x + 4y$ is an objective function and the corner points of the feasible region are $(5,4), (0,0)$, $(4,0)$ and $(0,6)$ then the function is maximum at:

(a) (4,0) (b) (0,6) (c) $(0,0)$ (d) $(5,4)$

vii. _____ are the entities whose values are to be determined from the solution of the LP problem.

(a) Objective function (b) Decision Variables

(c) Constraints (d) Opportunity cost

viii. __________ specifies the objective or goal of solving the LP problem.

(a) Objective function (b) Decision Variables

(c) Constraints (d) Opportunity cost

ix. _______ are the restrictions or limitations imposed on the LP problem:

(a) Variables (b)Costs (c) Profits (d)Constraints

x. The region of solution in LP problem is called ____.

(a) Infeasible region (b) Unbounded region

(c) Infinite region (d) Feasible region

xi. In case of an '______' constraint, the feasible region is a straight line.

(a) Less than or equal to (b) Greater than or equal to

(c) Mixed (d) Equal to

xii. $ax + b \leq 0, \ a \neq 0, \ a, b \in \mathbb{R}$, is called ____________.

(a) Non-linear inequality (b) Linear inequality

(c) Linear equality (d) Complex inequality

xiii. A point of a solution region where two of its boundary lines intersect, is called ____________.

(a) Middle point (b) Origin

(c) Corner point (d) Feasible point

xiv. The solution region of an inequality restricted to the first quadrant is called __________ region.

(a) Combined (b) Unbounded

(c) Infeasible (d) Feasible

xv. A function which is to be maximized or minimized is called ________.

(a)Linear function (b) Equal function

(c) Objective function (d) Non linear function

xvi. The feasible solution which maximizes or minimizes the objective function is called the ___________.

(a) Optimal solution (b) Corner solution

(c) Initial solution (d) Complex solution

2. Draw graph of the following system of linear inequalities and find feasible region.

$$2x + 3y \geq 10; \quad 3x + 2y \geq 12; \quad x \geq 0, y \geq 0$$

3. Find the feasible region graphically subject to the following constraints and also find its corner points.

$$4x + y \geq 9; \quad 2x + 3y \leq 14; \quad x - y \leq 5; \quad x \geq 0, y \geq 0$$

4. Find the maximum and minimum values of the function $f(x, y) = 4x - 3y$ subject to the constraints

$$2x + 3y \geq 5; \quad 3x + 2y \leq 12; \quad x \geq 0, y \geq 0$$

5. A dealer wishes to purchase some of fans and sewing machines. He has only ¥ 5760 to invest and space of at most 20 items. The costs of a fan and a sewing machine are ¥ 360 and ¥ 240 respectively. He wants profit on a fan and a sewing machine ¥ 22 and ¥ 18 respectively. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit?