Unit 10 Differential Equations

10.1 Introduction

In previous chapters of differentiation, we discussed how to differentiate a given function f with respect to an independent variable i.e., how to find $f'(x)$ for a given function f at each x in its domain of definition. Further, in the chapter of integration, we discussed how to find a function f whose derivative is the function g, which may also be formulated as follows.

For a given function g, find a function f such that

$$\frac{dy}{dx} = g(x) \quad \text{where} \quad y = f(x) \qquad \text{... (i)}$$

An equation of the form (i) is known as differential equation. It is defined as an equation containing the derivatives of one or more dependent variables with respect to one independent variables.

10.1.1 Define ordinary differential equation (DE), order of a DE, degree of a DE, solution of a DE – general solution and particular solution

An equation involving derivatives ordinary derivatives of one or more dependent variables with respect to a single independent variable is called ordinary differential equation.

(i) $\dfrac{dy}{dx} + 5y = e^x$ (ii) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 6y = 0$

(iii) $\dfrac{dx}{dt} + \dfrac{dy}{dt} = 2x + y$ (iv) $\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 + xy = 0$

Order and degree of the differential equation

The order of a differential equation is the order of the highest derivative appearing in it.

The degree of the differential equation is the degree of the highest order derivative occurring in it, after the equation has been expressed in a form free from radicals and non-integer powers of derivatives.

Solution of differential equation

A solution of a differential equation is a relation between the variables free from derivatives, such that this relation and the derivatives obtained from it satisfies the given differential equation.

Example: Find the order and degree of following differential equation.

(i) $\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^3 + x = e^x$

The order of differential equation is 2 and its degree is 1.

(ii) $\left(\frac{d^3y}{dx^3}\right)^2 + 3\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + y = x$

The order of differential equation is 3 and its degree is 2.

(iii) $\left(\frac{dy}{dx}\right)^3 = \sqrt{1 + \frac{d^2y}{dx^2}}$

The equation contains fraction power of derivative. First, we reduce it into integer power by squaring the whole equation.

We get

$$\left(\frac{dy}{dx}\right)^6 = 1 + \frac{d^2y}{dx^2}$$

Now the order of differential equation is 2 and its degree is 1.

(iv) $\frac{d^2y}{dx^2} = 2y^{\frac{1}{3}}$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)^3 = 2y$$

order of differential equation is 2 and its degree is 3.

Example 1. Show that $y = Ae^{2x}$ is the solution of differential equation $\frac{dy}{dx} - 2y = 0$.

Solution: The given differential equation is

$$\frac{dy}{dx} - 2y = 0 \qquad \ldots(i)$$

Now, to verify $y = Ae^{2x}$ is the solution of the differential equation. We differentiate y w.r.t x

$$\frac{dy}{dx} = 2Ae^{2x}$$

Now by substituting the values of y and $\frac{dy}{dx}$ in equation (i), we get

$$2Ae^{2x} - 2Ae^{2x} = 0$$
$$0 = 0$$

Hence $y = Ae^{2x}$ is the solution of differential equation $\frac{dy}{dx} - 2y = 0$.

Example 2. Show that $y = A\sin x + B\cos x$ is the solutions of differential equation $\frac{d^2y}{dx^2} + y = 0$.

Verification: The given differential equation is

$$\frac{d^2y}{dx^2} + y = 0 \qquad \ldots(i)$$

Now to verify, $y = A\sin x + B\cos x$ is the solution of the given differential equation,

We differentiate y w.r.t x

We have

$$\frac{dy}{dx} = \frac{d}{dx}(A\sin x + B\cos x)$$

$$\frac{dy}{dx} = A\cos x - B\sin x$$

Again, differentiate w.r.t x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(A\cos x - B\sin x)$$

$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$$

Now by substituting the values of y and $\frac{d^2y}{dx^2}$ in equation (i), we get

$$-A\sin x - B\cos x + (A\sin x + B\cos x) = 0$$

$$-A\sin x - B\cos x + A\sin x + B\cos x = 0$$

$$0 = 0$$

Hence, $y = A\sin x + B\cos x$ is the solution of differential equation $\frac{d^2y}{dx^2} + y = 0$.

General and Particular Solution:

The general solution (complete solution) of a differential equation is the one in which the number of arbitrary constants is equal to the order of the differential equation.

A solution obtained from the general solution by giving particular values to the arbitrary constants is called particular solution.

For example, the differential equation $\frac{d^2y}{dx^2} + y = 0$ has the general solutions $y = A\sin x + B\cos x$ whose A & B are arbitrary constants. When we assign fixed values to arbitrary constants according to given condition. For example, at $y(0) = 1$ and $y'(0) = 2$, we get $A = 2$ and $B = 1$, then the solution will be $y = 2\sin x + \cos x$ known as particular solution.

Example 1. Verify that $y = Ae^x + Be^{2x}$ is the solution of differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$. Also, find the particular solution of the equation satisfying the conditions $y(0) = 1$ and $y'(0) = -1$.

Solution: Since $y = Ae^x + Be^{2x}$

$$\frac{dy}{dx} = Ae^x + 2Be^{2x}$$

$$\frac{d^2y}{dx^2} = Ae^x + 4Be^{2x}$$

Putting the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in given differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

We get $Ae^x + 4Be^{2x} - 3Ae^x - 6Be^{2x} + 2Ae^x + 2Be^{2x} = 0$ ...(i)

$0 = 0$ Hence proved.

To find the particular solution we use the given conditions:

$y(0) = 1$ (mean when $x = 0, y = 1$)

and $y'(0) = -1$ (means when $x = 0, \frac{dy}{dx} = -1$)

$y(0) = 1 \quad \Rightarrow \quad 1 = Ae^0 + Be^0$

$\Rightarrow \quad A + B = 1$ ...(i)

$y'(0) = -1 \quad \Rightarrow \quad -2 = Ae^0 + 2Be^0$

$\Rightarrow \quad A + 2B = -1$ ...(ii)

Solving (i) and (ii) we get $A = 3, B = -2$.

Hence the particular solution of given differential equation is $y = 3e^x - 2e^{2x}$.

10.2 Formation of differential Equation

10.2.1 Demonstrate the concept of formation of a differential equation

If the relation between the dependent variable and independent variable involves some arbitrary constants, we can form a differential equation by eliminating arbitrary constants from the relation by differentiating with respect to the independent variable successively as many times as the number of arbitrary constants. We illustrate by the following examples.

Example: Form the differential equation

(a) $y = A\sin 2x + B\cos 2x$ (b) $y = C_1e^x + C_2e^{-x}$

(c) $y = x + Ce^x$ (d) $x^2 + y^2 = r^2$

Solution: (a) $y = A\sin 2x + B\cos 2x$ ...(i)

As there are two arbitrary constants, so we differentiate two times. Differentiate (i) with respect to x

$$\frac{dy}{dx} = A\cos 2x\,(2) - B\sin 2x\,(2) \quad \text{...(ii)}$$

Again, differentiate with respect to x

$$\frac{d^2y}{dx^2} = -A\sin 2x\,(4) - B\cos 2x\,(4)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -4\big(A\sin 2x + B\cos 2x\big)$$

$$\frac{d^2y}{dx^2} = -4y \qquad \because y = A\sin 2x + B\cos 2x$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

Solution: (b) $y = C_1e^x + C_2e^{-x}$ ...(i)

As there are two arbitrary constants, so we differentiate two times. Differentiate (i) with w.r.t x

$$\frac{dy}{dx} = C_1e^x - C_2e^{-x} \qquad \text{...(ii)}$$

Again, differentiate w.r.t x

$$\frac{d^2y}{dx^2} = C_1e^x - C_2e^{-x}\big(-1\big) = c_1e^x + c_2e^{-x} \qquad \text{...(iii)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = y \qquad \text{(Using i)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - y = 0 \text{ is required differential equation.}$$

Solution: (c) $y = x + Ce^x$ ...(i)

As there are only one constant, so we differentiate one time. Differentiate (i) with w.r.t x

$$\frac{dy}{dx} = 1 + Ce^x \qquad \text{...(ii)}$$

Now by using equation (i)

$$\frac{dy}{dx} = 1 + y - x$$

This is the required differential equation.

Solution: (d) $x^2 + y^2 = r^2$ ...(i)

Since there is only one arbitrary constant, so we differentiate one time. Differentiate (i) w.r.t x

$$\frac{d}{dx}(x)^2 + \frac{d}{dx}(y)^2 = \frac{d}{dx}(r^2)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

which is the required differential equation.

Exercise 10.1

1. Find the order and degree of each of the following differential equation.

(i) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + xy = 0$ (ii) $x^3dx + y^3dy = 0$

(iii) $\frac{dy}{dx} = \sqrt[3]{\left(\frac{d^2y}{dx^2} + 1\right)^2}$ (iv) $\frac{d^3y}{dx^3} - 5\left(\frac{d^2y}{dx^2}\right)^3 + 7\left(\frac{dy}{dx}\right)^8 = 0$

(v) $\frac{d^4y}{dx^4} - \left(\frac{dy}{dx}\right)^{\frac{1}{2}} = 0$

2. Show that $y = x - x\ln x$ is the solution of the differential equation $x\frac{dy}{dx} + x - y = 0$.

3. Show that $y = Ae^{2x} + Be^{3x}$ is the general solution of $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

4. Obtain the differential equation by eliminating arbitrary constant from the relation.

(i) $y = A\cos x + B\sin x$ (ii) $y = A\sin(x+1)$

(iii) $y = ax^2 + bx$ (iv) $y = C_1e^x + C_2e^{-2x}$

5. Find the particular solution of:

(i) $\frac{dy}{dx} = \frac{\sqrt{1+\cos y}}{\sin y}, y(3) = \frac{\pi}{2}$, given that $x + 2\sqrt{1+\cos y} + c = 0$ is the general solution of the differential equation.

(ii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$, $y(0) = 1, y'(0) = 3$, given that $y = Ae^{2x} + Be^{-x}$ is the general solution of the differential equation.

10.3 Solution of Differential Equation

10.3.1 Solve differential equations of first order and first degree of the form:

- **separable variables,**
- **homogeneous equations,**
- **equations reducible to homogeneous form.**

(i) Separable variables

If the differential equation $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ …(i)

Where $f(x)$ is the function of x only and $g(y)$ is the function of y only, then we can write equation (i) as below

$$g(y)dy = f(x)dx$$

The equation is in separable variable form. To find the solution we integrate both sides,

i.e., $\int g(y)dy = \int f(x)dx + C$

Where C is an arbitrary constant, called constant of integration.

Example 1. $\dfrac{dy}{dx} - x^2 \cos^2 y = 0$

Solution: We have $\dfrac{dy}{dx} - x^2 \cos^2 y = 0$

$$\frac{dy}{dx} = x^2 \cos^2 y$$

By separable variable, We have

$$\frac{dy}{\cos^2 y} = x^2 dx$$

$\Longrightarrow$ $\sec^2 y\, dy = x^2 dx$

Integrate both sides

$$\int \sec^2 y\, dy = \int x^2 dx$$

$$\tan y = \frac{x^3}{3} + C$$

or $y = \tan^{-1}\left(\dfrac{x^3}{3} + C\right)$ is the general solution of differential equations.

Example 2. $\dfrac{dy}{dx} = 1 + x + y + xy$

Solution: We have $\dfrac{dy}{dx} = 1 + x + y + xy$

or $\dfrac{dy}{dx} = (1 + x) + y(1 + x)$

$$\frac{dy}{dx} = (1 + x)(1 + y)$$

separating the variables,

$$\frac{dy}{1 + y} = (1 + x)dx$$

By integrating both sides

$$\int \frac{dy}{1 + y} = \int (1 + x)dx$$

$\ln(1 + y) = x + \dfrac{x^2}{2} + C$ is the general solution of given differential equation.

Example 3. Solve $\frac{dy}{dx} = e^{x-y}$; $y(0) = 2$

Solution: The given equation can be written as

$$\frac{dy}{dx} = e^x \cdot e^{-y}$$

By separating the variables

$$\frac{dy}{e^{-y}} = e^x dx \quad \Longrightarrow \quad e^y dy = e^x dx$$

Integrating both sides

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C \qquad \ldots\text{(i)}$$

Apply $y(0) = 2$, $y = 2$ where $x = 0$, we get $e^2 = e^0 + C$

$$e^2 - 1 = C$$

Substituting the values of C in equation (i)

Hence particular solution is $e^y = e^x + e^2 - 1$.

(ii) Homogenous Differential Equation

Before going to discuss the definition of homogeneous differential equation, first we define homogeneous function.

A function $f(x, y)$ is said to be homogeneous function of degree n if it can be expressed in the form.

$$f(x, y) = x^n f\left(\frac{y}{x}\right)$$

or $\quad f(\lambda x, \lambda y) = \lambda^n f(x, y)$

For example, let $f(x, y) = \frac{x^3+y^3}{x^2-y^2}$

Replacing x *by* λx and y *by* λy

$$f(\lambda x, \lambda y) = \frac{\lambda^3 x^3 + \lambda^3 y^3}{\lambda^2 x^2 - \lambda^2 y^2} = \frac{\lambda^3(x^3 + y^3)}{\lambda^2(x^2 - y^2)} = \lambda\, f(x, y)$$

Thus $f(x, y)$ is homogeneous function of degree 1.

Homogeneous differential equation: A differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is said to be homogeneous differential equation, if $f(x, y)$ and $g(x, y)$ are the homogenous functions of the same degree in x and y. For example, $\frac{dy}{dx} = \frac{x^2+y^2}{x^2-y^2}$ is homogeneous differential equation.

To solve the homogeneous differential equation, we reduce it into the separable variable form by putting

$$y = vx \quad \Longrightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx},$$ where v is a new variable.

Example 1. Solve $(x^2+y^2)dx-2xydy=0$

Solution: We have $(x^2+y^2)dx-2xydy=0$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{x^2+y^2}{2xy}=\frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x^2\left(\frac{2y}{x}\right)} \qquad \ldots(i)$$

Thus, given differential equation is homogeneous differential equation.

Let $y=vx$

Differentiate w.r.t x

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

So equation (i) becomes

$$v+x\frac{dv}{dx}=\frac{x^2+v^2x^2}{2x(vx)}=\frac{x^2(1+v^2)}{2x^2v}=\frac{1+v^2}{2v}$$

$$x\frac{dv}{dx}=\frac{1+v^2}{2v}-v=\frac{1+v^2-2v^2}{2v}=\frac{1-v^2}{2v}$$

$$x\frac{dv}{dx}=\frac{1-v^2}{2v}$$

By separating the variable and then integrating

$$\int\frac{2v}{1-v^2}dv=\int\frac{dx}{x}$$

$$-\ln(1-v^2)=\ln x+d$$

$$\Rightarrow \qquad -\ln(1-v^2)=\ln x+\ln C \qquad \text{where } d=\ln C$$

$$\ln(1-v^2)^{-1}=\ln(Cx)$$

$$\Rightarrow \qquad (1-v^2)^{-1}=Cx$$

Replacing v by $\frac{y}{x}$

$$\left(1-\frac{y^2}{x^2}\right)^{-1}=Cx$$

$$\left(\frac{x^2-y^2}{x^2}\right)^{-1}=Cx$$

$$\Rightarrow \qquad \frac{x^2}{x^2-y^2}=Cx$$

$$\Rightarrow \qquad x^2=Cx(x^2-y^2)$$

$$\Rightarrow \qquad x=C(x^2-y^2)$$

This is the general solution of given differential equation.

Example 2. Solve $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$

Solution: First we check the given differential equation is homogeneous or not.

Here, $f(x) = \dfrac{y}{x} + \tan\dfrac{y}{x}$

$\Longrightarrow \quad f(tx, ty) = \dfrac{ty}{tx} + \tan\dfrac{ty}{tx}$

$\because \quad f(x, y) = \dfrac{y}{x} + \tan\dfrac{y}{x}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$ is homogeneous differential equation.

$\Longrightarrow \quad f(tx, ty) = f(x, y)$

Let $\quad y = vx$

$\Longrightarrow \quad \dfrac{y}{x} = v$

Differentiate w.r.t x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Given equation becomes

$$v + x\frac{dv}{dx} = v + \tan v$$

$$x\frac{dv}{dx} = \tan v$$

By separation the variables and integrating

$$\int \frac{dv}{\tan v} = \int \frac{dx}{x}$$

$$\int \cot v\, dv = \ln x + \ln c$$

$$\ln \sin v = \ln(c)$$

$\Longrightarrow \quad \sin v = cx$

Replacing v by $\dfrac{y}{x}$

$$\sin\frac{y}{x} = cx$$

$$\frac{y}{x} = \sin^{-1}(cx)$$

$$y = x\sin^{-1}(cx)$$

(iii) Equations reducible to homogeneous form

The differential equation of the form $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is not homogenous, but can be reduced to the homogeneous form, when $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

- $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ for the differential equation $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$,

We put $x = X + h,\ y = Y + k$

$dx = dX$ and $dy = dY$

Then given differential equation becomes homogeneous and then we reduce it into separable variable form. We explain the method by the following example.

Example: $\frac{dy}{dx} = \frac{x-2y+2}{2x+y-1}$.

Solution: Here $\frac{a_1}{a_2} = \frac{1}{2}$ and $\frac{b_1}{b_2} = \frac{-2}{1} \quad \Rightarrow \quad \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

So, we put $x = X + h,\ y = Y + k$

$dx = dX \ \ , dy = dY$

Given equation becomes'

$$\frac{dY}{dX} = \frac{(X+h)-2(Y+k)+2}{2(X+h)+(Y+k)-1} = \frac{X+h-2Y-2k+2}{2X+2h+Y+k-1}$$

$$\frac{dY}{dX} = \frac{(X-2Y)+(h-2k+2)}{(2X+Y)+(2h+k-1)} \qquad \ldots(i)$$

To convert the equation (i) into homogeneous we assume

Let $h - 2k + 2 = 0$ and $2h + k - 1 = 0$

$\Rightarrow$ $h = 0$ and $k = 1$.

Now (i) becomes

$$\frac{dY}{dX} = \frac{X-2Y}{2X+Y}$$

Put $Y = VX \quad \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$

$$V + X\frac{dV}{dX} = \frac{X-2VX}{2X+VX} = \frac{X(1-2V)}{X(2+V)} = \frac{1-2V}{2+V}$$

$$X\frac{dV}{dX} = \frac{1-2V}{2+V} - V$$

$$X\frac{dV}{dX} = \frac{1-2V-2V-V^2}{2+V} = \frac{-(V^2+4V-1)}{2+V}$$

Separating the variable and integrating

$$\int \frac{2+V}{V^2+4V-1}dV = -\int \frac{dX}{X}$$

$$\frac{1}{2}\int \frac{2V+4}{V^2+4V-1}dV = -\ln X + \ln C$$

$$\frac{1}{2}\ln(V^2+4V-1)=\ln\left(\frac{C}{X}\right)$$

$$\ln\sqrt{V^2+4V-1}=\ln\left(\frac{C}{X}\right)$$

Replace V by $\frac{Y}{X}$

$$\Rightarrow \quad \sqrt{\frac{Y^2}{X^2}+\frac{4Y}{X}-1}=\frac{C}{X}$$

$$\Rightarrow \quad \sqrt{\frac{Y^2+4XY-X^2}{X^2}}=\frac{C}{X}$$

$$\Rightarrow \quad \sqrt{Y^2+4XY-X^2}=C$$

$$\Rightarrow \quad Y^2+4XY-X^2=C^2$$

Replace $\quad X=x-h=x-0=x$ and $\quad Y=y-k=y-1$

$$(y-1)^2+4x(y-1)-(x)^2=C$$

$$y^2-2y+1+4xy-4x-x^2=C$$

This is the general solution of given differential equation.

- When $\quad \dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{1}{m}$.

then $\quad \dfrac{dy}{dx}=\dfrac{a_1x+b_1y+c_1}{m(a_1x+b_1y)+c_2}$

We put $z=a_1x+b_1y$ and $\dfrac{dz}{dx}=a_1+b_1\dfrac{dy}{dx}$

$$\Rightarrow \quad \frac{1}{b}\left(\frac{dz}{dx}-0\right)=\frac{dy}{dx}=\frac{z+c_1}{m(z)+c_2}$$

Then the given differential equation reduced to separable variable form. We explain by the following example.

Example: Solve $\dfrac{dy}{dx}=\dfrac{x-y-1}{x-y-5}$.

Solution: Here, $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=1$, so, we put $z=x-y$

$$\frac{dz}{dx}=1-\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx}=1-\frac{dz}{dx}$$

Given equation becomes

$$1-\frac{dz}{dx}=\frac{z-1}{z-5} \quad \Rightarrow \quad -\frac{dz}{dx}=\frac{z-1}{z-5}-1$$

$$-\frac{dz}{dx}=\frac{z-1-z+5}{z-5}=\frac{4}{z-5}$$
$$\frac{dz}{dx}=-\frac{4}{(z-5)}$$

It is in separable variable form, so separating the variables and then integrating

$$\int (z-5)\,dz=\int -4dx$$
$$\frac{(z-5)^2}{2}=-4x+d$$

Replace z by $x-y$

We have $(x-y-5)^2=-8x+2d$

$$x^2-2xy+y^2-10x+10y+25=4x+c$$
$$x^2-2xy+y^2-14x+16y+25=c$$

is required solution.

10.3.2 Solve real life problems related to differential equation

Example 1. If the population of a certain town doubles in 10 years, in how many years will it triple. Under the assumption that the rate of increase in population is proportional to the number of inhabitants.

Solution: Let y denote the population at time t years and y_0 at time $t=0$.

According to the given condition $\frac{dy}{dt}\propto y \implies \frac{dy}{dt}=ky$ where k is the constant.

Separating the variables and integrating

$$\int\frac{dy}{y}=\int k\,dt$$

$\ln y=kt+C$ ...(i)

Apply $t=0$ and $y=y_0$, we get $\ln y_0=C$

$\ln y=kt+\ln y_0$ ...(ii)

Apply $t=10,\ y=2y_0$ (Double given)

$\ln 2y_0=10k+\ln y_0$

$\ln 2y_0-\ln y_0=10k \implies \ln\left(\frac{2y_0}{y_0}\right)=10k$

$\frac{ln2}{10}=k \implies k=0.06931$

Hence $\ln y=0.06931\,t+\ln y_0$ ...(iii)

To find time t for triple population, we put $y=3y_0$

$\ln 3y_0=0.06931\,t+\ln y_0$

$$\ln 3y_0 - \ln y_0 = 0.06931\, t$$

$$\ln\left(\frac{3y_0}{y_0}\right) = 0.06931t \quad \Rightarrow \quad t = \frac{\ln 3}{0.06931} = 15.85$$

Hence the population will be triple in 15.85 years.

Example 2. According to Newton's law of cooling, the rate at which a substance cools in air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 300K and the substance cools from 370K to 340K in 15 minutes. Find the time when the temperature will be 310K.

Solution: Let T be the temperature of the substance at the time t minutes.

Then, $$\frac{dT}{dt} \propto (T - 300)$$

$$\frac{dT}{dt} = -k(T - 300) \quad \Rightarrow \quad \frac{dT}{T - 300} = -kdt$$

Integrating with the given limit $t = 0$, when $T = 370$ and $t = 15$, when $T = 340$

$$\int_{370}^{340} \frac{dT}{T - 300} = -k\int_{0}^{15} dt$$

$$[\ln(T - 300)]_{370}^{340} = [-kt]_0^{15}$$

$$\ln 40 - \ln 70 = -15k$$

Now, $\Rightarrow$ $$\ln\left(\frac{7}{4}\right) = 15k$$

$\Rightarrow$ $$15k = 0.56 \quad \Rightarrow \quad k = 0.0373$$

$$\int_{370}^{310} \frac{dT}{T - 300} = -k\int_{0}^{t} dt$$

$$\ln(T - 300) \int_{370}^{310} = -kt$$

$$\ln 10 - \ln 70 = -kt \quad \Rightarrow \quad \ln 7 = 0.0373t$$

$\Rightarrow$ $$t = 52.2 \text{ min}$$

Example 3. A capacitor of 0.1 farads and a resistor of 10 ohm are connected in series with 100 volts battery. Assume that there is no charge and current in the circuit initially. Find the charge and current in the circuit at any time.

Solution: Potential difference at resistor

$$V = IR \qquad \text{(by ohm law)}$$

$$V = 10I$$

Potential difference at capacitor

$$Q = CV \quad \Rightarrow \quad V = \frac{Q}{0.1} = 10Q$$

By Kirchhoff's law

$$10I + 10Q = 100 \qquad \left(\because I = \frac{dQ}{dt}\right)$$

$$\frac{dQ}{dt} + Q = 10$$

$$\Rightarrow \quad \frac{dQ}{dt} = (10 - Q)$$

Separating the variables and integrating

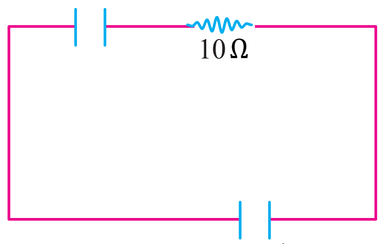


Fig. 10.1

$$\int \frac{dQ}{Q-10} = -\int dt$$

$$\ln|Q - 10| = -t + C$$

Apply $t = 0$, when $Q = 0$, we get

$$C = \ln|-10|$$

Hence $\ln(Q - 10) = -t + \ln|-10|$

$$\ln\left|\frac{Q-10}{-10}\right| = -t$$

$$\Rightarrow \quad \frac{Q-10}{-10} = e^{-t}$$

$$Q - 10 = -10e^{-t}$$

$$\Rightarrow \quad Q = 10 - 10e^{-t} = 10(1 - e^{-t})$$

Charge on the capacitor at any time.

To find current, differentiate Q w.r.t time

$$I = \frac{dQ}{dt} = 0 - 10e^{-t}(-1) = +10e^{-t}$$

$$I = 10e^{-t}$$

Example 4. A ball is thrown upward vertically with velocity 49 m/s. Find (i) the time when the body at maximum height (ii) Find the height with $t = 3$ sec (iii) Find the maximum height.

Solution: $g = -9.8\ m/s^2$

$$\frac{d^2h}{dt^2} = -9.8$$

Integrating both sides

$$\Rightarrow \quad \frac{dh}{dt} = -9.8t + c_1$$

$$\text{V} = \frac{dh}{dt} = -9.8t + c_1$$

Put $t = 0$, when $V = 49 \quad \Rightarrow \quad c_1 = 49$

$$V = -9.8t + 49 \quad \ldots(i)$$

$$\frac{dh}{dt} = -9.8t + 49$$

Integrating both sides

$$\Rightarrow \quad h = -9.8\frac{t^2}{2} + 49t + c_2$$

$t = 0, h = 0 \quad \Rightarrow \quad c_2 = 0$

$$h = -4.9t^2 + 49t \quad \ldots(ii)$$

(i) At maximum height V = 0 put in equation (i)

$0 = -9.8t + 49 \quad \Rightarrow \quad t = 5$ second

(ii) Put $t = 3$ in equation (ii) height

$h = -4.9(3)^2 + 49 = 102.9\, m$

(iii) Put $t = 5$ in equation (ii)

$h = -4.9(25) + 49(5) = 122.5\, m$

Exercise 10.2

1. Solve the following differential equation by separating the variables.

(i) $x \tan y\, dy = dx$ (ii) $x \sin y\, dx + (x^2 + 1) \cos y\, dy = 0$

(iii) $y(1 + x)dx + x(1 + y)dy = 0$ (iv) $(1 + x^3)dy - x^2dx = 0$

(v) $xy\, dy = (y + 1)(1 - x)dx$ (vi) $\frac{dy}{dx} = 3y^2 - y^2 \sin x$

(vii) $(y^2 - 1)\frac{dy}{dx} = 4xy^2$ (viii) $x \cos^2 y\, dx + \tan y\, dy = 0$

2. Solve the following homogenous differential equation.

(i) $(x + y)dy - (x - y)dx = 0$ (ii) $(6x^2 + 2y^2)dx - (x^2 + 4xy)dy = 0$

(iii) $(x^2 + 3y^2)dx - 2xy\, dy = 0$ (iv) $(x^2 + y^2)dx - 2xy\, dy = 0$

(v) $\frac{dy}{dx} = \left(\frac{y}{x}\right) + \sin\left(\frac{y}{x}\right)$

3. Solve the following differential equation.

(i) $\frac{dy}{dx} = \frac{x+y+1}{x-y}$ (ii) $(2x + y + 1)\, dx + (2x + y - 1)dy = 0$

(iii) $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ (iv) $\frac{dy}{dx} = \frac{2x+y-2}{2x+y+3}$

4. A body moves in a straight line, so that its velocity exceeds by 2 its distance from a fixed point of the line. If $V = 5$ m/s when $t = 0$, find the equation of the motion.

5. When the temperature of the air is 290 K a certain substance cools from 400 K to 350 K in 20 minutes. Find
   (i) the temperature after 40 minutes
   (ii) After how much time temperature is $300°C$
6. A resistor of 5 ohms and a capacitor of 0.02 farads are connected with 10 volts battery. Assume that initially charge on capacitor is 5 (coulombs). Find the charge and current in the circuit at any time.
7. The population of a certain town is directly proportional to the square root of the present population at any time. If the population initially is 20000.
   (i) How much the population after 10 years?
   (ii) After how much time the population be doubled?

10.4 Orthogonal Trajectories

10.4.1 Define and find orthogonal trajectories (rectangular coordinates) of the given family of curves.

Any family of curves $\emptyset(x, y, c) = 0$ which cuts every member of a given family of curves $f(x, y, c) = 0$ at right angles, is called an orthogonal trajectory of the given family.

For example, family of straight lines passes through origin $y = mx$ cuts every circle whose centre is at origin $x^2 + y^2 = r^2$. Hence straight right line passes through origin is orthogonal trajectory of family of circles whose centre is at origin. As shown in the figure 10.2.

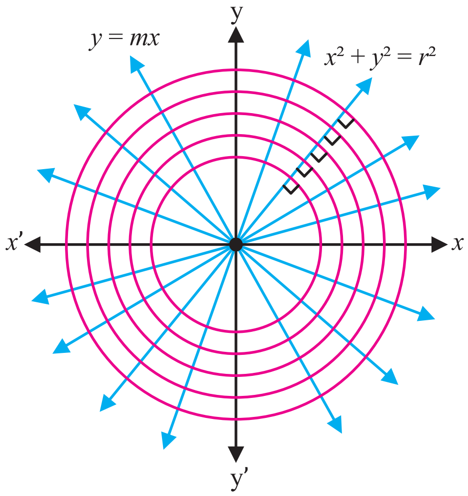


Fig. 10.2

Procedure to find orthogonal trajectories.

Step 1: Let $f(x, y, c) = 0$ be the equation of given family of curves. Where c s an arbitrary constant.

Step 2: Form the differential equations of the given family of curves.

Step 3: Substitute $-\dfrac{dx}{dy}$ for $\dfrac{dy}{dx}$ in equation obtained in step 2.

Step 4: Solve the differential equation obtained from step 3.

Example 1. Find the orthogonal trajectory of family of straight lines passing through the origin.

Solution: Family of straight line passing through the origin is $y = mx$.

Where m is an obituary constant.

$y = mx$ ...(i)

Differentiating w.r.t x

We get

$\frac{dy}{dx} = m$ ...(ii)

Eliminating 'm' from equation (i) and equation (ii), we get

$\frac{dy}{dx} = \frac{y}{x}$ ...(iii)

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$y\,dy = -x\,dx$$

Integrating on both sides

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$y^2 + x^2 = 2d$$

$$y^2 + x^2 = r^2 \quad [\text{Assume } r^2 = 2d]$$

$$x^2 + y^2 = r^2$$

Which is orthogonal trajectory.

Example 2. Find the orthogonal trajectories of the curves $xy = c$.

Solution: The equation of the given family of curves is $xy = c$ ...(i)

Differentiating equation (i) w.r.t x,

We get

$$x\frac{dy}{dx} + y = 0$$

$\frac{dy}{dx} = \frac{-y}{x}$ ...(ii)

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{-y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx$$

Integrating on both sides

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\frac{y^2}{2} - \frac{x^2}{2} + c$$

$$\Rightarrow y^2 - x^2 = 2c$$

Which is required orthogonal trajectory.

Example 3. Find the orthogonal trajectories of the circles $x^2 + y^2 - ay = 0$ where a is a parameter.

Solution: Here, $x^2 + y^2 - ay = 0$ ...(i)

is the given family of curves.

Differentiating equation (i) w.r.t to x,

We get

$$2x + 2y\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(2y - a\right) = -2x \qquad \text{...(ii)}$$

Eliminating 'a' from equation (i) and equation (ii), we get

$$\frac{dy}{dx}\left(2y - \frac{x^2+y^2}{y}\right) = -2x \qquad \text{...(iii)}$$

$$\frac{dy}{dx}(y^2 - x^2) = -2xy$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in equation (iii)

$$-\frac{dx}{dy} = \frac{-2xy}{y^2 - x^2}$$

$$\Rightarrow \quad -\frac{dy}{dx} = \frac{y^2-x^2}{2xy} \qquad \text{...(iv)}$$

It is homogeneous differential equation.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (iv) becomes

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2vx^2}$$

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$

$$x\frac{dv}{dx} = -\frac{(1 + v^2)}{2v}$$

By separating the variable

$$\frac{2v}{1 + v^2} = -\frac{dx}{x}$$

Integrating on both sides, we get

$$\ln|1 + v^2| = -\ln|x| + \ln c$$

$$\ln|1 + v^2| + \ln|x| = \ln c$$

$$\Longrightarrow \qquad x(1 + v^2) = c$$

Replacing v by $\frac{y}{x}$

$$x\left(1 + \frac{y^2}{x^2}\right) = c$$

$$x(x^2 + y^2) = x^2c$$

$$x^3 + y^2x - x^2c = 0$$

Which is the required equation of orthogonal trajectory.

10.4.2 Use MAPLE graphic commands to view the graphs of given family of curves and its orthogonal trajectories

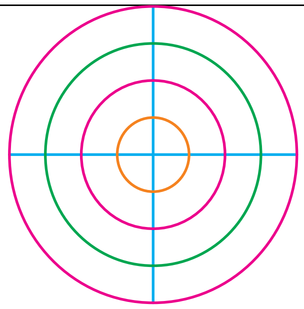
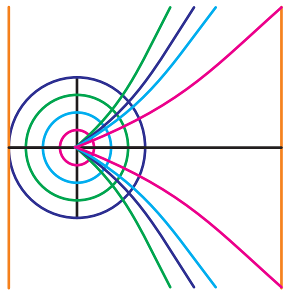
To view the graphs of given family of curves and its orthogonal trajectories following steps are to be considered:

Steps for orthogonal families of curves:

1. Differentiate the implicit function
2. Eliminate constant k of the function from the differential equation, slop of the original family
3. Form opposite reciprocal- slope of the orthogonal family
4. Separation of variables to find y the orthogonal family
5. Plot several versions of the original function and the orthogonal family

Maple Command Format	Description
$y^2 = kx^3$ $y^2 = kx^3$	To calculate or compute the orthogonal trajectories for the function $y^2 = kx^3$

Maple Command Format	Description
$\xrightarrow{\text{solve for } k}$ $\left[\left[k=\frac{y^2}{x^3}\right]\right]$ $y^2 = kx^3$ $y^2 = kx^3$ $\xrightarrow{\text{implicit differentiation}}$ $\frac{3kx^2}{2y}$ $\frac{3\frac{y^2}{x^3}x^2}{2y}$ $\frac{3y}{2x}$ $y' = -\frac{2x}{3y}$ $\frac{d}{dx}y(x) = -\frac{2x}{3y(x)}$ $\xrightarrow{\text{solve DE}}$ $y(x) = -\frac{\sqrt{-6x^2+9c_1}}{3},$ $y(x) = \frac{\sqrt{-6x^2+9c_1}}{3}$ $y = \frac{\sqrt{-6x^2+c}}{3}$ $\xrightarrow{\text{cross multiply}}$ $3y = \sqrt{-6x^2+c}$ $9y^2 = -6x^2+c$ $\xrightarrow{\text{solve for } c}$ $[[c = 6x^2+9y^2]]$ $c = 6x^2+9y^2$	**We follow the following steps in Maple command:** 1. First, we implicitly differentiate the given function. To do this we select the function and use the differentiate option from the command pallet 2. Here we choose y a dependent variable and x an independent variable in check box and get $\frac{3kx^2}{2y}$ 3. From the function we solve for k by solve option command and get $\left[\left[k=\frac{y^2}{x^3}\right]\right]$ 4. The value of k will be replaced by copy and paste command and get $\frac{3y}{2x}$ which is the differential function of the given function. 5. And write its negative reciprocal to get orthogonal prime function as $y' = -\frac{2x}{3y}$ 6. By using command pallet, from differential function command we get $\frac{d}{dx}y(x) = -\frac{2x}{3y(x)}$ 7. By using DE command from command pallet we get solution of this differential equation as $y(x) = -\frac{\sqrt{-6x^2+9c_1}}{3}, y(x) = \frac{\sqrt{-6x^2+9c_1}}{3}$ 8. Here $9c$ in square root is just a constant it can be replaced be another constant c and by selecting one of the solution i.e., $y = \frac{\sqrt{-6x^2+c}}{3}$ 9. By cross multiply command from the pallet and square both sider we get the solution as $[[c = 6x^2+9y^2]]$. 10. Now taking different value of c i.e., 1,3,5,9 we can plot family of curves (ellipses) whose concentric center is origin. 11. Through Maple >plot 2D command from the command pallet, first we draw all the ellipses from

<table>
<tr><th>Maple Command Format</th><th>Description</th></tr>
<tr><td>
</td><td>following solution
$$1 = 6x^2 + 9y^2, 3 = 6x^2 + 9y^2, 5 = 6x^2 + 9y^2, 9 = 6x^2 + 9y^2$$
12. We also want to graph the original family. i.e.,
$$y^2 = kx^3$$
By taking different values of k i.e., 1,3,5,9 we have the following equations
$$y^2 = x^3, y^2 = 3x^3, y^2 = 5x^3, y^2 = 9.x^3$$
13. we can draw the graphs of these function and drag them on its differential functions graph (ellipses) by using Maple which will add them in. And we'll drag and drop one at a time so we can see that they are curves, these families of curves intersect everywhere at 90 degree angles. And so, the trajectories, the paths of those functions are orthogonal.</td></tr>
<tr><td colspan="2">$1 = 6x^2 + 9y^2, 3 = 6x^2 + 9y^2, 5 = 6x^2 + 9y^2, 9 = 6x^2 + 9y^2$
$$1 = 6x^2 + 9y^2, 3 = 6x^2 + 9y^2, 5 = 6x^2 + 9y^2, 9 = 6x^2 + 9y^2$$
$y^2 = 1x^3, y^2 = 3x^3, y^2 = 5x^3, y^2 = 9.x^3$
$$y^2 = x^3, y^2 = 3x^3, y^2 = 5x^3, y^2 = 9.x^3$$</td></tr>
</table>

Exercise 10.3

1. Find the orthogonal trajectory of the curves $y = ax^2$.
2. Find the orthogonal trajectories of the hyperbola $xy = c$.
3. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$.
4. Find the orthogonal trajectories of the family of curves $y = \frac{x}{1+c_1x}$.
5. Find the general equation of family of curves perpendicular to the $y = c_1 \sin x$.
6. Find the general equation of family of curves perpendicular to the $x^{\frac{1}{3}} + y^{\frac{1}{3}} = c$.

Review Exercise 10

1. Tick the correct answer.

(i) The order and degree of the differential equation $1 + \frac{d^2y}{dx^2} = x\frac{dy}{dx}$ is ________.

(a) order 2, degree 2 (b) order 2, degree 1

(c) order 1, degree 2 (d) order 1, degree 1

(ii) The degree of differential equation $\left(\frac{dy}{dx}\right)^4 + 3\frac{d^2y}{dx^2}$ is ________.

(a) 2 (b) 4 (c) 1 (d) 3

(iii) The order and degree of differential equation $\left(\frac{d^3y}{dx^3}\right)^2 = \sqrt{\frac{dy}{dx}}$ is ________.

(a) order 3, degree 4 (b) order 4, degree 3
(c) order 2, degree 1 (d) order 1, degree 2

(iv) The degree of differential equation $y = x\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)$ is ________.

(a) 1 (b) 2 (c) 3 (d) 4

(v) The order and degree of differential equation $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^3}$ is ________.

(a) 2, 2 (b) 2, 3 (c) 3, 2 (d) 2, 1

(vi) Differential equation $xdy - ydx = 0, y(1) = 2$ has a solution given by y. Then $y(-1)$ is ________.

(a) -1 (b) -2 (c) 2 (d) 1

(vii) The solution of differential equation $\frac{dy}{dx} + y^2 = 0$ is ________.

(a) $y = ce^n$ (b) $y = \frac{1}{x+c}$
(c) $y = -\frac{x^3}{3} + c$ (d) $y = \frac{x^2}{3} + c$

(viii) The general solution of the differential equation $9y\frac{dy}{dx} + 4x = 0$ is ________.

(a) $4x^2 + 9y^2 = c$ (b) $\frac{x^2}{9} - \frac{y^2}{4} = c$
(c) $4x^2 + y^2 = c$ (d) $9x^2 - 4y^2 = 0$

2. Show that $y = Ae^{3x} + Be^{4x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$.

3. Solve the following differential equation

(i) $\cos(x+y)\,dy = dx$ (ii) $x^2\frac{dy}{dx} = x^2 + xy + y^2$

(iii) $(xy + y^2)dx = (x^2 - xy)dy$ (iv) $x\frac{dy}{dx} = y + x\tan\frac{y}{x}$

(v) $(2x + 3y - 5)dy + (3x + 2y + 1)dx = 0$

(vi) $(2x + y + 1)dx + (4x + 2y - 1)dy = 0$

(vii) $\frac{dy}{dx} = \frac{y}{x} + \cos\left(\frac{y}{x}\right)$ (viii) $\frac{dy}{dx} = e^{x+y}$

4. Given $y = (A + Bx)\,e^{-2x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$, $y(0) = 2$, $y'(0) = 5$. Find the particular solution.

5. Form / obtain the differential equation by eliminating arbitrary constants from the given relation

(i) $y = \frac{9}{x} + b$ (ii) $y = e^x(a\cos x + b\sin x)$

(iii) $y = \ln\cos(x - a) + b$ (iv) $y = A\cos(\ln x) + B\sin(\ln x)$

6. A body moves along a straight line, its acceleration after t sec is given by $\frac{1}{\sqrt{t}}$. At $t = 9$ sec, $V = 25$ m/s. Find the velocity at any time and at $t = 20$ sec.

7. Find the orthogonal trajectories of the family of the curve $3x + 4y = c$.

8. Find the general equation of the family of the curves perpendicular to the $y = \ln(\tan x + c_1)$.