Unit 11 Partial Differentiation

11.1 Differentiation of function of two variables

We have already studied the differentiation of function of one variable. Now, in this section, we will focus on differentiation of function of two variables.

11.1.1 Define a function of two variables

If a quantity z has unique and finite value for every pair of values x and y, then z is called function of two independent variables x and y.

i.e., $$z = f(x, y)$$

Here, z possesses unique and finite value for each ordered pair $(x, y) \in \mathbb{R}^2$.

For example, $f(x, y) = x^2 + xy + y^2$ is a function of two variables, because for different values of x and y, f has a unique and finite value.

11.1.2 Define partial derivative

The concept of partial derivative arises when function is of two or more variables.

Definition:

Let f is the function of two variables x and y, denoted by $f(x, y)$, then partial derivative of f with respect to x is the ordinary derivative of $f(x, y)$ with respect to x by taking y as a constant. It is denoted as $\dfrac{\partial f}{\partial x}$ or f_x. Similarly, partial derivative of $f(x, y)$ with respect to y can be defined, and is denoted by $\dfrac{\partial f}{\partial y}$ or f_y.

11.1.3 Find partial derivatives of a function of two variables

Example 1. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, given that

(i) $f(x, y) = x^2 + xy + y^2$ (ii) $f(x, y) = ye^x$

(iii) $f(x, y) = \ln y\,, y > 0$

(i) As $f(x, y) = x^2 + xy + y^2$

Differentiating f partially with respect to x, we get

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + xy + y^2)$$

$$\frac{\partial f}{\partial x} = 2x + y(1) + 0$$

$$\frac{\partial f}{\partial x} = 2x + y$$

Similarly, differentiating f partially with respect to y, we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy + y^2)$$

$$\frac{\partial f}{\partial y} = 0 + x(1) + 2y$$

$$\frac{\partial f}{\partial y} = x + 2y$$

(ii) $\boldsymbol{f(x, y) = ye^x}$

Differentiating f partially with respect to x

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(ye^x)$$

$$\frac{\partial f}{\partial x} = y\left(\frac{\partial e^x}{\partial x}\right)$$

$$\frac{\partial f}{\partial x} = ye^x$$

Similarly, differentiating f partially with respect to y

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(ye^x)$$

$$\frac{\partial f}{\partial y} = e^x(1)$$

$$\frac{\partial f}{\partial y} = e^x$$

(iii) $\boldsymbol{f(x, y) = \ln y, y > 0}$

Differentiating partially with respect to x, we get

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\ln y)$$

$$\frac{\partial f}{\partial x} = \ln y \frac{\partial}{\partial x}(1)$$

$$\frac{\partial f}{\partial x} = \ln y \times 0$$

$$\frac{\partial f}{\partial x} = 0$$

Similarly, differentiating partially with respect to y, we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\ln y)$$

$$\frac{\partial f}{\partial y} = \frac{1}{y}$$

Example 2. Find the partial derivative of the area of triangle by base as well as height of the triangle.

Solution: The area of triangle is defined as $A = \frac{1}{2}bh$

Here b and h are base and height of the triangle respectively.

Now, partial derivative of Area A with respect to base b is

$$A = \frac{1}{2}bh$$

Differentiating A partially w.r.t b, we get

$$\frac{\partial A}{\partial b} = \frac{\partial}{\partial b}\left(\frac{1}{2}bh\right)$$

$$\frac{\partial A}{\partial b} = \frac{1}{2}h\,(1) \qquad \text{(here } h \text{ is treated as constant coefficient)}$$

$$\frac{\partial A}{\partial b} = \frac{1}{2}h$$

Similarly, differentiating A partially w.r.t h, we get

$$\frac{1}{2}bh$$

$$\frac{\partial A}{\partial h} = \frac{\partial}{\partial h}\left(\frac{1}{2}bh\right)$$

$$\frac{\partial A}{\partial h} = \frac{1}{2}b(1) = \frac{b}{2} \qquad \text{(here } b \text{ is treated as constant coefficient)}$$

11.2 Euler's Theorem

Euler's theorem is one of the most important theorems of calculus, which contains homogeneous function and its partial derivative.

11.2.1 Define a homogeneous function of degree n

Definition: A function $f(x, y)$ is said to be a homogeneous function of degree n if it can be written in the form of $f(tx, ty) = t^n f(x, y)$

or $\quad f(x, y) = x^n f\left(\frac{y}{x}\right)$

Example 1. Show that the polynomial function in two variables $p(x, y) = x^3 + x^2y + xy^2 + y^3$ is the homogeneous function of degree 3.

Solution: As $p(x, y) = x^3 + x^2y + xy^2 + y^3$

By taking highest power of x as common

$$p(x, y) = x^3\left[1 + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{x^3}\right]$$

$$p(x,y) = x^3\left[\left(\frac{y}{x}\right)^0 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3\right]$$

$$p(x,y) = x^3\, p\left[\left(\frac{y}{x}\right)\right]$$

Hence $p(x,y)$ is the homogeneous function of degree 3.

Example 2. Show that the function $f(x,y) = \frac{x^4+y^4}{x-y}$ is homogeneous function of degree 3.

Solution: Replacing x by tx and y by ty

$$f(tx,ty) = \frac{(tx)^4 + (ty)^4}{tx - ty}$$

$$f(tx,ty) = \frac{t^4x^4 + t^4y^4}{t(x-y)}$$

$$= \frac{t^4(x^4 + y^4)}{t(x-y)}$$

$$f(tx,ty) = t^3\left(\frac{x^4+y^4}{x-y}\right)$$

$$f(tx,ty) = t^3 f(x,y)$$

$\therefore \quad f(x,y) = \frac{x^4+y^4}{x-y}$ is homogeneous function of degree 3. Hence shown.

Alternatively, $f(x,y) = \frac{x^4+y^4}{x-y} = \frac{x^4\left(1+\frac{y^4}{x^4}\right)}{x\left(1-\frac{y}{x}\right)}$

$$= x^3\left(\frac{1+\left(\frac{y}{x}\right)^4}{1-\left(\frac{y}{x}\right)}\right)$$

$$f(x,y) = x^3 f\left(\frac{y}{x}\right)$$

This shows that $f(x,y) = \frac{x^4+y^4}{x-y}$ is the homogeneous function degree 3.

Example 3. Show that the function $f(x,y) = \sin\left(\frac{x^2+y^2}{x-y}\right)$ is not a homogeneous function.

Solution: Here $f(x,y) = \sin\left(\frac{x^2+y^2}{x-y}\right)$

Replacing x by tx and y by ty , we get

$$f(tx,ty) = \sin\left(\frac{(tx)^2 + (ty)^2}{(tx) - (ty)}\right)$$

$$= \sin\left(\frac{t^2x^2 + t^2y^2}{tx - ty}\right)$$

$$= \sin\left(\frac{t^2(x^2 + y^2)}{t(x - y)}\right)$$

$$f(tx, ty) = \sin\left(t\left(\frac{x^2 + y^2}{x - y}\right)\right)$$

$$\because \sin\left(t\left(\frac{x^2 + y^2}{x - y}\right)\right) \neq t\sin\left(\frac{x^2 + y^2}{x - y}\right)$$

$$\therefore \quad f(tx, ty) \neq t\, f(x, y)$$

Hence, $f(x, y)$ is not homogeneous a function.

Example 4. Show that the function $f(x, y) = \frac{\sqrt{x}+\sqrt{y}}{x^2-y^2}$ is homogeneous of degree $-\frac{3}{2}$.

Solution: As $f(x, y) = \frac{\sqrt{x}+\sqrt{y}}{x^2-y^2}$

$$= \frac{\sqrt{x}\left(1 + \sqrt{\frac{y}{x}}\right)}{x^2\left(1 - \frac{y^2}{x^2}\right)}$$

$$= x^{-\frac{3}{2}} \frac{\left(1 + \left(\frac{y}{x}\right)^{\frac{1}{2}}\right)}{\left(1 - \left(\frac{y}{x}\right)^2\right)} \qquad \left[\because f(x, y) = x^n f\left(\frac{y}{x}\right)\right]$$

$$f(x, y) = x^{-\frac{3}{2}} f\left(\frac{y}{x}\right)$$

Hence $f(x, y) = \frac{\sqrt{x}+\sqrt{y}}{x^2-y^2}$ is the homogeneous function of degree $-\frac{3}{2}$.

11.2.2 State and prove Euler's theorem on homogeneous functions

Let $z = f(x, y)$ is a homogeneous function of degree n, then by Euler's theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Proof: It is given that $z = f(x, y)$ is the homogeneous function of degree n. So, it can be written as

$$z = f(x, y) = x^n f\left(\frac{y}{x}\right) \qquad \text{... (i)}$$

From the statement of Euler's theorem, we need the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Differentiating partially (i) with respect to x, we get

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[x^n f\left(\frac{y}{x}\right)\right]$$

By applying product rule of derivative, we get

$$\frac{\partial z}{\partial x} = f\left(\frac{y}{x}\right)\frac{\partial}{\partial x}(x^n) + x^n\frac{\partial}{\partial x} f\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right)$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) - x^{n-2}yf'\left(\frac{y}{x}\right)$$

Multiplying both sides by x, we get

$$x\frac{\partial z}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1}yf'\left(\frac{y}{x}\right) \qquad \text{... (ii)}$$

Similarly, differentiating (i) partially with respect to y

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^n f\left(\frac{y}{x}\right)\right)$$

$$\frac{\partial z}{\partial y} = x^n\frac{\partial}{\partial y} f\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^{n-1} f'\left(\frac{y}{x}\right)$$

Multiplying both sides by y, we get

$$y\frac{\partial z}{\partial y} = x^{n-1}yf'\left(\frac{y}{x}\right) \qquad \text{... (iii)}$$

By adding equations (ii) and (iii), we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \left[nx^n f\left(\frac{y}{x}\right) - x^{n-1}yf'\left(\frac{y}{x}\right)\right] + x^{n-1}yf'\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n f\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nf(x,y) \qquad \left[\because f(x,y) = x^n f\left(\frac{y}{x}\right)\right]$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Hence proved.

11.2.3 Verify Euler's theorem for homogeneous functions of different degrees (simple cases)

Example 1. Let $z = \frac{x^2y^2}{x^2+y^2}$ then verify Euler's theorem.

As, $z = \frac{x^2y^2}{x^2+y^2}$ is homogeneous function of degree 2. Then by Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \qquad \text{... (i)}$$

To verify this, we find partial derivatives of z.

$$z = \frac{x^2y^2}{x^2+y^2}$$

Differentiating partially with respect to x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2y^2}{x^2+y^2}\right)$$

$$\frac{\partial z}{\partial x} = \frac{(x^2+y^2)\,\frac{\partial}{\partial x}(x^2y^2) - (x^2y^2)\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial x} = \frac{(x^2+y^2)\,(2xy^2) - (x^2y^2)(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2x^3y^2 + 2xy^4 - 2x^3y^2}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2xy^4}{(x^2+y^2)^2}$$

$$x\frac{\partial z}{\partial x} = \frac{2x^2y^4}{(x^2+y^2)^2} \qquad \text{... (ii)}$$

Similarly, differentiating partially with respect to y, we get

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2y^2}{x^2+y^2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{(x^2+y^2)\,\frac{\partial}{\partial y}(x^2y^2) - (x^2y^2)\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x^2+y^2)\,(2x^2y) - (x^2y^2)(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2x^4y + 2x^2y^3 - 2x^2y^3}{(x^2+y^2)^2}$$

$$y\frac{\partial z}{\partial y} = \frac{2x^4y^2}{(x^2+y^2)^2} \qquad \text{... (iii)}$$

By adding equations (ii) and (iii), we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2x^4y^2}{(x^2+y^2)^2} + \frac{2x^2y^4}{(x^2+y^2)^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2x^2y^2(x^2+y^2)}{(x^2+y^2)^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2x^2y^2}{x^2+y^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \qquad \left[\because z = \frac{x^2y^2}{x^2+y^2}\right]$$

Hence verified.

Example 2. Given that $p(x,y) = ax^2 + bxy + cy^2$ be the homogeneous function of degree 2. Then verify Euler's theorem for it.

Proof: As $p(x,y) = ax^2 + bxy + cy^2$ is the homogeneous function of degree 2. Then by Euler's theorem.

$$x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} = 2p \qquad \text{... (i)}$$

To verity this, first we find partial derivatives of $p(x,y)$.

$$p(x,y) = ax^2 + bxy + cy^2$$

$$\frac{\partial p}{\partial x} = 2ax + by$$

Multiplying both sides by x, we get

$$x\frac{\partial p}{\partial x} = 2ax^2 + bxy \qquad \text{... (ii)}$$

Similarly,

$$\frac{\partial p}{\partial y} = bx + 2cy$$

Multiplying both sides by y, we get

$$y\frac{\partial p}{\partial y} = bxy + 2cy^2 \qquad \text{... (iii)}$$

By adding equation (ii) and equation (iii)

We get,

$$x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} = 2ax^2 + bxy + bxy + 2cy^2$$

$$x\frac{\partial p}{\partial x}+y\frac{\partial p}{\partial y}=2(ax^2+bxy+cy^2)$$

$$x\frac{\partial p}{\partial x}+y\frac{\partial p}{\partial y}=2p(x,y)$$

Hence verified.

Example 3. Verify Euler's theorem for the function $z=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$.

Solution: Let us check the homogeneity and degree of the function.

Here $\quad f(x,y)=z=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$

Replacing x by tx and y by ty, we get

$$f(tx,ty)=\sin^{-1}\left(\frac{tx}{ty}\right)+\tan^{-1}\left(\frac{ty}{tx}\right)$$

$$f(tx,ty)=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$$

$$f(tx,ty)=t^0\left[\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}\right]$$

Hence $z=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$ is the homogeneous function of degree 0.

$\therefore$ By Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0.z=0 \qquad \text{... (i)}$$

To verify Euler's theorem, we find the partial derivatives of $z=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$ w.r.t their independent variables.

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left[\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}\right]$$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y}+\frac{x^2}{x^2+y^2}\cdot\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}$$

Multiplying both sides by x, we get

$$x\frac{\partial z}{\partial x}=\frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2} \qquad \text{... (ii)}$$

Similarly,

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left[\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}\right]$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{y^2-x^2}}\left(\frac{-x}{y^2}\right) + \frac{x^2}{x^2+y^2}\cdot\frac{1}{x}$$

Multiplying both sides by y, we get

$$y\frac{\partial z}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \qquad \text{... (iii)}$$

By adding equations (ii) and (iii), we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

Hence verified.

11.2.4 Use MAPLE command diff to find partial derivative

The format of diff command to partial derivative of a function in MAPLE is as under:

$> diff(f, x, y)$ is equivalent to the command $\frac{\partial}{\partial x} f$ in Maple version 2022.

Where,

f stands for function whose partial derivative is to be evaluated

X, Y stands for the variable x and y, the partial derivative with respect to x or y.

$\frac{\partial}{\partial x}$ means 1st order partial derivative with respect to variable x

Note: All above operators should be taken from the Maple calculus pallet.

Use MAPLE command **diff** or $\left(\frac{\partial}{\partial x} f\right)$ to differentiate a function:

Partial Derivative of functions:

$> f := (x, y) \rightarrow (x^2y + 5xy + xy^2)$

$f := (x, y) \rightarrow x^2y + 5xy + xy^2$

$> diff(f(x, y), x)$

$2xy + y^2 + 5y$

$> diff(f(x, y), y)$

$x^2 + 2xy + 5y$

$> diff(f(x, y), x, y)$

$2x + 2y + 5$

$> f := (x, y) \rightarrow (x + \ln(xy) + 2x\sin(y)^2)$

$f := (x, y) \rightarrow x + \ln(yx) + 2x\sin y^2$

$> diff(f(x, y), x)$

$1 + \frac{1}{x} + 2\sin y^2$

$> diff(f(x, y), y)$

Partial Derivative of functions:

$> diff(f(x,y),y,x)$

$$2x+2y+5$$

$> diff(f(x,y),x,x)$

$$2y$$

$> diff(f(x,y),y,y)$

$$2x$$

$$\frac{1}{y}+4x\sin y$$

$> diff(f(x,y),x,y)$

$$4\sin y$$

$> diff(f(x,y),x,x)$

$$-\frac{1}{x^2}$$

$> diff(f(x,y),y,y)$

$$-\frac{1}{y^2}+4\sin x$$

$> f := (x,y) \longrightarrow (x+y+ye^x)$

$$f := (x,y) \longrightarrow x+y+ye^x$$

$> diff(f(x,y),x)$

$$1+ye^x$$

$> diff(f(x,y),y)$

$$1+e^x$$

$> diff(f(x,y),x,x)$

$$ye^x$$

$> diff(f(x,y),y,y)$

$$0$$

$> f := (x,y) \longrightarrow (\ln(x+1)+y+ye^x)$

$$f := (x,y) \longrightarrow \ln(x+1)+y+ye^x$$

$> diff(f(x,y),x)$

$$\frac{1}{1+x}+ye^x$$

$> diff(f(x,y),y)$

$$1+e^x$$

$> diff(f(x,y),x,x)$

$$-\frac{1}{(1+x)^2}+ye^x$$

$> diff(f(x,y),y,y)$

$$0$$

Exercise 11

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ when $f(x,y)$ is given by

   (i) $f(x,y)=3x^3+y^2-6x+2y-7$

   (ii) $f(x,y)=x^2+xy-y^2-2x-2y-8$ (iii) $f(x,y)=\sin(x+y)$

   (iv) $f(x,y)=e^x\cos y$ (v) $f(x,y)=\frac{1}{2}\ln(x^2+y^2)$

2. The volume of the cone is given by formula $V=\frac{1}{3}\pi r^2 h$. Differentiate V with respect to their independent variables.

3. Check whether the following functions are homogeneous or not. Find the degree in case of homogeneous function.

   (i) $f(x,y)=\frac{x^3-5x^2y+7xy^2+y^3}{xy^2}$ (ii) $f(x,y)=\tan\left(\frac{x+y}{y^2}\right)$

(iii) $f(x,y) = x^3 + 3x^2y + 2y^2x + y^3$ (iv) $f(x,y) = \cos^{-1}\left(\frac{x^2-y^2}{xy}\right)$

(v) $f(x,y) = \frac{x^2-xy+y^2}{x+y^2}$ (vi) $f(x,y) = \sqrt{x^8 - 3x^2y^6}$

(vii) $f(x,y) = x^3 \sin\left(\frac{y^2}{x}\right)$ (viii) $f(x,y) = \ln\left(\frac{x^2+y^2}{x+y}\right)$

4. Verify Euler's theorem for the following homogeneous function.

(i) $f(x,y) = xy + y^2$ (ii) $f(x,y) = \cos\left(\frac{x}{y}\right)$

(iii) $f(x,y) = \sqrt{xy} - x$ (iv) $f(x,y) = \ln\left(\frac{x+y}{y}\right)$

5. If $u = x^2(y-x) + y^2(x-y)$ then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -2(x-y)^2$.

6. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

7. Use MAPLE command >**diff** or $\left(\frac{\partial}{\partial x} f\right)$ to partial differentiate with respect to *x* and y of the following functions:

(i) $f(x,y) = x^2y + xy + xy^2$ (ii) $f(x,y) = y + x\cos(y)$

(iii) $f(x,y) = \frac{x+\sqrt{x}}{y+\sqrt{y}}$

Review Exercise 11

1. Multiple choice questions (MCQs)

(i) Given that $f(x,y) = e^{xy}$ then $\frac{f_x}{f_y} =$ ____________.

(a) $\frac{x}{y}$ (b) 1 (c) $\frac{y}{x}$ (d) $\frac{y+x\,f'(x,y)}{x+y\,f'(x,y)}$

(ii) Surface area of a cube is a function of __________ variables.

(a) 1 (b) 2 (c) 3 (d) 4

(iii) Given that $g(x,y) = \cos\left(\frac{x}{y}\right)$ then $\frac{g_y}{g_x} =$ ___________.

(a) $-\frac{x}{y}$ (b) $\frac{x}{y}$ (c) $-\frac{x}{y^3}$ (d) $-\frac{y}{x}$

(iv) A function $\tan\left(\frac{2x}{3y}\right)$ is a homogeneous function of degree ___________.

(a) undefined (b) $\frac{2}{3}$ (c) 1 (d) 0

(v) The perimeter of rectangle is given by a function $P(x,y) = 2(x+y)$, where x and y are respectively its length and breadth. Then sum of partial derivatives w.r.t their independent variables is ___________.

(a) $2x$ (b) $2y$ (c) $2(x+y)$ (d) 4

(vi) Given that $z = f(x,y)$ is a homogeneous function of degree 0 then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} =$ __.

(a) $(x+y)\,f'(x,y)$ (b) $x+y$ (c) 0 (d) $f(x,y)$

(vii) The area of trapezium is a function of ________________ variables.

(a) 1 (b) 2 (c) 3 (d) 4

(viii) Given $w = f(u,v)$ is a homogeneous function of degree $\frac{2}{3}$ then $u\frac{\partial w}{\partial u} + v\frac{\partial w}{\partial v} =$ ____.

(a) $(u+v)f'(u,v)$ (b) 0 (c) $\frac{2}{3}$ (d) $\frac{2}{3}w$

(ix) Given that $z = y\left(\frac{u}{v}\right)$ is a homogeneous function of degree 0 then $v\frac{\partial z}{\partial v} =$ _______.

(a) $u\frac{\partial z}{\partial u}$ (b) 0 (c) $-u\frac{\partial z}{\partial u}$ (d) -1

(x) Let $f(x,y)$ and $g(x,y)$ are homogeneous functions of degrees 2 and 3 respectively, then degree of homogeneous functions $\frac{f(x,y)}{g(x,y)}$ is _____________.

(a) 6 (b) 1 (c) $\frac{2}{3}$ (d) -1

2. Let $f(x,y) = xy$ and $g(x,y) = \frac{1}{2}xy$ be the homogeneous functions for the areas of rectangle and triangle respectively, where x and y are their independent variables. Are $f+g, f-g, fg$ and $\frac{f}{g}$ homogenous? If yes, what are their degrees?

3. Verify Euler's theorem for the function $z = \sqrt{x^2+y^2}$

4. Given that $z = g(x,y)$ is a homogeneous function of degree 3 then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$.

5. Given that $y = f(u,v)$ is a homogeneous function of degree $-\frac{3}{2}$ then show that $u\frac{\partial y}{\partial u} + v\frac{\partial y}{\partial v} = -\frac{3}{2}y$.

6. Given that $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

7. Given that $u = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$.