Introduction to Numerical Methods

Unit 12

12.1 Numerical Solution of Non-linear Equations

12.1.1 Describe importance of numerical methods

Till now, all the methods we have learnt to solve the non-linear equation and finding of the derivative or integration of functions are analytical methods.

When analytic approaches are failed to find the solution of a non-linear equation or require too many tedious computations then, mathematicians used numerical methods to compute approximate solution. Therefore, numerical methods have great importance in the field of mathematics.

12.1.2 Explain the basic principles of solving a non-linear equation in one variable

The basic principle of solving a non-linear equation is to find the interval (values of a and b) for a function $f(x)$, where $f(a)$ and $f(b)$ are of opposite signs such that $f(a).f(b) < 0$, then the root of $f(x) = 0$ lies in the interval $[a, b]$.

For example, $f(x) = x^3 - 2x - 5$, put the values of $x = 0,1,2,3$ in $f(x)$, we get

$$f(0) = (0)^3 - 2(0) - 5 = -5$$
$$f(1) = (1)^3 - 2(1) - 5 = -6$$
$$f(2) = (2)^3 - 2(2) - 5 = -1$$
$$f(3) = (2)^3 - 2(3) - 5 = 16$$

Here $f(2) = -1 < 0(-ve)$ and $f(3) = 16 > 0(+ve)$,
such that $f(2).f(3) = (-1).(16) = -16 < 0$,
Now, root lies in the interval $[a, b] = [2, 3]$.

12.1.3 Calculate the real roots of a non-linear equation in one variable by

- bisection method
- regula-falsi method
- Newton-Raphson method

- **Bisection Method**

The bisection technique is a root-finding method which repeatedly bisects an interval and then selects a sub-interval in which a root must lie for further processing. It is a very easy and reliable procedure.

Algorithm of a Bisection method

If $f(x)$ is a continuous function over an interval, then to find the root of $f(x) = 0$ by bisection method, following steps are taken.

Step 1. Choose two approximations a and $b(b > a)$ such that $f(a).f(b) < 0$.

Step 2. Evaluate the midpoint c of an interval $[a, b]$ given by $c = \frac{a+b}{2}$.

Step 3. Now there are three possibilities:

(i) If $f(c) = 0$, , then c is a root of $f(x) = 0$.

(ii) If $f(c) < 0$, and $f(a)f(c) < 0$ then root lies in the $[a, c]$ else if $f(b)f(c) < 0$ then root lies in the $[c, b]$

Step 4. Continue the process till the root is found to the desired accuracy, that is two decimal places or three decimal places or four decimal places etc.

Example 1. Use Bisection method to find a root of an equation $x^2 - 3 = 0$ up to four iteration.

Solution:

Let $f(x) = x^2 - 3 = 0$

Taking $a = 1$ and $b = 2$.

Here $f(1) = 1 - 3 = -2 < 0$ and $f(2) = 4 - 3 = 1 > 0$

Since $f(1)f(2) < 0$

Therefore, root lies between 1 and 2

1st iteration

Now,

$$c = \frac{1 + 2}{2} = 1.5$$

$$f(c) = f(1.5) = (1.5)^2 - 3 = -0.75 < 0$$

Since $f(2)f(1.5) < 0$

therefore, root lies between 1.5 and 2

2nd iteration

Taking $a = 1.5$ and $b = 2$

$$c = \frac{(1.5 + 2)}{2} = 1.75$$

$$f(c) = f(1.75) = (1.752)^2 - 3 = 0.062 > 0$$

Since $f(1.5)f(1.75) < 0$

therefore, root lies between 1.5 and 1.75

3rd iteration

Taking $a = 1.5$ and $b = 1.75$

$$c = \frac{(1.5 + 1.75)}{2} = 1.625$$

$$f(c) = f(1.625) = (1.6252)^2 - 3 = -0.359 < 0$$

Since $f(1.75)f(1.625) < 0$

therefore, root lies between 1.625 and 1.75

4th iteration

Taking $a = 1.625$ and $b = 1.75$

$$c = \frac{(1.625 + 1.75)}{2} = 1.688$$

$$f(c) = f(1.688) = (1.6882)^2 - 3 = -0.152 < 0$$

Hence 1.688 is the approximate root of $x^2 - 3 = 0$ after four iterations.

Example 2. Find a root of an equation $f(x) = x^3 + 2x^2 + x - 1$, using Bisection method correct to two decimal places.

Solution:

Here $f(x) = x^3 + 2x^2 + x - 1 = 0$

Find the value of $f(x)$ at $x = 0,1$

$f(0) = (0)^3 + 2(0)^2 + (0) - 1 = -1$

$f(1) = (1)^3 + 2(1)^2 + (1) - 1 = 3$

Here $f(0) = -1 < 0$ and $f(1) = 3 > 0$

Since $f(0)f(1) < 0$

therefore, root lies between 0 and 1

1st iteration

Taking $a = 0$ and $b = 1$

$$c = \frac{0 + 1}{2} = 0.5$$

$$f(0.5) = (0.5)^3 + 2(0.5)^2 + (0.5) - 1 = 0.125 > 0$$

Here $f(0) = -1 < 0$ and $f(0.5) = 0.125 > 0$

Since $f(0)f(0.5) < 0$

therefore, root lies between 0 and 0.5

2nd iteration

Taking $a = 0$ and $b = 0.5$

$$c = \frac{0 + 0.5}{2} = 0.25$$

$$f(0.25) = (0.25)^3 + 2(0.25)^2 + (0.25) - 1 = -0.6094 < 0$$

Here $f(0.25) = -0.6094 < 0$ and $f(0.5) = 0.125 > 0$

Since $f(0.25)f(0.5) < 0$

therefore, root lies between 0.25 and 0.5

3rd iteration

Taking $a = 0.25$ and $b = 0.5$

$$c = \frac{0.25 + 0.5}{2} = 0.375$$

$$f(0.375) = (0.375)^3 + 2(0.375)^2 + (0.375) - 1 = -0.291 < 0$$

$$f(0.375) = -0.291 < 0 \text{ and } f(0.5) = 0.125 > 0$$

Since $f(0.375)f(0.5) < 0$

therefore, root lies between 0.375 and 0.5

4th iteration

Taking $a = 0.375$ and $b = 0.5$

$$c = \frac{0.375 + 0.5}{2} = 0.4375$$

$$f(0.4375) = (0.4375)^3 + 2(0.4375)^2 + (0.4375) - 1 = -0.0959 < 0$$

Since $f(0.4375) < 0$ and $f(0.5) > 0$.

therefore, root lie in the interval.

5th iteration

Taking $a = 0.4375$ and $b = 0.5$

$$c = \frac{0.4375 + 0.5}{2}$$

$$c = 0.4688$$

$$f(0.4688) = (0.4688)^3 + 2(0.4688)^2 + 0.4688 - 1$$

$$f(0.4688) = 0.112 > 0$$

Since $f(0.4688)f(0.4375) < 0$

therefore, root lie between 0.4375 and 0.4688.

6th iteration

Taking $a = 0.4375$ and $b = 0.4688$

$$c = \frac{0.4375 + 0.4688}{2}$$

$$c = 0.4531$$

$$f(0.4531) = (0.4531)^3 + 2(0.4531)^2 + 0.4531 - 1$$

$$f(0.4531) = -0.0432 < 0$$

Since $f(0.4531)f(0.4688) < 0$

therefore, root lie between 0.4531 and 0.4688.

7th iteration

Taking $a = 0.4531$ and $b = 0.4688$

$$c = \frac{0.4531 + 0.4688}{2} = 0.4609$$

$$f(0.4609) = (0.4609)^3 + 2(0.4609)^2 + 0.4609 - 1$$

$$f(0.4609) = -0.0162 < 0$$

Since $f(0.4609)f(0.4688) < 0$

therefore, root lie between the 0.4609 and 0.4688.

8th iteration

Taking $a = 0.4609$ and $b = 0.4688$

$$c = \frac{0.4609 + 0.4688}{2} = 0.4648$$

$$f(0.4648) = (0.4648)^3 + 2(0.4648)^2 + 0.4648 - 1$$

$$f(0.4648) = -0.0026$$

Hence, we obtained accuracy up to two decimal places. Therefore 0.4648 is required approximate root.

12.1.4 Use MAPLE command fsolve to find numerical solution of an equation and demonstrate through examples

The fsolve command is the numeric equivalent of solve. The fsolve command finds the roots of the equation(s), producing approximate (floating-point) solutions.

Examples:

$> polynomial := 3x^4 - 16x^3 - 3x^2 + 13x + 16$

$> fsolve(polynomial)$

$$1.324717957, 5.333333333$$

$> polynomial := x^6 - x - 1$

$> fsolve(polynomial)$

$$-0.7780895987, 1.134724138$$

$> fsolve(2x + y = 17, x^2 - y^2 = 20, x, y)$

$$\{x = 16.37758198, y = -15.75516397\}$$

$> f := sin(x + y) - exp(x) * y = 0$

$> g := x^2 - y = 2$

$> fsolve(f, g, x = -1..1, y = -2..0)$

$$\{x = -0.6687012050, y = -1.552838698\}$$

$> \mathrm{fsolve(cos(x) - x = 0, x);}$

$$\{\mathrm{x} = .7390851332\}$$

Exercise 12.1

1. Use Bisection method to find a real root of the following equations.
   (i) $f(x) = 2x^3 - 2x - 5, [1,2]$ up to three iterations
   (ii) $f(x) = x^3 - 2x - 5, [1.5, 2.5]$ up to four iterations
   (iii) $f(x) = x^3 - x + 1, [-2, -1]$ up to five iterations
   (iv) $f(x) = \cos x, [1, 2]$ up to one decimal place (five iterations)
   (v) $f(x) = 3x - e^x, [0, 1]$ up to three decimal places (eleven iterations)
   (vi) $f(x) = 3x - \sqrt{1 + \sin x}, [0,1]$ up to three decimal places (thirteen iterations)
2. Write MAPLE command fsolve to find numerical solution the following:
   (i) polynomial :$3x^4y^2 = 17, x^2y - 5xy^2 - 2y = 1$
   (ii) polynomial :$3x^3 - 27x + 3$
   (iii) polynomial :$3x^3 + 9x + 3$
   (iv) polynomial :$2x^3 + 4x + 2$

- **Regula Falsi method**

This approach is also known as the false position method. It is an iterative method for determining the real root of a nonlinear equation $f(x) = 0$. This method gives a better approximation for the roots of the equation than bisection method.

- **Algorithm of Regula Falsi method:**

Let $f(x)$ is a continuous function over the interval, to find the approximate root of $f(x) = 0$ by Regula Falsi Method following steps are taken.

Step 1. Find points a and b such that $a < b$ and $f(a).f(b) < 0$.

Step 2. Take the interval $[a, b]$ and find next value using

formula: $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$

Step 3. If $f(x_1) = 0$ then x_1 is an exact root.

else if, $f(x_1).f(b) < 0$ then approximate root lies in $[x, b]$

else if, $f(a).f(x_1) < 0$ then let $b = x_1$ approximate root lies in $[a, x_1]$

Step 4. Repeat steps 2 and 3 until desired accuracy is obtained.

Example 1. Find a root of an equation $f(x) = x^2 - 3$ using Regula Falsi Method up to four iterations.

Solution:

here $f(x) = x^2 - 3$

so, $f(1) = -2 < 0$ and $f(2) = 1 > 0$

therefore, root lies between $a = 1$ and $b = 2$

since $f(1)f(2) < 0$

1st iteration

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{(1)(1) - 2(-2)}{1 - (-2)} = \frac{1+4}{3} = 1.6667$$

$$f(x_1) = f(1.6667) = (1.6667)^2 - 3 = -0.2222 < 0$$

2*nd* iteration:

Here $f(1.6667) = -0.2222 < 0$ and $f(2) = 1 > 0$

root lies between $a = 1.6667$ and $b = 2$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_2 = \frac{(1.6667)(1) - 2(-0.2222)}{1 - (-0.2222)} = 1.7273$$

$$f(x_2) = f(1.7272) = (1.7273)^2 - 3 = -0.0165 < 0$$

3*rd* iteration:

Here $f(1.7273) = -0.0165 < 0$ and $f(2) = 1 > 0$

Root lies between $a = 1.7273$ and $b = 2$

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_3 = \frac{(1.7273)(1) - 2(-0.0156)}{1 - (-0.0156)} = 1.7317$$

$$f(x_3) = f(1.7317) = (1.7317)^2 - 3 = -0.0012 < 0$$

4*th* iteration:

Here $f(1.7317) = -0.0012 < 0$ and $f(2) = 1 > 0$

root lies between $a = 1.7317$ and $b = 2$

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_4 = \frac{(1.7317)(1) - 2(-0.0012)}{1 - (-0.0012)} = 1.732$$

$$f(x_4) = f(1.732) = (1.732)^2 - 3 = -0.000176 < 0$$

Approximate root of the equation $x^2 - 3 = 0$ using False Position method is 1.732 (After 4 iterations).

Example 2. Find a root of the equation $2e^x \sin x = 3$ using the false position method and correct it up to two decimal places.

Solution:

Let $f(x) = 2e^x \sin x - 3 = 0$

so, $f(0) = 2e^0 \sin 0 - 3 = 0$

$f(0) = -3 < 0$

also $f(1) = 2e^1 \sin(1) - 3$

$f(1) = 1.574770$

since $f(0)f(1) < 0$

therefore, root lies between 0 and 1.

First Iteration

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

here $a = 0$ and $b = 1$

$$x_1 = \frac{0 \times (1.5747) - 1 \times (-3)}{1.5747 + 3}$$

$x_1 = 0.6557$

Now $f(x_1) = f(0.6557) = 2e^{0.6557} \sin(0.6557) - 3$

$f(x_1) = -0.6507 < 0$

since $f(0.6557f)f(1) < 0$

Therefore, roots lie between 0.6557 and 1.

Second Iteration

$a = 0.6557$ and $b = 1$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_2 = \frac{0.6557(1.5747) - (0.6557)(-0.6507)}{1.5747 - (-0.6507)}$$

$x_2 = 0.7563$

Now $f(x_2) = f(0.7563) = -0.0761 < 0$

since $f(0.7563)f(1) < 0$

therefore, roots lie between 0.7563 and 1.

Third Iteration

$$x_3 = \frac{(0.7563)(1.5747) - 1(-0.0761)}{1.5747 - (0.0761)}$$

$x_3 = 0.7675$

Then the best approximation of the roots up to two decimal places is 0.768.

Exercise 12.2

Use Regula Falsi method to find a root of following equations:

1. $x^3 - 2x - 5 = 0, [2, 3]$ up to three iterations.
2. $\sin(2x) - e^{x-1} = 0, [-2, -1]$ up to four iterations.
3. $x^4 - x - 10 = 0, [1,2]$ up to two decimal places.
4. $3x + \sin(x) - e^x = 0, [0,1]$ up to three decimal places.
5. $f(x) = 2\cos x - x = 0, [1, 2]$ up to five decimal places.

- **Newton's Raphson method**

The Newton Raphson Method is also commonly known as Newton's Method. It is an iterative procedure for determining a better approximation for the root of a continuous, differentiable function $f(x) = 0$ at $x = x_0$

Algorithm

Step 1.

Let $f(x)$ is differentiable function over (a, b) then to find the approximate root of $f(x) = 0$ by Newton's Rephson method, we have to take initial guess $x_0 \in (a, b)$ such that $f(x_0) \neq 0$. Following steps are taken to find approximate root by bisection method.

Step 2.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Step 3.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Step 4.

By continuing this process, desired accuracy is obtained.

Example 1. Find a root of an equation $x^2 - 3 = 0$, using Newton Raphson method up to three iteration.

Solution:

Let $f(x) = x^2 - 3 = 0$

$f'(x) = 2x$

For our simplicity, we take initial guess $x_0 = 1.5$

1*st* iteration:

$f(x_0) = f(1.5) = (1.5)^2 - 3 = -0.75$

$f'(1.5) = 2(1.5) = 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1.5 - \frac{-0.75}{3} = 1.75$$

2*nd* iteration:

$$f(x_1) = f(1.75) = (1.75)^2 - 3 = 0.0625$$
$$f'(1.75) = 2(1.75) = 3.5$$

$$x_2 = x_2 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 1.75 - \frac{0.0625}{3.5} = 1.7321$$

3*rd* iteration:

$$f(x_2) = f(1.7321) = (1.7321)^2 - 3 = 0.0003$$
$$f'(1.7321) = 2(1.7321) = 3.4643$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_2 = 1.7321 - \frac{0.0003}{3.4643} = 1.7321$$

Approximate root of the equation $x^2 - 3 = 0$, using Newton's method is 1.7321 (After 3 iterations).

Example 2. Find a root of $3x - \cos x - 1 = 0$ by Newton's Raphson method, correct up to 4 decimal places.

Solution:

Let $f(x) = 3x - \cos x - 1 = 0$

$f'(x) = 3 + \sin x$

We take initial guess $x_0 = 0$

Here $f(0) = 3(0) - \cos 0 - 1$

$f(0) = -2$

and $f'(0) = 3 + \sin(0)$

$f'(0) = 3$

1st iteration

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0 - \frac{(-2)}{3}$$

$$x_1 = 0.6667$$

2*nd* iteration:

$$f(x_1) = f(0.6667) = 3(0.6667) - \cos 0.6667 - 1$$
$$f(x_1) = 0.000167$$
$$f(x_1) = 3 + \sin(0.6667)$$
$$f(x_1) = 3.01163$$

Now

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$x_2 = 0.6667 - \frac{0.000167}{3.01163}$$
$$x_2 = 0.6667 - 0.0000554$$
$$x_2 = 0.6666$$
$$f(0.6666) = 3(0.6666) - \cos(0.6666) - 1$$
$$f(0.6666) = 0.000032$$

Hence 0.6666 is the approximate root correct up to four decimal places.

Exercise 12.3

Use Newton Raphson method to find a real root of following functions:

1. $f(x) = 2x^3 - 2x - 5$ up to three iterations with initial guess $x_0 = 2$.
2. $f(x) = x^3 - x - 1$ up to three iterations with initial guess $x_0 = 1$.
3. $f(x) = x^3 - 2x - 5$ up to two iterations with initial guess $x_0 = 1$.
4. $f(x) = 2\,cos\,x - x, x_0 = 0$
5. $f(x) = 2^x - x - 1.7, x_0 = 1.5$
6. $f(x) = 3x - e^x, x_0 = 0$
7. $f(x) = 3x - \sqrt{1 + sin\,x}, x_0 = 1$

12.2 Numerical Quadrature

Quadrature refers to any method for numerically approximating the value of a definite integral

$$\int_a^b f(x)dx.$$

The estimated calculation of an integral using numerical technique is known as numerical integration.

12.2.1 Define numerical quadrature. Use:

- **Trapezoidal rule,**
- **Simpson's rule, to compute the approximate value of definite integrals without error terms.**

Trapezoidal Rule

The Trapezoidal rule is an integration rule that evaluates the area under the curve by dividing the total area into smaller trapezoids rather than using rectangles.

Trapezoidal Rule Formula

To prove the trapezoidal rule, consider a curve as shown in the figure 12.1 above and divide the area under that curve into trapezoids.

Let $f(x)$ be a continuous function on the interval $[a,b]$. Now divide the interval $[a,b]$ into n sub intervals with each of equal width Δx.

We use here formula to calculate the width $\Delta x = h$ of each subinterval.

$\Delta x = h = \frac{b-a}{n}$ such that $a = x_0 < x_1 < x_2 < x_3 <, \ldots, < x_n = b$.

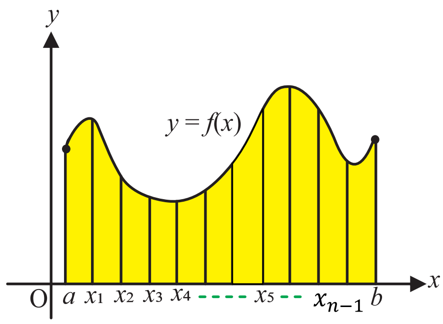

Fig. 12.1

We see that the first trapezoid has a height Δx and length of parallel base is the sum are $f(x_0)$ or y_0 and $f(x_1)$ or y_1 respectively.

Thus, the area of the first trapezoid in the above figure can be given as,

$$\frac{1}{2}\Delta x[f(x_0) + f(x_1)]$$

The areas of the remaining trapezoids are

$$\frac{1}{2}\Delta x[f(x_1) + f(x_2)], \frac{1}{2}\Delta x[f(x_2) + f(x_3)]$$

and so on.

Consequently,

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(f(x_0) + f(x_1)) + \frac{\Delta x}{2}(f(x_1) + f(x_2)) + f(x_0) +$$
$$\frac{\Delta x}{2}(f(x_2) + f(x_3)) + \ldots + \frac{\Delta x}{2}(f(x_{n-1}) + f(x_n))$$

After taking out a common factor of $\frac{\Delta x}{2}$ and combining like terms, we have,

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}[f(x_0) + 2(f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1})) + f(x_n)]$$

Then the Trapezoidal Rule formula for area approximating the definite integral $\int_a^b f(x)dx$ is given by:

If we take $y = f(x)$ and $\Delta x = h$ then,

$$\int_a^b ydx \approx T_n = \frac{h}{2}[y_0 + 2(y_1 + 2y_2 + \ldots + 2y_{n-1}) + y_n]$$

Example 1. Approximate the area under the curve $y = f(x)$ between $x = 0$ and $x = 8$ using Trapezoidal Rule with $n = 4$ subintervals. A function $f(x)$ and x values are given in the following table:

x	0	2	4	6	8
$f(x)$	3	7	11	9	3

Solution: As $f(x)$ and x are given in the table:

The Trapezoidal Rule formula for $n = 4$ subintervals is given as:

$$T_4 = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$$T_4 = \frac{h}{2}[f(x_0) + 2(f(x_1) + f(x_2) + f(x_3)) + f(x_4)]$$

Here the subinterval width $\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$

Now, substitute the values from the table to find the approximate value of the area under the curve.

$$A \approx T_4 = \frac{2}{2}[3 + 2(7 + 11 + 9) + 3]$$

$$A \approx T_4 = [6 + 2(27)]$$

$$A \approx T_4 = 60 \text{ unit square.}$$

Therefore, the approximate value of area under the curve using Trapezoidal Rule is 60.

Example 2. Evaluate $\int_2^7 \frac{1}{x} dx$ using Trapezoidal rule, taking subintervals $n = 5$.

Solution:

$$f(x) = \frac{1}{x}$$

Here $a = 2$ and $b = 7$, subintervals $n = 5$

Now, $h = \frac{b-a}{n} = \frac{7-2}{5} = 1$

The values of x and $f(x)$ are given in the following table:

x	2	3	4	5	6	7
$f(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$

OR

x	2	3	4	5	6	7
$f(x)$	0.5	0.3333	0.25	0.2	0.1667	0.1429

Using Trapezoidal Rule

$$T_5 = \frac{h}{2}[f(x_0) + 2(f(x_1) + f(x_2) + f(x_3) + f(x_4)) + f(x_5)]$$

$T_5 = \frac{1}{2}[0.5 + 2(0.3333 + 0.25 + 0.2 + 0.1667) + 0.1429]$

$T_5 = \frac{1}{2}[0.5 + 2(0.95) + 0.1429]$

$T_5 = 1.2714$

Example 3. Evaluate $\int_1^2 e^{\frac{1}{x}}\, dx$ using Trapezoidal rule with subintervals $n = 6$.

Solution:

$$f(x) = \int_1^2 e^{\frac{1}{x}}\, dx$$

Here $a = 1$ and $b = 2$, subintervals $n = 6$

Now, $h = \frac{b-a}{n} = \frac{2-1}{6} = 0.1667$

The values of x and $f(x)$ are given in the following table:

x	1	1.1667	1.3333	1.5	1.6667	1.8333	2
$f(x)$	2.7183	2.3564	2.117	1.9477	1.8221	1.7254	1.6487

Using Trapezoidal Rule

$T_6 = \frac{h}{2}[f(x_0) + 2(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)) + f(x_6)]$

$T_6 = \frac{0.1667}{2}[2.7183 + 2(2.3564 + 2.117 + 1.9477 + 1.8221 + 1.7254) + 1.6487]$

$T_6 = \frac{0.1667}{2}[2.7183 + 2(0.99687) + 1.6487] = 2.0254$ unit square.

Simpson's Rule

Simpson's rule is an extension of Trapezoid rule. It contains two different schemes Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule. Here, we discuss each one separately.

Simpson's $\frac{1}{3}$ Rule

Let $f(x)$ be a continuous function on $[a, b]$, then value of $\int_a^b f(x)\, dx$ by Simpson's $\frac{1}{3}$ rule is calculated by

$$\text{Simpson's } \frac{1}{3} = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\right]$$

Here h is the width of each interval and calculated by $h = \frac{b-a}{n}$

Where n denotes number of subintervals.

Note: The method is only valid if n is the multiple of 2.

Example 1. Evaluate $\int_0^2 2x\,dx$ using Simpson's 1/3 rule with $n = 6$.

Solution:

Let $f(x) = \int_0^2 2x\,dx$,

Take $a = 0, b = 2$, and $n = 6$.

$$h = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3} = 0.3333$$

The values of $f(x)$ at x are given in the following table:

x	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$
$f(x)$	0	0.6667	1.3333	2	2.6667	3.3333	4

Use Simpson's 1/3 rule by taking $y = f(x)$ for $n = 6$

$$S\frac{1}{3} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$S\frac{1}{3} = \frac{0.3333}{3}[(0+4) + 4(0.6667 + 2 + 3.3333) + 2(1.3333 + 2.6667)]$$

$$S\frac{1}{3} = \frac{0.3333}{3}[(0+4) + 4(6) + 2(4)] = 4 \text{ square unit.}$$

Example 2. Evaluate $\int_0^1 e^x\,dx$ by Simpson's 1/3 rule with $n = 6$.

Solution:

Let $f(x) = \int_0^1 e^x\,dx$,

Take $a = 0, b = 1$, and $n = 6$.

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6} = 0.667$$

The values of $f(x)$ at x are given in the following table:

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$
$f(x)$	1	1.1814	1.3956	1.6487	1.9477	2.301	2.7183

Use Simpson's 1/3 rule by taking $y = f(x)$

$$S\frac{1}{3} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$S\frac{1}{3} = \frac{0.1667}{3}[(1 + 2.7183) + 4(1.1814 + 1.6487 + 2.301) + 2(1.3956 + 1.9477)]$$

$$S\frac{1}{3} = \frac{0.1667}{3}[(1 + 2.7183) + 4(5.1311) + 2(3.3433)]$$

$$S\frac{1}{3} = 1.7183 \text{ square units.}$$

Example 3. Evaluate $\int_1^2 e^{x^3}\,dx$ using Simpson's 1/3 rule by taking $n = 4$.

Solution:

Let $f(x) = \int_1^2 e^{x^3}\,dx$

Take $a = 1, b = 2$, and $n = 4$.

$$h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

The values of $f(x)$ at x are given in the following table:

x	1	1.25	1.5	1.75	2
$f(x)$	2.7183	7.0507	29.2243	212.592	2980.958

Use Simpson's 1/3 rule

$$S\frac{3}{8} = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$$

$$S\frac{3}{8} = \frac{0.25}{3}[(2.7183 + 2980.958) + 4(7.0507 + 212.592) + 2(29.2243)]$$

$$S\frac{3}{8} = \frac{0.25}{3}[(2.7183 + 2980.958) + 4(219.6427) + 2(29.2243)]$$

$$S\frac{3}{8} = 326.7246 \text{ square units.}$$

Simpson's $\frac{3}{8}$ Rule

Let $f(x)$ be a continuous function on $[a, b]$ there values of $\int_a^b f(x)$ by Simpson's $\frac{3}{8}$ rule is calculated by

$$S\frac{3}{8} = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})]$$

Here h is the length of each interval and calculated by $h = \frac{b-a}{n}$

Where n is the number of subintervals. The formula is used when n is the multiple of 3.

Example 1. Evaluate $\int_0^1 e^x\,dx$ by Simpson's 3/8 rule with $n = 6$.

Solution:

Let $f(x) = \int_0^1 e^x\,dx$,

Take $a = 0, b = 1$, and $n = 6$.

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6} = 0.667$$

The values of $f(x)$ at x are given in the following table:

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$
$f(x)$	1	1.1814	1.3956	1.6487	1.9477	2.301	2.7183

Use Simpson's 1/3 rule by taking $y = f(x)$

$$S\frac{3}{8} = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$S\frac{3}{8} = \frac{3(0.1667)}{8}[(1 + 2.7183) + 2(1.6487) + 3(1.1814 + 1.3956 + 1.9477 + 2.301)]$$

$$S\frac{3}{8} = \frac{3(0.1667)}{8}[(1 + 2.7183) + 2(1.6487) + 3(6.8257)]$$

$$S\frac{3}{8} = 1.7183 \text{ unit square}$$

Example 2. Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ by Simpson's 3/8 rule with $n = 6$.

Solution:

$$f(x) = \int_0^6 \frac{1}{1+x^2} dx$$

Here $a = 0$ and $b = 6$, Subintervals $n = 6$

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

The values of x and $f(x)$ are given in the following table:

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027

Using Simpson's 3/8 Rule

$$S\frac{3}{8} = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$S\frac{3}{8} = \frac{3 \times 1}{8}[(1 + 0.027) + 2(0.1) + 3(0.5 + 0.2 + 0.0588 + 0.0385)]$$

$$S\frac{3}{8} = \frac{3 \times 1}{8}[(1 + 0.027) + 2(0.1) + 3(0.7973)]$$

$$S\frac{3}{8} = 1.3571 \text{ unit space}$$

12.2.2 Use MAPLE command Trapezoid for trapezoidal rule and SIMPSON for Simpson rule and demonstrate through examples

The trapezoidal rule to compute an approximation to a definite integral. The call trapezoid $(f(x), x, n)$ finds an approximation to the definite integral $\int_a^b f(x)dx$ using n subdivisions of the interval $[a, b]$. We use the trapezoidal rule with $n = 12$ to find an approximation to $\int_a^b \frac{1}{\sqrt{1+x^4}}\,dx$,

where, $f(x)$ stands for an algebraic expression in x

x variable of integration

a lower bound on the range of integration

b upper bound on the range of integration

n stands for the number of trapezoids to use(optional)

Note: All above operators should be taken from the Maple calculus pallets.

Trapezoidal Rule

$> with(Student[Calculus1]):$

$> ApproximateInt\left(\frac{1}{\sqrt{1+(x)^4}}, x = 0..1, method = trapezoid, output = plot\right);$

An approximation of $\int_0^1 f(x)\,dx$ using Trapezoid rule, where $f(x) = \frac{1}{\sqrt{x^4+1}}$ and the partition is uniform. The approximate value of the integral is 0.9264474916. Number of subintervals used: 10.

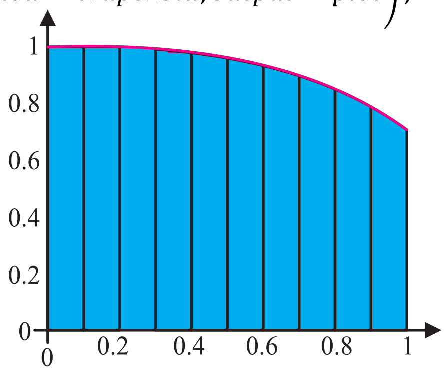

Fig. 12.2

$> ApproximateInt$
$\left(1 + exp(x), x = 0..3, method = trapezoid, output = plot, partition = 10\right)$

An approximation of $\int_0^3 f(x)\,dx$ using Trapezoid rule, where $f(x) = 1 + e^x$ and the partition is uniform. The approximate value of the integral is 22.22846420. Number of subintervals used: 10.

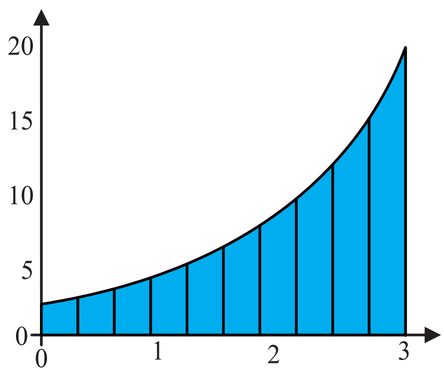

Fig. 12.3

$> ApproximateInt$
$\left(cos(x) - exp(-x), x = 0.5..3.5, method = trapezoid, output = plot, partition = 10\right)$

An approximation of

$$\int_{0.5}^{3.5} f(x)\,dx$$

using Trapezoid rule, where

$$f(x) = \cos(x) - e^{-x}$$

and the partition is uniform. The approximate value of the integral is -1.404622147. Number of subintervals used: 10.

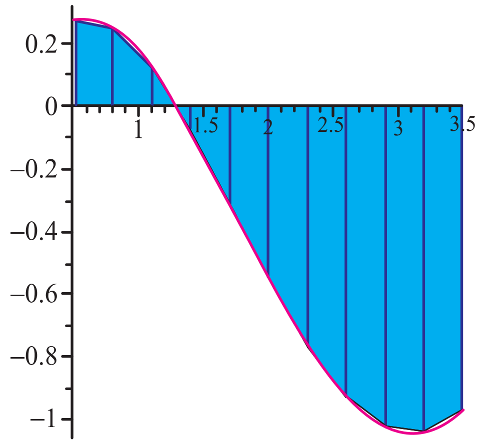

Fig. 12.4

Simpson Rule

$> ApproximateInt\left(1/(x^2+3*x+2), x = 0..3, method = simpson, output = plot, partition = 10\right)$

An approximation of $\int_0^3 f(x)\,dx$ using Simpson's rule, where $f(x) = \frac{1}{x^2+3x+2}$ and the partition is uniform. The approximate value of the integral is 0.4700185982. Number of subintervals used: 10.

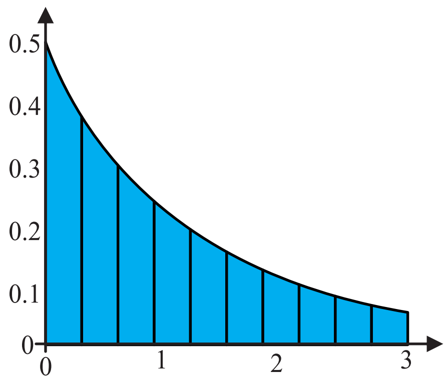

Fig. 12.5

$> ApproximateInt\left(exp(x^2), x = -1..1, method = simpson, output = plot, partition = 10\right)$

An approximation of $\int_{-1}^{1} f(x)\,dx$ using Simpson's rule, where $f(x) = e^{x^2}$ and the partition is uniform. The approximate value of the integral is 2.925362800. Number of subintervals used: 10.

Note: Before executing above commands, it is important to write with (student [calculus1]).

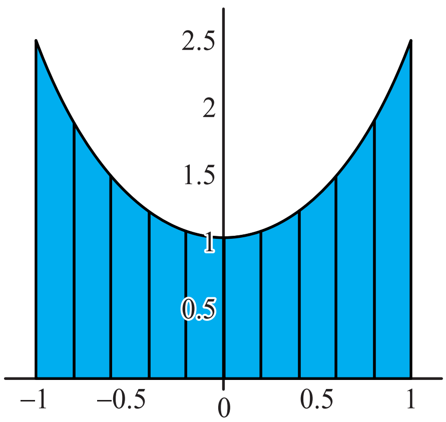

Fig. 12.6

Exercise 12.4

1. Evaluate the following integrals by Trapezoidal rule

(i) $\int_0^2 e^x\, dx$ with 6 intervals (ii) $\int_1^3 \frac{1}{\sqrt{x}} dx$ with 5 intervals

(iii) $\int_0^{\frac{\pi}{2}} \sin x\, dx$ with 7 intervals (iv) $\int_0^{1.4} e^{-x^2} dx$ with 5 intervals

2. Evaluate the following integrals by Simpson $\frac{1}{3}$ rule

(i) $\int_2^3 (4x^2+6)dx$, with 4 intervals (ii) $\int_0^2 \frac{1}{e^x} dx$, with 6 intervals

(iii) $\int_0^{\frac{\pi}{3}} \sqrt{\sin x}\, dx$ with 6 intervals (iv) $\int_0^1 \frac{2}{1+x^2} dx$ with 8 intervals

3. Evaluate the following integrals by Simpson $\frac{3}{8}$ rule

(i) $\int_1^3 \sqrt{x}\, dx$, with 6 intervals (ii) $\int_1^6 \frac{\ln x}{x} dx$, with three sub intervals

(iii) $\int_0^2 \frac{e^{2x}}{1+x^2} dx$ with 9 sub intervals (iv) $\int_0^{\frac{\pi}{4}} \sin x\, dx$ with 6 sub intervals

4. Write MAPLE Command Trapezoid for trapezoidal rule and SIMPSON for Simpson rule

(i) $\frac{x^2-2}{2}$, $x = 0..1$ $n = 10$, method trapezoiadal rule

(ii) $\sqrt{9+x^2}$, $x = 0..4$ $n = 10$, method trapezoiadal rule

(iii) $\frac{1}{x^2+4x+3}$, , $x = 0..2$ $n = 10$, method simpson's rule

(iv) e^{-x^2} , $x = 0..2$ $n = 10$, method simpson's rule

Review Exercise 12

1. Select the correct option.

(i) If real root of an equation $f(x) = 0$ lies in the interval $[a, b]$ then $f(a)f(b)$ will be

(a) > 0 (b) < 0 (c) $= 0$ (d) All of them

(ii) In bisection method, the approximate root is a/an __________ of end point of an interval in which actual root lies

(a) Arithmetic mean (b) Geometric mean (c) Sum (d) Product

(iii) Iterative formula for False Position Method to solve the equation $f(x) = 0$ at interval $[a, b]$ is

(a) $\frac{af(a)-bf(b)}{f(a)-f(b)}$ (b) $\frac{af(b)-bf(a)}{a-b}$

(c) $\frac{af(a)-bf(b)}{f(b)-f(a)}$ (d) $\frac{af(b)-bf(a)}{f(b)-f(a)}$

(iv) The fastest method to solve the nonlinear equation numerically is
(a) Bisection Method (b) False Position Method
(c) Newton Raphson Method (d) Both a and b

(v) Newton Raphson Method fails when derivative value of $f(x)$ becomes
(a) > 0 (b) < 0 (c) $= 0$ (d) All of them

(vi) Iterative formula of Newton Raphson method of solve $f(x) = 0$ is _______________
(a) $x_{n+1} = x_n + \frac{f'(x_n)}{f(x_n)}$ (b) $x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}$
(c) $x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$ (d) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

(vii) Numerical integration comprises a broad family of algorithms for calculating the numerical value of a?
(a) Definite integral (b) Indefinite integral
(c) Simple integral (d) Compound integral

(viii) In Trapezoidal rule the number of sub interval is the multiple of:
(a) 0 (b) 1 (c) 2 (d) 3

(ix) In Simpson One Third Method, the number of subinterval is the multiple of
(a) 4 (b) 1 (c) 2 (d) 3

(x) The fastest method to solve the definite integral numerically is
(a) Trapezoidal Rule (b) Simpson One Third Rule
(c) Simpson Three Eight Rule (d) Both a and b

2. Using Bisection method find the root of $cosx - xe^x$ with a = 0 and b = 1, by taking 5 iterations.

3. Find a root for the equation $2e^x sinx = 1$ using the false position method and correct it to three decimal places with three iterations, taking $[1, 2]$ as an interval.

4. Find the cube root of 12 using the Newton Raphson method assuming $x_0 = 2.5$.

5. Solve $\int_0^1 cosx^2 dx$ using trapezoidal rule for $n = 5$.

6. Solve $\int_{-2}^4 e^{x^2} dx$ using Simpson one Third as well as Simpson Three Eight rule for $n = 6$.