Functions and Limits

Unit 2

2.1 Functions

We know that function is a rule or correspondence between two non-empty sets X and Y in such a way that, each element of X corresponds to one and only one element of Y. Here X is called the domain of the function and the set of corresponding elements of Y is called the range of the function.

2.1.1 Identify through graph the domain and range of a function

Graph of a function is useful to identify the domain and range of a function. The domain of the function consists of all the input values shown on the x-axis. The range is the set of possible output values shown on the y-axis.

For example, consider the graph of function as shown in Fig. 2.1. We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $\{x|x \in \mathbb{R} \land x \geq -5\}$. The graph extends vertically from 5 to downward without bound, so the range of the function is $\{y|y \in \mathbb{R} \land y \leq 5\}$.

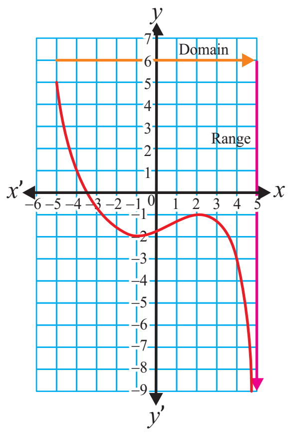


Fig. 2.1

Example 1. Identify the domain and range of the function through given graph.

Solution: The given graph is shown in Fig. 2.2.

We can observe that the graph extends horizontally from -3 to 1. So, the domain is $\{x|x \in \mathbb{R} \land -3 \leq x \leq 1\}$.

The graph extends vertically from 0 to -4. So, the range is $\{y|y \in \mathbb{R} \land -4 \leq y \leq 0\}$.

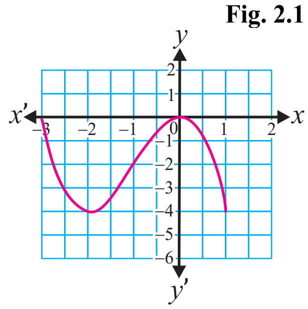


Fig. 2.2

Example 2. Identify the domain and range of the function through given graph.

Solution: The given graph is shown in Fig. 2.3.

The graph extends horizontally and vertically without and any bound. Thus, the domain and range of the function is $\{\mathbb{R}\}$. (Fig. 2.3)

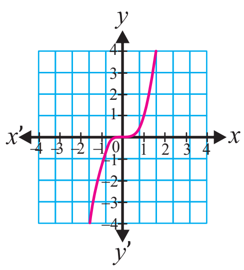


Fig. 2.3

2.1.2 Draw the graph of modulus function (i.e., $y = |x|$) and identify its domain and range

The modulus function $y = |x|$is defined as $|x| = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -x & \text{when } x < 0 \end{cases}$.

First, we draw the graph with the help of following table.

x	0	1	-1	2	-2	3	-3	...
$y = \|x\|$	0	1	1	2	2	3	3	...

By plotting these points on coordinate axes. We get, the graph of modulus function (Fig. 2.4). Now, we identify its domain and range with the help of graph.

The arrows indicate that the graph extends horizontally without any bound, so the domain is ℝ. While, the graph extends vertically from 0 to upward without any bound. So, its range is $\{y | y \in \mathbb{R} \wedge y \geq 0\}$.

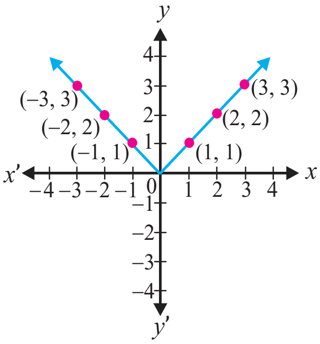


Fig. 2.4

2.2 Composition of Functions

Composition of functions is an operation or process where two functions f and g produce a new function h by replacing the variable of one function with other function.

2.2.1 Recognize the composition of functions

Let $f: \text{A} \to \text{B}$ and $g: \text{B} \to \text{C}$ be two functions. Then the composition of f and g, denoted by $\boldsymbol{gof}$, is defined as the function

$gof: \text{A} \to \text{C}$, given by $gof(x) = g(f(x)), \forall\, x \in \text{A}$.

The composition gof of functions f and g exists when Range $f \subseteq$ domain of g. The domain and range of composite function gof will be domain of f and range of g respectively as shown in the Fig. 2.5.

The order of function is an important while dealing with the composition of functions since $gof(x)$ is not equal to $fog(x)$ in general.

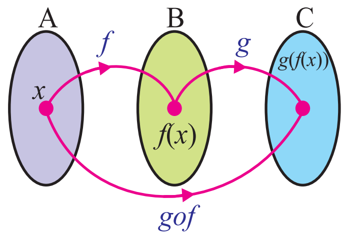


Fig. 2.5

2.2.2 Find the composition of two given functions

Example 1. If $f: R \longrightarrow R$ is a function which is defined as $f(x) = 3x + 1$ and $g: R \longrightarrow R$ is another function which is defined as $g(x) = x^2$. Find $fog(x)$.

Solution: Since range $g = R \subseteq$ Domain f, therefore fog exists.

The composition of f and g will be

$$fog(x) = f(g(x)) = f(x^2)$$
$$= 3(x^2) + 1.$$
$$fog(x) = 3x^2 + 1.$$

Example 2. $f(x) = 2x + 1$ and $g(x) = -x^2$, then find $gof(x)$ for $x = 2$.

Solution:

The composition of g and f will be

$$gof(x) = g(f(x)) = g(2x + 1)$$
$$= -(2x + 1)^2$$

Now,

$$gof(2) = -[2(2) + 1]^2$$
$$= -(5)^2$$
$$= -25$$

Example 3. If $R \longrightarrow [-1, 1]$ is sine function i.e., $s(x) = \sin x$ and $p(x)$ is a polynomial function i.e., $p(x) = x^2 + 5x + 7$ then find pos.

Solution:

$$pos(x) = p(s(x))$$
$$= p(\sin x)$$
$$= (\sin x)^2 + 5(\sin x) + 7$$
$$= \sin^2 x + 5\sin x + 7$$

2.3 Inverse of Composition of Functions

2.3.1 Describe the inverse of composition of two given functions

Let f and g are bijective functions then inverse of composition of f and g is the composition of g^{-1} and f^{-1}. Mathematically, $(fog)^{-1} = g^{-1}of^{-1}$.

Example: If $f(x) = \frac{x+1}{2}$ and $g(x) = 2x - 1$ are two given bijective functions then find the inverse of composition of f and g, also show that $(gof)^{-1} = f^{-1}og^{-1}$

Solution:

Here $f(x) = \frac{x+1}{2}$ and $g(x) = 2x - 1$

Now, we find $gof(x) = g(f(x))$

$$= g\left(\frac{x+1}{2}\right)$$
$$gof(x) = 2\left(\frac{x+1}{2}\right) - 1 = x$$
$$\Rightarrow \quad (gof)^{-1} = x$$

Now, we verify $(gof)^{-1} = f^{-1}og^{-1}$

$$f(x) = \frac{x+1}{2}$$

$\Rightarrow$ $\quad f^{-1}(x) = 2x - 1$

and $\quad g(x) = 2x - 1$

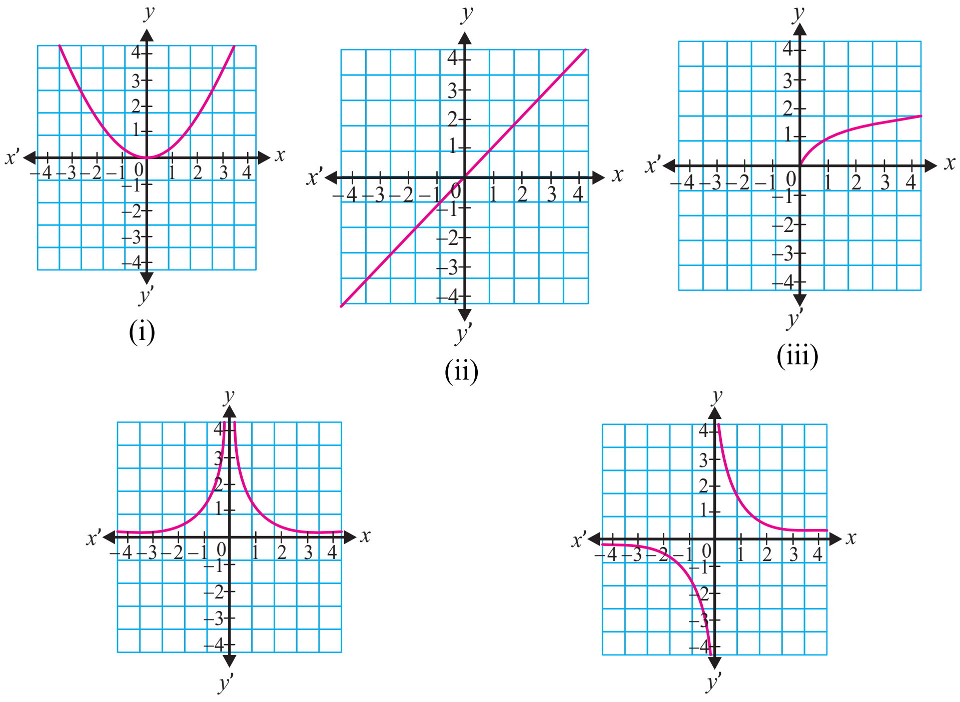
$\Rightarrow$ $\quad g^{-1}(x) = \frac{x+1}{2}$

$$f^{-1}og^{-1} = f^{-1}(g^{-1}(x)) = f^{-1}\left(\frac{x+1}{2}\right) = 2\left(\frac{x+1}{2}\right) - 1 = x$$

Hence $f^{-1}og^{-1} = (gof)^{-1}$ shown.

Exercise 2.1

1. Identify the domain and range of the functions through following graph.



(i) (ii) (iii) (iv) (v)

2. If $f(x) = 5x + 2$ and $g(x) = 2x^2 - 3$, then find

   (i) fog (ii) gof (iii) fof (iv) gog

3. If $f(x) = 2x$ and $g(x) = x + 1$, then find $fog(x)$ for $x = -5$.
4. If $f(x) = x + 3$ and $g(x) = x^2$, then find $gof(x)$ for $x = 1$.
5. If $c(x) = \cos x$ and $p(x) = x^3 + 1$ then find $poc(x)$.

6. Given that $f(x) = x + 2$ and $g(x) = 3x - 2$ are two given functions then find $(fog)^{-1}$ and $(gof)^{-1}$ also show that $(gof)^{-1} = f^{-1}og^{-1}$.

7. Given that $h(x) = x - 3$ and $k(x) = 2x + 5$ are two functions then verify that:
(i) $hok \neq koh$ (ii) $(hok)^{-1} = k^{-1}oh^{-1}$ (iii) $(koh)^{-1} = h^{-1}ok^{-1}$

2.4 Transcendental Functions

All the functions other than algebraic functions are transcendental functions. Like, $\sin x, \cos^{-1} x, \ln x, e^x$ and sinh$x$ etc.

2.4.1 Recognize algebraic, trigonometric, inverse trigonometric, exponential, logarithmic, hyperbolic (and their identities), explicit and implicit functions, and parametric representation of functions.

Some important types of functions are as under:

(a) Algebraic functions
(b) Transcendental functions
(c) Explicit and Implicit functions
(d) Parametric functions

(a) Algebraic functions:

Algebraic function is a function which is defined by algebraic expression that contain only algebraic operations. For example, $p(x) = x^2 + 5x + 7, q(x) = \frac{x+1}{x-2} + 7$ and $r(x) = \sqrt{x+1} + 8x^2 + 9$ are algebraic function.

(b) Transcendental functions: Exponential functions, logarithmic functions, trigonometric functions, hyperbolic functions, and inverse of all these functions are called transcendental functions.

(I) Exponential functions: If $f(x) = a^x$ where $a \in R^+$ and $a \neq 1$ then $f(x)$ is called an exponential function of x to the base a. For example, $f(x) = 3^x$; $f(x) = \left(\frac{1}{2}\right)^x ; h(x) = \left(\sqrt{5}\right)^x$ and $k(x) = (7)^{-x}$ are exponential functions.

The function e^x is called the natural exponential function where $e = 2.718281 \ldots$

(II) Logarithmic function: If $y = a^x$ where $a \in R^+$ and $a \neq 1$ then $\log_a y = x$ is called logarithmic function of y to the base a.

Note:

(i) $y = \log_{10} x$ is a Logarithmic function of base 10 which is called common logarithmic function.

(ii) $y = \log_e x$ or $y = \ln x$ is a Logarithmic function of base e which is called natural logarithmic function.

(iii) The Logarithmic function is the inverse of the exponential function.

(iv) The domain of Logarithmic function is $\mathbb{R}^+$ and its range is $\mathbb{R}$.

(v) $a^y = e^{y \cdot \ln a}$ (vi) $\log_a a = 1$; $\log_a 1 = 0$

(III) Trigonometric functions: We have already studied the six trigonometric functions in previous classes which are $\sin x, \cos x, \tan x, \sec x, \csc x$ and $\cot x$ are the trigonometric functions.

(IV) Inverse trigonometric functions: We have already studied the inverse trigonometric function in previous class. $\text{Sin}^{-1} x, \text{Cos}^{-1} x, \text{Tan}^{-1} x, \text{Sec}^{-1} x, \text{Csc}^{-1} x$ and $\text{Cot}^{-1} x$ are the inverse trigonometric functions.

(V) Hyperbolic functions: Hyperbolic functions are defined in a way similar to trigonometric functions. As the name suggests, the graph of a hyperbolic function represents a hyperbola. They are expressed in terms of exponential function e^x. There are six hyperbolic functions which are defined as under:

(i) $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$

is called sine hyperbolic function of x its domain and range are $\mathbb{R}$.

(ii) $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

is called cosine hyperbolic function of x its domain is $\mathbb{R}$ and range is $\{y | y \in \mathbb{R} \wedge y \geq 1\}$.

(iii) $y = \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

is called tangent hyperbolic function, its domain is $\mathbb{R}$ and range is $\{y | y \in \mathbb{R} \wedge -1 \leq y \leq 1\}$.

(iv) $y = \text{sech}\, x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

is called secant hyperbolic function of x its domain is $\mathbb{R}$ and range $\{y | y \in \mathbb{R} \wedge 0 < y \leq 1\}$.

(v) $y = \text{csch}\, x = \dfrac{1}{sinh\, x} = \dfrac{2}{e^x - e^{-x}}$

is called cosecant hyperbolic function of x, its domain and range $\{y | y \in \mathbb{R} \wedge y \neq 0\}$.

(vi) $y = \coth x = \dfrac{1}{\tanh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

is called cotangent hyperbolic function of x, its domain is $\{x | x \in \mathbb{R} \wedge x \neq 0\}$ and range is $\{y | y \in \mathbb{R} \wedge y \leq -1 \wedge y \geq 1\}$.

(VI) Inverse Hyperbolic functions:

The inverse hyperbolic functions are defined as under:

(i) $y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$ is inverse sine hyperbolic function its domain and range is $\mathbb{R}$.

(ii) $y = \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$, is inverse cosine hyperbolic function, its domain is $\{x|x \in \mathbb{R} \wedge x > 1\}$ and range is $\{y|y \in \mathbb{R} \wedge y \geq 0\}$.

(iii) $y = \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ is inverse tangent hyperbolic function, its domain is $\{x|x \in \mathbb{R} \wedge -1 < x < 1\}$ and range is $\mathbb{R}$.

(iv) $y = \text{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ is inverse secant hyperbolic function, its domain is $\{x|x \in \mathbb{R} \wedge 0 < x \leq 1\}$ and range is $\{y|y \in \mathbb{R} \wedge y \geq 0\}$.

(v) $y = \text{csch}^{-1} x = \text{In}\left(\frac{1}{x} + \frac{1+\sqrt{1+x^2}}{|x|}\right)$ is inverse cosecant hyperbolic function, its domain and range are $\{y|y \in \mathbb{R}, y \neq 0\}$.

(vi) $y = \coth^{-1} x = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|$

is inverse cotangent hyperbolic functions. Its domain is $\{x|x \in \mathbb{R} \wedge x \neq 1\}$ and range is $\{y|y \in \mathbb{R} \wedge y \neq 0\}$.

Identities of trigonometric and hyperbolic functions

	Trigonometric Identities	Hyperbolic Identities
i.	$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
ii.	$1 + \tan^2 x = \sec^2 x$	$1 - \tanh^2 x = \text{sech}^2 x$
iii.	$1 + \cot^2 x = \text{cosec}^2 x$	$\coth^2 x - 1 = \text{cosech}^2 x$
iv.	$\sin 2x = 2\sin x \cos x$	$\sinh 2x = 2\sinh x \cosh x$
v.	$\cos 2x = \cos^2 x - \sin^2 x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
vi.	$\cos 2x = 2\cos^2 x - 1$	$\cosh 2x = 2\cosh^2 x - 1$
vii.	$\cos 2x = 1 - 2\sin^2 x$	$\cosh 2x = 2\sinh^2 x + 1$
viii.	$\sin 3x = 3\sin x - 4\sin^3 x$	$\sinh 3x = 3\sinh x + 4\sinh^3 x$
ix.	$\cos 3x = 4\cos^3 x - 3\cos x$	$\cosh 3x = 4\cosh^3 x - 3\cosh x$
x.	$\sin(x \pm y)$ $= \sin x \cos y \pm \cos x \sin y$	$\sin\text{h}(x \pm y)$ $= \sinh x \cosh y \pm \cosh x \sinh y$
xi.	$\cos(x \pm y)$ $= \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y)$ $= \cosh x \cosh y \pm \sinh x \sinh y$
xii.	$\sin(-x) = -\sin x$	$\sinh(-x) = -\sinh x$
xiii.	$\cos(-x) = \cos x$	$\cosh(-x) = \cosh x$

(c) Explicit and Implicit functions:

(I) Explicit function: An explicit function is a function in which dependent variable y can be written explicitly only in terms of the independent variable x. Mathematically, it is written as $y = f(x)$. For example, $y = x - 1, y = e^x + \sin x$ etc.

(II) Implicit function: A function in which dependent variable y can not be expressed explicitly in terms of independent variable x. Both dependent variable y and independent variable x are mixed with each other where y cannot be expressed isolately as the function of x.

For example, $x^2 + xy + y^2 = 0$, where y is the implicit function of x.

(d) Parametric Representation of Function: A function can be represented parametrically by expressing the both dependent and independent variable as the functions of parameter such as t.

For example, $x = \cos t$ and $y = \sin t$ are the parametric representation of $x^2 + y^2 = 1$, here t is a parameter.

2.5 Graphical Representations

2.5.1 Display graphically:

The explicitly defined functions like $y = f(x)$, where

- $f(x) = e^x$,
- $f(x) = a^x$,
- $f(x) = log_a x$
- $f(x) = log_e x$

(i) $f(x) = e^x$

The graph of $y = e^x \ \forall \ x \in \mathbb{R}$.

It is observed from the graph of e^x, as shown in Fig. 2.6, the function e^x is increasing and its graph cuts the y-axis at $(0, 1)$.

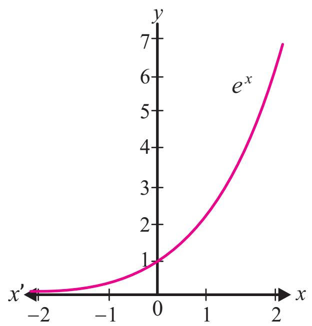


Fig. 2.6

(ii) $f(x) = a^x$

The graph of the function $f(x) = a^x$ is similar to the graph of e^x and its shape depends on the changing value of 'a'. The graph of $f(x) = a^x$ for different value of 'a' is show in Fig. 2.7.

The graph indicates that:

- If $0 < a < 1$ then $y = a^x$ is decreasing function.
- If $a > 1$ then $y = a^x$ is increasing function.

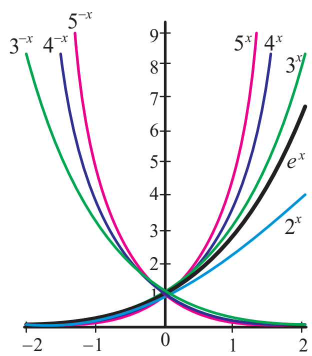


Fig. 2.7

- The curves $y = a^x$ close to the positive y-axis as $a > 0$ increases.

(iii) $f(x) = log_a x$

A function of the form $y = f(x) = log_a x$ where $a > 1$ is known as logarithmic function with base $'a'$. Its graph depends on the value of $'a'$.

Since the inverse function of $y = log_a x$ is the exponential function $y = a^x$, the graph of $y = log_a x$ is the reflection of $y = a^x$ with the line $y = x$ as shown in the Fig. 2.8.

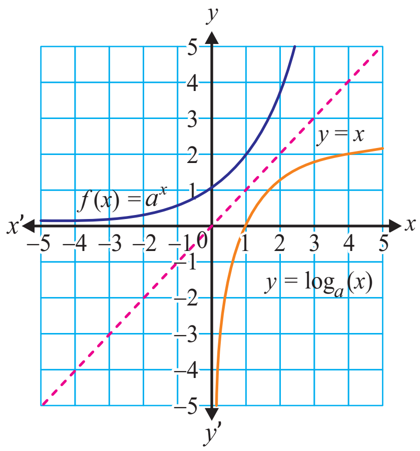


Fig. 2.8

The graph indicates that:

- The curve $y = log_a x$ cuts the x-axis at the point (1,0).
- the domain of the curve $y = log_a x$ is ℝ.
- the curves $y = log_a x$ approaches to negative y-axis as $x \in (0,1)$ as shown in Fig. 2.29.

- Changing the base b in $f(x) = log_b x$ can affect the graphs of $f(x)$. It is observed that the graph compresses vertically as the value of the base increases.
- $f(x)$ increases if $b > 1$, Fig. (2.10)
- $f(x)$ decreases if $0 < b < 1$, Fig.(2.11)

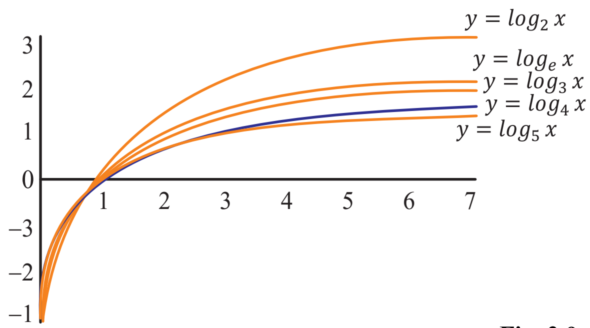


Fig. 2.9

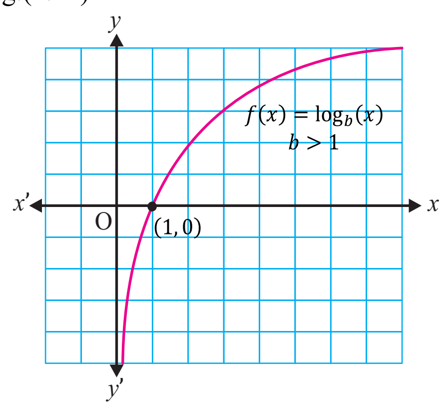


Fig. 2.10

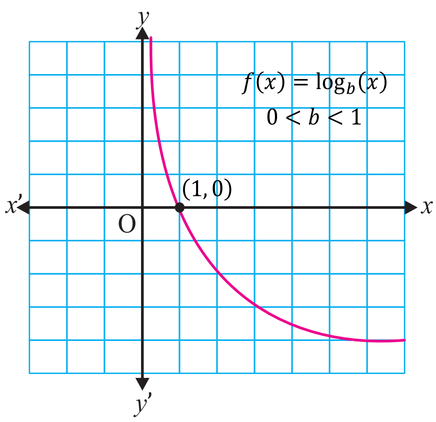


Fig. 2.11

(iv) $f(x) = log_e x$

Logarithmic function $f(x) = log_e x$ is known as natural Logarithmic function and represented by $\ln x$, which is displayed as in Fig. 2.12.

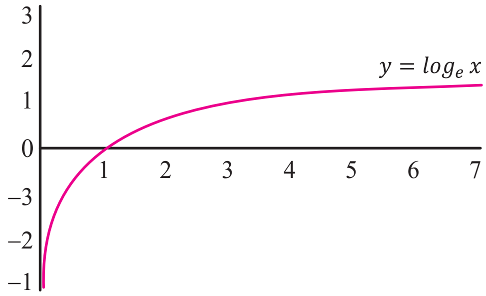


Fig. 2.12

- **Display graphically the implicitly defined functions such as $x^2 + y^2 = a^2$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and distinguish between graph of a function and an equation**

To display graphically the implicitly defined function, we solve the equation $f(x, y) = 0$ for y in terms of x where more than one function may be obtained.

Now, we draw the graphs of each function separately. Finally, by combining both graph, the graph of $F(x, y) = 0$ can be obtained.

For example, to display graphically $x^2 + y^2 = a^2$, first we solve y for x,

We get

$$y = \sqrt{a^2 - x^2} \text{ or } y = -\sqrt{a^2 - x^2}$$

Now, we separately draw the graph of each function as shown in Fig. 2.13 and 2.14.

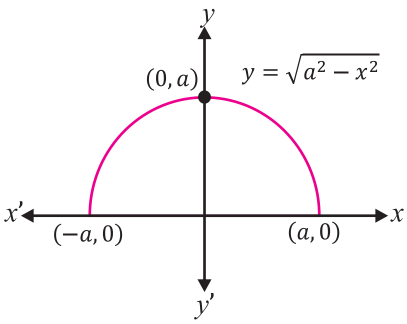


Fig. 2.13

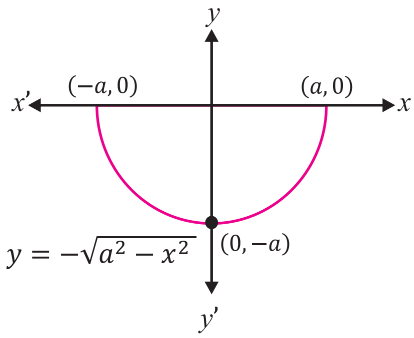


Fig. 2.14

Now, by combining both graphs, we get the graph of $x^2 + y^2 = a^2$ as shown in Fig. 2.15.

It is circle whose radius is 'a' unit and centre at the origin.

Similarly, we can display the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

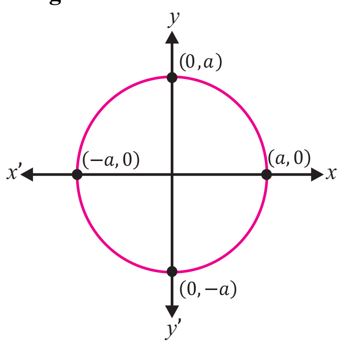


Fig. 2.15

It is an ellipse whose major axis is along x-axis and minor axis is along y-axis as shown in Fig 2.16.

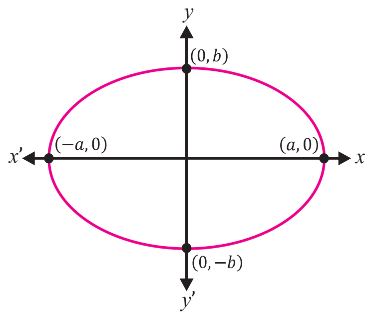


Fig. 2.16

Distinguish between graph of a function and an equation:

The graph of a function and an equation (implicitly defined function) can be distinguished by drawing vertical line on the same plane. If vertical line cuts the graph at only one point, then it is the graph of the function and if it cuts the graph at more than one point then it is the graph of the equation.

Example: Distinguish between the following graphs of function and equation.

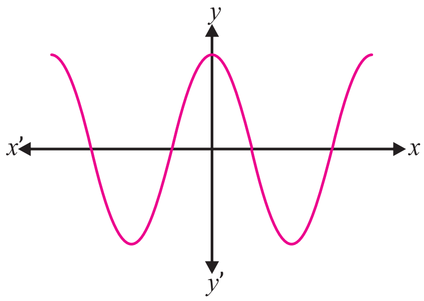


Fig. 2.17 (a)

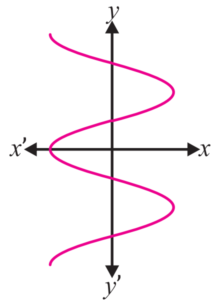


Fig. 2.17 (b)

We check it through vertical line test.

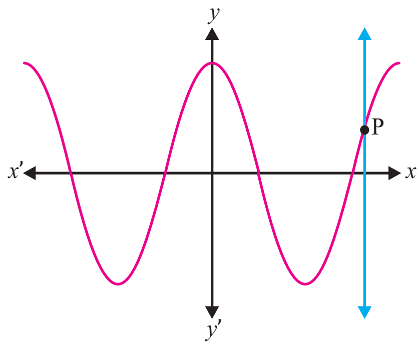


Fig. 2.18 (a)

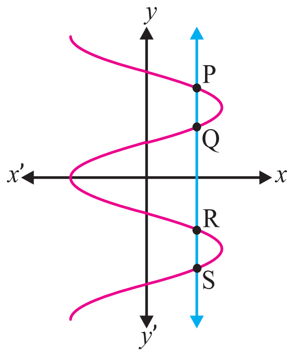


Fig. 2.18 (b)

In Fig. 2.18 (a) vertical line touches the graph at only one point P, so the Fig. 2.18 (a) represents the graph of the function.

In Fig. 2.18 (b) vertical line touches the graph at more than one point, so Fig. 2.18 (b) it is the graph of the equation.

- **Display graphically the parametric equations of functions such as $x = at^2,\ y = 2at\ ;\ x = a\sec\theta,\ y = b\tan\theta$**

Example 1. Draw the graph of parametric equations of function $x = at^2,\ y = 2at$, when $a = 2$ and $-3 \le t \le 3$

Solution: Parametric equations for $a = 2$ are

$$x = 2t^2 \qquad \ldots(i)$$

$$y = 4t \qquad \ldots(ii)$$

By constructing a table as $-3 \le t \le 3$

t	-3	-2	-1	0	1	2	3
$\boldsymbol{x = 2t^2}$	18	8	2	0	2	8	18
$\boldsymbol{y = 4t}$	-12	-8	-4	0	4	8	12

By plotting the points (x, y) on coordinate axes, we get the required graph Fig. 2.19.

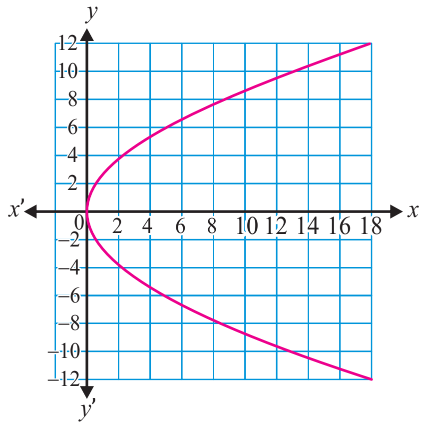


Fig. 2.19

Example 2. Draw the graph of parametric equations of function $x = a\sec\theta,\ y = b\tan\theta$, where $a = 5,\ b = 3$ and $-\pi \le \theta \le \pi$.

Solution:

Parametric equations for $a = 5$ and $b = 3$ are

$$x = 5\sec\theta \qquad \ldots(i)$$

$$y = 3\tan\theta \qquad \ldots(ii)$$

By constructing a table as $-\pi \le \theta \le \pi$

θ	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$x = 5\sec\theta$	-5	-5.8	-10	∞	10	5.8	5	5.8	10	∞	-10	-5.8	-5
$y = 3\tan\theta$	0	1.7	5.2	$-\infty$	-5.2	-1.7	0	1.7	5.2	∞	-5.2	-1.7	0

By plotting the points (x, y) on coordinate axes, we get the required graph of Fig. 2.20.

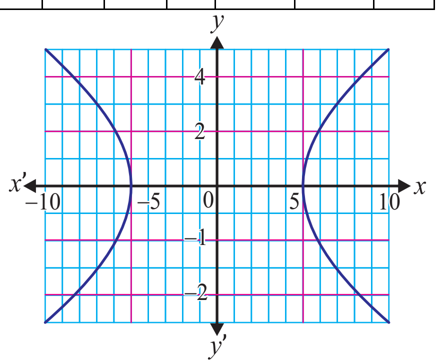


Fig. 2.20

- **Display graphically the discontinuous functions of the type**

$$y = \begin{cases} x & \text{when } 0 \le x < 1 \\ x - 1 & \text{when } 1 \le x \le 2 \end{cases}$$

Here we have a discontinuous function. Let us draw both the function at their respective interval.

- For $y = x$ when $0 \le x < 1$

By constructing a table as $0 \le x \le 1$

x	0	0.2	0.4	0.6	0.8	0.99
$y = x$	0	0.2	0.4	0.6	0.8	0.99

- For $y = x - 1$ when $1 \le x \le 2$

By constructing a table as $1 \le x \le 2$

x	1	1.2	1.4	1.6	1.8	2
$y = x - 1$	0	0.2	0.4	0.6	0.8	1

The graph of discontinuous function as shown in Fig. 2.21.

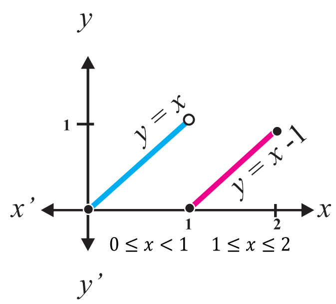


Fig. 2.21

2.5.2 Use MAPLE graphic commands for two-dimensional plot of

- **an expression (or a function),**
- **parameterized form**
- **implicit function,**

by restricting domain and range of a function

(a) An expression or a function (2D plot using MAPLE)

The standard scale for a Maple plot is x (horizontal axis) ranging from -10 to 10 and the vertical axis is based on the value of the function when x ranges from -10 to 10. The view option allows you to scale the axes in order to see details of interest.

$$> f2 := x \to x^4 + x^3 - 2.x^2 - 3$$

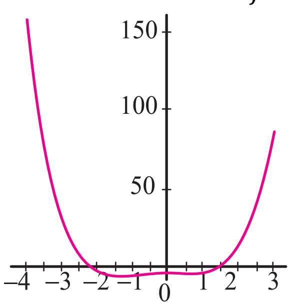


Fig. 2.22

$> plot\ (f2)$

2D plot of the function without x and y ranges

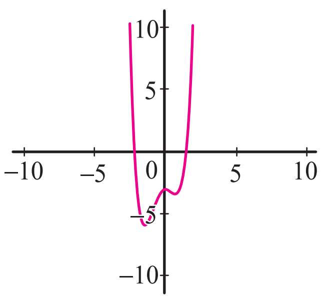


Fig. 2.23

$> plot(f2, view = [-10\,..\,10, -10\,..\,10])$

2D plot of the function with x and y ranges. (Restricted Domain)

(b) Parameterized form (2D plot using MAPLE)

Maple Command format for 2D plot of Parametric function is as under:

$> \mathrm{plot}([\mathrm{x}(\mathrm{t}), \mathrm{y}(\mathrm{t}), \mathrm{t}$
$= \text{range of t}], \mathrm{h}, \mathrm{v}, \text{options})$

Where,

$[x, y, \text{range}]$ is the parametric specifications

h and v are the horizontal and vertical ranges

Example:

$> \mathrm{plot}([\mathrm{t}^2, 2\mathrm{t}, \mathrm{t} = -3..3)])$

$\mathrm{x} = \mathrm{t}^2, \mathrm{y} = 2\mathrm{t}$ and range $\mathrm{t} = 3..3$

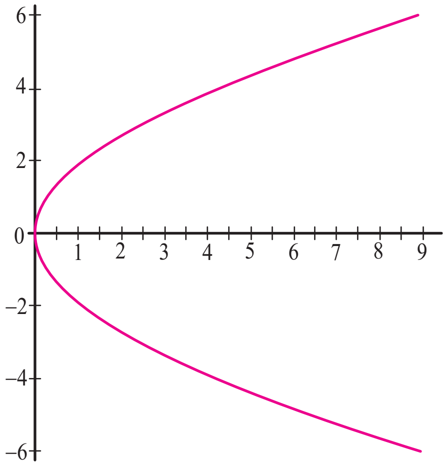


Fig. 2.24

(c) Implicit function form (2D plot using MAPLE)

Maple Command format for 2D plot of implicit function is as under:

$> \mathrm{with}(\mathrm{plots}, \mathrm{implicitplot})$

$> \mathrm{mplicitplot}(\mathrm{f}, \mathrm{x} = \mathrm{a}..\mathrm{b}.\ \ \mathrm{y} = \ \mathrm{c}..\mathrm{d}, \text{options})$

Where, f is the implicit function

$x = a..b$ and $y = c..d$ are the range on x and y-axis.

Example:

$> \mathrm{with}(\mathrm{plots}, \mathrm{implicitplot})$

$> \mathrm{impliciitplot}$

$\left(\left[\frac{x^2}{16} - \frac{y^2}{9} = 1, \mathrm{x} = -10..10, \mathrm{y} = -6..6\right)\right]\right)$

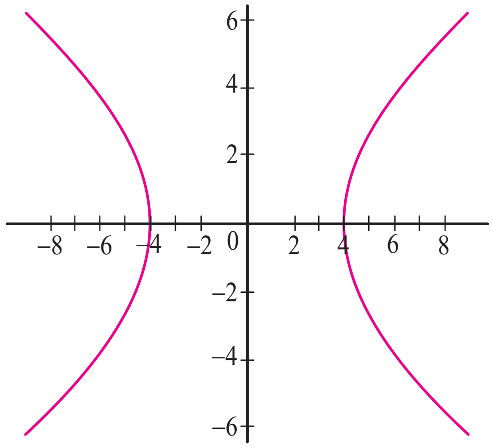


Fig. 2.25

2.5.2 Use MAPLE package plots for plotting different types of functions

Different type of functions is plotted with Maple package. The Maple command format is as under:

$> plot(f, x = x_0..x_1)$

f is a function and $x = x_0..x_1$ is the interval on x-axis.

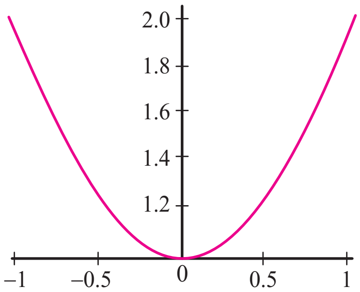


Fig. 2.26

Plot $(1 + x^2, x = -1..1)$

Algebraic function

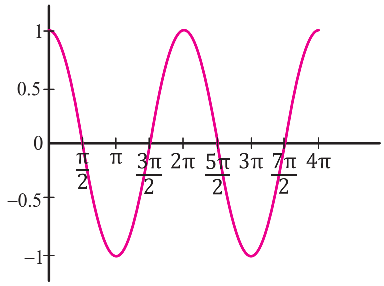


Fig. 2.27

Plot $(\cos(x), x = 0..4\pi)$

Trigonometric function

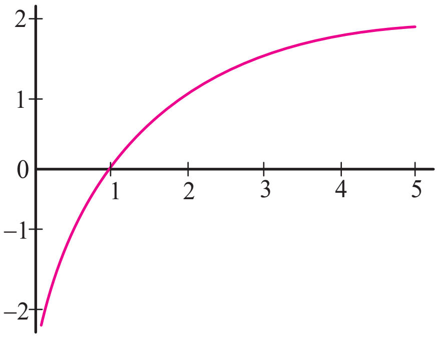


Fig. 2.28

Plot $([\ln(x)], x = -5..5, color = ["Red"])$

Logarithmic function

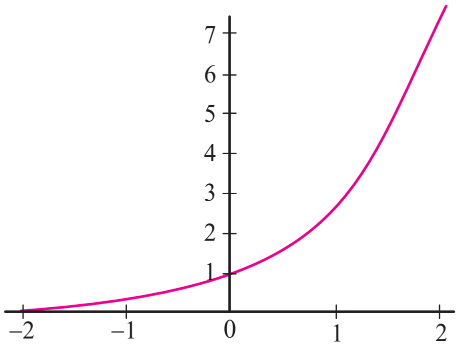


Fig. 2.29

Plot $([\exp(x)], x = -2..2, color = ["Red"])$

Exponential function

Exercise 2.2

1. Which of the following are algebraic, exponential, logarithmic, trigonometric, inverse trigonometric, hyperbolic and inverse hyperbolic functions.

(i) $y = x^2 + 5x + 6$	(ii) $f(x) = \tan^{-1}x$
(iii) $y = 2^{x+1}$	(iv) $y = \log_5(x + 2)$
(v) $f(x) = 3\sin x$	(vi) $y = a^{\sin x}$
(vii) $f(x) = \dfrac{x^2 + 5x + 7}{x + 9}$	(viii) $f(x) = \dfrac{\sin x}{\sec x}$
(ix) $y = \log_a \sin x$	(x) $f(x) = \text{cosec}^{-1}\sqrt{x^2 - 1}$
(xi) $f(x) = \tan(\sin x)$	(xii) $y = \dfrac{x}{x + 3}$
(xiii) $f(x) = \sinh x$	(xiv) $y = \ln \cosh x$
(xv) $y = \tan h^{-1}x$	(xvi) $y = \cos^{-1}(\ln x)$

2. Identify, whether the y is the explicit or implicit function of independent variable x if:

(i) $xy^2 + 5xy + 7 = 0$ (ii) $y = 3x^2 - 3x + 5$

(iii) $yx^2 + y^2x = 3 - 5y$ (iv) $x^2 + xy^2 = 2 + 3xy$

(v) $y = \dfrac{x+3}{x^2+5}$ (vi) $\dfrac{x}{y} = 3x^3y - 5$

3. Draw the graph of the following functions:

(i) $f(x) = e^{3x}$ (ii) $f(x) = 3log_{10}x$

(iii) $y = \sqrt{36 - x^2}$ (iv) $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

4. Draw the graph of parametric equations of function

$x = at^2,\ y = 2at$, when $a = 4$ and $-5 \le t \le 5$

5. Draw the graph of parametric equations of function

$x = asec\,\theta,\ y = btan\,\theta$, when $a = 3, b = 4$ and $-\pi \le \theta \le \pi$

6. Draw the graph of the $f(x) = \begin{cases} x^2 & x \le 1 \\ 2x & x > 1 \end{cases}$

2.6 Limit of a Function

2.6.1 Identify a real number by a point on the number line

A number which does not involve the square root of negative number is called real number, any real number x can be represented on a straight line by a point P such that the distance of P from a fixed-point O on the line is equal to $|x|$. The straight line is called the number line (Fig. 2.30).

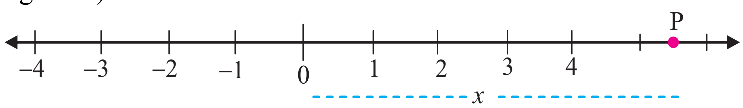


Fig. 2.30

For each real number there is a unique point and conversely for each point of the line, there is a real number, i.e., there is one to one correspondence between the set of real numbers and the set of point on the number line. So, every real number can be identified through a point on the number line.

2.6.2 Define and represent

- open interval
- closed interval
- semi open and semi-closed intervals, on the number line

- **Open interval:**

Let p and q be two real numbers with $p < q$ then the set of all real numbers x such that $p < x < q$ is called an *open interval* and denoted by $]p, q[$ or (p, q) (i.e., it does not include the endpoints p and q).

i.e., $(p, q) = \{x | x \in \mathbb{R} \wedge p < x < q\}$

and geometrically it is the set of points on the number line between p and q as shown in the Fig. 2.31.

Fig. 2.31

- **Closed interval:**

Closed interval is the set of all real numbers x such that $p \leq x \leq q$ and denoted by $[p, q]$ (i.e, it includes the endpoints p and q),

i.e., $[p, q] = \{x | x \in \mathbb{R} \wedge p \leq x \leq q\}$

and geometrically it is the line segment with end points p and q on the number line as shown in the Fig. 2.32.

Fig. 2.32

- **Semi open-Semi closed interval**

Semi open- semi closed interval is the set of all real numbers x such that $p < x \leq q$ and is denoted by $(p, q]$. It includes the end point q but not p,

i.e., $(p, q] = \{x | x \in R \wedge p < x \leq q\}$

and geometrically it is the set of all points between p and q where end point q is included and the end point p is excluded on the number line as shown in the Fig. 2.33.

Fig. 2.33

Similarly for $p \leq x < q$, we denote the interval by $[p, q)$ (i.e., it includes the end point p but not q), also defined as: $[p, q) = \{x | x \in R \wedge p \leq x < q\}$

and geometrically it is the set of all points between p and q where end point p is included and the end point q is excluded as shown in the Fig. 2.34.

Fig. 2.34

Note:

(i)	$[p, q] - (p, q) = \{p, q\}$	(ii)	$(p, q) - [p, q] = \{\}$
(iii)	$[p, q] \cup (p, q) = [p, q]$	(iv)	$[p, q] \cap (p, q) = (p, q)$
(v)	$\{p, q\} \cup (p, q) = [p, q]$	(vi)	$(-\infty, \infty) = \mathbb{R}$

Examples: Find the following

(i) $[-4, \infty) \cup (-2, 7)$ (ii) $[3, \infty) - (2, \infty)$ (iii) $(2, \infty) \cap (1, 3)$

(iv) $(-\infty, 4] - (2, \infty)$, $(-\infty, 3)$, $(-4, \infty)$ and $(-\infty, \infty)$

Solution:

(i) $[-4, \infty) \cup (-2,7) = [-4, \infty)$
(ii) $[3, \infty) - (2, \infty) = \{\ \}$
(iii) $(2, \infty] \cap (1,3) = (2,3)$
(iv) $(-\infty, 4] - (2, \infty) = (-\infty, 2]$
(v) $(-\infty, 4] - (-\infty, 3) = [3,4]$
(vi) $(-\infty, 4] - (-4, \infty] = (-\infty, -4]$
(vii) $(-\infty, 4] - (-\infty, \infty) = \{\ \}$

2.6.3 Explain the meaning of phrase

- x **tends to zero** $(x \longrightarrow 0)$,
- x **tends to** a $(x \longrightarrow a)$
- x **tends to infinity** $(x \longrightarrow \infty)$

Before the definition of the limit of a function, it is necessary to know the clear understanding of the meaning of the following phrases:

- x **tends to zero** $(x \longrightarrow 0)$

x tends to zero means x varies in such a way that its numerical value becomes 'closer' to 0 but not 0. Symbolically we write as $x \longrightarrow 0$.

- x **tends to** a $(x \longrightarrow a)$

x tends to a mean x varies in such a way that the numerical difference of x and a tends to 0. Symbolically $|x - a| \longrightarrow 0 \ \Rightarrow \ x \longrightarrow a$.

- x **tends to infinity** $(x \longrightarrow \infty)$

x tends to infinity means x increases without any bound in such a way that no real number exists which is greater than or equal to x. Symbolically, we write $x \longrightarrow \infty$.

2.6.4 Define limit of the sequence

Recall the definition of the sequence, it is a function whose domain is the set of natural numbers. Consider the sequence $a_1, a_2, a_3, \ldots a_n, \ldots$ denoted by $\{a_n\}$. If the terms of the sequence $\{a_n\}$ getting closer to a specific real number l as n tends to infinity, then l is called the limit of the sequence and is written as

$$\lim_{n\to\infty} a_n = l \text{ or } \lim a_n = l$$

If the value of a_n gets larger and larger without bound as n tends to infinity, then we say limit does not exist and we write

$$\lim_{n\to\infty} a_n = \infty$$

Nevertheless, if value of a_n gets smaller and smaller without bound as n tends to infinity, then limit also does not exist and we write

$$\lim_{n\to\infty} a_n = -\infty$$

For example, the limit of the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ will be 0 as each next term of the sequence decreases and becomes closer to 0.

Theorems:

- $\lim(c) = c$ where c is constant
- $\lim(c.a_n) = c.\lim\ a_n$
- $\lim(a_n + b_n) = \lim a_n + \lim b_n$
- $\lim(a_n - b_n) = \lim a_n - \lim\ b_n$
- $\lim(a_n b_n) = \lim a_n \lim\ b_n$
- $\lim\left(\frac{a_n}{b_n}\right) = \frac{lim\ a_n}{lim\ b_n}$

2.6.5 Find the limit of a sequence whose nth term is given

Example 1. Find the limit of the sequence $a_n = \frac{3n^2+5n+7}{5n^2-8n-11}$.

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2 + 5n + 7}{5n^2 - 8n - 11}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2\left(3 + \frac{5}{n} + \frac{7}{n^2}\right)}{n^2\left(5 - \frac{8}{n} - \frac{11}{n^2}\right)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3 + \frac{5}{n} + \frac{7}{n^2}}{5 - \frac{8}{n} - \frac{11}{n^2}}$$

$$= \frac{\lim_{n\to\infty}(3) + \lim_{n\to\infty}\left(\frac{5}{n}\right) + \lim_{n\to\infty}\left(\frac{7}{n^2}\right)}{\lim_{n\to\infty} 5 - \lim_{n\to\infty}\frac{8}{n} - \lim_{n\to\infty}\frac{11}{n^2}}$$

(Applying limit)

$$= \frac{3 + 0 + 0}{5 - 0 - 0}$$

$$\lim_{n\to\infty} a_n = \frac{3}{5}$$

Example 2. Find the limit of the sequence $a_n = \frac{5n+7}{9n^2+11}$

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n + 7}{9n^2 + 11}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n\left(5 + \frac{7}{n}\right)}{n^2\left(9 + \frac{11}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{5+\frac{7}{n}}{n\left(9+\frac{11}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{5+0}{n(9+0)} = 0$$

2.6.6 Define limit of a function

Limit of a function $f(x)$ at point a is the number L such that the values of the function get close to L as long as x becomes close enough to the point a.

Mathematically it is written as

$$\lim_{x\to a} f(x) = L$$

For example, to find the limit of the function $f(x) = \frac{1}{4}(x+1)(x-1)(x-5)$ at $x = 3$.

We find all the values of the function when x approaches to 3.

x	2.5	2.55	2.6	2.65	2.7	2.75	2.9	2.95
$f(x)$	−3.28	−3.37	−3.46	−3.54	−3.62	−3.69	−3.89	−3.95

When x approaches to 3, the values of function become close to -4, as shown in the above table.

Hence, the limit of the function at 3 is -4.

i.e., $$\lim_{x\to 3}\left[\frac{1}{4}(x+1)(x-1)(x-5)\right] = -4$$

Note: Let $p(x)$ is polynomial function, then $\lim_{x\to a} p(x) = p(a)$

2.6.7 State the theorems on limits of sum, difference, product and quotient of functions and demonstrate through examples

Let $f(x)$ and $g(x)$ be two functions defined on an open interval containing the number "a". If x approaches "a" both from left and right side of "a", $f(x)$ and $g(x)$ approaches, a specific numbers c and d, called the limit of the function $f(x)$ and $g(x)$ respectively. The same may be written as:

i.e., $\lim_{x\to a} f(x) = c$ and $\lim_{x\to a} g(x) = d$

Following theorems of limits or properties may be applied for finding the limit of the functions:

Theorem 1. (Limit of Sum of Functions)

The limit of the sum of functions is equal to the sum of their limits

i.e., $$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = c + d$$

Theorem 2. (Limit of Difference of Functions)

The limit of the difference of functions is equal to the difference of their limits

i.e., $$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = c - d$$

Theorem 3. (Limit of Product of Functions)

The limit on the Product of functions is equal to the product of their limits

i.e., $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x) = c.d$

Theorem 4. (Limit of Quotient of Functions)

The limit on the Quotient of functions is equal to the Quotient of their limits

i.e., $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{c}{d}$ where $g(x) \neq 0$

2.7 Important limits

2.7.1 Evaluate the limits of functions of the following types:

- $\frac{x^2-a^2}{x-a}$ and $\frac{x-a}{\sqrt{x}-\sqrt{a}}$ when $x \to a$
- $\left(1+\frac{1}{x}\right)^x$ when $x \to \infty$
- $(1+x)^{\frac{1}{x}}$, $\frac{\sqrt{x+a}-\sqrt{a}}{x}$, $\frac{a^x-1}{x}$, $\frac{(1+x)^n-1}{x}$ and $\frac{\sin x}{x}$ when $x \to a$

- **$\frac{x^2-a^2}{x-a}, \frac{x-a}{\sqrt{x}-\sqrt{a}}$ when $x \to a$**

(i) $\lim_{x\to a}\frac{x^2-a^2}{x-a} = \lim_{x\to a}\frac{(x-a)(x+a)}{x-a}$

$= \lim_{x\to a}(x+a) = 2a$

(ii) $\lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}} = \lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}}\,\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$

$= \lim_{x\to a}\frac{(x-a)}{\sqrt{x}-\sqrt{a}}\,\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim_{x\to a}\frac{(x-a)}{(\sqrt{x})^2-(\sqrt{a})^2}\left(\sqrt{x}+\sqrt{a}\right)$

$= \lim_{x\to a}\frac{(x-a)}{(x-a)}\left(\sqrt{x}+\sqrt{a}\right) = \lim_{x\to a}\left(\sqrt{x}+\sqrt{a}\right)$

$= 2\sqrt{a}$

- **$\left(1+\frac{1}{x}\right)^x$ when $x \to \infty$**

$$\left(1+\frac{1}{x}\right)^x \quad \text{when} \quad x \to \infty$$

By using Binomial Series

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left[1 + x\left(\frac{1}{x}\right) + \frac{x(x-1)}{2!}\left(\frac{1}{x}\right)^2 + \frac{x(x-1)(x-2)}{3!}\left(\frac{1}{x}\right)^3 + \cdots\right]$$

$$= \lim_{x\to\infty}\left[1+1+\frac{1}{2!}x^2\left(1-\frac{1}{x}\right).\frac{1}{x^2}+\frac{1}{3!}x^3\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right).\frac{1}{x^3}+\cdots\right]$$

$$= \lim_{x\to\infty}\left[1-1+\frac{1}{2!}\left(1-\frac{1}{x}\right)+\frac{1}{3!}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)+\cdots\right]$$

when $x \to \infty$, All $\frac{1}{x}, \frac{2}{x}, \frac{3}{x}$, ... tends to zero.

$$= \left[1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots\right] \qquad \because \; e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

$$e^1 = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$ [Approximate value if e is 2.718281]

- $(1+x)^{\frac{1}{x}}$, $\frac{\sqrt{x+a}-\sqrt{a}}{x}$, $\frac{a^x-1}{x}$, $\frac{(1+x)^n-1}{x}$ **and** $\frac{\sin x}{x}$ **when** $x \to 0$

(i) $(1+x)^{\frac{1}{x}}$ **when** $x \to 0$

Let $x = \frac{1}{y}$

As $x \to 0$ then $y \to \infty$

Now, $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y = e$

(ii) $\frac{\sqrt{x+a}-\sqrt{a}}{x}$ **when** $x \to 0$

$$\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x} = \lim_{x\to 0}\frac{(\sqrt{x+a}-\sqrt{a})}{x}.\frac{(\sqrt{x+a}+\sqrt{a})}{(\sqrt{x+a}+\sqrt{a})}$$

$$= \lim_{x\to 0}\frac{x+a-a}{x}.\frac{1}{(\sqrt{x+a}+\sqrt{a})}$$

$$= \lim_{x\to 0}\frac{1}{(\sqrt{x+a}+\sqrt{a})} = \frac{1}{2\sqrt{a}}$$

(iii) $\frac{a^x-1}{x}$ **when** $x \to 0$, $a > 0$, $a \neq 1$

To find $\lim_{x\to 0}\frac{a^x-1}{x}$

Let $a^x - 1 = y$ …(i)

$\Rightarrow$ $a^x = 1 + y$

$\Rightarrow$ $x = \log_a(1+y)$

From (i) when $x \to 0$ then $y \to 0$ we have

$$\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{y\to 0}\frac{y}{\log_a(1+y)}$$

$$= \lim_{y\to 0}\frac{1}{\frac{1}{y}\ \log_a(1+y)} = \lim_{y\to 0}\frac{1}{\log_a(1+y)^{\frac{1}{y}}}$$

$$= \frac{\lim\limits_{y\to 0}(1)}{\lim\limits_{y\to 0}\left[\ \log_a(1+y)^{\frac{1}{y}}\right]}$$

$$= \frac{1}{\log_a \lim\limits_{y\to 0}\left[(1+y)^{\frac{1}{y}}\right]} \qquad \because \lim_{x\to a}\log_b(f(x)) = \log_b\left(\lim_{x\to a}f(x)\right)$$

$$\lim_{x\to 0}\frac{a^x-1}{x} = \frac{1}{\log_a e} = \log_e a = \ln a \qquad \because \left[\lim_{y\to 0}(1+y)^{\frac{1}{y}} = e\right]$$

Corollary: If a is replaced by e, then above formula reduces to

$$\lim_{x\to 0}\frac{e^x-1}{x} = \ln e = 1$$

i.e., $$\boxed{\lim_{x\to 0}\frac{e^x-1}{x} = 1}$$

(iv) $\dfrac{(1+x)^n-1}{x}$ **when** $x \to 0$ **and** $n \in Q$

By using Binomial series

$$(1+x)^n = 1 + n(x) + \frac{n(n-1)}{2!}(x)^2 + \frac{n(n-1)(n-2)}{3!}(x)^3 + \cdots$$

$$\lim_{x\to 0}\frac{(1+x)^n-1}{x} = \lim_{x\to 0}\frac{\left[1+n(x)+\frac{n(n-1)}{2!}(x)^2+\frac{n(n-1)(n-2)}{3!}(x)^3+\cdots\right]-1}{x}$$

$$= \lim_{x\to 0}\frac{n(x)+\frac{n(n-1)}{2!}(x)^2+\frac{n(n-1)(n-2)}{3!}(x)^3+\cdots}{x}$$

$$= \lim_{x\to 0}\frac{x\left[n+\frac{n(n-1)}{2!}x+\frac{n(n-1)(n-2)}{3!}x^2+\cdots\right]}{x}$$

$$= \lim_{x\to 0}\left[n+\frac{n(n-1)}{2!}x+\frac{n(n-1)(n-2)}{3!}x^2+\cdots\right]$$

$$= n$$

(v) $\dfrac{\sin x}{x}$ **when** $x \to 0$

Consider a unit circle with circular sector OAB. If x is the angle measured in radian between the radial segment OA and OB, then it follows from the definitions of the trigonometric functions that $\overline{BD} = \sin x, \overline{OD} = \cos x,$ and $AC = \tan x$,. Also, x is the length of the arc AB from figure 2.35.

We have

Area of ΔODB <Area of sector OAB < Area of ΔOAC

i.e., $\frac{sinx\ cosx}{2} < \frac{x}{2} < \frac{tanx}{2}$

$cosx < \frac{x}{sinx} < \frac{1}{cosx}$ $\left[\text{dividing by } \frac{sinx}{2}\right]$

$\Rightarrow \frac{1}{cosx} > \frac{sinx}{x} > cosx$

or $\cos x < \frac{\sin x}{x} < \frac{1}{cosx}$

$\because \lim_{x\to 0} \frac{1}{\cos x} = 1 = \lim_{x\to 0} \cos x$

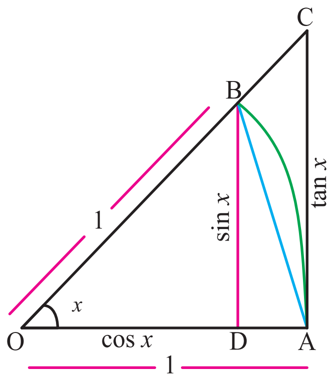


Fig. 2.35

∴ By sandwich theorem, we have

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

2.7.2 Evaluate limits of different algebraic, exponential and trigonometric functions

(a) The evaluation of the limits of algebraic and exponential functions:

(i) $\lim_{x\to 0} \frac{1}{x-1} = \frac{1}{0-1} = -1$

(ii) $\lim_{x\to \infty} \frac{1}{x+3} = 0$

(iii) $\lim_{x\to 0} (1 + \frac{x}{5})^{\frac{3}{x}} = \lim_{x\to 0} \left(1 + \frac{x}{5}\right)^{\frac{5}{x}\times\frac{3}{5}}$

$$= \left[\lim_{x\to 0} \left(1 + \frac{x}{5}\right)^{\frac{5}{x}}\right]^{\frac{3}{5}} = e^{\frac{3}{5}} \qquad \because \lim_{x\to 0} [1 + x]^{\frac{1}{x}} = e$$

(iv) $\lim_{x\to \infty} \left(1 + \frac{3}{x}\right)^{\frac{x}{5}} = \lim_{x\to \infty} \left(1 + \frac{3}{x}\right)^{\frac{x}{3}\times\frac{3}{5}}$

$$= \left[\lim_{x\to \infty} \left(1 + \frac{3}{x}\right)^{\frac{x}{3}}\right]^{\frac{3}{5}} = e^{\frac{3}{5}} \qquad \because \left[\lim_{x\to \infty} \left[1 + \frac{1}{x}\right]^{x} = e\right]$$

(v) $\lim_{x\to 0} \frac{3^x - 1}{x}$

As we know

$\because \lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$

$\therefore \lim_{x\to 0} \frac{3^x - 1}{x} = \ln 3$

(vi) $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{5x}$

We know

$$\because \quad \lim_{x\to 0}\frac{(1+x)^n-1}{x}=n$$

$$\therefore \quad \lim_{x\to 0}\frac{(1+x)^n-1}{5x}=\frac{1}{5}\lim_{x\to 0}\frac{(1+x)^n-1}{x}$$

$$=\frac{1}{5}(n)=\frac{n}{5}$$

(vii) $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$ where $n\in Q$

Let $x=a+h$, as $x\longrightarrow a$, we have $h\longrightarrow 0$

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{h\to 0}\frac{(a+h)^n-a^n}{h}=\lim_{h\to 0}\frac{a^n\left(1+\frac{h}{a}\right)^n-a^n}{h}$$

$$=a^n\lim_{h\to 0}\frac{1}{h}\left(1+n\left(\frac{h}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{a}\right)^2+\cdots-1\right)$$

$$=a^n\lim_{h\to 0}\left(\frac{n}{a}+\frac{n(n-1)}{2!}\frac{h}{a^2}+\cdots\right)$$

$$=na^{n-1} \qquad \text{(by applying limit)}$$

(b) The evaluation of the limits of trigonometric Functions:

(i) $$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{2\sin 2x}{2x}=2\lim_{2x\to 0}\frac{\sin 2x}{2x}=2$$

(ii) $$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{1-\cos x}{x}\times\frac{1+\cos x}{1+\cos x}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{x\sin^2 x}{x^2(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{x}{1+\cos x}.\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2=0\times 1=0$$

(iii) $$\lim_{x\to 0}\frac{\sin 5x}{\sin 3x}=\lim_{x\to 0}\frac{\frac{\sin 5x}{x}}{\frac{\sin 3x}{x}}=\frac{\lim\limits_{x\to 0}\frac{\sin 5x}{x}}{\lim\limits_{x\to 0}\frac{\sin 3x}{x}}=\frac{\lim\limits_{x\to 0}\frac{5\sin 5x}{5x}}{\lim\limits_{x\to 0}\frac{3\sin 3x}{3x}}=\frac{5\lim\limits_{5x\to 0}\frac{\sin 5x}{5x}}{3\lim\limits_{3x\to 0}\frac{\sin 3x}{3x}}=\frac{5}{3}$$

(iv) $$\lim_{x\to 0}\frac{\pi-x}{\cos(\pi-x)}=\frac{\lim\limits_{x\to 0}(\pi-x)}{\lim\limits_{x\to 0}[\cos(\pi-x)]}=\frac{\pi}{-1}=-\pi$$

2.7.3 Use MAPLE command limit to evaluate limit of a function

The format of limit command to evaluate limit of a function in MAPLE are as under:

> limit(f, x = a) for $\left[\lim_{x\to a} f\right]$

> Limit(f, x = a, dir)

Where,

f stands for function whose limit is to be evaluated

X=a stands for $x \to a$

dir means direction i.e., real/complex or left/right of a in $x \to a$

Examples:

$> limit\left(\frac{1}{x}, x = 5\right)$

$\frac{1}{5}$

$> limit\left(\frac{\sin(x)}{x}, x = 0\right)$

1

Directional limits are:

$> limit\left(\frac{1}{x}, x = 3\right)$

$\frac{1}{3}$

$> limit\left(\frac{1}{x}, x = 0, real\right)$

$undefined$

$> limit\left(\frac{1}{x}, x = 0, right\right)$

∞

$> limit\left(\frac{1}{x}, x = 0, left\right)$

$-\infty$

Limit of Piecewise functions:

$> g := piecewise(x < 3, x^2 - 6, 3 \le x, 2x - 1)$

$$g := \begin{cases} x^2 - 6 & x < 3 \\ 2x - 1 & 3 \le x \end{cases}$$

$> limit\ (g, x = 3)$

$undefined$

$> limit\ (g, x = 3, right)$

5

$> limit\ (g, x = 3, left)$

3

Exercise 2.3

1. Find the following:

(i) $[2,\infty) \cup (3,5)$ (ii) $[-1,1] - (2,\infty)$ (iii) $(5,\infty) \cap (-3,6)$

(iv) $[3,5] - (3,5)$ (v) $[1,10] \cap [3,11]$ (vi) $(-\infty,5) - (-\infty,3)$

2. Find the nth term and limit of the following sequences

(i) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ (ii) $\frac{1.2}{3.4}, \frac{3.4}{5.6}, \frac{5.6}{7.8}, \ldots$

3. Find the limit of the following sequences whose nth terms are:

(i) $a_n = \frac{1+5n}{7n}$ (ii) $a_n = \frac{(3n-1)(n^4-n)}{(n^2+5)(n^3-7)}$ (iii) $a_n = \frac{(n+1)!}{n!-(n+1)!}$

4. Find limit of the function $y = \frac{5x}{x+1}$ for $x \to \infty$.

5. Evaluate:

(i) $\lim_{x\to 2}(x^5 + x^2 + x + 1)$ (ii) $\lim_{x\to 5}\left(\frac{1+x}{x^2}\right)$

(iii) $\lim_{x\to 1}[(2x^3 + 3x^2)(x+1)]$ (iv) $\lim_{x\to 5}\{(x+1) - (x^2 + 2x + 3)\}$

6. Evaluate the limits of following algebraic and exponential functions:

(i) $\lim_{x\to 1}(x^2+4)^3$ (ii) $\lim_{x\to 0}\frac{4}{x-4}$ (iii) $\lim_{x\to\infty}\frac{5}{5x+10}$

(iv) $\lim_{x\to 7}\frac{x^2-49}{x-7}$ (v) $\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}$ (vi) $\lim_{x\to 7}\frac{x-7}{\sqrt{x}-\sqrt{7}}$

(vii) $\lim_{x\to 0}\left(1+\frac{x}{3}\right)^{\frac{5}{x}}$ (viii) $\lim_{x\to 0}(1+ax)^{\frac{a}{x}}$ (ix) $\lim_{x\to\infty}\left(1+\frac{p}{x}\right)^{px}$

(x) $\lim_{x\to\infty}\left(1-\frac{pq}{x}\right)^{\frac{x}{p}}$ (xi) $\lim_{x\to 0}\frac{17^x-1}{x}$ (xii) $\lim_{h\to 0}\frac{(1+2h)^n-1}{5h}$

(xiii) $\lim_{x\to 0}\frac{2^x-4^x-8^x-1}{x+2}$ (xiv) $\lim_{x\to 0}\left(\frac{1+7x}{1-9x}\right)^{\frac{1}{x}}$ (xv) $\lim_{x\to 0}\frac{a^x-b^x}{x}$

(xvi) $\lim_{x\to 0}\frac{e^{-2x}-e^{-11x}}{x}$ (xvii) $\lim_{x\to 0}\frac{3e^{-5x}-5e^{-2x}+2}{x}$

(xviii) $\lim_{x\to 0}(1+3\tan x)^{\cot x}$

7. Evaluate the limits of following trigonometric functions:

(i) $\lim_{x\to 0}\frac{a\sin ax}{x}$ (ii) $\lim_{x\to 0}\frac{\sin\sqrt{a}x}{\frac{x}{\sqrt{a}}}$ (iii) $\lim_{x\to 0}(3\cos x + 2\tan x)^3$

(iv) $\lim_{x\to 0}\frac{3\sin x - x^3}{2x}$ (v) $\lim_{x\to 0}\frac{\sin px}{\sin qx}$ (vi) $\lim_{x\to 0}\frac{(2\pi-x)\sec(\pi-x)}{\frac{\pi}{2}}$

(vii) $\lim_{x\to 0}\frac{\sin x^0}{x}$ (viii) $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}$ (ix) $\lim_{x\to 0}\frac{\sin 3x\sin 5x}{7x^2}$

8. Evaluate the limits of the following functions:

(i) $\lim_{x\to 0}\frac{\sin^2\left(\frac{x}{2}\right)}{4x^2}$ (ii) $\lim_{x\to\infty}x\sin\frac{1}{x}$ (iii) $\lim_{x\to\infty}\left[\sqrt{x^2+x+1}-x\right]$

(iv) $\lim_{x\to 0}\frac{1-\cos^3 x}{\sin^2 x}$ (v) $\lim_{x\to 0}\frac{a^x+a^{-x}-2}{x^2}$ (vi) $\lim_{x\to 0}\frac{6^x-3^x-2^x+1}{x^2}$

(vii) $\lim_{x\to 3}\frac{\frac{x}{x+2}-\frac{3}{5}}{x-3}$ (viii) $\lim_{y\to 4}\frac{y^{\frac{5}{2}}-16y^{\frac{1}{2}}}{y-4}$ (ix) $\lim_{x\to 1}\frac{\frac{1}{\sqrt{x}}-1}{1-x}$

(x) $\lim_{x\to\pi}\frac{\sqrt{5+\cos x}-2}{\pi-x}$ (xi) $\lim_{x\to 1}x^{\frac{1}{(x-1)}}$ (xii) $\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}$

(xiii) $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2}$ (xiv) $\lim_{x\to e}\frac{\ln x-1}{x-e}$

2.8 Continuous and Discontinuous Functions

If the graph of function takes a sudden jump or has a break at $x = x_o$, it is said to be discontinuous function at that point (Fig 2.36), if on the other hand, no such jump occurs then the function is said to be continuous (Fig 2.37).

Discontinuous function

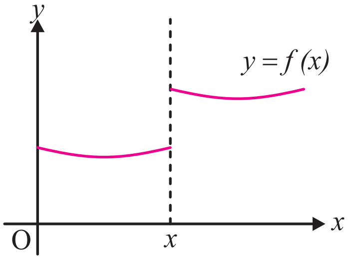


Fig. 2.36

Continuous function

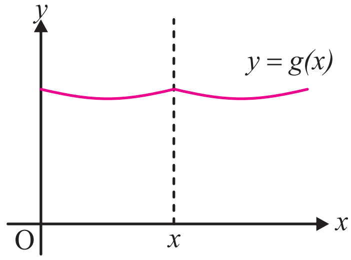


Fig. 2.37

2.8.1 Recognize left hand and right-hand limits and demonstrate through examples

There are two possible limits of the function at any point a. They are left hand limit and right-hand limit. When x approaches "a" from left side, the obtained limit is called left hand limit. It is written as

$$\lim_{x \to a^-} f(x) = m$$

Here m is the left-hand limit of function $f(x)$ at $x = a$.

Similarly, when x approaches "a" from right side, the obtained limit is called right hand limit. It is written as

$$\lim_{x \to a^+} f(x) = n$$

Here n is the right-hand limit of $f(x)$ at $x = a$.

The limit of the function $f(x)$ exists at $x = a$ if both left hand and right-hand limit exist and are equal i.e.,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = m = n = L$$

Example 1. Find left and right-hand limit of $f(x) = \frac{|x|}{x}$ at $x = 0$ and check the existence of the limit.

Solution:

Left hand limit $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} \qquad x < 0$

$$= \frac{-x}{x} = -1$$

Right hand limit $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x} \qquad x > 0$

$$= \frac{x}{x} = 1$$

$\because$ Left hand limit $\neq$ Right hand limit

Limit of the $f(x) = \frac{|x|}{x}$ does not exist at $x = 0$.

Example 2. Find left and right-hand limit of $f(x) = \begin{cases} 2x+1, & x < 2 \\ x, & x = 2 \\ 3x-1, & x > 2 \end{cases}$ at $x = 2$ and check the existence of the limit.

Solution:

Left hand limit $= \lim_{x \to 2^-} f(x)$ as $x \longrightarrow 2^-$ $\Rightarrow$ $x < 2$

$\therefore$ $= \lim_{x \to 2^-} (2x + 1)$

$= 2(2) + 1 = 5$

Right hand limit $= \lim_{x \to 2^+} f(x)$ as $x \longrightarrow 2^+$ $\Rightarrow$ $x > 2$

$\therefore$ $= \lim_{x \to 2^+} (3x - 1)$

$= 3(2) - 1 = 5$

$\because$ Left hand limit $=$ Right hand limit

$\therefore$ $= \lim_{x \to 2} f(x)$ exists.

Example 3. Check the existence of the limit for $f(x) = \frac{x}{\sqrt{1 - \cos x}}$ at $x = 0$.

Solution:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x\sqrt{1 + \cos x}}{\sqrt{1 - \cos x} \cdot \sqrt{1 + \cos x}} \quad \left[\text{Multiplying and dividing by } \sqrt{1 + \cos x}\right]$$

$$= \lim_{x \to 0} \frac{x\sqrt{1 + \cos x}}{\sqrt{(1 - \cos x)(1 + \cos x)}}$$

$$= \lim_{x \to 0} \frac{x\sqrt{1 + \cos x}}{\sqrt{1 - \cos^2 x}}$$

$$= \lim_{x \to 0} \frac{x\sqrt{1 + \cos x}}{\sqrt{\sin^2 x}}$$

$$= \lim_{x \to 0} \frac{x\sqrt{1 + \cos x}}{|\sin x|}$$

Now, we find the Left hand and Right hand limits to check the existence of the limit.

$$\text{LHL} = \lim_{x \to 0^-} \frac{x\sqrt{1 + \cos x}}{|\sin x|} = \lim_{x \to 0^-} \frac{x\sqrt{1 + \cos x}}{-\sin x} \qquad (\because x < 0 \Rightarrow \sin x < 0)$$

$$= -\lim_{x \to 0^-} \left(\frac{x}{\sin x}\right) \lim_{x \to 0^-} \sqrt{1 + \cos x} \qquad \left(\because \lim_{x \to 0^-} \frac{x}{\sin x} = 1\right)$$

$\therefore \quad = (-1)\sqrt{2} = -\sqrt{2}$

$$\text{RHL} = \lim_{x\to 0^+} \frac{x\sqrt{1+cos\,x}}{|\sin x|} = \lim_{x\to 0^+} \frac{x\sqrt{1+cos\,x}}{\sin x} \qquad (\because x > 0 \Rightarrow \sin x > 0)$$

$$= \lim_{x\to 0^+}\left(\frac{x}{\sin x}\right) \lim_{x\to 0^+}\sqrt{1+cos\,x} \qquad \left(\because \lim_{x\to 0^+}\frac{x}{\sin x} = 1\right)$$

$\therefore \quad = (1)\sqrt{2} = \sqrt{2}$

$\because LHL \neq RHL$

$\therefore \quad \lim_{x\to 0}\frac{x}{\sqrt{1-cos\,x}}$ does not exist.

2.8.2 Define continuity of a function at a point and in an interval

(a) Continuity of a function at a point.

A function $f(x)$ is continuous at the point $x = a$ if it satisfies the following conditions:

(i) $f(a)$ is defined i.e, a is in the domain of $f(x)$.

(ii) $\lim_{x\to a} f(x)$ exists.

(iii) $\lim_{x\to a} f(x) = f(a)$

(b) Continuity of a function in an interval:

A function f is continuous over the open interval (a, b) iff it is continuous on every point in (a, b). The function $f(x)$ is continuous over the closed interval $[a, b]$ iff it is continuous on (a, b), the right-hand limit of f at $x = a$ is $f(a)$ and the left-hand limit of f at $x = b$ is $f(b)$.

(c) Discontinuity of a function at point

If a function f is not continuous at a point a then it is said to be discontinuity at a point a. Similarly, if a function is not continuous on interval, then it is called discontinuous on interval.

2.8.3 Test continuity and discontinuity of a function at a point and in an interval.

Example: Test the continuity and discontinuity of the following functions:

(i) $f(x) = \tan x + x^2 + 3x$ at a point $x = 0$

Solution:

Left hand limit $\lim_{x\to 0-} f(x) = \lim_{x\to 0-}(\tan x + x^2 + 3x), \qquad x < 0$

$= (\lim_{x\to 0-}\tan x + \lim_{x\to 0-} x^2 + 3\lim_{x\to 0-} x) \qquad \because \tan 0 = 0$

$= 0+0+0=0$ [Applying limit]

Right hand limit: $\lim_{x\to 0+} f(x) = \lim_{x\to 0+}(\tan x + x^2 + 3x), \qquad x > 0$

$$= (\lim_{x\to0+} \tan x + \lim_{x\to0+} x^2 + 3\lim_{x\to0+} x)$$

$$= 0 + 0 + 0 = 0 \qquad \text{[Applying limit]}$$

Left hand limit = Right hand limit

Limit exists at $x = 0$. Now the value of the function we have $f(x)$ at $x = 0$

$$f(x) = \tan x + x^2 + 3x$$

$$f(0) = \tan 0 + (0)^2 + 3(0)$$

$$f(0) = 0 + 0 + 0 = 0$$

Thus $\lim_{x\to0} f(x) = f(0)$

So, function is continuous at $x = 0$

(ii) $f(x) = \begin{cases} 2 + x, & \text{when } x < 3 \\ 5 - 2x, & \text{when } x \geq 3 \end{cases}$ at $x = 3$

Solution:

Left hand limit $\lim_{x\to3-} f(x) = \lim_{x\to3-} (2 + x), \quad x < 3$

$$= 2 + 3 = 5 \qquad \text{[Applying limit]}$$

Right hand limit $\lim_{x\to3+} f(x) = \lim_{x\to3+} (5 - 2x), \quad x \geq 3$

$$= 5 - 2(3) = -1 \qquad \text{[Applying limit]}$$

Left hand limit ≠ Right hand limit

Limit does not exist at point $x = 3$.

So, the function is discontinuous at $x = 3$

(iii) $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

Solution:

Left hand limit $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \left(\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right), \quad x < 0$

$$= \lim_{x\to0^-} \left[\frac{e^{\frac{1}{x}}\left(1 - \frac{1}{e^{\frac{1}{x}}}\right)}{e^{\frac{1}{x}}\left(1 + \frac{1}{e^{\frac{1}{x}}}\right)}\right] = \lim_{x\to0-} \frac{\left(1 - \frac{1}{e^{\frac{1}{x}}}\right)}{\left(1 + \frac{1}{e^{\frac{1}{x}}}\right)}$$

$$= \frac{(1-0)}{(1+0)} = 1 \qquad \text{[Applying limit]}$$

Similarly Right-hand limit $\lim_{x\to0^+} f(x) = \lim_{x\to0^-} \left(\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right) = 1, \quad x > 0$

∵ Left hand limit of $f(x)$ at $x = 0$ is equal to Right hand limit of $f(x)$ at $x = 0$

∵ limit exists at $x = 0$.

Now the value of the function we have $f(x)$ at $x = 0$

$f(x) = 1$ at $x = 0$

$f(0) = 1$

Thus $\lim_{x \to 0} f(x) = f(0)$

So, function is continuous at $x = 0$

Note: Polynomial functions are continuous on $(-\infty, \infty)$.

2.8.4 Use MAPLE command iscont to test continuity of a function at a point and in a given interval

In MAPLE we use following commands to test whether the expression or function is continuous or discontinuous at a point and in a given interval.

```
>iscont (expr,  x = a..b)
>iscont (expr,  x = a..b,  'closed' )
>iscont (expr,  x = a..b,  'open' )
```

Where,

expr is an algebraic expression

X is a variable name

a..b is a real interval

'closed' is (optional) indicates that endpoints should be checked

'open' is (optional) indicates that endpoints should not be checked(default)

Examples:

$> iscont\left(\frac{2}{x+1}, x = 1..2\right)$

$true$

$> iscont\left(\frac{2}{x+1}, x = -2..1\right)$

$false$

$> iscont\left(\frac{2}{x+1}, x = -1..1\right)$

$true$

$x = -1..1$ is an open interval, so at point $x = -1$ function is discontinuous but other points of the interval it is continuous. So, it is true.

$> iscont\left(\frac{1}{x-1}, x = 1..2\right)$

$true$

$> iscont\left(\frac{1}{x-1}, x = -1..2\right)$

$false$

$> iscont\left(\frac{1}{x-1}, x = -\infty..\infty\right)$

$false$

$> iscont\left(\frac{1}{x-1}, x = 0..1\right)$

$true$

$> iscont\left(\frac{1}{x-1}, x = 0..1\,'close'\right)$

$false$

$> iscont\ (\sec(x), x = 0..1)$

$true$

$> iscont\ (\sec(x), x = 0..2\pi)$

$false$

$> iscont\ (\text{piecewise}(x < 3, x + 8, 3 \leq x, x^2 + 2), x = 0..\infty)$

$true$

$> iscont\ (\text{piecewise}(x < 3, x + 2, 3 \leq x, x^2 + 2), x = 0..\infty)$

$false$

2.8.5 Application of continuity and discontinuity

We have numerous applications of continuity and discontinuity in our daily life. Few are given below.

(i) If we drop an ice cube in a glass of warm water the temperature of water continuously changes with the time and eventually approaches the room temperature where the glass is stored.

(ii) The human heart is also an example of application of continuity as it beats continuously even the person sleeps.

(iii) The continuous spreading of corona virus however, be controlled or discontinued through precautionary measures such as social distancing, wearing mask and vaccination.

(iv) Population growth is a continuous process and be measured by an exponential function known as population growth model.

Example: The profit obtained by wholesaler of biscuits is given by continuous function $p(x) = \frac{x^2-4}{x-2}$, here x denotes the number of packets of biscuits. Find the profit of wholesaler for selling two packets of biscuits.

Solution: Since, the function $p(x) = \frac{x^2-4}{x-2}$ is continuous, therefore $\lim_{x\to 2} \frac{x^2-4}{x-2} = p(2)$

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)}$$

$$= \lim_{x\to 2} x + 2 = 4 = p(2)$$

Hence, the wholesaler would obtain profit of Rs. 4 for selling of two packets of biscuits to retailer.

Application of discontinuity of function

Discontinuity of function plays a significant role in various real-life scenarios. Here are some practical applications of discontinuity of function in daily life.

1. Electrical circuits: In electronics and electrical engineering, functions often describe the relationship between voltage, current, and resistance in a circuit. Discontinuities in these functions can represent abrupt changes, such as a switch turning on or off, or a diode transitioning between conducting and non-conducting states.

2. Stock market and finance: Financial data often exhibits discontinuities due to sudden price changes, market openings and closings.

3. Population growth and decay: In demography and biography, functions that describe population growth or decay may experience discontinuities due to sudden events line disease outbreaks, natural disasters, or population control measures.

4. Internet and network traffic: Data transmission rates in computer networks can experience discontinuities when network congestion occurs or when there are abrupt changes in data flow, such as a sudden spike in website traffic.

Exercise 2.4

1. Evaluate the following limits.

(i) $\lim\limits_{x\to 2^+} \frac{x-2}{|x-2|}$ (ii) $\lim\limits_{x\to 1^-} \frac{x^2+2x-3}{|x-1|}$ (iii) $\lim\limits_{x\to 2} \frac{x^2+4x-12}{|x-2|}$

2. Determine whether $\lim\limits_{x\to 1} f(x)$, $\lim\limits_{x\to 2} f(x)$, $\lim\limits_{x\to 3} f(x)$ and $\lim\limits_{x\to 4} f(x)$ exit, when

$$f(x) = \begin{cases} 2x+1 & if\ 0 \le x \le 2 \\ x-7 & if\ 2 \le x \le 4 \\ x & if\ 4 \le x \le 6 \end{cases}$$

3. Test the continuity and discontinuity of the following functions.

(i) $f(x) = \sin(x^2 + \pi x) + 7x^2 + x$ at a point $x = 0$

(ii) $f(x) = \frac{2-\cos 3x - \cos 4x}{x}$ at a point $x = 0$

(iii) $f(x) = \begin{cases} 7+3x, & \text{when } x < 1 \\ 1-5x, & \text{when } x \ge 1 \end{cases}$ at $x = 1$

4. Determine whether the following function are continuous at $x = 2$

(i) $f(x) = \frac{x^2-4}{x-2}$ (ii) $g(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{when } x \ne 2 \\ 3 & \text{when } x = 2 \end{cases}$

(iii) $h(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{when } x \ne 2 \\ 4 & \text{when } x = 2 \end{cases}$

5. Suppose that $f(x) = \begin{cases} -x^4+3 & \text{when } x \le 2 \\ x^2+9 & \text{when } x > 2 \end{cases}$

Is continuous everywhere justify your conclusion?

6. Find the value of k if $f(x) = \begin{cases} \frac{\sin kx}{x} & , x \ne 0 \\ 2 & , x = 0 \end{cases}$ is continuous at $x = 0$.

7. Find the value of k if $f(x)=\begin{cases} kx-9 & x<5 \\ 9x-k & x>5 \\ 36 & x=5 \end{cases}$ is continuous at $x=5$.

8. Find the values of m and n, so that given function f is continuous at $x=3$.

(i) $f(x)=\begin{cases} mx & if\ x<3 \\ n & if\ x=3 \\ -2x+9 & if\ x>3 \end{cases}$ (ii) $f(x)=\begin{cases} mx & if\ x<3 \\ x^2 & if\ x\geq 3 \end{cases}$

9. If $f(x)=\begin{cases} \frac{\sqrt{2x+5}-\sqrt{x+7}}{2} & , x\neq 2 \\ k & , x=2 \end{cases}$

Find the value of k so that f is continuous at $x=2$.

Review Exercise 2

1. Choose the correct Answer

(i) The function $f(x)=x$ is called -------------

(a) Quadratic function (b) Cubic function
(c) Identity function (d) None of these

(ii) The set $\{x|\ x\in R\}$ can be written as

(a) $[-\infty,\infty]$ (b) $(-\infty,a)$ (c) $(-\infty,\infty)$ (d) (a,∞)

(iii) $[3,5]-(3,5)=$ -------------

(a) $\emptyset$ (b) $(3,5)$ (c) $[3,5]$ (d) $\{3,5\}$

(iv) Let $R=\{(x,y)|\ x,y\in R$ and $y=5\}$

(a) Set of real numbers (b) $\{5\}$ (c) $\emptyset$ (d) $\{1,5\}$

(v) A function $f(x)=\frac{x}{|x|}, x\neq 0$ is an:

(a) odd function (b) even function
(c) linear function (d) none of these

(vi) If $f(x)=4^x$ then $f(x+1)-f(x)=$ -------------

(a) 4 (b) $f(x)$ (c) $3f(x)$ (d) $2f(x)$

(vii) The limit of the sequence having general term $a_n=\frac{1}{2^n}$ is:

(a) 0 (b) 1 (c) ∞ (d) None of these

(viii) $\lim\limits_{x\to\infty}\frac{\sin ax}{x}=?$

(a) $\frac{1}{a}$ (b) a (c) ∞ (d) None of these

(ix) $\lim\limits_{x\to 3}\frac{x^2-9}{x-3}=?$

(a) 3 (b) ∞ (c) 0 (d) 6

(x) If $f:[-1,4]\to\mathbb{R}$ is given by $f(x) = x^2$ then $f(-3)$ is:

(a) 9 (b) -9 (c) does not exit (d) 6

(xi) $\lim_{h\to 0}\sin\left(\frac{\pi}{2}+h\right)$

(a) 1 (b) -1 (c) 0 (d) $\frac{1}{2}$

(xii) $\lim_{x\to 0} e^{\frac{-1}{x}}$

(a) 0 (b) 1 (c) ∞ (d) $-\infty$

(xiii) $\lim_{x\to c} f(x)$ exits if and only if

(a) $\lim_{x\to c^+} f(x)$ exist (b) $\lim_{x\to c^-} f(x)$ exist

(c) $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$ (d) $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$

(xiv) $\lim_{n\to\infty}\left[1+\frac{1}{n}\right]^{-7n}$

(a) e^{-7} (b) e (c) 1 (d) ∞

(xv) The limit of the sequence $1, \pi^{-1}, \pi^{-2}, \pi^{-3}, \ldots$ is

(a) 1 (b) π (c) ∞ (d) 0

(xvi) Which of the following represents parametric function

(a) $y = f(x)$ (b) $f(x,y) = 0$

(c) $x = f(t), y = g(t)$ (d) None of these

(xvii) If $g(x) = 3x+2$ and $g\big(f(x)\big) = x$ then $f(2) =$ ___

(a) 2 (b) 6 (c) 0 (d) 8

(xviii) The value of k for which the function $f(x) = \begin{cases} \frac{x}{\tan 3x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous is

(a) 0 (b) 3 (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

(xix) $\lim_{x\to 0}(1-x)^{\frac{1}{x}}$

(a) e^3 (b) $e^{\frac{-1}{2}}$ (c) e (d) e^{-1}

(xx) $\text{sech}^{-1} x =$ -----------------

(a) $\ln\left(x+\sqrt{x^2+1}\right)$ (b) $\ln\left(\frac{1}{x}+\frac{\sqrt{1-x^2}}{x}\right)$

(c) $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ (d) $\frac{1}{2}\ln\left(\frac{1-x}{1+x}\right)$

2. If $h(x) = \sqrt{x^2 + 3}$ and $k(x) = x^2 - 2$, then find composition of function

   (i) $hok(x)$ (ii) koh (iii) hoh (iv) kok

3. If $f(x) = x + 3$ and $g(x) = x^2$, then find $gof(x)$ for $x = 1$.

4. $f(x) = \frac{x+1}{3}$ and $g(x) = 3x + 5$ are two given functions then verify that:

   (i) $(fog)^{-1} = g^{-1}of^{-1}$ (ii) $(gof)^{-1} = f^{-1}og^{-1}$

5. Recognize in the following as explicit or implicit functions and expressed the implicit function as explicit function if possible.

   (i) $x^3 + 2xy = 5x^2 - 3y$ (ii) $3y = 5x^2 - 3x$

   (iii) $x^2y + xy = 5y - 3$ (iv) $2x^2y - xy = 5 + 3xy$

6. Show that the parametric equation $x = a\sec\theta$, $y = b\tan\theta$. represent the equation of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

7. Draw and explain the graph of the following functions:

   (i) $f(x) = e^{5x}$ (ii) $y = \sqrt{4 - x^2}$ (iii) $\frac{x^2}{9} + \frac{y^2}{16} = 1$

8. Draw the graph of parametric equations of function

   $x = at^2$, $y = 2at$, when $a = 6$ and $-5 \le t \le 5$

9. Draw the graph of parametric equations of function

   $x = a\sec\theta$, $y = b\tan\theta$, when $a = 4, b = 3$ and $-\pi \le \theta \le \pi$

10. Solve the following:

    (i) $[1, \infty) \cup (2,3)$ (ii) $[-1,1] \cap (2, \infty)$

    (iii) $(4, \infty) \cap (-3,3)$ (iv) $[2,3] - (2,3)$

    (v) $[1,9] \cap [3,12]$ (vi) $(-\infty, 7) - (-\infty, 2)$

11. Find the limit of the sequence $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$

12. Find the limits of $a_n = -1 + \left(\frac{1}{5}\right)^n$

13. Find limit of the function $y = \frac{1}{x^3}$ for $x \to \infty$

14. Find the value of the following:

    (i) $\lim_{x\to 3} (x^3 + x^4 + x + 5)$ (ii) $\lim_{x\to 7} \left(\frac{1+x}{x^7}\right)$

    (iii) $\lim_{x\to 1} [(2x^4 + 3x^3)(x + 3)]$ (iv) $\lim_{x\to 3} [(x + 3) - (x^3 + 5x + 5)]$

    (v) $\lim_{x\to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$ (vi) $\lim_{x\to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

(vi) $\lim_{x\to 0}(1+2x)^{\frac{5}{x}}$ (viii) $\lim_{x\to 0}\frac{3^{2x}-1}{\sin x}$

15. Determine whether the following functions are continuous at $x = 3$.

(i) $f(x) = \frac{x^2-9}{x-3}$ (ii) $g(x) = \begin{cases} \frac{x^2-9}{x-3}, & \text{when } x \neq 3 \\ 5 & \text{when } x = 3 \end{cases}$

(iii) $h(x) = \begin{cases} \frac{x^2-9}{x-3}, & \text{when } x \neq 3 \\ 6 & \text{when } x = 3 \end{cases}$

16. If $f(x) = \begin{cases} 3x & if\ x \leq -2 \\ x^2 - 1 & if\ -2 < x < 2 \\ 3 & if\ x \geq 2 \end{cases}$. Discuss the continuity or discontinuity at $x = 2$ and $x = -2$.

(i) $f(x) = \frac{x^2-9}{x-3}$ (ii) $g(x) = \begin{cases} \frac{x^2-9}{x-3}, & \text{when } x \neq 3 \\ 5 & \text{when } x = 3 \end{cases}$

(iii) $h(x) = \begin{cases} \frac{x^2-9}{x-3}, & \text{when } x \neq 3 \\ 6 & \text{when } x = 3 \end{cases}$