Differentiation

Unit 3

Introduction

The differential calculus is the branch of mathematics developed by Isaac Newton and Gottfried Wilhelm Leibniz (G. W. Leibniz). This branch is concerned with the problems of finding the rate of change of function with respect to the variable on which it depends.

3.1 Derivative of a Function

3.1.1 Distinguish between independent and dependent variables

An independent variable is a variable whose value never depends on another variable, whereas a dependent variable is a variable whose values depends on another variable.

The equation $y = f(x)$ is a general notation which expresses the relation between the two variables x and y, where y depends on x.

e.g., in function $y = f(x) = 3x + 4$, x is the **independent variable** and y is **dependent variable**.

3.1.2 Estimate corresponding change in the dependent variable when independent variable is incremented (or decremented)

Let $y = f(x)$ is a function with dependent variable y and independent variable x. If Δx is the small change in the independent variable x then corresponding change in y will be Δy

i.e., $\Delta y = f(x + \Delta x) - f(x)$

Similarly, when independent variable x is decremented then corresponding change in y will be Δy

i.e., $\Delta y = f(x) - f(x - \Delta x)$

Example 1. $y = x^3 + 1$ then calculate the corresponding change in y when x is incremented from 1 to 1.01.

Solution: Since x is incremented from 1 to 1.01, therefore the change in independent variable x is $\Delta x = 1.01 - 1$

$\Delta x = 0.01$

Now, the corresponding change in y will be

$\Delta y = f(x + \Delta x) - f(x)$

$\Delta y = f(1 + 0.01) - f(1)$

$\Delta y = f(1.01) - f(1)$

$$= ((1.01)^3 + 1) - ((1)^3 + 1)$$
$$= 2.03031 - 2$$
$$\Delta y = 0.030301$$

Thus, the corresponding change in y when x is incremented from 1 to 1.01 is 0.030301.

Example 2. If $y = e^x$ calculate the corresponding change in y, when x is decremented from 2 to 1.98.

Solution: Since, x is decremented from 2 to 1.98, therefore

$$\Delta x = 2 - 1.98$$
$$\Delta x = 0.02$$

Now, the corresponding change in y will be

$$\Delta y = f(x) - f(x - \Delta x)$$
$$\Delta y = f(2) - f(2 - 0.02)$$
$$\Delta y = f(2) - f(1.98)$$
$$\Delta y = e^2 - e^{1.98}$$
$$\Delta y \approx 7.38905 - 7.24274$$
$$\Delta y \approx 0.14631.$$

Thus, the corresponding change in y when x is decrements from 2 to 1.98 is 0.14631.

3.1.3 Explain the concept of rate of change

The rate of change is the speed at which a dependent variable changes with respect to an independent variable. It can generally be expressed as a ratio of change in dependent variable and change in independent variable. Let $y = f(x)$ is function and Δx and Δy are the changes in independent variable x and dependent variable y respectievly. Now,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is the rate of change of y with respect to x, which is commonly known as average rate of change. However, when Δx is very small that is $\Delta x \longrightarrow 0$, then such rate of change is called instantaneous rate of change.

Example 1. If $y = x^2 - 6x + 8$ determine the average rate of change of y which respect to x when x varies from 1 to 1.3.

Solution: Given function is

$$y = f(x) = x^2 - 6x + 8$$

and $\Delta x = 1.3 - 1 = 0.3$

Now, average rate of change of y with respect of x is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$\frac{\Delta y}{\Delta x} = \frac{f(1 + 0.3) - f(1)}{0.3}$$

$$\frac{\Delta y}{\Delta x} = \frac{f(1.3) - f(1)}{0.3}$$
$$\frac{\Delta y}{\Delta x} = \frac{((1.3)^2 - 6(1.3) + 8) - ((1)^2 - 6(1) + 8)}{0.3}$$
$$\frac{\Delta y}{\Delta x} = \frac{1.89 - 3}{0.3}$$
$$\frac{\Delta y}{\Delta x} = \frac{-1.11}{3} = -3.7$$

Thus, the required rate of change is -3.7.

Exercise 3.1

1. Find the average rate of change of the following functions when x varies from a to b.
   (i) $y = f(x) = x^2 + 4;$ $a = 2, b = 2.3$
   (ii) $y = f(x) = x^3 - 4;$ $a = 2, b = 2.3$
   (iii) $y = f(x) = x^3 - 8;$ $a = 3, b = 2.5$
2. Find out the average rate of change when x changes from a to b.
   (i) $A = \pi x^2$, where x is the radius of the sphere; $a = 3, b = 3.1$
   (ii) $V = \frac{4}{3}\pi x^3$, where x is the radius of the circle; $a = 2, b = 1.9$
3. The price p in rupees after "t" years is given by $p(t) = 3t^2 + t + 1$. Find the average rate of change of inflation from $t = 3$ to $t = 3.5$ years.
4. A ball is thrown vertically up, its height in metres after t seconds is given by the formula $h(t) = -16t^2 + 80t$. Find the average velocity when t changes from a to b.
   (a) $a = 2, b = 2.1$ (b) $a = 2, b = 2.01$

3.1.4 Define derivative of a function as an instantaneous rate of change of a variable with respect to another variable

The instantaneous rate of change of dependent variable y with respect to x is called the derivative of the function $y = f(x)$.

For example, consider displacement of an object is the function of time i.e., $s = f(t)$. Now, instantaneous rate of displacement with respect to time is called velocity and it is the derivative of displacement with respect to time.

3.1.5 Define derivative or differential coefficient of a function

Let $y = f(x)$ the derivative of $f(x)$ is the limit of ratio of increments δy and δx at zero i.e., $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x}$ and it is denoted by $f'(x) \frac{dy}{dx}, \frac{d}{dx} f(x)$ or y'.

A real valued function $f(x)$ is said to be derivable or differentiable at x, iff $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x}$ exists where δy and δx are the increments in y and x respectively.

i.e., $f(x) = \lim\limits_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$ exists ...(i)

The derivative of a function $f(x)$ at any point a is denoted by $f'(a)$ is defined as:

$$f'(a) = \lim_{\delta x \to 0} \frac{f(a+\delta x) - f(x)}{(a+\delta x) - a},$$

Now, if we substitute, $x = a + \delta x$ and $x = a$, with new limits $x \to a$ as $\delta x \to 0$, then;

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \ldots(ii)$$

Note: The process of finding derivative is called the differentiation and to find the derivative by either (i) or by (ii) is called ab-initio method/ first principle or by method of definition.

Note: $\frac{dy}{dx}$ does not mean the ratio of dy and dx i.e., $\frac{dy}{dx} \neq dy \div dx$

$\frac{dy}{dx}$ means derivative of y w.r.t. x, i.e., $\frac{d}{dx}(y)$, $\frac{d}{dx}$ is a differential operator.

Example 1. Find derivative of $y = x^2 + 2$ w.r.t x by definition

Solution: Given that

$$y = f(x) = x^2 + 2$$

$$\therefore f(x+\delta x) = (x+\delta x)^2 + 2$$

By definition, we mean that:

$$f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - f(x)}{\delta x}$$

$$\Rightarrow f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{x^2 + 2x.\delta x + (\delta x)^2 + 2 - x^2 - 2}{\delta x}$$

$$\Rightarrow f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta x\,(2x + \delta x)}{\delta x}$$

$$\Rightarrow f'(x) = \frac{dy}{dx} = 2x + (0) = 2x$$

Thus, derivative of $x^2 + 2$ is $2x$.

Example 2. Find the derivative of $\sqrt{x}$ by definition.

Solution: Given that

$$y = f(x) = \sqrt{x}$$

$$\therefore\ y + \delta y = f(x+\delta x) = \sqrt{x+\delta x}$$

By definition

$$f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore\ f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sqrt{x+\delta x} - \sqrt{x}}{\delta x}$$

$$\Rightarrow f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\left(\sqrt{x+\delta x} - \sqrt{x}\right)\left(\sqrt{x+\delta x} + \sqrt{x}\right)}{\delta x\left(\sqrt{x+\delta x} + \sqrt{x}\right)}$$

$$\Rightarrow f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{x + \delta x - x}{\delta x\left(\sqrt{x+\delta x} + \sqrt{x}\right)}$$

$$\Rightarrow f'(x) = \frac{dy}{dx} = \frac{1}{\left(\sqrt{x+0} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

Thus, derivative of $\sqrt{x}$ is $\dfrac{1}{2\sqrt{x}}$

3.1.6 Differentiate $y = x^n$, where $n \in Z$ (the set of integers), from first principle (the derivation of power rule)

Case-I: Let $y = f(x) = x^n$, where n is positive integer

$$\therefore\ y + \delta y = f(x+\delta x) = (x+\delta x)^n\ ,$$

By definition of derivative,

$$\because\ \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore\ \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{x^n + \binom{n}{1}x^{n-1}(\delta x) + \binom{n}{2}x^{n-2}(\delta x)^2 + \cdots + (\delta x)^n - x^n}{\delta x}$$

(apply binomial theorem)

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{\delta x}{\delta x}[nx^{n-1} + \cdots + (\delta x)^{n-1}]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = nx^{n-1} + 0 + 0 + \cdots + 0 = nx^{n-1}$$

Thus, $\boxed{f'(x) = \frac{d}{dx}(x^n) = nx^{n-1}}$

Case-II: Let $y = f(x) = x^n$ when n is negative integer

$$\therefore y + \delta y = f(x+\delta x) = (x+\delta x)^n$$

By definition of derivative,

$$\because\ \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore\ \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{x^n\left(1+\frac{\delta x}{x}\right)^n - x^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^n . \lim_{\delta x \to 0} \frac{\left(1+\frac{\delta x}{x}\right)^n - 1}{\delta x}$$

using binomial series, we have,

$$\frac{dy}{dx} = f'(x) = x^n . \lim_{\delta x \to 0} \frac{\left[1 + n\left(\frac{\delta x}{x}\right) + \frac{n\,(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right] - 1}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^n . \lim_{\delta x \to 0} \left(\frac{\delta x}{x} . \frac{1}{\delta x}\right)\left[n + \frac{n\,(n-1)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^{n-1} . \lim_{\delta x \to 0} \left[n + \frac{n\,(n-1)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^{n-1} . \lim_{\delta x \to 0} [n + 0 + 0 + \cdots] = nx^{n-1}$$

Thus, $f'(x) = \frac{d}{dx}(x^n) = nx^{n-1} \quad \forall\, n \in Z$

3.1.7 Differentiate $y = (ax+b)^n$, where $n = \frac{p}{q} \in$ Q and p & q are integers such that $q \neq 0$, from first principle.

Let $\quad y = f(x) = (ax+b)^n$, where $n = \frac{p}{q} \in$ Q and $q \neq 0$.

$\therefore y + \delta y = f(x+\delta x) = [a(x+\delta x)+b]^n$,

By definition of derivative,

$$\because \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{[a(x+\delta x) + b]^n - (ax+b)^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{[(ax+b) + a\delta x]^n - (ax+b)^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{(ax+b)^n\left[1 + \frac{a\delta x}{(ax+b)}\right]^n - (ax+b)^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = (ax+b)^n . \lim_{\delta x \to 0} \frac{[1 + \frac{a\delta x}{(ax+b)}]^n - 1}{\delta x}$$

Using binomial series, we have

$$\Rightarrow \frac{dy}{dx} = f'(x) = (ax+b)^n . \lim_{\delta x \to 0} \frac{[1 + n\frac{a\delta x}{ax+b} + \frac{n(n-1)}{2!} . \left(\frac{a\delta x}{ax+b}\right)^2 + \ldots - 1]}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = (ax+b)^n . \lim_{\delta x \to 0} \frac{a\delta x}{(ax+b)} . \frac{1}{\delta x}\left[n + \frac{n(n-1)}{2!} . \frac{a\delta x}{(ax+b)^2} + \cdots\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = (ax+b)^{n-1} . a \lim_{\delta x \to 0} \left[n + \frac{n(n-1)}{2!} . \frac{a\delta x}{(ax+b)} + \cdots\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = (ax+b)^{n-1} . a \lim_{\delta x \to 0} [n + 0 + 0 + \cdots]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = na(ax+b)^{n-1}$$

Thus, $\frac{d}{dx}\left(ax+b\right)^n = na\left(ax+b\right)^{n-1}$.

Examples: Find derivative of the following the w.r.t. x by first principle.

(a) $2x^5 + 1$ (b) x^{-3} (c) $(2x+5)^{\frac{5}{2}}$

Solutions (a): Let $y = f(x) = 2x^5 + 1$

$\therefore y + \delta y = f(x+\delta x) = 2(x+\delta x)^5 + 1$

definition of derivative, we have

$$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{[2\,(x+\delta x)^5 + 1] - (2x^5 + 1)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{2\,(x+\delta x)^5 - 2x^5}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 2. \lim_{\delta x \to 0} \frac{(x+\delta x)^5 - x^5}{\delta x}$$

Using binomial theorem, we have,

$$\Rightarrow \frac{dy}{dx} = f'(x) = 2. \lim_{\delta x \to 0} \left[x^5 + 5x^4(\delta x) + \frac{5.4}{2!} x^3(\delta x)^2 + \cdots + (\delta x)^5 - x^5\right] . \frac{1}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 2. \frac{(5.\delta x)}{\delta x} \lim_{\delta x \to 0} \left[x^4 + \frac{4}{2!} x^3 . \delta x + \cdots + (\delta x)^4\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 10. \lim_{\delta x \to 0} \left[x^4 + \frac{4}{2!} x^3 . \delta x + \cdots + (\delta x)^4\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 10(x^4 + 0 + 0 + \cdots +)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 10x^4$$

Thus, $\frac{d}{dx}(2x^5 + 1) = 10x^4$

Solution (b): Let $y = f(x) = x^{-3}$

$$\therefore y + \delta y = f(x + \delta x) = (x + \delta x)^{-3}$$

By definition of derivative, we have,

$$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{(x + \delta x)^{-3} - x^{-3}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{x^{-3}\left(1 + \frac{\delta x}{x}\right)^{-3} - x^{-3}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^{-3}. \lim_{\delta x \to 0} \frac{\left(1 + \frac{\delta x}{x}\right)^{-3} - 1}{\delta x}$$

Using binomial series, we have

$$\therefore \frac{dy}{dx} = f'(x) = x^{-3}. \lim_{\delta x \to 0} \left[\left\{1 + (-3)\left(\frac{\delta x}{x}\right) + \frac{(-3)(-4)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right\} - 1\right] . \frac{1}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^{-3}. \left(\frac{-3\delta x}{x} . \frac{1}{\delta x}\right) . \lim_{\delta x \to 0} \left[1 + \frac{(-4)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = -3x^{-4}$$

Thus, $\boxed{\frac{d}{dx}\left(x^{-3}\right) = -3x^{-4}}$

Solution (c): Given that

$$f(x) = y = (2x + 5)^{\frac{5}{2}}$$

$$f(x + \delta x) = (2x + 5 + 2\delta x)^{\frac{5}{2}}$$

Now, by using the definition of derivatives

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + 2\delta x) - f(x)}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(2x + 5 + 2\delta x)^{\frac{5}{2}} - (2x + 5)^{\frac{5}{2}}}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(2x + 5)^{\frac{5}{2}}\left[\left(1 + \frac{2\delta x}{2x + 5}\right)^{\frac{5}{2}} - 1\right]}{\delta x}$$

$$= (2x + 5)^{\frac{5}{2}} \lim_{\delta x \to 0} \frac{1}{\delta x}\left[1 + \frac{5}{2}\left(\frac{2\delta x}{2x + 5}\right) + \frac{5}{2} \cdot \frac{3}{2}\left(\frac{2\delta x}{2x + 5}\right)^2 + \cdots - 1\right]$$

$$= (2x+5)^{\frac{5}{2}} \lim_{\delta x \to 0} \frac{\delta x}{\delta x}\left(\frac{5(2)}{2(2x+5)} + \frac{5}{2}\cdot\frac{3}{2}\left(\frac{\delta x}{2x+5}\right)^{1} + \cdots\right)$$

$$= (2x+5)^{\frac{5}{2}} \cdot 5 \frac{1}{(2x+5)} = 5(2x+5)^{\frac{3}{2}}$$

Exercise 3.2

1. Find by definition (ab-initio) the derivatives w.r.t "x" of the following functions defined as:

(i) $f(x) = 2x$ (ii) $f(x) = 1 - \sqrt{x}$ (iii) $\frac{1}{\sqrt{x}}$

(iv) $f(x) = 3 - x^2$ (v) $f(x) = x(x+1)$ (vi) $f(x) = x^2 - 3$

(vii) $f(x) = x^3 + 5$ (viii) $f(x) = 4x^2 - 3x$

(ix) $f(x) = \frac{1}{x+2}$ (x) $f(x) = \frac{3}{2x+5}$

2. Find $f'(x)$ for the following functions using definition:

(i) $f(x) = \sqrt[3]{2x+1}$ (ii) $f(x) = (2x-1)^{-\frac{1}{2}}$

(iii) $f(x) = (6x+7)^{\frac{5}{2}}$ (iv) $f(x) = (3x-5)^{-\frac{3}{2}}$

3.2 Theorems on differentiation

We will prove different theorems for differentiation.

- **The derivative of a constant is zero.**

Proof: Let $y = f(x) = c$ ... (i), where c is constant

$\therefore\ y + \delta y = f(x + \delta x) = c$

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{c - c}{\delta x} = 0$$

Thus, $\frac{d}{dx}(c) = 0$

Hence proved.

- **The derivative of any constant multiple of a function is equal to the product of that constant and derivative of the function.**

i.e., $\frac{d}{dx}[a\,f(x)] = a\frac{d}{dx}\,f(x) = af'(x)$.

Proof: Let $y = af(x) = g(x)$, (say)

$\therefore y + \delta y = af(x+\delta x) = g(x+\delta x)$

$$\because \frac{dy}{dx} = g'(x) = \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = g'(x) = \lim_{\delta x \to 0} \frac{a\,f(x+\delta x) - a\,f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = g'(x) = a\,.\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x},$$

$$\Rightarrow \frac{dy}{dx} = g'(x) = af'(x), \qquad \left[\because f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}\right]$$

Thus, $\boxed{\frac{d}{dx}[af(x)] = af'(x)}$

Hence proved.

- **The derivative of a sum (or difference) of two functions is equal to the sum (or difference) of their derivatives.**

i.e., $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$.

Proof: Let $y = h(x) = f(x) \pm g(x)$

$$\therefore\ y + \delta y = h(x+\delta x) = f(x+\delta x) \pm g(x+\delta x)$$

$$\because \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \frac{h(x+\delta x) - h(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) \pm g(x+\delta x) \pm [f(x) \pm g(x)]}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0}\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] \pm \lim_{\delta x \to 0}\left[\frac{g(x+\delta x) - g(x)}{\delta x}\right]$$

$$\frac{dy}{dx} = h'(x) = f'(x) \pm g'(x)$$

Thus, $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$.

Hence proved.

- **The derivative of a product of two functions is equal to (The first function) × (derivative of second function) plus (derivative of the first function) × (the second function).**

i.e., $\frac{d}{dx}[f(x).g(x)] = f(x).\frac{d}{dx}g(x) + g(x).\frac{d}{dx}f(x)$

$$= f(x).g'(x) + g(x).f'(x)$$

Proof: Let $y = h(x) = f(x) \cdot g(x)$, (say)

$$\therefore\ y + \delta y = h(x+\delta x) = f(x+\delta x).g(x+\delta x)$$

$$\because \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{h(x+\delta x) - h(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x).g(x+\delta x) - f(x).g(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x).g(x+\delta x) - f(x+\delta x).g(x) + f(x+\delta x).g(x) - f(x).g(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} f(x+\delta x) . \lim_{\delta x \to 0} \left[\frac{g(x+\delta x) - g(x)}{\delta x}\right] + \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] . \lim_{\delta x \to 0} g(x)$$

$$\Rightarrow \frac{dy}{dx} = f(x+0) . g'(x) + f'(x) . g(x)$$

$$\Rightarrow \frac{dy}{dx} = f(x). g'(x) + g(x) . f'(x)$$

Thus, $\frac{d}{dx}[f(x) . g(x)] = f(x). g'(x) + f'(x). g(x)$

This is known as product rule for differentiation of two functions.

Hence Proved.

- **The derivative of a quotient of two functions is equal to denominator times the derivative of the numerator minus the numerator times the derivative of the denominator and all divided by the square of the denominator.**

i.e.,
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) . \frac{d}{dx} f(x) - f(x) . \frac{d}{dx} g(x)}{[g(x)]^2} = \frac{g(x) . f'(x) - f(x) . g'(x)}{[g(x)]^2}$$

Proof: Let $y = h(x) = \frac{f(x)}{g(x)}$ (say)

Let $y = h(x) =$

$\therefore$ $y + \delta y = h(x + \delta x) = \frac{f(x+\delta x)}{g(x+\delta x)}$

$\because$ $\frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \frac{h(x+\delta x) - h(x)}{\delta x}$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \left[\frac{f(x+\delta x)}{g(x+\delta x)} - \frac{f(x)}{g(x)}\right] . \frac{1}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \left[\frac{g(x) . f(x+\delta x) - g(x+\delta x) . f(x)}{\delta x . g(x+\delta x) . g(x)}\right]$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \left[\frac{g(x) . f(x+\delta x) - f(x).g(x) + f(x).g(x) - g(x+\delta x) . f(x)}{\delta x . g(x+\delta x) . g(x)}\right]$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \frac{g(x) \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] - f(x) . \lim_{\delta x \to 0} \left[\frac{g(x+\delta x) - g(x))}{\delta x}\right]}{\lim_{\delta x \to 0} . g(x+\delta x) . g(x)}$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \frac{g(x)\,.\frac{d}{dx}\,f(x)\; - \;f(x).\frac{d}{dx}\,g(x)}{g(x+0).\,g(x)}$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \frac{g(x).\,f'(x) - f(x).\,g'(x)}{[g(x)]^2}$$

Thus, $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2}$$

This is known as quotient rule of differentiation

Hence proved.

3.3 Application of Theorems on Differentiation

- **Differentiate constant multiple of x^n**

Let us try to understand by solving the following examples:

Example 1. Differentiate the following w.r.t. "x"

(a) $4x^5$

Solution (a) Let $y = 4x^5$

Differentiating w.r.t. $'x'$, we have

$$\frac{dy}{dx} = \frac{d}{dx}(4x^5)$$

$$\Rightarrow \frac{dy}{dx} = 4\,.\,\frac{d}{dx}(x^5)$$

$$\Rightarrow \frac{dy}{dx} = 4 \times 5x^{5-1}, \quad [\because \frac{d}{dx}(x^n) = nx^{n-1}]$$

$$\Rightarrow \frac{dy}{dx} = 20x^4$$

(b) $\frac{5}{2}x^{\frac{2}{5}}$

Solution (b) Let $y = \frac{5}{2}x^{\frac{2}{5}}$

Differentiating w.r.t. $'x'$, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{5}{2}x^{\frac{2}{5}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{5}{2}\,.\,\frac{d}{dx}x^{\frac{2}{5}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{5}{2} \times \frac{2}{5}x^{\frac{2}{5}-1}$$

$$\Rightarrow \frac{dy}{dx} = x^{-\frac{3}{5}}.$$

- **Differentiate sum (or difference) of functions**

Example 2. Differentiate the following functions w.r.t. 'x'

(a) $y = (2x^2 - 3x + 1) + (4x + 1)$ (b) $y = (2x^3 - 1) - (1 + \frac{1}{x^4})$

Solution (a): given that

$$y = h(x) = (2x^2 - 3x + 1) + (4x - 1)$$

Differentiating w.r.t.'x', we have,

$$\frac{dy}{dx} = \frac{d}{dx}[(2x^2 - 3x + 1) + (4x + 1)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(2x^2 - 3x + 1) + \frac{d}{dx}(4x + 1)$$

$$\Rightarrow \frac{dy}{dx} = 4x^{2-1} - 3(1) + 0 + 4(1) + 0$$

$$\Rightarrow \frac{dy}{dx} = 4x - 3 + 4$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = 4x + 1$$

Solution (b): Given that

$$y = h(x) = (2x^3 - 1) - \left(1 + \frac{1}{x^4}\right)$$

Differentiating w.r.t. "x", we have

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[(2x^3 - 1) - (1 + x^{-4})\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(2x^3 - 1) - \frac{d}{dx}(1 + x^{-4})$$

$$\Rightarrow \frac{dy}{dx} = 3\text{x}2x^2 - 0 - 0 - (-4).x^{-5}$$

$$\Rightarrow \frac{dy}{dx} = 6x^2 + 4x^{-5} = 6x^2 + \frac{4}{x^5}$$

- **Differentiate the polynomials**

Example 3. Differentiate w.r.t. "x" the following polynomial functions.

(a) $f(x) = 2x^3 - 4x^2 + 3x + 1$

(b) $f(x) = -5x^3 + 2x^2 + 3x + 5$

Solution (a): Given that

$$f(x) = 2x^3 - 4x^2 + 3x + 1$$

Differentiating w.r.t "x", we have

$$\therefore \frac{d}{dx}f(x) = \frac{d}{dx}(2x^3) - \frac{d}{dx}(4x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(1)$$

$$\Rightarrow f'(x) = 2\frac{d}{dx}(x^3) - 4\frac{d}{dx}(x^2) + 3\frac{d}{dx}(x) + 0$$

$$\Rightarrow f'(x) = 2 \times 3x^2 - 4 \times 2x + 3(1)$$

$$\Rightarrow f'(x) = 6x^2 - 8x + 3$$

Solution (b): Given that;

$$f(x) = -5x^3 + 2x^2 + 3x + 5$$

Differentiating w.r.t "x", we have

$$\frac{d}{dx}f(x) = \frac{d}{dx}(-5x^3) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(5)$$

$$\Rightarrow f'(x) = -5\frac{d}{dx}x^3 + 2\frac{d}{dx}x^2 + 3\frac{d}{dx}(x) + 0$$
$$\Rightarrow f'(x) = -5 \times 3x^2 + 2 \times 2x + 3(1)$$
$$\Rightarrow f'(x) = -15x^2 + 4x + 3$$

- **Differentiate product of functions**

Example 4. Differentiate w.r.t "x" the following product functions using product rule.

(a) $h(x) = (x^2 + 1)(5x^2 + 6)$

(b) $h(x) = (x + 1)(x + 2)(x + 3)$

Solution (a): Given that;

$$h(x) = (x^2 + 1)(5x^2 + 6) = f(x).g(x) \text{ (say)}$$

Let $f(x) = x^2 + 1$ and $g(x) = 5x^2 + 6$ …(ii)

Differentiating equations both sides of (i) and (ii) w.r.t "x",

$$\therefore \frac{d}{dx}f(x) = \frac{d}{dx}(x^2 + 1) \text{ and } \frac{d}{dx}g(x) = \frac{d}{dx}(5x^2 + 6),$$
$$\Longrightarrow f'(x) = 2x \text{ and } g'(x) = 10x,$$
$$\because \frac{d}{dx}[f(x).g(x)] = f(x).g'(x) + g(x)f'(x),$$
$$\therefore \frac{d}{dx}h(x) = (x^2 + 1)(10x) + (5x^2 + 6)(2x),$$
$$\Longrightarrow h'(x) = 10x^3 + 10x + 10x^3 + 12x,$$
$$\Longrightarrow h'(x) = 20x^3 + 22x,$$
$$\Longrightarrow h'(x) = 2x(10x^2 + 11).$$

Solution (b): Given that;

$$h(x) = (x + 1)(x + 2)(x + 3)$$
$$\Longrightarrow h(x) = (x + 1)(x^2 + 5x + 6) = f(x).g(x), \quad \text{(Say)} \quad \ldots\text{(i)}$$
$$\because \frac{d}{dx}h(x) = h'(x) = f(x).g'(x) + g(x).f'(x)$$

Now differentiate both sides of the equations (i) w.r.t "x"

$$\therefore h'(x) = (x + 1).\frac{d}{dx}(x^2 + 5x + 6) + (x^2 + 5x + 6).\frac{d}{dx}(x + 1),$$
$$\Longrightarrow h'(x) = (x + 1).(2x + 5 + 0) + (x^2 + 5x + 6)(1 + 0),$$
$$\Longrightarrow h'(x) = (x + 1).(2x + 5) + (x^2 + 5x + 6)(1),$$
$$\Longrightarrow h'(x) = 2x^2 + 7x + 5 + x^2 + 5x + 6,$$
$$\Longrightarrow h'(x) = 3x^2 + 12x + 11.$$

- **Differentiate Quotient of two functions.**

Example 5. Differentiate w.r.t "x" the following quotients (rational) functions using quotient rule;

(a) $h(x) = \frac{x+1}{x^2-2x+3}$ (b) $h(x) = \frac{(x-1)(x+2)}{(x+2)(x+3)}$

Solution (a): Given that

$$h(x) = \frac{x+1}{x^2-2x+3} = \frac{f(x)}{g(x)} \quad \text{Provided } g(x) \neq 0, \forall x \in \mathbb{R} \quad \ldots\text{(i)}$$

$$f(x) = x + 1 \quad \ldots\text{(ii)}$$

$$g(x) = x^2 - 2x + 3 \quad \ldots\text{(iii)}$$

Now,

$$h'(x) = \frac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2} \quad \ldots\text{(iv)}$$

$$f'(x) = \frac{d}{dx}(x+1) = 1 + 0 = 1 \text{ and } g'(x) = \frac{d}{dx}(x^2 + 2x + 3) = 2x - 2$$

Now, substitute the values in (iv), we have,

$$h'(x) = \frac{(x^2 - 2x + 3)(1) - (x+1)(2x-2)}{(x^2 - 2x + 3)^2}$$

$$h'(x) = \frac{x^2 - 2x + 3 - (2x^2 - 2x + 2x - 2)}{(x^2 - 2x + 3)^2}$$

$$h'(x) = \frac{-x^2 - 2x + 5}{(x^2 - 2x + 3)^2}$$

Solution (b): Given that;

$$h(x) = \frac{(x-1)(x+2)}{(x-2)(x+3)},$$

$$\Rightarrow h(x) = \frac{x^2+x-2}{x^2+x-6} = \frac{f(x)}{g(x)} \quad \text{Provided } g(x) \neq 0 \quad \ldots\text{(i)}$$

$$\because h'(x) = \left[\frac{f(x)}{g(x)}\right]' = \frac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2}$$

$$\therefore h'(x) = \frac{(x^2 + x - 6)\frac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\frac{d}{dx}(x^2 + x - 6)}{(x^2 + x - 6)^2}$$

$$\Rightarrow h'(x) = \frac{(x^2 + x - 6)(2x+1) - (x^2 + x - 2)(2x+1)}{(x^2 + x - 6)^2}$$

$$\Rightarrow h'(x) = \frac{2x^3 + 3x^2 - 11x - 6 - (2x^3 + 3x^2 - 3x - 2)}{(x^2 + x - 6)^2}$$

$$\Rightarrow h'(x) = \frac{2x^3 + 3x^2 - 11x - 6 - 2x^3 - 3x^2 + 3x + 2}{(x^2 + x - 6)^2}$$

$$\Rightarrow h'(x) = \frac{-8x - 4}{(x^2 + x - 6)^2}$$

$$\Rightarrow h'(x) = -\frac{4(2x + 1)}{(x^2 + x - 6)^2}$$

Note: The derivative of an even function is always an odd function and viceversa. i.e., if $f(-x) = f(x) \Rightarrow f'(-x) = -f'(x)$ and $f(-x) = -f(x) \Rightarrow f'(-x) = f'(x)$.

Exercise 3.3

1. Differentiate the following w.r.t "x".

(i) $5x^5$ (ii) $\frac{7}{9}x^9$ (iii) $-25x^{\frac{-3}{5}}$ (iv) $124\sqrt{x}$

(v) $\frac{1}{22}x^{22}$ (vi) x^{-100} (vii) $15\sqrt[3]{x}$ (viii) $16\sqrt[4]{x^{\frac{3}{4}}}$

(ix) $\frac{-4}{x^{\frac{3}{4}}}$ (x) $\frac{3}{\sqrt[3]{x^2}}$

2. Differentiate the following w.r.t "x".

(i) $\frac{x^5}{a^2+b^2} + \frac{x^2}{a^2-b^2}$ (ii) $2x + \frac{1}{2}x^6$ (iii) $\sqrt[3]{x^2} + \sqrt{x}$

(iv) $\frac{1}{21}x^{21} + \frac{1}{22}x^{22}$ (v) $-\frac{5}{4}x^{\frac{-4}{5}} + \frac{2}{3}x^{\frac{-3}{2}}$

3. Differentiate the following w.r.t "x".

(i) $2ax^3 - \frac{x^2}{b} + 6$ (ii) $x^3 - \frac{3}{7}x^{\frac{7}{3}}$ (iii) $5x^{\frac{3}{5}} - \frac{1}{7}x^{\frac{7}{3}}$

(iv) $x^{10} - 10x^{15}$ (v) $3\left(\sqrt[3]{x^2}\right) - 4\left(\sqrt[4]{x}\right)$

4. Differentiate the following polynomial function w.r.t "x".

(i) $p(x) = x^3 - 3x^2 + 2x + 1$ (ii) $p(x) = x^4 - 3x^2 + 2x - 3$

(iii) $p(x) = x^6 - x^4 + x^3 + x$ (iv) $p(x) = 9x^9 + 7x^7 + \frac{1}{5}x^5 - \frac{1}{4}x^4 + x + 1$

(v) $p(x) = x^3 + x^2 + x + 1$

5. Find the derivative of the following functions using product rule.

(i) $h(x) = (2x - 5).(5x + 7)$ (ii) $h(x) = x.\sqrt{3x^2 + 4}$

(iii) $h(x) = \sqrt[3]{x + 1}.\sqrt[5]{x^2 + 1}$ (iv) $h(x) = x^2\left(\sqrt{x} + 1\right)$

(v) $h(x) = (x + 1)^3.x^{\frac{-3}{2}}$

6. Find the derivative of the following functions using quotient rule.

(i) $h(x) = \frac{3x+4}{2x-3}$ (ii) $h(x) = (x^2 - 1).(x^2 + 1)^{-1}$

(iii) $h(x) = \frac{x^2-x+1}{x^2+x-1}$ (iv) $h(x) = \frac{2x^4}{b^2-x^2}$

(v) $h(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}$

3.4. Chain Rule

The rule for differentiating composite function is called chain rule. In this rule we take the derivative of the outer function and then multiply it with the derivative of the inner function.

The derivative of the composite function fog is $(fog)' = f'(g(x)).g'(x)$ which is called chain rule.

3.4.1 Prove that $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$, when $y = f(u)$ and $u = g(x)$. (chain rule)

Proof:

Let $y = f[g(x)] = f(u)$, where $u = g(x)$

$\therefore \delta y = f[g(x+\delta x)] - f[g(x)] = f(u+\delta u) - f(u)$,

$$\because \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{f[g(x+\delta x)] - f[g(x)]}{\delta x}, \text{provided } \delta x \neq 0$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{f[g(x+\delta x)] - f[g(x)]}{g(x+\delta x) - g(x)}.\lim_{\delta x\to 0}\frac{g(x+\delta x) - g(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta u\to 0}\frac{f(u+\delta u) - f(u)}{\delta u}.\lim_{\delta x\to 0}\frac{\delta u}{\delta x},$$

[where $\delta u = g(x+\delta x) - g(x)$ as $\delta x \to 0$, $\delta u \to 0$]

$$\Rightarrow \frac{dy}{dx} = \frac{d}{du}f(u).\frac{d}{dx}g(x) = \frac{dy}{du}.\frac{du}{dx}$$ [$\therefore y = f(u)$ and $u = g(x)$]

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Hence proved.

Example: Differentiate $y = (5x-4)^5$.

Solution:

$y = (5x^2-4)^5$ and let $u = 5x^2 - 4$

then $y = u^5$ ...(i)

$u = 5x^2 - 4$ ...(ii)

Differentiating equation (i) w.r.t "u" and equation (ii) w.r.t "x"

$$\therefore \frac{dy}{du} = 5u^4 \; and \; \frac{du}{dx} = 2(5)x - 0 = 10x$$

By chain rule is:

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = 5u^4.10x = 50x\,(5x^2 - 4)^4$$

3.4.2 Show that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Proof: If $y = f(x)$ is any differentiable function in the domain of x, then its inverse function is defined as $x = g(y)$, such that:

$$(gof)(x) = g\big(f(x)\big) = g(y) = x \qquad \ldots\text{(i)}$$

Differentiating both sides of equation (i) w.r.t x

$g'(y).f'(x) = 1$ (by chain rule)

$$\Rightarrow f'(x) = \frac{1}{g'(y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \left[\because f'(x) = \frac{dy}{dx} \text{ and } g'(y) = \frac{dx}{dy}\right]$$

Hence Showed.

3.4.3 Use chain rule to show that $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}.f'(x)$.

Let $y = [f(x)]^n, \quad \forall x \in \mathbb{R}$

Suppose that

$u = f(x)$ … (i)

then $y = u^n$ … (ii)

From equation (i) and (ii) by differentiating

$$\therefore \frac{du}{dx} = \frac{d}{dx}f(x) = f'(x)$$

and $\frac{dy}{du} = \frac{d}{du}\left(u^n\right)$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{du}(u^n).\frac{d}{dx}(u) = nu^{n-1}.f'(x)$$

$$\Rightarrow \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}.f'(x)$$

Hence Showed.

Example: Differentiate $(x^4 - 4x^2 + 5)^{\frac{5}{2}}$.

Solution: Let $y = (x^4 - 4x^2 + 5)^{\frac{5}{2}}$

Here $f(x) = x^4 - 4x^2 + 5$ and $n = \frac{5}{2}$.

$$\because \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}.f'(x)$$

$$\therefore \frac{d}{dx}(x^4 - 4x^2 + 5)^{\frac{5}{2}} = \frac{5}{2}(x^4 - 4x^2 + 5)^{\frac{5}{2}-1}.\frac{d}{dx}(x^4 - 4x^2 + 5)$$

$$\Longrightarrow \frac{d}{dx}(x^4 - 4x^2 + 5)^{\frac{5}{2}} = \frac{5}{2}(x^4 - 4x^2 + 5)^{\frac{3}{2}}.(4x^3 - 8x + 0)$$

$$\Longrightarrow \frac{d}{dx}(x^4 - 4x^2 + 5)^{\frac{5}{2}} = \frac{20}{2}.(x^4 - 4x^2 + 5)^{\frac{3}{2}}(x^3 - 2x)$$

$$\Longrightarrow \frac{d}{dx}(x^4 - 4x^2 + 5)^{\frac{5}{2}} = 10x(x^2 - 2)(x^4 - 4x^2 + 5)^{\frac{3}{2}}$$

3.4.4 Find the derivative of implicit functions

The chain rule will help us to find the derivative of implicit functions.

Example 1. Find $\frac{dy}{dx}$, if $x^2 + y^2 + 2gx + 2fy + c = 0$.

Solution: Given that

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Differentiating both the sides w.r.t. "x", keeping y as a function of x ,

$$\therefore \frac{d}{dx}(x^2 + y^2 + 2gx + 2fy + c) = \frac{d}{dx}(0)$$

$$\Longrightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(2gx) + \frac{d}{dx}(2fy) + \frac{d}{dx}(c) = 0$$

$$\Longrightarrow 2x + 2y.\frac{dy}{dx} + 2g(1) + 2f.\frac{dy}{dx} + 0 = 0$$

$$\Longrightarrow (y + f).\frac{dy}{dx} = -(x + g)$$

$$\Longrightarrow \boxed{\frac{dy}{dx} = -\frac{(x + g)}{(y + f)} = -\frac{x + g}{y + f}}$$

Example 2. Find $\frac{dy}{dx}$, if $x^2y + 2y^3 = 3x + 2y$

Solution: Given that

$$x^2y + 2y^3 = 3x + 2y$$

It is an implicit equation

$\therefore$ diff: both the sides w.r.t. "x" regarding y as function of x,

$$\therefore \frac{d}{dx}(x^2y + 2y^3) = \frac{d}{dx}(3x + 2y)$$

$$\Longrightarrow \frac{d}{dx}(x^2y) + \frac{d}{dx}(2y^3) = \frac{d}{dx}(3x) + \frac{d}{dx}(2y)$$

$$\Longrightarrow x^2.\frac{dy}{dx} + y.\frac{d}{dx}(x^2) + 3 \times 2y^2.\frac{dy}{dx} = 3(1) + 2\frac{dy}{dx}$$

$$\Longrightarrow x^2.\frac{dy}{dx} + 6y^2\frac{dy}{dx} - 2\frac{dy}{dx} = 3 - 2xy$$

$$\Longrightarrow (x^2 + 6y^2 - 2)\frac{dy}{dx} = 3 - 2xy$$

$$\frac{dy}{dx} = \frac{3 - 2xy}{x^2 + 6y^2 - 2}$$

3.5 Differentiation of trigonometric and Inverse Trigonometric Functions

3.5.1 Differentiate Trigonometric functions ($\sin x, \cos x, \tan x, \csc x, \sec x$ *and* $\cot x$) from the first principle

In the process of finding the derivative of trigonometric functions, we assume that x is measured in radians.

- **Differentiate $\sin x$ from the first principle.**

Consider the sine function $s: \mathbb{R} \to \mathbb{R}$, where $s(x) = \sin x, \forall x \in \mathbb{R}$.

Let $s(x) = \sin x$

$$\therefore \quad y + \delta y = s(x + \delta x) = \sin(x + \delta x)$$

Using the first principle, i.e.,

$$\therefore \quad s'(x) = \lim_{\delta x \to 0} \frac{s(x+\delta x) - s(x)}{\delta x}$$

$$\therefore \quad s'(x) = \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin(x)}{\delta x}$$

$$\Rightarrow \quad s'(x) = \lim_{\delta x \to 0} \frac{2\cos\left(\frac{x + \delta x + x}{2}\right) \cdot \sin\left(\frac{x + \delta x - x}{2}\right)}{\delta x}$$

$$\left[\begin{array}{c} \because \sin a - \sin b \\ = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right) \end{array}\right]$$

$$\Longrightarrow \quad s'(x) = \lim_{\frac{\delta x}{2} \to 0} 2\cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\frac{\delta x}{2} \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{2 \cdot \frac{\delta x}{2}}$$

$$\Longrightarrow \quad s'(x) = \lim_{\frac{\delta x}{2} \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\frac{\delta x}{2} \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

$$\Rightarrow \quad s'(x) = \cos(x+0)\cdot 1 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$\Rightarrow \quad s'(x) = \cos x$$

Thus, $\dfrac{d}{dx}(\sin x) = \cos x$

Note:
- $\dfrac{d}{dx}\sin ax = \cos ax.\dfrac{d}{dx}(ax) = a\cos ax$
- $\dfrac{d}{dx}\sin^n x = n\sin^{n-1} x\cdot \dfrac{d}{dx}(\sin x) = n\sin^{n-1} x.\cos x$

- **Differentiate $\tan x$ from first Principle**

Let $\quad t(x) = \tan x$

Using first principle,

i.e., $\quad f'(x) = \lim_{\delta x\to 0}\frac{t(x+\delta x)-t(x)}{\delta x}$

$$\therefore \quad t'(x) = \lim_{\delta x\to 0}\frac{\tan(x+\delta x)-\tan x}{\delta x}$$

$$\Rightarrow \quad t'(x) = \lim_{\delta x\to 0}\left[\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right]\cdot\frac{1}{\delta x}$$

$$\Rightarrow \quad t'(x) \lim_{\delta x\to 0}\left[\frac{\sin(x+\delta x).\cos x - \cos(x+\delta x).\sin x}{\delta x\cos(x+\delta x).\cos x}\right]$$

$$\Rightarrow \quad t'(x) = \lim_{\delta x\to 0}\frac{\sin(x+\delta x-x)}{\delta x}\cdot\frac{1}{\lim_{\delta x\to 0}\cos(x+\delta x)\cdot\cos x}$$

$$\Rightarrow \quad t'(x) = \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}\cdot\frac{1}{\lim_{\delta x\to 0}\cos(x+\delta x)\cdot\cos x}$$

$$\Rightarrow \quad t'(x) = 1\cdot\frac{1}{\cos(x+0)\cos x} \qquad \because \lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\Rightarrow \quad t'(x) = \frac{1}{\cos x\cdot\cos x} = \frac{1}{\cos^2 x} = \sec^2 x \qquad [\text{Provided } \cos x \neq 0]$$

Thus, $\boxed{\dfrac{d}{dx}(\tan x) = \sec^2 x}$

Note:
- $\dfrac{d}{dx}(\tan ax) = \sec^2 ax\cdot\dfrac{d}{dx}(ax) = a\sec^2 ax$
- $\dfrac{d}{dx}(\tan^n x) = n\tan^{n-1} x\cdot\dfrac{d}{dx}(\tan x) = n\tan_x^{n-1}\cdot\sec^2 x$

- **Differentiate $\sec x$ from first Principle.**

Let $\quad y = \sec x$

$$\therefore \quad y+\delta y = \sec(x+\delta x)$$

$$\because \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sec(x+\delta x) - \sec(x)}{\delta x}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sec(x+\delta x) - \sec x}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{1}{\cos(x+\delta x)} - \frac{1}{\cos x}\right] \cdot \frac{1}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{\cos x - \cos(x+\delta x)}{\cos(x+\delta x)\ \cdot\ \cos x\ \cdot\ \delta x}\right]$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{2\sin\left(\frac{x+x+\delta x}{2}\right) \cdot \sin\left(\frac{x+\delta x - x}{2}\right)}{\cos(x+\delta x)\ \cdot \cos x\ \cdot\ \delta x}\right]$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{2\sin\left(x+\frac{\delta x}{2}\right)}{\cos(x+\delta x) \cdot \cos x}\right] . \lim_{\frac{\delta x}{2} \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{2 . \frac{\delta x}{2}}$$

$$\left[\because \cos a - \cos b = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin(x+0)}{\cos(x+0)\ \cdot\ \cos x} \cdot \frac{1}{2}\ 1 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{\cos x} . \frac{1}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \tan x\ .\sec x$$

$$\Rightarrow \frac{dy}{dx} = \sec x\ .\tan x\ ,$$

Thus, $\boxed{\frac{d}{dx}(\sec x) = \sec x . \tan x}$

In general cases:

- $\frac{d}{dx}(\sec ax) = \sec ax . \tan ax . \frac{d}{dx}(ax)$
  $= a \sec ax . \tan ax$
- $\frac{d}{dx}(\sec^n x) = n \sec^{n-1} x . \frac{d}{dx}(\sec x)$
  $= n \sec^{n-1} x . \sec x . \tan x$
  $= n \sec^n x . \tan x$

Note: The derivative of cosine, cosecant and cotangent are left as an exercise for readers.

Examples 1. Differentiate "x" $\cos\sqrt{x}$ by ab-intio/first principle.

Solution: Let $y = f(x) = \cos\sqrt{x}$

$$\therefore \quad y + \delta y = f(x+\delta x) = \cos\sqrt{x+\delta x}\,,$$

$$\because \quad \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{\cos\sqrt{x+\delta x} - \cos\sqrt{x}}{\delta x}$$

Using trigonometric formula

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right).\ \sin\left(\frac{a-b}{2}\right)$$

we have,

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{-2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\ \sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = -2\lim_{\delta x \to 0} \sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\lim_{\delta x \to 0} \frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{x+\delta x - x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = -2\lim_{\delta x \to 0} \sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\lim_{\delta x \to 0} \frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\frac{2\left[\left(\sqrt{x+\delta x}\right)^2-\left(\sqrt{x}\right)^2\right]}{2}}$$

$$= -\lim_{\delta x \to 0} \sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\lim_{\theta \to 0}\frac{\sin\theta}{\theta}.\frac{1}{\lim_{\delta x \to 0}\left(\sqrt{x+\delta x}+\sqrt{x}\right)}$$

$$\left[\text{where } \theta = \frac{\left(\sqrt{x+\delta x}-\sqrt{x}\right)}{2}\right], \text{as } \delta x \to 0, \theta \to 0$$

$$= -\sin\left(\frac{\sqrt{x+0}+\sqrt{x}}{2}\right).1.\frac{1}{(\sqrt{x+0}+\sqrt{x}} \qquad \left[\because \lim_{\delta x \to 0}\frac{\sin x}{x} = 1\right]$$

$$= -\sin\left(\frac{2\sqrt{x}}{2}\right).\frac{1}{2\sqrt{x}}$$

$$= -\frac{\sin(\sqrt{x})}{2\sqrt{x}},$$

Thus, $\boxed{\frac{d}{dx}(\cos\sqrt{x}) = \frac{-\sin\sqrt{x}}{2\sqrt{x}}}$

Example 2. Differentiate:

(i) $y = \frac{x^2+\tan x}{3x+2\tan x}$ w.r.t x (ii) $y = \cos^2 x$ w.r.t $\sin^2 x$

Solution (i): Given that

$$y = \frac{x^2 + \tan x}{3x + 2\tan x}$$

Differentiating w.r.t "x" by using quotient rule, we have

$$\therefore \frac{dy}{dx} = \frac{(3x + 2\tan x).\frac{d}{dx}(x^2 + \tan x) - (x^2 + \tan x).\frac{d}{dx}(3x + 2\tan x)}{(3x + 2\tan x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(3x + 2\tan x)(2x + \sec^2 x) - (x^2 + \tan x)(3 + 2\sec^2 x)}{(3x + 2\tan x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x^2 + 3x\sec^2 x + 4x\tan x + 2\tan x\sec^2 x - (3x^2 + 2x^2\sec^2 x + 3\tan x + 2\tan x.\sec^2 x)}{(3x + 2\tan x)^2}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{3x^2 + (4x - 3)\tan x + x(3 - 2x)\sec^2 x}{(3x + 2\tan x)^2}}$$

Solution (ii): Given that:

$$y = \cos^2 x, \qquad \ldots(i)$$

and let

$$u = \sin^2 x, \qquad \ldots(ii)$$

In this case, we have to find $\frac{dy}{du}$

Here,

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\cos^2 x) = 2\cos x.\frac{d}{dx}(\cos x) = -2\sin x.\cos x$$

and $$\frac{du}{dx} = \frac{d}{dx}(\sin^2 x) = 2\sin x.\frac{d}{dx}(\sin x) = 2\sin x.\cos x$$

Using chain rule:

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$

$$\frac{dy}{du} = \frac{dy}{dx} \div \frac{du}{dx}$$

$$\therefore \frac{dy}{du} = \frac{-2\sin x\ \cos x}{2\sin x\cos x}$$

$$\Rightarrow \boxed{\frac{dy}{du} = -1}$$ Provided $\sin x \neq 0$ and $\cos x \neq 0$

3.5.2 Differentiate inverse trigonometric functions ($arc\ \sin x$, $arc \cos x$, $arc \tan x$, $arc\ \csc x$, $arc \sec x$ and $arc\ \cot x$) using differentiation formulae.

- **Differentiate arc $\sin x$ or $(\sin^{-1} x)$**

Let $\quad y = \sin^{-1} x \quad \forall\, x \in (-1, 1)$

$\Rightarrow \sin y = x$

∴ Differentiating w.r.t "x", regarding y as a function of x, we have

$$\therefore \frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dy}(\sin y).\frac{dy}{dx} = 1$$

$$\Rightarrow \cos y.\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} \qquad [\text{Provided } \cos y \neq 0]$$

$$\therefore \quad \cos y = \pm\sqrt{1-\sin^2 y}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\pm\sqrt{1-\sin^2 y}}$$

The principal domain of $\sin y$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in which $\cos y$ is +ve.

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad (\because x = \sin y)$$

Thus,

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad \forall x \in (-1,1)}$$

Note: $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right) = \frac{a}{a\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-x^2}}$

- **Differentiate $arc \tan$ or $\tan^{-1} x$**

Let $\quad y = \tan^{-1} x \qquad \forall x \in \mathbb{R}$

$\Longrightarrow \quad \tan y = x$

Differentiating w.r.t "x" regarding y as a function of x,

$$\therefore \frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dy}(\tan y)\,.\,\frac{dy}{dx} = 1$$

$$\Rightarrow \sec^2 y\,.\,\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} \qquad \forall y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y} \qquad [\because \sec^2 y = 1 + \tan^2 y]$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{1+x^2} \qquad \forall x \in \mathbb{R}$$

Thus, $\boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad \forall x \in \mathbb{R}}$

Note: $\frac{d}{dx}(\tan^{-1}\frac{x}{a}) = \frac{1}{1+\frac{x^2}{a^2}}\,.\,\frac{d}{dx}\left(\frac{x}{a}\right) = \frac{a^2}{(a^2+x^2)}.\frac{1}{a} = \frac{a}{a^2+x^2}$

- **Differentiate *arc* sec x, $\forall x \in \mathbb{R} - [-1, 1]$ $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$**

Let $y = sec^{-1}x, \forall x \in \mathbb{R} - [-1,1] \longrightarrow \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

then, $x = sec\ y$, $\quad \forall y \in [0, \pi]\ and\ y \neq \frac{\pi}{2}$,

It is an implicit equation:

Differentiating w.r.t "x" regarding y as function of x

$$\therefore \frac{d}{dx}(x) = \frac{d}{dx}(\sec y)$$

$$\Longrightarrow 1 = \frac{d}{dy}(\sec y).\frac{d}{dx}(y)$$

$$\Longrightarrow 1 = \sec y.\,tan\ y\,.\frac{dy}{dx}$$

or

$$\sec y.\,tan\ y\,.\frac{dy}{dx} = 1$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{sec\ y\,.\,tan\ y} \qquad \forall y \in (0, \pi)\ and\ y \neq \frac{\pi}{2}$$

when, $y \in \left(0, \frac{\pi}{2}\right)$, sec y and tan y are positive, so that sec $y = x$, i.e., x is positive in this case,

and $tan\ y = \sqrt{sec^2 y - 1} = \sqrt{x^2 - 1}$,

Thus, $\boxed{\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}, \text{ when } x > 0}$

When, $y \in \left(\frac{\pi}{2}, \pi\right)$, $\sec y$ and $\tan y$ are negative so that x is negative and in this case.

$\tan y = -\sqrt{x^2-1}$, when $x < 0$,

Thus,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{(-x)(-\sqrt{x^2-1})} = \frac{1}{x\sqrt{x^2-1}} \qquad \text{... (ii)}$$

Combining equations (i) and (ii) we have,

Thus, $$\boxed{\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}},} \qquad \forall x \in \mathbb{R} - [-1,1]$$

Note: The derivatives of $cos^{-1}x, csc^{-1}x$ and $cot^{-1}x$ are left as an exercise for readers.

- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

Example 1. If $y = x\, sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2}$, find $\frac{dy}{dx}$

Solution: Given that

$$y = x\, sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2-x^2}$$

diff: w.r.t "x" we have,

$$\frac{dy}{dx} = \frac{d}{dx}\left[x\, sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2}\,\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(x\, sin^{-1}\frac{x}{a}\right) + \frac{d}{dx}\left(\sqrt{a^2-x^2}\,\right)$$

$$\Rightarrow \frac{dy}{dx} = x\frac{d}{dx}\left(sin^{-1}\frac{x}{a}\right) + sin^{-1}\left(\frac{x}{a}\right).\frac{d}{dx}(x) + \frac{1}{2\sqrt{a^2-x^2}}.\frac{d}{dx}(a^2-x^2\,)$$

$$\Rightarrow \frac{dy}{dx} = x.\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}.\frac{d}{dx}\left(\frac{x}{a}\right) + sin^{-1}\left(\frac{x}{a}\right) \times 1 + \frac{0-2x}{2\sqrt{a^2-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x \times a}{\sqrt{a^2-x^2}} \times \frac{1}{a} + sin^{-1}\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2-x^2}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = sin^{-1}\left(\frac{x}{a}\right)}$$

Example 2. If $y = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$, find $\frac{dy}{dx}$.

Solution: Given that:

$$y = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$$

Differentiating w.r.t 'x'

$$\therefore \frac{dy}{dx} = \frac{d}{dx}cos^{-1}\left(\frac{x^2-1}{x^2+1}\right),$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}.\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right), \qquad \left[\frac{d}{dx}cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x^2+1)}{\sqrt{(x^2+1)^2-(x^2-1)^2}} \times \frac{(x^2+1)\frac{d}{dx}(x^2-1)-(x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{x^2+1} \times \frac{(x^2+1)(2x)-(x^2-1)(2x)}{\sqrt{x^4+2x^{2+}1-x^4+2x^2-1}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{x^2+1} \times \frac{2x^3+2x-2x^3+2x}{\sqrt{4x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-4x}{2x(x^2+1)} \qquad \text{provided } x \neq 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-2}{(x^2+1)}}$$

Exercise 3.4

1. Find the derivatives of the following using chain rule.

(i) $y = (x^4+5x^2+6)^{\frac{3}{2}}$ (ii) $y = \left(\frac{x-1}{x+1}\right)^{\frac{3}{4}}$ (iii) $y = \sqrt{\frac{2+x}{3+x}}$

(iv) $y = \left(x+\sqrt{x^2-1}\right)^n$ (v) $y = \sqrt[3]{\frac{x^3+1}{x^3-1}}$

2. Differentiate $\frac{x^3}{1+x^3}$ w.r.t x^3.

3. If f is a function with $y = f(x)$, given implicitly, find $\frac{dy}{dx}$, where it exists in the following cases.

(i) $y - xy - \sin y = 0$ (ii) $y^3 - 3y + 2x = 0$

(iii) $x^2 + y^2 + 4x + 6y - 12 = 0$ (iv) $\sin x\, y + \sec x = 2$

(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$ (vi) $y(x^2+1) = x(y^2+1)$

4. If $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$, where a and b are non-zero constants, find $\dfrac{du}{dv}$ and $\dfrac{dv}{du}$.

5. Find the slope of the tangent to the curve $3x^2 - 7y^2 + 14y - 27 = 0$ at the point $(-3,0)$.

6. Differentiate the following by using first principle method.

(i) $\sin 4x$ (ii) $\cos^2 2x$ (iii) $\sec\sqrt{x}$

(iv) $\sqrt{\tan x}$ (v) $\csc 3x$ (vi) $\cot 2x$

7. Using differentiations rules, differentiate w.r.t. their involved variables:

(i) $f(x) = (x+2).\sin x$ (ii) $f(\theta) = \tan^2\theta.\sec^3\theta$

(iii) $f(t) = \sin^2 3t.\ \cos^3 t$ (iv) $f(x) = \sqrt{\dfrac{\sin 2x}{\cos x}}$

(v) $f(\theta) = \dfrac{\tan\theta - 1}{\sec\theta}$ (vi) $f(x) = \sin x^2 + \sin^2 x$

8. Differentiate $\dfrac{1+\tan^2 x}{1-\tan^2 x}$ w.r.t. $\tan^2 x$.

9. Find $\dfrac{dy}{dx}$ of the following:

(i) $y = \sin^{-1}\sqrt{\dfrac{1-\cos x}{2}}$ (ii) $y = \cot^{-1}\left(\sqrt{\dfrac{1+\cos x}{1-\cos x}}\right)$

(iii) $y = \tan^{-1}\left(\dfrac{\sin 2x}{1+\cos 2x}\right)$ (iv) $y = x + \cos^{-1}x\ .\sqrt{1-x^2}$

(v) $y = \dfrac{x\tan^{-1}x}{1+x^2}$ (vi) $y = \tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

10. If $y = \tan^{-1}\left(2\tan\frac{x}{2}\right)$, then prove that $\dfrac{dy}{dx} = 4\left(\dfrac{1+y^2}{4+x^2}\right)$

11. If $\dfrac{y}{x} = \tan^{-1}\left(\dfrac{x}{y}\right)$, show that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

12. If $y = \tan(a\tan^{-1}x)$, show that $(1+x^2)\dfrac{dy}{dx} - a(1+y^2) = 0$.

13. Find $\dfrac{dy}{dx}$

(i) $x = a\sin\theta$ and $y = a\cos\theta$ (ii) $x = t + \dfrac{1}{t}$ and $y = t+1$

(iii) $x = \frac{a(1-t^2)}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$, (a and b are constant)

(iv) $x = a\theta^2$ and $y = 2a\theta$, (a is constant)

14. If $y = \sqrt{tanx + \sqrt{tanx + \sqrt{tanx} + .. + \infty}}$, prove that $(2y-1)\frac{dy}{dx} = \sec^2 x$.

15. Find the derivatives of $\cos^{-1} x$, $\csc^{-1} x$ and $\cot^{-1} x$ by using differentiation formula.

3.6 Differentiation of Exponential and Logarithmic Functions

3.6.1 Find the derivatives of e^x and a^x from first principle

(a) Derivative of e^x

Let $y = f(x) = e^x$

$\therefore y + \delta y = f(x+\delta x) = e^{x+\delta x}$

$\because \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{e^{x+\delta x} - e^x}{\delta x}$

$= \lim_{\delta x \to 0} \frac{e^x(e^{\delta x} - 1)}{\delta x}$

$= e^x \lim_{\delta x \to 0} \frac{e^{\delta x} - 1}{\delta x}$ $\left(\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1\right)$

$= e^x . 1 = e^x$

Thus, $\Rightarrow \frac{d}{dx}(e^x) = e^x$

Note: $\frac{d}{dx}(e^{ax}) = e^{ax}.\frac{d}{dx}(ax) = ae^{ax}$

(b) Derivatives of $a^x, \forall\, a > 0$ and $a \neq 1$

Let $y = f(x) = a^x, \forall a > 0$ and $x \in \mathbb{R}$

$\therefore y + \delta y = f(x+\delta x) = a^{x+\delta x}$,

$\because \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$, Provided $\delta x \neq 0$ and its limit exists

$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{a^{x+\delta x} - a^x}{\delta x}$,

$\Longrightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} a^x \left(\frac{a^{\delta x} - 1}{\delta x}\right)$,

$$\Rightarrow \frac{dy}{dx} = a^x . \lim_{\delta x \to 0}\left(\frac{a^{\delta x}-1}{\delta x}\right),$$

$$\Rightarrow \frac{dy}{dx} = a^x . \ln a, \qquad \left(\because \lim_{x\to 0}\frac{a^x-1}{x} = \ln a\right)$$

Thus, $\Rightarrow$ $\boxed{\frac{d}{dx}\left(a^x\right) = a^x \ln a}$

Note: $\frac{d}{dx}\left(a^{bx}\right) = a^{bx}. b\ln a\,(bx) = ba^x \ln a$

Example 1. Find $\frac{dy}{dx}$ when $y = e^{\sin x} + a^{\cos x}$

Solution: Given that

$$y = e^{\sin x} + a^{\cos x}$$

diff: w.r.t "x"

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin x} + a^{\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin x}\right) + \frac{d}{dx}\left(a^{\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin x}\ \frac{d}{dx}(\sin x) + a^{\cos x}\ \ln a\ \frac{d}{dx}(\cos x)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin x}\ \cos x + a^{\cos x} \ln a\,(-\sin x)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = e^{\sin x}\ \cos x - a^{\cos x}\ \ln a \sin x}$$

3.6.2 Find the derivative of $\ln x$ and $\log_a x$ from first principle

- **Derivative of $\ln x$ from first principle**

Let $y = f(x) = \ln x\ ,\ \forall\ x > 0$

$$\therefore y + \delta y = f(x+\delta x) = \ln(x+\delta x)$$

$$\because \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{ln(x+\delta x) - ln(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\ln\left(\frac{x+\delta x}{x}\right)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x\to 0}\ \frac{1}{x}.\frac{x}{\delta x}.\ln\left(1+\frac{\delta x}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \to 0} \ln\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \qquad \left[\because \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \times 1 = \frac{1}{x} \qquad [\because \ln e = 1]$$

Thus, $\frac{d}{dx}(\ln x) = \frac{1}{x} \quad , \quad \forall \ x > 0$ ...(i)

Similarly, $\frac{d}{dx}\ln(-x) = \frac{1}{-x} \times (-1) = \frac{1}{x} \quad , \quad \forall \, x < 0$ ...(ii)

Combining (i) and (ii), we have

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

Note: $\frac{d}{dx}(\ln a x) = \frac{1}{ax} \cdot \frac{d}{dx}(ax) = \frac{1}{x} \ , \ \forall \ x > 0$

- **Derivative of $\log_a x$ by first principle**

Let $\quad y = f(x) = \log_a x$

$\therefore \quad y + \delta y = f(x + \delta x) = \log_a(x + \delta x)$

$\because \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$, (Provided $\delta x \neq 0$ and limit exist)

$$\Longrightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\log_a(x+\delta x) - \log_a x}{\delta x}$$

$$\Longrightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{x} \cdot \frac{x}{\delta x} \log_a\left(\frac{x + \delta x}{x}\right)$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{x} \cdot \lim_{\delta x \to 0} \log_a\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{x} \cdot \log_a\left[\lim_{\delta x \to 0} \left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{x} \log_a e \ , \qquad \left[\because \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e\right]$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{x \ln a} , \qquad \left[\because \log_a e = \frac{1}{\ln a}\right]$$

Thus, $\boxed{\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}}$

Note: $\frac{d}{dx}(\log_a bx) = \frac{1}{bx \ln a} \cdot \frac{d}{dx}(bx) = \frac{1}{x \ln x} \ , \ \forall \ x > 0$

Example 1. Find $\frac{dy}{dx}$, when: $y = \ln(x^2 + 4)$

Solution: Given that

$$y = \ln(x^2 + 4)$$

Differentiating w.r.t "x"

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\ln(x^2 + 4), \qquad \left[\because \frac{d}{dx}\ln x = \frac{1}{x}\right]$$

$$\frac{dy}{dx} = \frac{1}{x^2 + 4} \cdot \frac{d}{dx}(x^2 + 4) = \frac{2x}{x^2 + 4}$$

Example 2. Find $\frac{dy}{dx}$, when: $y = \log_{10}\sqrt{x^2 + 2x} - 4x^4$

Solution: Given that

$$y = \log_{10}\sqrt{x^2 + 2x} - 4x^4$$

$$\Rightarrow \quad y = \log_e \sqrt{x^2 + 2x}.\log_{10} e - 4x^4 \quad [\because \log_{10}^x = \log_e^x.\log_{10}^e \text{ and } \log_e^x = \ln x]$$

$$\Rightarrow \quad y = \ln\left(\sqrt{x^2 + 2x}\right).\log_{10} e - 4x^4$$

$$y = \frac{\log_{10} e}{2}.\ln(x^2 + 2x) - 4x^4$$

Differentiating w.r.t "x"

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\frac{\log_{10} e}{2}.\ln(x^2 + 2x) - 4x^4\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log_{10} e}{2}.\frac{d}{dx}\ln(x^2 + 2x) - 4\frac{d}{dx}(x^4)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\ln 10} \cdot \frac{1}{x^2 + 2x}\frac{d}{dx}(x^2 + 2x) - 4 \times 4x^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\ln 100} \times \frac{(2x + 2 \times 1)}{x^2 + 2x} - 16x^3$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{(x + 1)}{x(x + 2)\ln 100} - 16x^3}$$

3.6.3 Use logarithmic differentiation to find derivative of algebraic expression involving product, quotient and power.

Example 1. Differentiate $\left[\frac{x(x+1)}{(x+2)}\right]$, w.r.t. "$x$"

Solution: Let $y = \ln\frac{x(x+1)}{(x+2)}$

Taking ln on both sides

$$\ln y = \ln\left(\frac{x(x + 1)}{(x + 2)}\right)$$

$$\ln y = \ln x + \ln(x+1) - \ln(x+2)$$

Differentiating w.r.t x

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{1}{x+1} - \frac{1}{x+2}$$

$$\frac{dy}{dx} = \frac{x(x+1)}{(x+2)}\left[\frac{(x+1)(x+2) + x(x+2) - x(x+1)}{x(x+1)(x+2)}\right]$$

$$\frac{dy}{dx} = \frac{x(x+1)}{(x+2)}\frac{(x^2+3x+2+x^2+2x-x^2-x)}{x(x+1)(x+2)}$$

$$\frac{dy}{dx} = \frac{x^2+4x+2}{(x+2)^2}$$

Example 2. If $x^y = y^x$, find $\frac{dy}{dx}$.

Solution: $x^y = y^x$ Given that

Taking natural logarithm of both sides, we have

$\therefore \quad \ln x^y = \ln y^x$

$\Longrightarrow \quad y\ln x = x\ln y \qquad [\therefore \ln a^x = x\ln a]$

Differentiating both sides w.r.t. "x" regarding y as a function of x.

$$\therefore \frac{d}{dx}(y\ln x) = \frac{d}{dx}(x\ln y)$$

$$\Rightarrow \quad y\frac{d}{dx}(\ln x) + \ln x\,.\frac{d}{dx}(y) = x.\frac{d}{dx}(\ln y) + \ln y\,.\frac{d}{dx}(x)$$

$$\Rightarrow \frac{y}{x} + \ln x\,.\frac{dy}{dx} = x.\frac{d}{dy}(\ln y).\frac{dy}{dx} + \ln y \times 1 \qquad \left[\because \frac{d}{dx}(\ln x) = \frac{1}{x}\right]$$

$$\Rightarrow \ln x\,.\frac{dy}{dx} - \frac{x}{y}\,.\frac{dy}{dx} = \left(\ln y - \frac{y}{x}\right)$$

$$\Rightarrow \left(lnx - \frac{x}{y}\right)\frac{dy}{dx} = \left(\ln y - \frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\ln y - \frac{y}{x})}{(lnx - \frac{x}{y})}$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{y(x\ln y - y)}{x(y\ln x - x)}}$$

List of Derivatives of the basic standard functions shown in the list 1.

Sr No.	$y = f(x)$	Derivative $\frac{dy}{dx}$
1.	$y = \sin x$	$\cos x$
2.	$y = \cos x$	$-\sin x$
3.	$y = \tan x$	$\sec^2 x$
4.	$y = \sec x$	$\sec x \tan x$
5.	$y = \cot x$	$-\text{cosec}^2 x$
6.	$y = \text{cosec}\, x$	$-\text{cosec}\, x \cot x$
7.	$y = \sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
8.	$y = \cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
9.	$y = \tan^{-1} x$	$\frac{1}{1+x^2}$
10.	$y = \cot^{-1} x$	$-\frac{1}{1+x^2}$
11.	$y = \csc^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$
12.	$y = \sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
13.	$y = \log_a x$	$\frac{1}{x \ln a}, \quad \forall a > 0$
14.	$y = \ln x$	$\frac{1}{x}, \quad \forall x > 0$

Exercise 3.5

1. Differentiate the following w.r.t. "x"

(i) $x^2 + 2^x$ (ii) $4^x + 5^x$ (iii) $e^{\tan x + \cot x}$

(iv) $e^{\tan x^2}$ (v) $e^{2\ln(2x+1)}$ (vi) $\log_{10} x$

(vii) $\frac{e^x}{x^2+1}$ (viii) $x^2 + 2^x + a^{2x}$ (ix) $(\ln x)^x$

(x) $\ln\left(\sqrt{e^{3x} + e^{-3x}}\right)$ (xi) $\ln(\sin(\ln x))$ (xii) $\ln\left[\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right]$

2. Using logarithmic differentiation to find $\frac{dy}{dx}$ if

(i) $y=\sqrt{\frac{x^2-1}{x^2+1}}$ (ii) $y=x^3\sqrt{x}$ (iii) $y=xe^{\cos x}$

(iv) $y=e^{-2x}(x^2+2x+1)$ (v) $y=\ln\left(\frac{e^x}{1+e^x}\right)$ (vi) $y=\sqrt{\frac{1+e^x}{1-e^x}}$

3. Find $\frac{dy}{dx}$ if

(i) $y=\frac{1-x^2}{\sqrt{1+x^2}}$ (ii) $y=\sqrt{\frac{1-x}{1+x}}$

4. Find $\frac{dy}{dx}$ if

(i) $y=x^{\sin x}$ (ii) $y=(\sin^{-1}x)^{\ln x}$ (iii) $y=(\tan^{-1}x)^{\sin x+\cos x}$

(iv) $y=(\ln x)^{cosx}$ (v) $y=x^x$ (vi) $y=\ln\left(\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x}\right)$

5. Find $\frac{dy}{dx}$, when:

(i) $x^y.y^x=1$ (ii) $\ln(xy)=x^2+y^2$

(iii) $y=\sin^{-1}(\cos x)+\cos^{-1}(\sin x)$ (iv) $y=x^y$

(v) $y=\cos x\ln(\sin^{-1}x)$ (vi) $x^n.y^n=a^n$

3.7 Differentiation of Hyperbolic and Inverse Hyperbolic Functions

3.7.1 (a) Differentiation of hyperbolic functions

(i) Differentiate $\sinh x$ by first principle

Let $y=f(x)=\sinh x=\frac{e^x-e^{-x}}{2}$

Differentiating w.r.t. "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}\left(\frac{e^x}{2}\right)-\frac{d}{dx}\left(\frac{e^{-x}}{2}\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{e^x}{2}\times 1-\frac{e^{-x}}{2}\times(-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x + e^{-x}}{2} = \cosh x$$

Thus, $\boxed{\frac{d}{dx}(\sinh x) = \cosh x}$

$$\boxed{\frac{d}{dx}(\sinh ax) = a \cos h\, ax}$$

Derivatives of $\cosh x, \tanh x, \operatorname{csch} x, \operatorname{sech} x$ *and* $\coth x$ are explained below:

- $$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right] = \frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$
$$= \frac{e^{2x} + e^{-2x} + 2 - (e^{2x} + e^{-2x} - 2)}{(e^x + e^{-x})^2}$$
$$= \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^2 + e^{-x}}\right)^2 = \sec h^2 x$$

- $$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\left(\frac{2}{e^x + e^{-x}}\right) = \frac{(e^x+e^{-x})(0) - 2(e^x + e^{-x} \times -1)}{(e^x+e^{-x})^2}$$
$$= \frac{-2}{(e^x + e^{-x})} \cdot \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$= -\sec hx \,.\, \tan hx$$

Derivative of $\cosh x$, cosech x and $\coth x$ are left as an exercise for the readers.

Thus, we have the list of derivatives of hyperbolic functions.

• $\frac{d}{dx}(\sinh x) = \cosh x$	• $\frac{d}{dx}(\cosh x) = \sinh x$
• $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$	• $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \,.\, \tanh x$
• $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$	• $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Example 1. Differentiate the following w.r.t. "x"

(i) $x \cosh 2x$ (ii) $\frac{\operatorname{csch} 2x}{x^2}$

Solution (i): Let $y = x \cosh 2x$

Differentiating w.r.t x, we have

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}(x \cosh 2x)$$

$$\frac{dy}{dx} = x\frac{d}{dx}(\cosh 2x) + \cosh 2x \frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x \times \sinh 2x \frac{d}{dx}(2x) + \cosh 2x \times 1$$

$$\frac{dy}{dx} = 2x \sinh 2x + \cosh 2x$$

Thus, $\frac{d}{dx}\left(x \cosh 2x\right) = 2x \sinh 2x + \cosh 2x$

Solution (ii): Let $y = \frac{\operatorname{csch} 2x}{x^2}$

Differentiating w.r.t x, we have

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{\operatorname{csch} 2x}{x^2}\right)$$

$$\frac{dy}{dx} = \left[x^2 . \frac{d}{dx}(\operatorname{csch} 2x) - \operatorname{csch} 2x \frac{d}{dx}(x^2)\right] \cdot \frac{1}{(x^2)^2}$$

$$\frac{dy}{dx} = \left[x^2 - \operatorname{csch} 2x \,.\, \coth 2x \frac{d}{dx}(2x) - \operatorname{csch} 2x \,.\, 2x\right] \cdot \frac{1}{x^4}$$

$$\frac{dy}{dx} = [-2x^2 \operatorname{csch} 2x \coth 2x - 2x \operatorname{csch} 2x] \cdot \frac{1}{x^4}$$

$$\frac{dy}{dx} = [-2x \operatorname{csch} 2x\,(x \coth 2x - 1)] \cdot \frac{1}{x^4}$$

$$\frac{dy}{dx} = \frac{-2 \operatorname{csch} 2x\,(x \coth 2x - 1)}{x^3}$$

Thus, $\frac{d}{dx}\left(\frac{\operatorname{csch} 2x}{x^2}\right) = \frac{-2 \operatorname{csch} 2x(x \coth 2x - 1)}{x^3}$

- **Differentiation of inverse hyperbolic functions**
  $\sinh^{-1} x, \cosh^{-1} x, \tanh^{-1} x, \operatorname{csch}^{-1} x, \operatorname{sech}^{-1} x, \coth^{-1} x$

- Find derivative of $\sinh^{-1} x$, w.r.t "x"

Solution: Let $y = \sinh^{-1} x \quad , \quad \forall x \in \mathbb{R}$

then $\sinh y = x, \quad \forall \ y \in \mathbb{R}$

Differentiating w.r.t. "x" both sides, regarding y as a function of x

$$\therefore \frac{d}{dx}(\sinh y) = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dy}(\sin hy) . \frac{dy}{dx} = 1$$

$$\Rightarrow \cos hy . \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos hy} \qquad (\because \cosh y > 0)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+\sin h^2 y}} \qquad \left[\because \cosh y = \sqrt{1+\sin h^2 y}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \quad , \qquad [\because \sinh y = x]$$

Thus, $\boxed{\frac{dy}{dx}\left(\sin h^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}, \quad \forall\ x \in \mathbb{R}}$

(iii) Find the derivative of $\tanh^{-1}x$, w.r.t. "x".

Solution: Let $y = \tan h^{-1}x, \forall\ x \in (-1,1)$

Then $\tanh y = x, \qquad \forall\ y \in \mathbb{R}$

diff: w.r.t. "x" both sides, keeping y as a function of x

$$\therefore \frac{d}{dx}(\tan hy)\frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dy}(\tan hy)\frac{dy}{dx} = 1$$

$$\Rightarrow \sec h^2 y \,.\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{sec^2 hy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-\tanh^2 y} \qquad [\because \operatorname{sech}^2 y = 1-\tanh^2 y]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-x^2}, \forall\ x \in (-1,1)$$

Thus, $\boxed{\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, \forall\ x \in (-1,1)}$

(iv) Find the derivative of $\operatorname{csch}^{-1}x$.

Solution: Let $y = \operatorname{csch}^{-1}x, \forall\ x \in \mathbb{R}-\{0\}$,

then $csc\ hy = x, \forall\ y \in \mathbb{R}-\{0\}$.

Differentiating: w.r.t "x" both sides, regarding y as a function of x.

$$\therefore \frac{d}{dx}(\operatorname{csch} y) = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dy}\left(\csc h\, y\right).\frac{dy}{dx} = 1$$

$$\Rightarrow -\csc h\, y\,.\coth y\,.\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{csch} y\,.\coth y} \qquad (\because \operatorname{csch} y > 0 \text{ and } \coth y > 0)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\csc h\, y\sqrt{1+\csc h^2 y - 1}} \qquad \left(\because\ \coth y = \sqrt{1+\text{csch}^2\, y}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{1+x^2}}$$

Thus, $\frac{d}{dx}(\text{cosech}^2 x) = \frac{-1}{x\sqrt{1+x^2}}$, $\forall\, x \in \mathbb{R} - \{0\}$.

> **Note:** The derivatives of $\cosh^{-1} x$, $\text{sech}^{-1} x$ and $\coth^{-1} x$ are left as an exercise for readers.

Examples: Find $\frac{dy}{dx}$; when:

(i) $y = \sinh^{-1}(2x+5)$ (ii) $y = \cosh^{-1}(\sec x)$

Solution: (i) Given that

$$y = \sin h^{-1}(2x+5)$$

Differentiating w.r.t "x" by applying chain rule:

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\sin h^{-1}(2x+5)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+(2x+5)^2}} \cdot \frac{d}{dx}(2x+5) \qquad \left[\because\ \frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2}{\sqrt{4x^2+20x+26}}}$$

Solution: (ii) Given that

$$y = \cosh^{-1}(\sec x)$$

Differentiating w.r.t. "x" by applying chain rule

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\cosh^{-1}(\sec x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\sec^2 x - 1}} \cdot \frac{d}{dx}(\sec x)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\sec x\,.\,\tan x}{\sqrt{\tan^2 x}} = \sec x}$$

List of derivatives of basic standards functions shown in the list 2.

Sr. No	$y = f(x)$	$\frac{dy}{dx}$
1.	$y = \sinh x$	$\cosh x$
2.	$y = \cosh x$	$\sinh x$
3.	$y = \tanh x$	$\text{sech}^2 x$
4.	$y = \coth x$	$-\text{csch}^2 x$
5.	$y = \text{csch}\, x$	$-\text{csch}\, x\ .\coth x$
6.	$y = \text{sech}\, x$	$-\text{sech}\, x\ .\tanh x$
7.	$y = \sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}, -\infty < x < \infty$
8.	$y = \cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}},\ x > 1$
9.	$y = \tanh^{-1} x$	$\frac{1}{1-x^2}, \|x\| < 1$
10.	$y = \coth^{-1} x$	$\frac{1}{1-x^2}, \|x\| > 1$
11.	$y = \text{csch}^{-1} x$	$-\frac{1}{x\sqrt{1+x^2}}, x > 0$
12.	$y = \text{sech}^{-1} x$	$-\frac{1}{x\sqrt{1-x^2}}, 0 < x < 1$

3.7.3 Use MAPLE command diff differentiate a function:

- The **diff** command computers the partial derivatives of the expression with respect to $x_1, x_2, \ldots x_n$ respectively. The most frequent use is **diff** $(f(x), x)$, which computes the derivative of the function $f(x)$ which respect to x.
- You can enter the **diff** command using either the 1-D or 2-D calling sequence, e.g., **diff** (x, x) is equivalent to $\frac{d}{dx}x$.
- **diff** has a user interface that will call the user's own differentiation functions. If the procedure **"diff"** is defined, then the function call **diff** $(f(x,y,z),y)$ will invoke **diff**$/f(x,y,z)$ to compute the derivative.
- If the derivative cannot be expressed (if the expression is an undefined function), the **diff** function call itself is returned. The pretty printer display the **diff** function in a two-dimensional $\frac{d}{dx}$ format. The differential operator D is also defined in Maple.

Examples.

1 > **diff** $(2x,\ [x]) = 2$

2\. > **diff** $(\sin 2x,\ [x])2\cos 2x$

3\. > **diff** $(\sec^2(x), [x]) = 2\sec^2 x \,.\tan x$

4\. > **diff** $(\sqrt{x^3+3}, [x])\ \frac{3}{2}\cdot\frac{x^2}{\sqrt{x^3}}$

5\. > **diff** $(\cos(\sin(2x)), [x]) = -2\sin(\sin 2x).\cos 2x$

3.7.3 Use MAPLE command diff to differentiate a function

The format of diff command to differentiate a function in MAPLE are as under:

>diff(f, [x]) is equivalent to the command $\frac{d}{dx}f$ in Maple version 2015.

Where,

f stands for function whose derivative is to be evaluated

x stands for the variable x, the derivative with respect x

$\frac{d}{dx}$ means 1st order derivative with respect to variable x

Note: All above operators should be taken from the Maple calculus pallet

Use MAPLE command **diff** or $\left(\frac{d}{dx}f\right)$ to differentiate a function:

Derivative of functions:

$> \frac{d}{dx}(2x^3+3x^2+5x+42)$

$$6x^2+6x+5$$

$> \frac{d}{dx}3\sqrt{x+1}$

$$\frac{1}{3(x+1)^{\frac{2}{3}}}$$

$> \frac{d}{dx}e^{3x}$

$$3e^{3x}$$

$> \frac{d}{dx}\sin(x)$

$$\cos(x)$$

$> \frac{d}{dx}\cos(\sqrt{x})$

$$-\frac{1}{2}\frac{\sin(\sqrt{x})}{\sqrt{x}}$$

$> \frac{d}{dx}\ln(x)$

$$\frac{1}{x}$$

Derivative on Product form:

$> \frac{d}{dx}(e^x\sqrt{x})$

$$e^x\sqrt{x}+\frac{1}{2}\frac{e^x}{\sqrt{x}}$$

$> \frac{d}{dx}(e^x.(x^2+1))$

$$e^x.(x^2+1)+2(e^x.x)$$

Derivative on Quotient form:

$> \frac{d}{dx}\left(\frac{e^x}{x+3}\right)$

$$\frac{e^x}{x+3}-\frac{e^x}{(x+3)^2}$$

$> \frac{d}{dx}\left(\frac{\ln(x+1)}{\sin(x)}\right)$

$> \frac{d}{dx}(e^x.\sin(x))$

$e^x.\sin(x) + e^x.\cos(x)$

$$\frac{1}{(x+1)\sin(x)} - \frac{\ln(x+1)\cos(x)}{\sin^2(x)}$$

Exercise 3.6

1. Differentiate the following w.r.t. x:

(i)	$\sinh[\ln(x+3)]$	(ii)	$\sinh(e^{3x})$	(iii)	$\cosh(2x^2+3x)$
(iv)	$\frac{\tanh\sqrt{x}}{\sqrt{\cosh x}}$	(v)	$\tan(e^{\sinh^{-1}x})$	(vi)	$\frac{\sinh^{-1}x}{\text{sech}^{-1}x}$
(vii)	$\cosh x\,.\coth x^2$	(viii)	$\sinh x \tanh x^2$	(ix)	$\ln[\tanh(x^2+2x+1)]$

2. Find $\frac{dy}{dx}$, for the following functions:

(i)	$y = x\cosh^{-1}x - \sqrt{x^2-1}$	(ii)	$y = x\tanh^{-1}(3x)$
(iii)	$\ln(\cosh^{-1}x) + \sinh^{-1}y = C$	(iv)	$y = \ln(1-x^2) + 2x\tanh^{-1}x$
(v)	$y = \tanh^{-1}(\tan x^3)$	(vi)	$y = x\,\text{sech}^{-1}(\sqrt{x})$

3. Write MAPLE command **<u>diff</u>** to differentiate the following:

(i)	$f(x) = 2x^3 + 3x^2 + 6$	(ii)	$f(x) = \sin(2x+3)$
(iii)	$f(x) = (x+1)(x+2)$	(iv)	$f(x) = \frac{x^2-3x+2}{x^2-4}$

4. Write MAPLE command $> \frac{d}{dx}$ to differentiate the following functions:

(i)	$f(x) = x^3 + 5x^2 + 3x + 7$	(ii)	$f(x) = \sin x^2$
(iii)	$f(x) = \frac{\sqrt{x}+1}{x^2+1}$		

Review Exercise 3

1. Select the correct options:

(i) The derivative of $\frac{2}{x^3}$ is:

(a) $\frac{2}{3x^2}$ (b) $-\frac{2}{3x^2}$ (c) $\frac{6}{x^4}$ (d) $-\frac{6}{x^4}$

(ii) The derivative of $\sqrt{x} + x\sqrt{x}$ is:

(a) $\frac{1}{2\sqrt{x}} + \frac{3\sqrt{x}}{2}$ (b) $-\frac{1+x}{2\sqrt{x}}$ (c) $\frac{1}{\sqrt{x}} + 3\sqrt{x}$ (d) $\frac{1}{\sqrt{x}} + \frac{2\sqrt{x}}{3}$

(iii) If $y = (x+1)(x^2-2)$, then $\frac{dy}{dx}$ is:

(a) $x^3 + x^2 - 2x - 2$ (b) $3x^2 + 2x - 2$ (c) $3x^2 - 2x + 2$ (d) $3x^2 + 2x + 2$

(iv) If $ax^2 + by^2 = ab$, then $\frac{dy}{dx}$ is:

(a) $\frac{-2ax}{by}$ (b) $\frac{-bx}{ay}$ (c) $\frac{-ax}{by}$ (d) $\frac{-ax}{2by}$

(v) If $y = \sqrt{\tan x - y}$, then $\frac{dy}{dx} = ?$

(a) $\frac{\tan x}{2y+1}$ (b) $\frac{-\csc^2 x}{2y-1}$ (c) $\frac{\sec^2 x}{2y-1}$ (d) $\frac{\sec^2 x}{2y+1}$

(vi) If $y = \tan^{-1}\sqrt{x}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{x+\sqrt{x}}$ (c) $\frac{1}{2(x+x\sqrt{x})}$ (d) $\frac{1}{2(\sqrt{x}+x\sqrt{x})}$

(vii) The derivative of $\tan x$ w.r.t. $\cot x$ is:

(a) $\sec^2 x \csc^2 x$ (b) $-\tan^2 x$ (c) $\frac{\sec^2 x}{\csc^2 x}$ (d) $\tan^2 x$

(viii) The $f(x) = ax^2 - 3x - 5$ and $f'(2) = 9$, then a is equal to:

(a) -2 (b) 3 (c) 4 (d) 5

(ix) The derivative of x^2e^{2x} is:

(a) $x^2e^{2x} + 2x^2e^x$ (b) $2xe^{2x}$ (c) $2e^{2x}(x^2+x)$ (d) $2e^{2x}(x^2+1)$

(x) The derivative of a^x, if $a < 0$ is:

(a) $-a^x \ln a$ (b) $a^x . \ln a$ (c) $\frac{a^x}{\ln a}$ (d) Does not exist

(xi) If $y = \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$ then $\frac{dy}{dx} = ?$

(a) 1 (b) -1 (c) 2 (d) $\frac{1}{2}$

(xii) $\frac{d}{dx}(\sinh^{-1} x + \cosh^{-1} x)$ is:

(a) $\cosh^{-1} x - \sinh^{-1} x$ (b) $\frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1-x^2}}$

(c) $\frac{1}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2-1}}$ (d) $\frac{1}{\sqrt{x^2-1}} - \frac{1}{\sqrt{1+x^2}}$

(xiii) The derivative of $\tanh ax$ is:

(a) $\operatorname{sech}^2 ax$ (b) $a \operatorname{sech} ax$ (c) $a \operatorname{sech}^2 ax$ (d) $2a \operatorname{sech}^2 ax$

(xiv) The derivative of $\coth^{-1}(2x)$ is:

(a) $\frac{1}{1-4x^2}$ (b) $\frac{2}{1-4x^2}$ (c) $\frac{2x}{1-4x^2}$ (d) $\frac{2}{1-x^2}$

(xv) f is the function with rule $f(x) = \ln 2x$ $(x > 0)$, if g is the inverse of f, then $g'(x) =$

(a) $\frac{2}{x}$ (b) $\frac{1}{2x}$ (c) $\frac{2}{e^x}$ (d) $\frac{e^x}{2}$

(xvi) If $f(x) = a \cos 3x$ and $f'\left(\frac{\pi}{2}\right) = 6$, then $a =$

(a) -6 (b) -2 (c) 2 (d) 3

2. Find the derivative of $\sqrt{\cos x}$ and $\sec \sqrt{x}$ by first principle.

3. If $y = (\sin x)^{\ln x}$, find $\frac{dy}{dx}$.

4. Find $\frac{dy}{dx}$, if $ax^2 + 2hxy + by^2 = 0$.

5. Let $f(x) = \cot^{-1}\left(\frac{2x}{1-x^2}\right)$, find $f'(x)$ and $f'(-\sqrt{3})$.

6. $x = 4(t - \sin t)$ and $y = 4(1 + \cos t)$, find $\frac{dy}{dx}$.

7. Differentiate w.r.t. x:

(i) $\frac{x^2+x^{-1}}{x^2-x^{-1}}$ (ii) $\frac{3x-2}{\sqrt{x^2+1}}$

8. If $y = x^4 + 2x^2$, show that $\frac{dy}{dx} = 4x\sqrt{y+1}$.

9. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots}}}$, show that $(2y-1)y' = \cos x$.

10. Differentiate w.r.t. x:

(i) $\cosh(\cos^{-1}\sqrt{x})$ (ii) $\tanh^{-1}(\cos 2x)$.