Unit 4 Higher Order Derivatives and Applications

4.1 Higher Order Derivatives

The derivative of a function $y = f(x)$ is $\dfrac{dy}{dx} = f'(x)$, which is itself a function. Now the derivative of $\dfrac{dy}{dx} = f'(x)$, written as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y'' = f''(x),$$

Generally it is referred as the second order derivative of $f(x)$ and this differentiation process can be continued to find the third, fourth,…., nth order derivative as under, and are called **higher order derivatives** of $f(x)$.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y''' = f'''(x)$$

$$\frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d^4y}{dx^4} = y^4 = f^4(x)$$

$$\ldots \quad \ldots. \quad \ldots \quad \ldots. \quad \ldots \quad \ldots. \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x)$$

4.1.1 Find higher order derivatives of algebraic, trigonometric, exponential and logarithmic functions.

(i) Higher order derivatives of algebraic functions:

Example 1. Find the first, second, and third order derivatives of

$$y = 5x^4 - 3x^3 + 7x^2 - 9x + 2$$

Solution: We have $\quad y = 5x^4 - 3x^3 + 7x^2 - 9x + 2$

Differentiating w.r.t x, we get

$$\Rightarrow \quad \frac{dy}{dx} = 20x^3 - 9x^2 + 14x - 9$$

Again, differentiating w.r.t x, we get

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 60x^2 - 18x + 14$$

Differentiating 3rd time w.r.t x, we get

$$\Rightarrow \quad \frac{d^3y}{dx^3} = 120x - 18$$

Example 2. Find $f'''(4)$ if $y = f(x) = \sqrt{x}$

Solution: As $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

Differentiating successively three times, we have

$$\Rightarrow \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\Rightarrow \quad f''(x) = \frac{-1}{4}x^{-\frac{3}{2}}$$

$$\Rightarrow \quad f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$$

Replacing x by 4, we get

Hence, $$f'''(4) = \frac{3}{8}(4)^{-\frac{5}{2}}$$

$$= \frac{3}{8}\left(\frac{1}{32}\right) = \frac{3}{256}$$

Example 3. If $f(x) = \frac{2}{1-x}$ then find $f^{(n)}(x)$.

Solution:

$$f(x) = \frac{2}{1-x} = 2(1-x)^{-1}$$

Differentiating successively w.r.t x and patterning for nth derivative, we have

$$f'(x) = 2(-1)(1-x)^{-2}(-1) = 2(1!)(1-x)^{-2}$$

$$f''(x) = 2(1!)(-2)(1-x)^{-3}(-1) = 2(2!)(1-x)^{-3}$$

$$f'''(x) = 2(2!)(-3)(1-x)^{-4}(-1) = 2(3!)(1-x)^{-4}$$

$$f^{(4)}(x) = 2(3!)(-4)(1-x)^{-5}(-1) = 2(4!)(1-x)^{-5}$$

Therefore,

$$f^{(n)}(x) = 2(n!)(1-x)^{-(n+1)}$$

(ii) Higher order derivatives of trigonometric function:

The higher order derivatives of trigonometric functions are explained in the following examples.

Example 1. Find the third order derivative of $y = sin^2x$.

Solution: As $y = sin^2x$

Differentiating successively thrice times w.r.t x, we have

$$\Rightarrow \quad \frac{dy}{dx} = 2\sin x\cos x = \sin 2x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\cos 2x$$

$$\Rightarrow \quad \frac{d^3y}{dx^3} = 2(-\sin 2x)(2)$$

$$\Rightarrow \quad \frac{d^3y}{dx^3} = -4\sin 2x$$

Example 2. Find 2nd order derivative of $f(x) = \frac{\cos x}{1+\sin x}$

Solution: $f(x) = \frac{\cos x}{1+\sin x}$

Differentiating successively two times w.r.t x, we have

$$\Rightarrow \quad f'(x) = \frac{(1+\sin x)(-\sin x) - \cos x\ \cos x}{(1+\sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{-1}{1+\sin x}$$

$$\Rightarrow \quad f''(x) = -\frac{d}{dx}(1+\sin x)^{-1} = (1+\sin x)^{-2}\cos x$$

$$\Rightarrow \quad f''(x) = \frac{\cos x}{(1+\sin x)^2}$$

(iii) Higher order derivatives of exponential function:

Example 1. Find the 3rd order derivative of $y = a^x$

Solution: As $y = a^x$

$y = e^{xlna}$ $\qquad [\because a^x = e^{xlna}]$

Differentiating successively three times w.r.t x, we have

$$\Rightarrow \quad y' = e^{xlna}.(lna) = lna\, e^{xlna}$$

$$\Rightarrow \quad y'' = (lna)\, e^{xlna}.(lna) = (lna)^2\, e^{xlna}$$

$$\Rightarrow \quad y''' = (lna)^2\, e^{xlna}.(lna) = (lna)^3\, e^{xlna}$$

or $\quad y''' = a^x(\ln a)^3$

Example 2. Find the 2nd order derivative of $f(x) = e^{1+x^2}$

Solution: $f(x) = e^{(1+x^2)}$

Differentiating successively three times w.r.t x, we have

$$\Rightarrow \quad f'(x) = e^{(1+x^2)} \cdot 2x$$

$$\Rightarrow \quad f''(x) = 2\left[xe^{(1+x^2)} \cdot 2x + e^{(1+x^2)} \cdot 1\right]$$

$$\Rightarrow \qquad = 2\left[2x^2e^{(1+x^2)} + e^{(1+x^2)}\right]$$

$$\Rightarrow \qquad f''(x) = 2e^{(1+x^2)}(2x^2+1)$$

(iv) Higher order derivatives of logarithmic function:

The higher order derivatives of logarithmic functions are explained in the following examples.

Example 1. Find 3rd order derivative of $f(x) = \log_b x^2$

As $f(x) = \log_b x^2$

Solution: $y = \log_b x^2$

or $\quad = 2\log_b x$

Differentiating successively three times w.r.t x, we have

$$\Rightarrow \qquad y' = \frac{2}{x}\cdot\frac{1}{\ln b} \qquad\qquad \left[\because \frac{d}{dx}[log_b x] = \frac{1}{x}\cdot\frac{1}{\ln b}\right]$$

$$\Rightarrow \qquad y'' = \frac{2}{\ln b}\cdot\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \qquad y''' = \frac{2}{\ln b}\cdot\left(\frac{2}{x^3}\right) = \frac{4}{x^3\ln b}$$

Example 2. Find the 2nd order derivative of $f(x) = \ln(1+x^2)$

Solution: $\qquad f(x) = \ln(1+x^2)$

Differentiating successively two times w.r.t x, we have

$$\Rightarrow \qquad f'(x) = \frac{1}{(1+x^2)}\frac{d}{dx}(1+x^2) = \frac{1}{(1+x^2)}.2x$$

$$\Rightarrow \qquad f''(x) = 2\frac{d}{dx}\left(\frac{x}{1+x^2}\right)$$

$$= 2\left[\frac{(1+x^2)(1) - x\,(2x)}{(1+x^2)^2}\right] = 2\left[\frac{1+x^2-2x^2}{(1+x^2)^2}\right]$$

$$\Rightarrow \qquad f''(x) = 2\left[\frac{1-x^2}{(1+x^2)^2}\right]$$

4.1.2 Find the second derivative of implicit, inverse trigonometric and parametric functions

(i) 2nd order derivatives of implicit function:

The method of finding the second order derivatives of implicit functions is explained in the following examples.

Example: Find $\frac{d^2y}{dx^2}$ if $xy + x - 2y - 1 = 0$

Solution: As $xy + x - 2y - 1 = 0$

Differentiating w.r.t x, we have

$$\frac{d}{dx}(xy)+\frac{d}{dx}(x)-2\frac{d}{dx}(y)-\frac{d}{dx}(1)=\frac{d}{dx}(0)$$

$$\Rightarrow \quad xy'+y\cdot 1+1-2y'-0=0$$

$$\Rightarrow \quad y'(x-2)=-(y+1)$$

$$\Rightarrow \quad y'=\frac{-(y+1)}{(x-2)}$$

Differentiating y', to get 2nd order derivative, we have

$$\Rightarrow \quad y''=-\frac{(x-2)\frac{d}{dx}(y+1)-(y+1)\frac{d}{dx}(x-2)}{(x-2)^2}$$

$$\Rightarrow \quad y''=-\frac{(x-2)y'-(y+1)(1)}{(x-2)^2}$$

$$\Rightarrow \quad y''=-\frac{(x-2)\left[-\frac{(y+1)}{(x-2)}\right]-(y+1)}{(x-2)^2}$$

$$\Rightarrow \quad y''=\frac{2(y+1)}{(x-2)^2}$$

(ii) 2nd order derivatives of inverse trigonometric function:

The method of finding second order derivatives of inverse trigonometric functions is explained in the following examples.

Example: Find the second order derivative of $\tan^{-1}x$

Solution:

Let $y=\tan^{-1}x$

Differentiating y w.r.t x, we get

$$\frac{d}{dx}(y)=\frac{d}{dx}(\tan^{-1}x)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{1+x^2}=(1+x^2)^{-1}$$

Again, differentiating w.r.t x, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(1+x^2)^{-1}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-1(1+x^2)^{-2}(2x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=\frac{-2x}{(1+x^2)^2}$$

(iii) 2nd order derivatives of parametric function:

Let $y = f(x)$ is a function, $x = f(t)$ and $y = g(t)$ are the parametric equations of $y = f(x)$. Then, by using chain rule.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \ldots\text{(i)}$$

To find $\frac{d^2y}{dx^2}$, let $z = \frac{dy}{dx} = h(t)$.

Now, using chain rule

$$\frac{dz}{dx} = \frac{\frac{dz}{dt}}{\frac{dx}{dt}}$$

From equation (i), we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)} \qquad \because \ z = \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$$

Example: Find $\frac{d^2y}{dx^2}$ where $y = 1 + 5t^2$; $x = 5\text{t} + 3t^2$ are parametric equation of $y = f(x)$.

Solution:

By using formula $\quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

Here $\quad \frac{dy}{dt} = \frac{d}{dt}\left(1 + 5t^2\right) = 10t$

$$\frac{dx}{dt} = \frac{d}{dt}(5\text{t} + 3t^2) = 5 + 6t$$

Now $\quad y' = \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{10t}{5+6t}$

For second order derivative we use the following formula

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{10t}{5+6t}\right)$$
$$=\frac{(5+6t)(10)-(10t)(6)}{(5+6t)^2}=\frac{(50+60t-60t)}{(5+6t)^2}=\frac{50}{(5+6t)^2}$$

By using formula $\quad y''=\dfrac{d^2y}{dx^2}=\dfrac{\frac{50}{(5+6t)^2}}{5+6t}$

$$y''=\frac{50}{(5+6t)^3}$$

4.1.3 Use MAPLE command diff repeatedly to find higher order derivative of a function

The format of diff command to differentiate a function repeatedly in MAPLE is as under:

>diff (f ^{n}, [x]) is equivalent to the command $\dfrac{d^n}{dx^n} f$ in Maple version 2022.

Where,

f^n stands for function whose nth order derivative is to be evaluated

x stands for the variable x, for the required derivative with respect x.

$\dfrac{d^n}{dx^n}$ means nth order derivative with respect to variable x.

Note: All above operators should be taken from the Maple calculus pallet

Use MAPLE command **diff** repeatedly or $\dfrac{d^n}{dx^n} f$ to differentiate a function repeatedly:

nth order Derivative of functions:

$> \frac{dy}{dx}(x^3+2x^2+5x+7)$ $3x^2+4x+5$ $> \frac{d^2}{dx^2}(x^3+2x^2+5x+7)$ $6x+4$ $> \frac{d^3}{dx^3}(x^3+2x^2+5x+7)$ 6	$> \frac{d}{dx}\sqrt{x^2+4x+3}$ $\frac{1}{2}\frac{2x+4}{\sqrt{x^2+4x+3}}$ $> \frac{d^2}{dx^2}\left(\sqrt{x^2+4x+3}\right)$ $-\frac{1}{4}\frac{(2x+4)^2}{(x^2+4x+3)^{\frac{3}{2}}}+\frac{1}{\sqrt{x^2+4x+3}}$ $> \frac{d^3}{dx^3}\left(\sqrt{x^2+4x+3}\right)$ $\frac{3}{8}\frac{(2x+4)^3}{(x^2+4x+3)^{\frac{5}{2}}}-\frac{3}{2}\frac{2x+4}{(x^2+4x+3)^{\frac{3}{2}}}$

> $\frac{d}{dx}\sin(x)$ $\cos(x)$ > $\frac{d^2}{dx^2}\sin(x)$ $-\sin(x)$ > $\frac{d^3}{dx^3}\sin(x)$ $-\cos(x)$	> $\frac{d^3}{dx^3}(e^{2x+5})$ $8e^{2x+5}$ > $\frac{d^3}{dx^3}(\ln x^2)^3$ $120\ln^3 x^3$
Derivative on Product form:	**Derivative on Quotient form:**
> $\frac{d^2}{dx^2}(e^{2x+1})(\sqrt{x})$ $\frac{1}{4}\frac{D^{(2)}(e^{2x+1})(\sqrt{x})}{x} - \frac{1}{4}\frac{D(e^{2x+1})(\sqrt{x})}{x^{\frac{3}{2}}}$ > $\frac{d}{dx}\left(e^x.(x^2+1)\right)$ $e^x.(x^2+1)+2(e^x.x)$ > $\frac{d^3}{dx^3}(e^x\sin(x))$ $2e^x\cos(x)-2e^x\sin(x)$	> $\frac{d^3}{dx^3}\left(\frac{e^x}{x+3}\right)$ $\frac{e^x}{x+3}-\frac{3e^x}{(x+3)^2}+\frac{6e^x}{(x+3)^3}-\frac{6e^x}{(x+3)^4}$ > $\frac{d^3}{dx^3}\left(\frac{\ln(x+1)}{\sin x}\right)$ $\frac{2}{(x+1)^3\sin x}$ > $\frac{d^3}{dx^3}\left(\frac{1+\sin x}{\cos x}\right)$ $\frac{6\sin}{\cos x^3}-\frac{6(\sin x+1)}{\cos x^4}$

Exercise 4.1

1. Calculate the first, second and third order derivatives of $y=\cos^2 x$.
2. Find the 2nd order derivative of $f(x)=\frac{\cos x}{1+\sin x}$.
3. Find the fourth order derivatives of the given functions.
   (i) $h(t)=3t^7-6t^4+8t^3-12t+18$
   (ii) $f(x)=\sqrt[3]{x}-\frac{1}{8x^2}-\sqrt{x}$
4. Determine the fourth order derivative in each of the following function.
   (i) $r(t)=3t^2+8\sqrt{t}$ (ii) $y=\cos x$ (iii) $f(y)=\sin 3y+e^{-2y}+\ln(7y)$
5. If $x^2+y^2=10$, find y''.
6. Find $\frac{d^2y}{dx^2}$ if
   (i) $2y^2+6x^2=76$ (ii) $x^3+y^3=1$

7. Find $\frac{d^2y}{dx^2}$ if
   (i) $x = -5t^3 - 7$ and $y = 3t^2 + 16$
   (ii) $x = \cos\theta$ and $y = \sin\theta$
8. The derivative of function $r(t)$ is given by
   $r'(t) = 6t + 4t^{-1/2} + e^t$; find $r''(t), r'''(t)$ and $r^{(4)}(t)$
9. If $x^2 + y^2 = 25$ then find $\frac{d^2y}{dx^2}$ at point (4,3).
12. Write MAPLE Command to find higher order derivatives of the following functions:
   (i) $f(x) = x^3 + 3x^2 + 6x + 8$; (third order derivative)
   (ii) $f(x) = \cos\sqrt{2x+3}$; (second order derivative)
   (iii) $f(x) = e^{(x^2+5x+3)}$; (third order derivative)
   (iv) $f(x) = \ln\sqrt{3x+2}$; (second order derivative)
   (v) $f(x) = e^{\sin x}$; (third order derivative)

4.2 Maclaurin’s and Taylor's Expansions

4.2.1 State Maclaurin’s and Taylor's theorems (without remainder terms). Use these theorems to expand $\sin x$, $\cos x$, $\tan x$, a^x, e^x, $log_a(1+x)$ and $\ln(1+x)$

Maclaurin’s Theorem:

If $f(x)$ is nth order differentiable function at $x = 0$ then it can be expanded as the infinite sum of the terms of the polynomial centered at $x = 0$ that is

i.e., $$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

Taylor’s Theorem:

If $f(x)$ is nth order differentiable function at $x = a$ then it can be expanded as the infinite sum of the terms of the polynomial centered at $x = a$

i.e., $$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots$$

Example: Find the Maclaurin’s series of $\sin x$, $\cos x$, $\tan x$, a^x, e^x, $log_a(1+x)$ and $ln(1+x)$.

Solution:

(i) $\boldsymbol{f(x) = \sin x}$

The Maclaurin’s series is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

$f(x) = \sin x$ at $\Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = 1$

By putting the values of $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0), \dots,$ in Maclaurin's series we get,

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) - \cdots$$

or $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

In summation form,

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

This is the required Maclaurin's series of the function $f(x) = \sin x$.

(ii) $\boldsymbol{f(x) = \tan x}$

$f(x) = \tan x = y$ at $\Rightarrow f(0) = 0$

$f'(x) = y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2 \Rightarrow f'(0) = 1$

$f''(x) = y'' = 2yy' \Rightarrow f''(0) = 0$

$f'''(x) = y''' = 2[yy'' + y'^2] \Rightarrow f'''(0) = 2$

$f^{(4)}(x) = y^{\text{iv}} = 2[yy''' + y''y'] + 2.2y'y'' \Rightarrow f^{(4)}(0) = 0$

$= 2[yy''' + y''.y' + 2y'y'']$

$= 2[yy''' + 3y'y'']$

$f^{(5)}(x) = y^5 = 2\left[yy^{(4)} + y'''y'\right] + 6[y'y''' + y''y'']$

$= 2yy^{(4)} + 2y'''y' + 6y'y''' + 6y''y''$

$= 2yy^{(4)} + 8y'y''' + 6y''y''$

$= 2yy^{(4)} + 8y'(2) + 6y''y''$

$= 2yy^{(4)} + 16y' + 6y''y'' \Rightarrow f^5(0) = 16$

By putting the values of $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0)$ and $f^{(5)}(0)$ in Maclaurin's series, we get,

$$\tan x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(16) + \cdots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

This is the required Maclaurin's series of the function $f(x) = \tan x$

(iii) $\boldsymbol{f(x) = a^x}$

$$f(x) = a^x \qquad \text{at } x = 0 \quad \Rightarrow f(0) = 1$$
$$f'(x) = a^x \ln a \qquad \Rightarrow f'(0) = \ln a$$
$$f''(x) = \ln a\,(a^x \ln a) = (\ln a)^2 a^x \qquad \Rightarrow f''(0) = (\ln a)^2$$
$$f'''(x) = (\ln a)^2 (a^x \ln a) = (\ln a)^3 a^x \qquad \Rightarrow f'''(0) = (\ln a)^3$$
$$f^{(4)}(x) = (\ln a)^3 (a^x \ln a) = (\ln a)^4 a^x \qquad \Rightarrow f^{(4)}(0) = (\ln a)^4$$
$$f^{(5)}(x) = (\ln a)^4 (a^x \ln a) = (\ln a)^5 a^x \qquad \Rightarrow f^{(5)}(0) = (\ln a)^5$$

By putting the values of $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0)$ and $f^{(5)}(0)$ in Maclaurin's series, we get

$$a^x = 1 + x(\ln a) + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \frac{x^4}{4!}(\ln a)^4 + \frac{x^5}{5!}(\ln a)^5 + \cdots$$

$$\Rightarrow \quad a^x = 1 + (x \ln a) + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \frac{(x\ln a)^4}{4!} + \frac{(x\ln a)^5}{5!} + \cdots$$

In summation form,

$$a^x = \sum_{n=0}^{\infty} \frac{(x\ln a)^n}{n!}$$

This is the required Maclaurin's series of the function $f(x) = a^x$

(iv) $\boldsymbol{f(x) = \log_a(1+x)}$**, where** $\boldsymbol{a > 0}$ **and** $\boldsymbol{a \neq 1}$

$$f(x) = \log_a(1+x) \qquad \Rightarrow \qquad f(0) = 0$$
$$f'(x) = \frac{1}{(1+x)\ln a} \qquad \Rightarrow \qquad f'(0) = \frac{1}{\ln a}$$

or $$f'(x) = \frac{(1+x)^{-1}}{\ln a}$$

$$f''(x) = -\frac{(1+x)^{-2}}{\ln a} \qquad \Rightarrow \qquad f''(0) = -\frac{1}{\ln a}$$
$$f'''(x) = \frac{2(1+x)^{-3}}{\ln a} \qquad \Rightarrow \qquad f'''(0) = \frac{2}{\ln a}$$
$$f^{(4)}(x) = \frac{-6(1+x)^{-4}}{\ln a} \qquad \Rightarrow \qquad f^{(4)}(0) = \frac{-6}{\ln a}$$
$$f^{(5)}(x) = \frac{24(1+x)^{-5}}{\ln a} \qquad \Rightarrow \qquad f^{(5)}(0) = \frac{24}{\ln a}$$

Putting values of $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0), f^{(5)}(0)$ in Maclaurin's series, we get

$$\log_a(1+x) = 0 + x\left(\frac{1}{\ln a}\right) + \frac{x^2}{2!}\left(\frac{-1}{\ln a}\right) + \frac{x^3}{3!}\left(\frac{2}{\ln a}\right) + \frac{x^4}{4!}\left(\frac{-6}{\ln a}\right) + \frac{x^5}{5!}\left(\frac{24}{\ln a}\right) + \cdots$$

$$\Rightarrow \qquad \log_a(1+x) = \frac{1}{\ln a}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right]$$

In summation form,

$$\log_a(1+x) = \frac{1}{\ln a}\sum_{n=0}^{\infty}\frac{(-1)^n\, x^{n+1}}{n+1}$$

Which is the required Maclaurin's series.

Note: Maclurin's series of $\cos x$, e^x and $\ln(1+x)$ are left as an exercise for readers.

Find the Taylor's series of the expansion of sin$x$, cos x, tan x, a^x, e^x, $log_a(1+x)$ and $ln(1+x)$ at particular point a

Solution: (i) $f(x) = \sin x$ **at** $a = \frac{\pi}{6}$

The Taylor's series is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

At $a = \frac{\pi}{6}$; $f(x) = f\left(\frac{\pi}{6}\right) + \left(x - \frac{\pi}{6}\right)f'\left(\frac{\pi}{6}\right) + \frac{\left(x-\frac{\pi}{6}\right)^2}{2!}f''\left(\frac{\pi}{6}\right) + \frac{\left(x-\frac{\pi}{6}\right)^3}{3!}f'''\left(\frac{\pi}{6}\right) + \cdots$

$$f(x) = \sin x \quad \Rightarrow \quad f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2} \qquad \ldots(\text{i})$$

$$f'(x) = \cos x \quad \Rightarrow \quad f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$f''(x) = -\sin x \quad \Rightarrow \quad f''\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$$

$$f'''(x) = -\cos x \quad \Rightarrow \quad f'''\left(\frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

Putting these above values in equation (i), we get

$$f(x) = \frac{1}{2} + \left(x - \frac{\pi}{6}\right)\left(\frac{\sqrt{3}}{2}\right) + \frac{\left(x-\frac{\pi}{6}\right)^2}{2!}\left(-\frac{1}{2}\right) + \frac{\left(x-\frac{\pi}{6}\right)^3}{3!}\left(-\frac{\sqrt{3}}{2}\right) + \cdots$$

$$\Rightarrow \sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\frac{\left(x-\frac{\pi}{6}\right)^2}{2!} - \frac{\sqrt{3}}{2}\frac{\left(x-\frac{\pi}{6}\right)^3}{3!} + \cdots$$

Which is the required Taylor series of $\sin x$ at the point $\frac{\pi}{6}$.

(ii) $f(x) = \cos x$ **at** $a = \frac{\pi}{4}$

The Taylor's series at $a = \frac{\pi}{4}$ is given by:

$$f(x)=f\left(\frac{\pi}{4}\right)+\left(x-\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right)+\cdots \qquad \ldots(\text{i})$$

$$f(x)=\cos x \quad\Rightarrow\quad f\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$f'(x)=-\sin x \quad\Rightarrow\quad f'\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

$$f''(x)=-\cos x \quad\Rightarrow\quad f''\left(\frac{\pi}{4}\right)=-\cos\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

$$f'''(x)=\sin x \quad\Rightarrow\quad f'''\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

By putting values in equation (i), we get

$$f(x)=\frac{1}{\sqrt{2}}+\left(x-\frac{\pi}{4}\right)\left(-\frac{1}{\sqrt{2}}\right)+\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right)+\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \cos x=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right)+\frac{1}{\sqrt{2}}\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}-\frac{1}{\sqrt{2}}\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}+\cdots$$

Which is the required Taylor series of $\cos x$ at the point $\frac{\pi}{4}$.

(iii) $\boldsymbol{f(x)=\tan x}$ **at** $\boldsymbol{a=\frac{\pi}{4}}$

$f(x)=\tan x=y$ at $x=\frac{\pi}{4}\Rightarrow f\left(\frac{\pi}{4}\right)=1$

$f'(x)=y'=\sec^2 x=1+\tan^2 x=1+y^2$ $\Rightarrow f'\left(\frac{\pi}{4}\right)=2$

$f''(x)=y''=2yy'$ $\Rightarrow f''\left(\frac{\pi}{4}\right)=4$

$f'''(x)=y'''=2[yy''+y'^2]$ $\Rightarrow f'''\left(\frac{\pi}{4}\right)=16$

$$\begin{aligned}f^4(x)=y^4 &= 2[yy'''+y''y'+2y'y'']\\ &=2[yy'''+y''.y'+2y'y'']\\ &=2[yy'''+3y'y''] \qquad \Rightarrow f^4\left(\frac{\pi}{4}\right)=80\end{aligned}$$

By putting the values of $f(0), f'(0), f''(0), f'''(0), f^4(0), \ldots,$ in Taylor series we get,

$$f(x)=1+\frac{2}{1!}\left(x-\frac{\pi}{4}\right)+\frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2+\frac{16}{3!}\left(x-\frac{\pi}{4}\right)^3+\frac{80}{4!}\left(x-\frac{\pi}{4}\right)^4+\cdots$$

$$\tan x=1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\frac{\pi}{4}\right)^3+\frac{10}{3}\left(x-\frac{\pi}{4}\right)^4+\cdots$$

This is the required Taylor series of the function $f(x)=\tan x$ at $a=\frac{\pi}{4}$

(iv) $\boldsymbol{f(x) = a^x}$ **at** $\boldsymbol{b = 2}$

$f(x) = a^x$ at $x = 2 \Rightarrow f(2) = a^2$

$f'(x) = a^x \ln a$ $\Rightarrow f'(2) = a^2 \ln a$

$f''(x) = \ln a\,(a^x \ln a) = (\ln a)^2 a^x$ $\Rightarrow f''(2) = a^2 (\ln a)^2$

$f'''(x) = (\ln a)^2 (a^x \ln a) = (\ln a)^3 a^x$ $\Rightarrow f'''(2) = a^2 (\ln a)^3$

The Taylor's series of the function at point b is given by

$$f(x) = f(b) + (x-b)f'(b) + \frac{(x-b)^2}{2!}f''(b) + \frac{(x-b)^3}{3!}f'''(b) + \cdots$$

At $b = 2$, $f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots$

Putting values of $f(2), f'(2), f''(2), f'''(2), \ldots$ we get

$$f(x) = a^2 + (x-2)(a^2 \ln a) + \frac{(x-2)^2}{2!}(a^2(\ln a)^2) + \frac{(x-2)^3}{3!}(a^2(\ln a)^3) + \cdots$$

$$\Longrightarrow \quad a^x = a^2\left[1 + (x-2)\ln a + \frac{(x-2)^2}{2!}(\ln a)^2 + \frac{(x-2)^3}{3!}(\ln a)^3 + \cdots\right]$$

Which is the required Taylor's series of a^x at the point 2.

(v) $\boldsymbol{f(x) = e^x}$ **at** $\boldsymbol{a = 1}$

$f(x) = e^x$ at $x = 1 \Rightarrow f(1) = e$

$f'(x) = e^x . \ln e = e^x$ $\Rightarrow f'(1) = e$

$f''(x) = e^x$ $\Rightarrow f''(1) = e$

$f'''(x) = e^x$ $\Rightarrow f'''(1) = e$

By putting the values of $f(0), f'(0), f''(0), f'''(0), \ldots,$ in Taylor series we get,

$$\Longrightarrow \quad e^x = e + (x-1)(e) + \frac{(x-1)^2}{2!}(e) + \frac{(x-1)^3}{3!}(e) + \cdots$$

$$\Longrightarrow \quad e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right]$$

This is the required Taylor series of the function $f(x) = e^x$ at $a = 1$

(vi) $\boldsymbol{f(x) = \log_a(1+x)}$ **at** $\boldsymbol{b = 1}$

Solution: By Taylor's series we have,

$$f(x) = f(b) + (x-b)f'(b) + \frac{(x-b)^2}{2!}f''(b) + \frac{(x-b)^3}{3!}f'''(b) + \cdots$$

At $b = 1$, we have

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \cdots \qquad \ldots\text{(i)}$$

$\therefore$ $f(x) = \log_a(1+x)$ $\Rightarrow$ at $\Rightarrow f(1) = \log_a 2$

$$f'(x) = \frac{1}{(1+x)\,ln\,a} \qquad \Rightarrow \qquad f'(1) = \frac{1}{2\,ln\,a}$$

or $$f'(x) = \frac{(1+x)^{-1}}{ln\,a}$$

$$f''(x) = \frac{-(1+x)^{-2}}{ln\,a} \qquad \Rightarrow \qquad f''(1) = \frac{-1}{4\,ln\,a}$$

$$f'''(x) = \frac{2(1+x)^{-3}}{ln\,a} \qquad \Rightarrow \qquad f'''(1) = \frac{1}{4\,ln\,a}$$

Putting values in equation (i), we get

$$f(x) = \log_a 2 + (x-1) \times \frac{1}{2\,ln\,a} + \frac{(x-1)^2}{2!} \times \frac{(-1)}{4\,ln\,a} + \frac{(x-1)^3}{3!} \times \frac{1}{4\,ln\,a} + \cdots$$

$$\Rightarrow f(x) = \log_a 2 + \frac{(x-1)}{2\,ln\,a} - \frac{(x-1)^2}{4.2!\,(\ln a)} + \frac{(x-1)^3}{4.3!\,(\ln a)} + \cdots$$

Which is the required Taylor's series of the function

$$f(x) = \log_a(1+x) \quad \text{at} \quad b = 1$$

(vii) $\boldsymbol{f(x) = ln(1+x)}$ **at** $\boldsymbol{b = 2}$

$f(x) = ln(1+x)$ at $\Rightarrow f(2) = ln\,3$

$f'(x) = \frac{1}{(1+x)} = (1+x)^{-1}$ $\Rightarrow f'(2) = \frac{1}{3}$

$f''(x) = -(1+x)^{-2}$ $\Rightarrow f''(2) = -\frac{1}{9}$

$f'''(x) = 2(1+x)^{-3}$ $\Rightarrow f'''(2) = \frac{2}{27}$

$f^{iv}(x) = -6(1+x)^{-4}$ $\Rightarrow f^{iv}(2) = -\frac{2}{27}$

By putting the values of $f(0), f'(0), f''(0), f'''(0), f^{iv}(0), \ldots,$ in Taylor series we get,

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots$$

$$\ln(1+x) = ln\,3 + (x-2)\left(\frac{1}{3}\right) + \frac{(x-2)^2}{2!}\left(-\frac{1}{9}\right) + \frac{(x-2)^3}{3!}\left(\frac{2}{27}\right) + \frac{(x-2)^4}{4!}\left(-\frac{2}{27}\right) + \cdots$$

$$\ln(1+x) = ln\,3 + \frac{1}{3}(x-2) - \frac{1}{18}(x-2)^2 + \frac{1}{81}(x-2)^3 - \frac{1}{324}(x-2)^4 + \cdots$$

This is the required Taylor series of the function $f(x) = ln(1+x)$ at point 2.

4.2.2 Use MAPLE command Taylor to find Taylor's expansion for a given function

The format of Taylor's expansion command in MAPLE is as under:

> taylor (f(x), x = a,n)

where,

f(x) is the function whose Taylor's expansion is required x=a is about the point x=a, series is expanded

n is the number of terms series expanded.

In order to compute the Taylor series expansion following examples are given:

$> taylor\left(\frac{1}{\sqrt{1+x}}, x=0,5\right)$

$$1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\frac{35}{128}x^4+0(x^5)$$

$> taylor\left(e^x, x=0,5\right)$

$$1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+0(x^5)$$

$> taylor\left(e^x, x=2,5\right)$

$$e^2+e^2(x-2)+\frac{1}{2}e^2(x-2)^2+\frac{1}{6}e^2(x-2)^3+\frac{1}{24}e^2(x-2)^4+0((x-2)^5)$$

$> taylor\left(\frac{1}{\sqrt{1+x}}, x=0,6\right)$

$$1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\frac{35}{128}x^4-\frac{35}{128}x^5+0(x^6)$$

$> taylor\ (\sin(x), x=0,10)$

$$x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\frac{1}{362880}x^9+0(x^{11})$$

$> taylor(\ln(1+x), x=1,4)$

$$\ln(2)+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{24}(x-1)^3+0((x-1)^4)$$

$> taylor\ (\cos(x), x=0,10)$

$$1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\frac{1}{40320}x^8+0(x^{10})$$

$> taylor(ln(1+x), x=2,4)$

$$\ln(3)+\frac{1}{3}(x-2)-\frac{1}{18}(x-2)^2+\frac{1}{81}(x-2)^3+0((x-2)^4)$$

Exercise 4.2

1. Obtain the first three terms of the Maclaurin's series for

   (i) $\cos x$ (ii) e^x (iii) $\ln(1+x)$ (iv) $\sin^2 x$

   (v) $e^{\sin x}$ (vi) xe^{-x} (vii) $\frac{1}{1+x}$

2. Find the first four terms of the Taylor's series for the following functions

   (i) $\ln x$ centered at $a=1$ (ii) $\frac{1}{x}$ centered at $a=1$

   (iii) $sin\ x$ centered at $a=\frac{\pi}{4}$ (iv) $\cos x$ centered at $a=\frac{\pi}{2}$

3. Does Maclaurin's series of the functions $f(x)=\frac{1}{x}, g(x)=\operatorname{cosec} x$ and $h(x)=\sqrt{x}$ exist? If not why? Give appropriate justification.

4. Write MAPLE Command to find Taylor's Expression of the following functions:

(i) $f(x) = e^x$ at $x = 1$ upto 10 terms.

(ii) $f(x) = \sin x$ at $x = \pi$ upto 10 terms.

(iii) $f(x) = \cos x$ at $x = \pi$ upto 10 terms.

(iv) $f(x) = \ln(1 + x)$ at $x = 0$ upto 10 terms

(v) $f(x) = \frac{1}{x}$ at $x = 1$ upto 5 terms.

(vi) $f(x) = \frac{1}{x}$ at $x = 2$ upto 5 terms

4.3 Application of Derivatives

Derivatives have various important applications in Mathematics such as to find the Rate of Change of a Quantity, to find the Approximation Value, to find the equation of Tangent and Normal to a Curve, angle between two curves and to find the Minimum and Maximum Values of algebraic expressions. Derivatives are vastly used in the fields of science, engineering, physics, etc.

4.3.1 Give geometrical interpretation of derivative.

Let $P(x, y)$ be any point on the curve $y = f(x)$.

Referring to Figure 4.1, we have,

$y = f(x) = \overline{MP}$

$\delta x = \overline{MN} = \overline{PK}$

$y + \delta y = f(x + \delta x) = \overline{NQ}$

$\therefore \delta y = f(x + \delta x) - f(x) = \overline{KQ}$ ...(i)

For average rate of change, we divide both sides of equation (i), by δx,

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} = \frac{\overline{KQ}}{\overline{PK}} = \tan \emptyset$$

= gradient (slope) of secant PQ

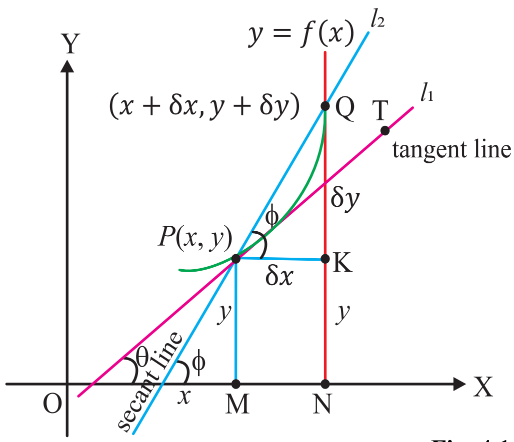

Fig. 4.1

Now, as δx approaches zero, the point Q will approach P along the curve, then secant will eventually becomes tangent

$$\text{So} \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

= gradient (slope) of the tangent PT = tan θ

where θ is the angle between the tangent at P and positive direction of x-axis.

Note: For derivative, $Q \longrightarrow P \Rightarrow \tan \emptyset \longrightarrow \tan \theta$.

i.e., $f'(x) = \frac{dy}{dx} = \tan\theta$, which is the slope of the tangent to the curve $y = f(x)$ at point.

We conclude that slope of the tangent to the curve is the derivative of the function of the curve at the point of tangency.

4.3.2 Find the equation of tangent and normal to the curve at a given point

Let $f(x, y) = 0$ is the equation of the curve. Then, to find the equation of the tangent at any given point (a, b) is found using following steps.

(i) Find slope of tangent at (a, b) i.e., $m = \left(\frac{dy}{dx}\right)_{(a,b)}$

(ii) By using point slope form, equation of tangent is $y - a = m(x - b)$.

Example 1. Find the equation of tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1, 1)

Solution: Given curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$

differentiate with respect to x regarding y as a function of x.

$$\therefore \quad \left(\frac{2}{3}x^{-\frac{1}{3}}\right) + \left(\frac{2}{3}y^{-\frac{1}{3}}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(x^{-\frac{1}{3}}\right) + \left(y^{-\frac{1}{3}}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Hence, the slope of the tangent at the point (1, 1) is $\left(\frac{dy}{dx}\right)_{(1,1)} = -1$

Now, by using point slope form of line

$y - 1 = -1(x - 1)$ or $y + x - 2 = 0$

To find the equation of normal, the slope of the normal at the point $(1,1)$ is equal to negative reciprocal of the slope of tangent. Therefore, the slope of the normal is 1.

Hence, the equation of the normal is $y - 1 = 1(x - 1)$ or $y - x = 0$

Example 2. Find the equation of the tangent to the curve $y = \frac{(x-7)}{(x-2)(x-3)}$ at the point where it cuts the x-axis.

Solution: The equation of curve is $y = \frac{(x-7)}{(x-2)(x-3)}$ …(i)

As curve cuts the x-axis, so $y = 0$.

Using $y = 0$ in (i) we get

$$0 = \frac{x - 7}{(x - 2)(x - 3)} \qquad \Rightarrow x = 7$$

Thus, the point where the curve cuts x-axis is $(7, 0)$.

Now, differentiating (i) with respect to x, we get

$$y = \frac{(x-7)}{(x-2)(x-3)} = \frac{(x-7)}{(x^2-5x+6)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x^2-5x+6)-(x-7)(2x-5)}{(x^2-5x+6)^2} = \frac{x^2-5x+6-2x^2+19x-35}{(x^2-5x+6)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x^2+14x-29}{(x^2-5x+6)^2}$$

$$\left[\frac{dy}{dx}\right]_{(7,0)} = \frac{-49+98-29}{(49-35+6)^2} = \frac{20}{400} = \frac{1}{20}$$

$\therefore$ slope of the tangent to the curve (i) at point $(7, 0)$ is $\frac{1}{20}$.

Equation of tangent is $y - y_1 = \frac{1}{20}(x - x_1)$

At (7,0) $\quad y - 0 = \frac{1}{20}(x-7) \Rightarrow 20y = x - 7$

$$\Rightarrow x - 20y - 7 = 0$$

4.3.3 Find the angle of intersection of the two curves.

Angle between two curves:

Let $y_1 = f(x)$ and $y_2 = g(x)$ be two curves which intersect each other at point $P(x_1, y_1)$ as shown in the figure 4.2. If we draw tangent line passing through intersecting point of the curves, then the angle between these tangent lines is called the angle between two curves.

To find the angle take m_1, m_2 be the slopes of tangent lines. By the definition of slope

$m_1 = \tan\alpha$

and $\quad m_2 = \tan\beta$

where α and β are the inclinations of the lines and can be calculated by using derivative.

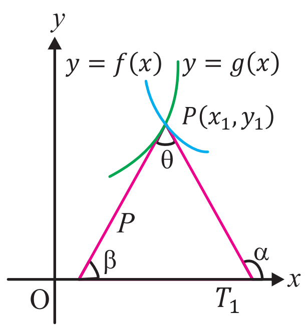

Fig. 4.2

The acute angle between the curves is given by

$$\theta = tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Steps to be followed to find the angle between two curves:

(i) Find point of intersection by solving the equations of both curves.

(ii) Find $\frac{dy}{dx}$ of both curves

(iii) Put value of point of intersection in $\frac{dy}{dx}$ and get m_1 and m_2.

(iv) Put value of m_1 and m_2 in $\theta = tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ and find θ.

Example: Find the angle between the curves $xy = 2$ and $y^2 = 4x$.

Solution: Given equations of curves are $xy = 2$ ..(i)

$y^2 = 4x$ ..(ii)

From equation (i) and (ii), we get

$$\frac{4}{x^2} = 4x$$

$$\Rightarrow \quad x^3 = 1 \qquad \Rightarrow x = 1$$

To get the value of y, put $x = 1$ in equation (i), we get $y = 2$, so the point of intersection of curves is $(1, 2)$.

Let m_1 be the slope of curve (i) at the point $(1,2)$.

By differentiating on both sides of equation (i) with respect to x we get,

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(1,2)} = \left(\frac{-2}{1}\right)_{(1,2)} = -2$$

Similarly, m_2 be the slope of curve (ii), at the point $(1,2)$ is given by

$$\frac{dy}{dx} = \frac{2}{y}$$

$$m_2 = \left(\frac{dy}{dx}\right)_{(1,2)} = \left(\frac{2}{y}\right)_{(1,2)} = 1$$

Angle between the given curves,

$$tan\,\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-2 - 1}{1 + (-2)(1)}\right| = \left|\frac{-3}{-1}\right| = 3$$

Hence $\theta = \tan^{-1}(3) = 71.56°$

4.3.4 Find the point on a curve where the tangent is parallel to the given line

Example 1. Find the point on the curve $xy = 12$, the tangent at the point is parallel to the given line $3x + y = 3$.

Solution: The slope of the line $3x + y = 3$, is $m_1 = -3$

From the equation of curve $xy = 12$ so, $y = \frac{12}{x}$

By differentiating with respect to x to get $\frac{dy}{dx} = \frac{-12}{x^2} = m_2$

As the given line is parallel to tangent to the curve so $\boldsymbol{m_1} = \boldsymbol{m_2}$ or $\boldsymbol{-3} = \dfrac{\boldsymbol{-12}}{\boldsymbol{x^2}}$

$$\Rightarrow \boldsymbol{x^2 = 4} \qquad \text{or} \quad \boldsymbol{x = \pm 2}$$

By substituting the value of $\boldsymbol{x}$ in the equation of curve $\boldsymbol{xy = 12}$ we have

$$\boldsymbol{y = \frac{12}{2} = 6} \qquad \text{at } \boldsymbol{x = 2}$$

$$\boldsymbol{y = \frac{12}{-2} = -6} \qquad \text{at } \boldsymbol{x = -2}$$

The points at which the tangent line is parallel to the given line are $(\boldsymbol{2, 6})$ and $(\boldsymbol{-2, -6})$.

Exercise 4.3

1. Determine the slope of tangent to the curve $y = x^3$ at the point $\left(\frac{3}{2}, \frac{27}{8}\right)$.
2. Find the slope of tangents to the curve $x^2 + y^2 = 25$ at the point on it whose x–coordinate is 2.
3. Find the equation of the tangent and the equation of the normal to the curve $y = x + \frac{1}{x}$ at the point where $x = 2$.
4. Given two curves $y = x^2$ and $y = (x-3)^2$. Find the angle between them
5. Prove that the tangent lines to the curve $y^2 = 4ax$ at points where $x = a$ are at right angles to each other.
6. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ the tangent is parallel to y-axis.

4.4 Maxima and Minima

4.4.1 Define increasing and decreasing functions

Maximum and minimum values of function are called maxima and minima of the function.

A function $f(x)$ is said to be increasing at a point $x = a$,

if $f(a-h) < f(a) < (a+h)$, where h is a positive change in x. (Fig. 4.3)

A function $f(x)$ is said to be decreasing at a point $x = a$, if

$$f(a-h) > f(a) > f(a+h) \text{ (Fig. 4.4)}$$

A function is said to be increasing or decreasing over an interval if it is increasing or decreasing at every point of that interval.

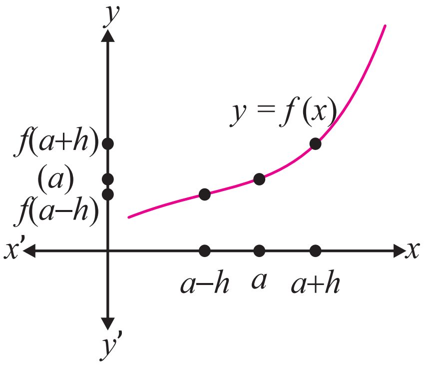

Fig. 4.3

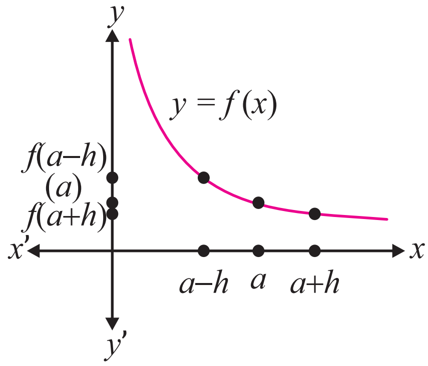

Fig. 4.4

The Fig. 4.3, represents the increasing function and the Fig. 4.4 represents the decreasing function.

4.4.2 Prove that if $f(x)$ is differentiable function on the open interval (a, b) then

- **$f(x)$ is increasing on (a, b) if $f'(x) > 0\ \forall x \in (a, b)$**
- **$f(x)$ is decreasing on (a, b) if $f'(x) < 0\ \forall x \in (a, b)$**

- **$f(x)$ is increasing on (a, b) if $f'(x) > 0\ \forall x \in (a, b)$**

Let $f(x)$ is increasing function at x where $x \in (a, b)$, then by the definition of increasing function.

$f(x+h) > f(x)$, where h is positive change in x.

Now, by definition of derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\because \quad f(x+h) - f(x) > 0$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} > 0$$

Hence, the function $f(x)$ is increasing on (a, b) if $f'(x) > 0\ \forall\ x \in (a, b)$.

- **$f(x)$ is decreasing on (a, b) if $f'(x) < 0\ \forall x \in (a, b)$**

Let $f(x)$ is decreasing function at x where $x \in (a, b)$, then by the definition of decreasing function.

$f(x+h) < f(x)$, where h is positive change in x.

Now, by definition of derivative,

We have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\because \quad f(x+h) - f(x) < 0$

$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} < 0$

Hence, the function $f(x)$ is decreasing on (a, b) if $f'(x) < 0 \; \forall \; x \in (a, b)$.

Example 1. Check whether the function $f(x) = x^2 + 5$ is increasing at $x = 3$ or not.

Solution:

$f(x) = x^2 + 5$

Differentiating w.r.t x,

We get

$f'(x) = 2x$

Put $x = 3$,

$f'(3) = 2(3)$

$f'(3) = 6 > 0$

Hence, the function $f(x) = x^2 + 5$ is increasing at $x = 3$.

Example 2. Check whether $y = \sin x$ is decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Solution:

$f(x) = \sin x$

Differentiating w.r.t to x

We get

$f'(x) = \cos x$

$\because \quad \cos x < 0 \; \forall \; x \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore \quad f(x) = \sin x$ is decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

4.4.3 Examine a given function for extreme values.

Let $y = f(x)$ be a function, and the graph of this function be shown in Fig. 4.5.

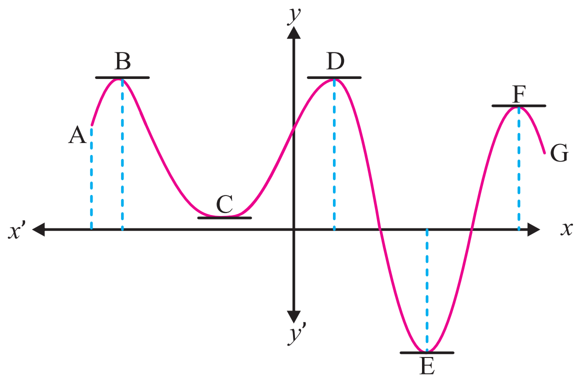

Fig. 4.5

From diagram F is increasing from A to B, C to D, E to F and d21ecreasing from B to C, D to E and F to G.

Points B, C, D, E and F are such points where the function is neither increasing nor decreasing. The tangents to the curve at these points are parallel to x-axis. The derivative of the function at these points vanishes, these points are called turning points or points of extreme values or extrema.

B, D and F are such turning points where the function changes from increasing to decreasing. These are called points of maximum values or maxima.

C and E are such turning points where the function changes from decreasing to increasing. These are called points of minimum values or minimas.

Maxima and Minima through first derivative

Let $y = f(x)$ be a function.

(i) Differentiate w.r.t 'x' and obtain $f'(x)$.

(ii) Put $f'(x) = 0$, solve it and obtain critical points.

(iii) Let $x = a$ be a critical point.

If $\left.\begin{array}{l} f'(a-h) < 0 \\ f'(a) = 0 \\ f'(a+h) > 0 \end{array}\right\} \Rightarrow x = a$ is point of minima.

If $\left.\begin{array}{l} f'(a-h) > 0 \\ f'(a) = 0 \\ f'(a+h) < 0 \end{array}\right\} \Rightarrow x = a$ is point of maxima.

If $\left.\begin{array}{l} f'(a-h) > 0 \\ f'(a) = 0 \\ f'(a+h) > 0 \end{array}\right\}$ or If $\left.\begin{array}{l} f'(a-h) < 0 \\ f'(a) = 0 \\ f'(a+h) < 0 \end{array}\right\} \Rightarrow x = a$ is point of inflection.

Note: Point of inflection is that point of curve which is nether point of minimum nor maximum.

4.4.4 State the second derivative rule to find the extreme values of a function at a point

Let $y = f(x)$ be a function.

(i) Differentiate w.r.t 'x' and obtain $f'(x)$.

(ii) Put $f'(x) = 0$, solve it and obtain critical function.

(iii) Differentiate again w.r.t 'x' of obtain $f''(x)$.

(iv) Let $x = a$ be a critical point.

If the $f''(a) < 0 \Rightarrow x = a$ is a point of maxima.

If the $f''(a) > 0 \Rightarrow x = a$ is point of minima.

If the $f''(a) = 0 \Rightarrow$ test fails.

4.4.5 Use second derivative rule to examine a given function for extreme values.

Example 1. Find extreme values of $f(x) = x^4 - 8x^2$ using the second derivative rule.

Solution: Here $f(x) = x^4 - 8x^2$

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$$

Put $\quad f'(x) = 0$

$4x(x^2 - 4) = 0$

$\Rightarrow \quad x = 0$ or $x = \pm 2$

Again Differentiate

$$f''(x) = 12x^2 - 16$$

Putting the values of $x = -2, 0$ and 2 into $f''(x)$.

$f''(-2) = 12(-2)^2 - 16 = 32 > 0$ that is function has a minimum at $x = -2$

$f''(0) = 12(0)^2 - 16 = -16 < 0$ function has a maximum at $x = 0$

$f''(2) = 12(2)^2 - 16 = 32 > 0$ function has a minimum at $x = 2$

Minimum value at $x = -2$

$$f(-2) = (-2)^4 - 8(-2)^2 = -16$$

Minimum value at $x = 2$

$$f(2) = (2)^4 - 8(2)^2 = -16$$

Maximum value at $x = 0$

$$f(0) = 0$$

Example 2. Find points of extrema of $f(x) = \sin x + \cos x$ on $[0,2\pi]$ using the Second Derivative Rule.

Solution: $\quad f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

As $\quad f'(x) = 0$

We have $\quad \cos x - \sin x = 0$

$\sin x = \cos x$

Dividing both side by cos x

$\tan\, x = 1$, $x = tan^{-1}(1) = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

so in the interval $[0,2\pi]$ we have $f'(x) = 0$ at $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

Again Differentiate

$$f''(x) = -\sin x - \cos x$$

Put the values of $x = 0, \frac{\pi}{4}, \frac{5\pi}{4}, 2\pi$

$f''(0) = -\sin(0) - \cos(0) = -1 < 0$ that is function has a maximum at $x = 0$

$f''\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$ function has a maximum at $x = \frac{\pi}{4}$

$f''\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$ function has a minimum at $\frac{5\pi}{4}$

$f''(2\pi) = -\sin(2\pi) - \cos(2\pi) = -1 < 0$ function has a maximum at 2π.

4.4.6 Solve real life problems related to extreme value.

Example 1. A farmer wishes to enclose a rectangular field using 1000 yards of fencing in such a way that the area of the field is maximized.

Solution: Let x and y be the length and breadth of the field and A be the area of the field. then $A = xy$.

For fencing we have an equation for perimeter

$$2x + 2y = 1000, \implies y = 500 - x$$

Area of rectangular field

$$A = x(500 - x) = 500x - x^2$$

Now $\frac{dA}{dx} = 500 - 2x,$

so $\frac{dA}{dx} = 0,$

$$500 - 2x = 0, \quad x = 250$$

when $x = 250.$

$$\frac{d^2A}{dx^2} = -2$$

Hence $\left(\frac{d^2A}{dx^2}\right)_{x=250} = -2 < 0$

The maximum area occurs at $x = 250$.

i.e., $A = 250(500 - 250) = 62500$ square and dimension of rectangular field is $x = 250$ and $y = 250$.

Example 2. A company finds that the cost of goods $C(x)$ is given by

$$C(x) = -x^3 + 9x^2 - 15x + 9$$

where x represents thousand of units. If the company can only make a minimum of 6000 units, what is the minimum cost company required. Here, cost is in dollar.

Solution: since x is in thousand of unit we must find the minimum cost in the interval $[0,6]$

$$C'(x) = -3x^2 + 18x - 15$$
$$= -3(x^2 - 6x + 5)$$
$$= -3(x - 5)(x - 1) = 0$$
$$\Rightarrow \quad x = 5, 1$$
$$\Rightarrow \quad C''(x) = -6x + 18$$
$$C''(1) = -6x + 18 = 12 > 0$$
$$C''(5) = -6(5) + 18 = -12 < 0$$
i.e., $C(1) = 12$.

The minimum cost for the company exist at $x = 1$. i.e., $C(1) = 12$.

4.4.7 Use MAPLE command Maximize (Minimize) to compute maximum (minimum) value of a function

The format of Maximize (Minimize) command in MAPLE is as under:

> maximize(f(x), x=a..b)

> minimize(f(x), x=a..b)

where,

f(x) is the function whose maximize (minimize) value is required

x = a..b is the interval for maximize (minimize) value

In order to compute the Maximize (Minimize) value of a function in the interval, following examples are given:

$> minimize\ (cos(x), x = -\pi..\pi)$

$$-1$$

$> maximize\left(cos(x), x = \frac{-\pi}{2}..\frac{\pi}{2}\right)$

$$1$$

$> minimize(sin(x), x = -\pi..\pi)$

$$-1$$

$> maximize\left(sin(x), x = \frac{-\pi}{2}..\frac{\pi}{2}\right)$

$$1$$

$> minimize(exp(x), x = 0..10);$

$$1$$

$> maximize(exp(x), x = 0..10);$

$$exp(10)$$

$> minimize(exp(x), x = 0..5);$

$$exp(1)$$

$> maximize(exp(x), x = 0..5);$

$$exp(5)$$

$> maximize\ (x^2 + y^2 - 2x + 2y + 2), x = 1..2, y = -2..2$, location);

$$10, \{[\{x = 2, y = 2\}, 10]\}$$

$> minimize\ (x^2 + y^2 - 2x + 2y + 2), x = 1..2, y = -2..2$, location);

$$10, \{[\{x = 2, y = 2\}, 0]\}$$

Exercise 4.4

1. Show that function $f(x) = -x^2 + 10x + 9$ is increasing at $x = 4$.
2. Show that $f(x) = \tan^2 x$ is decreasing at $x = \frac{3\pi}{4}$.
3. Find the maximum and minimum values, if any, of the function $f: \mathbb{R} \to \mathbb{R}$ in the following cases:

   (i) $f(x) = x^2 - 2x + 3$ (ii) $f(x) = x^3 - 9x^2 + 15x + 3$
   (iii) $f(x) = -x^4 + 2x^2$ (iv) $f(x) = e^x \sin x$
   (v) $f(x) = 2e^x + e^{-x}$ (vi) $f(x) = 2x - x^2$
4. A rectangular reservoir with a square bottom and open top is to be lined inside with lead. Find the dimensions of the reservoir to hold $\frac{1}{2}a^3$ cubic metres, such that the lead required is minimum.
5. Find a right-angled triangle of maximum area with a hypotenuse of length h.
6. A particle moves so that its distance s at time is given by $s = ut + \frac{1}{2}at^2$, where u a are fixed real numbers. Find its speed and magnitude of its accelerations at time t.
7. If the P and time t are connected by the formula $P = t^2 + 11 + 9$. Find force at $t = 5$ second.

Review Exercise 4

1. Choose the correct answer

   (i) If $y = \frac{1}{x}$, then $y'''(1) =$ --------------

   (a) 6 (b) -6 (c) 2 (d) None of these

   (ii) What will be the nth derivative of $2e^x$

   (a) $2ne^x$ (b) $2e^{nx}$ (c) $2e^x$ (d) $\frac{e^x}{2}$

   (iii) If $xy = k^2$, then $y'' =$ __

   (a) $\frac{2k^2}{x}$ (b) $\frac{-2k^2}{x^3}$ (c) $\frac{3k^2}{x^3}$ (d) $\frac{2k^2}{x^3}$

   (iv) 3rd order derivative of 2^x is ---------------

   (a) $(ln3)^2\, e^{xln2}$ (b) $(ln2)^3\, e^{xln3}$ (c) $(ln2)^3\, e^{xln2}$ (d) None of these

   (v) 2nd order derivative of $f(x) = A\sin x + B\cos x$ is -------------

   (a) $f(x)$ (b) $-f(x)$ (c) $\pm f(x)$ (d) None of these

(vi) If S is the distance covered by a car, then $\frac{d^2S}{dt^2}$ will be its -----------------

(a) Velocity (b) deceleration (c) acceleration (d)Average velocity

(vii) $y = \tan^{-1}x$, then $y''(1) =$ -----------------

(a) $\frac{-1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{-1}{4}$ (d) $\frac{1}{2}$

(viii) If $y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ $x \in R$, then y is the Mclaurin series of

(a) $\sin x$ (b)$\cos x$ (c) e^x (d) None of these

(ix) A function $f(x)$ is said to be a decreasing function when $x_1 < x_2$ and

(a) $f(x_1) < f(x_2)$ (b) $f(x_1) > f(x_2)$

(c) $f(x_1) \geq f(x_2)$ (d) $f(x_1) \leq f(x_2)$

(x) If a function $f(x)$ is such that $f'(c) = 0$ then the point $(c, f(c))$ is called

(a) Maximum point (b) Minimum point

(c) Stationary point (d) Critical point

2. Find 2nd order derivative of $f(x) = \ln(1 + x^2)$

3. Find the second derivative of following parametric functions

$x = 3u^2 + 1$ and $y = 3u^2 + 5u$, Find $\frac{d^2y}{dx^2}$

4. Evaluate the third derivatives of the given functions.

(i) $v(x) = x^3 - x^2 + x - 1$

(ii) $f(x) = 7\sin\left(\frac{x}{3}\right) + \cos(1 - 2x)$

(iii) $y = e^{-5x} + 8\ln(2x^4)$

5. Determine the second derivative of the given functions.

(i) $g(x) = \sin(2x^3 - 9x)$ (ii) $z(x) = \ln(7 - x^3)$

(iii) $q(x) = \frac{2}{(6+2x-x^2)^4}$ (iv) $h(t) = cos^2(7t)$

(v) $2x^3 + y^2 = 1 - 4y$ (vi) $6y - xy^2 = 1$

6. Given that

$y = \cos x$; Find y'; y''; y'''; $y^{(4)}$.

7. Find $\frac{d^2y}{dx^2}$; if $x = 4\sin t$, $y = 5\cos t$.

8. Find the rth derivative of $f(x) = x^n$ where $r \leq n$.

9. Find the Taylor series of the function $x^4 + x - 2$ centered at $a = 1$.

10. Obtain the Taylor series for the function $(x-1)e^{x}$ near $x=1$.

11. Find the McLaurin series for $\ln(1+x)$ and hence find for $\ln\left(\frac{1+x}{1-x}\right)$.

12. Find the equation of the tangent line to the curve $y=x^{3}-3x^{2}+x$ at the point $(2,-2)$.

13. At what point on the graph of $y=x^{2}$ where the tangent line is parallel to the line $3x-y=2$.

14. Determine the interval on which the function $f(x)=x^{2}-3x+1$ is increasing and decreasing.