Integration

Unit 6

6.1 Introduction

Integration is the reverse process of differentiation. It is used in dealing with problems in which the derivative of a function, or its rate of change is known and we want to find the function.

The principles of integration were formulated independently by Isaac Newton and Gottfried Wilhelm Leibnitz in the 17^{th} century.

Integration is usually used to find area under a curve and volume of solid of revolution.

6.1.1 Demonstrate the concept of integral as an accumulator

Integral is the outcome of the process of integration and is of two types definite and indefinite. It is an accumulator which is used to find the definite integral of a function $f(x)$, which is continuous on a closed interval $[a, b]$. In this process, the region bounded by the geometrical curve of function $f(x)$, x-axis and the vertical lines $x = a$ and $x = b$ is divided into infinitesimal vertical rectangles each of width Δx on x-axis and height $f(x_i)$ from x-axis, where $i = 1, 2, 3, \ldots, n$ as shown in figure 6.1. The accumulation of the product $f(x_i)\Delta x$ is approximately equal to the definite integral of $f(x)$ from $x = a$ to $x = b$.

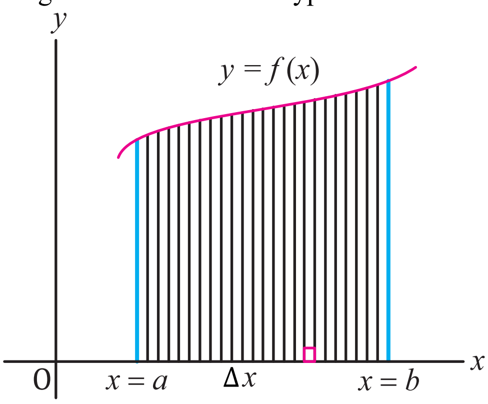

Fig. 6.1

Definite integral of $f(x)$ from a to b is Accumulation of $f(x_i)\Delta x$ as shown in the Fig. 6.1.

6.1.2 Know integration as inverse process of differentiation

In differentiation we are given an original function and we are required to find its derivative, while in integration we are required to find the original function whose derivative is given.

$f(x)$ Original function —differentiation→ $f'(x)$ derivative / derived function

$f'(x)$ Derived function —integration→ $f(x)$ original function

Thus, integration is the reverse or inverse process of differentiation.

Indefinite integral or antiderivative

Let f be a continuous function. A function F whose derivative is f is called integral of f, i.e., $F'(x) = f(x)$, $\forall\, x$ in the domain of $f(x)$.

As F is an integral or antiderivative or primitive of a function f, then

$$\int f(x)dx = F(x) + C$$

where $f(x)$ is called integrand and C is called the constant of integration. The solution $F(x) + C$, depends on the arbitrary constant C, so $\int f(x)dx$ has indefinite solution, called indefinite integral.

6.1.3 Explain Constant of integration

Since, the derivative of a constant is 0, therefore all the functions which differ by constant have the same derivative.

For example:

$$f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2$$
$$f(x) = x^3 \pm 1 \quad \Rightarrow \quad f'(x) = 3x^2$$
$$f(x) = x^3 \pm c \quad \Rightarrow \quad f'(x) = 3x^2$$

So, it is not possible to find original function through integration, therefore, we use a constant C in indefinite integral to represent the family of functions whose derivatives are the function $f(x)$. This arbitrary constant C is called constant of integration.

Note: In $\int f(x)\,dx = F(x) + C$,

- $\int$ is the integral sign (elongated S) which is used to represent the process of integration.
- $f(x)$ is the integrand; a function which is to be integrated or under the effect of integral sign.
- dx, x is the variable of integration that tells integrand is to be integrated w.r.t x.
- C is the integral constant or constant of integration.
- $F(x) + C$ represents family of integrals or antiderivatives or primitives whose derivatives are $f(x)$.

6.1.4 Know simple standard integrals which directly follow from standard differentiation formulae.

The basic integration formulae can be obtained directly from the differentiation formulae of the functions as given I the following table.

S. No.	Differentiation formulae	Integration formulae
1.	$\frac{d}{dx}(x+c)=1$	$\int dx = x + C$
2.	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}+c\right)=x^n$	$\int x^n dx = \frac{x^{n+1}}{n+1}+C, \quad n \neq -1$
3.	$\frac{d}{dx}\left\{\frac{(ax+b)^{n+1}}{a(n+1)}+c\right\}=(ax+b)^n$	$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}+C,$ $n \neq -1$
4.	$\frac{d}{dx}(\ln\|x\|+c)=\frac{1}{x}$	$\int \frac{1}{x}dx = \ln\|x\|+C$

Trigonometric Functions

5.	$\frac{d}{dx}(sin\, x+c)=cos\, x$	$\int cos\, x\, dx = sin\, x + C$
6.	$\frac{d}{dx}(cos\, x+c)=-sin\, x$	$\int sin\, x\, dx = -cos\, x + C$
7.	$\frac{d}{dx}(\ln\|\sec x\|+c)=tan\, x$	$\int tan\, x\, dx = \ln\|\sec x\|+C$
8.	$\frac{d}{dx}(\ln\|\sin x\|+c)=cot\, x$	$\int cot\, x\, dx = \ln\|\sin x\|+C$
9.	$\frac{d}{dx}[\ln\|\sec x+\tan x\|+c\,]=\sec x$	$\int sec\, x\, dx = \ln\|\sec x+\tan x\|+C$
10.	$\frac{d}{dx}[\ln\|\mathrm{cosec}\, x-\cot x\|+c\,]=cosec\, x$	$\int \mathrm{cosec}\, x\, dx = \ln\|\mathrm{cosec}\, x-\cot x\|+C$
11.	$\frac{d}{dx}(\tan x+c)=sec^2\, x$	$\int sec^2 x\, dx = \tan x + C$
12.	$\frac{d}{dx}(\cot x+c)=-cosec^2\, x$	$\int \mathrm{cosec}^2 x\, dx = -\cot x + C$
13.	$\frac{d}{dx}(\sec x+c)=sec\, x\tan x$	$\int sec\, x \tan x\, dx = \sec x + C$
14.	$\frac{d}{dx}(\mathrm{cosec}\, x+c)=-cosec\, x\cot x$	$\int \csc x \cot x\, dx = -\mathrm{cosec}\, x + C$

Inverse Trigonometric Functions

15.	$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}+c\right)=\frac{1}{\sqrt{a^2-x^2}}$	$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+C$
16.	$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}+c\right)=\frac{1}{a^2+x^2}$	$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$
17.	$\frac{d}{dx}\left(\frac{1}{a}\sec^{-1}\frac{x}{a}+c\right)=\frac{1}{x\sqrt{x^2-a^2}}$	$\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{x}{a}+C$

Exponential Functions

18.	$\frac{d}{dx}(e^x+c)=e^x$	$\int e^x dx=e^x+C$
19.	$\frac{d}{dx}\left(\frac{1}{a}e^{ax}+c\right)=e^{ax}$	$\int e^{ax}dx=\frac{1}{a}e^{ax}+C$
20.	$\frac{d}{dx}\left(\frac{a^x}{\ln a}+c\right)=a^x$	$\int a^x dx=\frac{a^x}{\ln a}+C$

Some other Integration Formulae

21.	$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\|+C$
22.	$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\|+C$
23.	$\int\frac{dx}{\sqrt{x^2+a^2}}=\ln\left\|x+\sqrt{x^2+a^2}\right\|+C$
24.	$\int\frac{dx}{\sqrt{x^2-a^2}}=\ln\left\|x+\sqrt{x^2-a^2}\right\|+C$
25.	$\int\sqrt{a^2-x^2}dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\text{Sin}^{-1}\frac{x}{a}+C$
26.	$\int\sqrt{a^2+x^2}dx=\frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}a^2\ln\left\|x+\sqrt{a^2+x^2}\right\|+C$
27.	$\int\sqrt{x^2-a^2}dx=\frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\ln\left\|x+\sqrt{x^2-a^2}\right\|+C$

6.2 Rules of integration

6.2.1 Recognize the following rules of integration

(i) $\int \frac{d}{dx}[f(x)]dx = \frac{d}{dx}\int f(x)dx = f(x) + C$ **where C is the constant**

As we know that integration is an inverse process of differentiation. So, the process of differentiation and integration neutralizes each other.

i.e., $$\int \frac{d}{dx}[f(x)]dx = \frac{d}{dx}\int f(x)\,dx = f(x) + C$$

Example: $$\int \left[\frac{d}{dx}\left(x^2 + 5x + 7\right)\right] dx = x^2 + 5x + 7 + C$$

(ii) The integral of the product of a constant and a function is the product of the constant and the integral of the function.

Let k be a constant and $f(x)$ be a function then,

$$\int k\,f(x)dx = k\int f(x)dx$$

Example: $$\int 4e^x\,dx = 4\int e^x dx$$
$$= 4\,e^x + c$$

(iii) The integral of the sum of a finite number of functions is equal to the sum of their integrals.

Let $f(x)$, $g(x)$ and $h(x)$ are three differentiable functions whose integrals exist then.

$$\int [f(x) + g(x) + h(x)]\,dx = \int f(x)dx + \int g(x)dx + \int h(x)dx$$

Example:

$$\int (9x^8 + 6x^5 + 5)dx = \int 9x^8\,dx + \int 6x^5\,dx + \int 5\,dx$$
$$= 9\int x^8\,dx + 6\int x^5\,dx + 5\int dx$$
$$= 9\left(\frac{x^9}{9}\right) + 6\left(\frac{x^6}{6}\right) + 5(x) + C$$
$$= x^9 + x^6 + 5x + C$$

6.2.2 Use standard differentiation formulae to prove the results for the following integrals

(i) $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, \qquad n \neq -1$

Proof: Consider the differentiation

$$\frac{d}{dx}\left\{\frac{[f(x)]^{n+1}}{n+1}+C\right\}=\frac{d}{dx}\left\{\frac{[f(x)]^{n+1}}{n+1}\right\}+\frac{d}{dx}(C)$$

$$=\frac{1}{n+1}\frac{d}{dx}[f(x)]^{n+1}+0$$

$$=\frac{1}{n+1}(n+1)[f(x)]^{n+1-1}\frac{d}{dx}f(x)$$

$$=[f(x)]^{n}f'(x)$$

By taking indefinite integral on both sides

$$\int\left[\frac{d}{dx}\left\{\frac{[f(x)]^{n+1}}{n+1}+C\right\}\right]dx=\int[f(x)]^{n}f'(x)dx$$

$$\frac{[f(x)]^{n+1}}{n+1}+C=\int[f(x)]^{n}f'(x)dx$$

$$\boxed{\int[f(x)]^{n}f'(x)dx=\frac{[f(x)]^{n+1}}{n+1}+C, n\neq-1}$$

(ii) $\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+C$

Proof: Consider the differentiation

$$\frac{d}{dx}(\ln|f(x)|+c)=\frac{d}{dx}\ln|f(x)|+\frac{d}{dx}(C) \qquad \left(\because \frac{d}{dx}\ln|x|=\frac{1}{x}\right)$$

$$=\frac{1}{f(x)}\frac{d}{dx}f(x)+0$$

$$=\frac{f'(x)}{f(x)}$$

By taking integrals on both sides

$$\int\frac{d}{dx}(\ln f(x)+c)dx=\int\frac{f'(x)}{f(x)}dx$$

$$\ln f(x)+c=\int\frac{f'(x)}{f(x)}dx$$

$$\boxed{\int\frac{f'(x)}{f(x)}dx=\ln f(x)+C}$$

(iii) $\int e^{ax}[af(x)+f'(x)]dx$

Proof: Consider the differentiation

$$\frac{d}{dx}[e^{ax}f(x)+c]=\frac{d}{dx}[e^{ax}f(x)]+\frac{d}{dx}(c)$$

By using product rule of differentiation

$$=f(x)\frac{d}{dx}e^{ax}+e^{ax}\frac{d}{dx}f(x)+0$$
$$=f(x)(ae^{ax})+e^{ax}f'(x)$$
$$=e^{ax}[af(x)+f'(x)]$$

By taking integrals on both sides

$$\int\frac{d}{dx}[e^{ax}f(x)+c]dx=\int e^{ax}[af(x)+f'(x)]dx$$

$$\boxed{\int e^{ax}[af(x)+f'(x)]dx=e^{ax}f(x)+C}$$

when $a=1$

$$\boxed{\int e^{x}[f(x)+f'(x)]dx=e^{x}f(x)+C}$$

Example: Evaluate

(i) $\int(10x^2+5)^8(20x)dx$

Solution:

Let $f(x)=10x^2+5$

$f'(x)=20x$

By the rule of integration

$$\int(10x^2+5)^8(20x)dx=\frac{(10x^2+5)^9}{9}+C$$

(ii) $\int\frac{2x+3}{x^2+3x-5}dx$

Solution: Let $f(x)=x^2+3x-5, f'(x)=2x+3$

By the rule of integration

$$\int\frac{2x+3}{x^2+3x-5}dx=\ln|x^2+3x-5|+C$$

(iii) $\int e^x(2x^2+4x)dx$

Let $f(x)=2x^2, f'(x)=4x$

By the rule of integration

$$\int e^x(2x^2+4x)dx=2e^xx^2+C$$

Exercise 6.1

1. Evaluate the following indefinite integrals by using standard formulae.

(i) $\int 9x^5\,dx$ (ii) $\int \frac{15}{x^3}dx$

(iii) $\int \frac{a}{\sqrt{bx}}dx$ (iv) $\int by^{\frac{2}{3}}dy$

(v) $\int(3x^2-9x+5)\,dx$ (vi) $\int(2x^{-5}+3x^{-2})\,dx$

(vii) $\int \frac{(x^5+3x^3-5x+6)}{x^4}dx$ (viii) $\int(\cos x+3\sin x)\,dx$

(ix) $\int(3\sec x-\operatorname{cosec} x)\,dx$ (x) $\int(2\tan x-5\sec x)\,dx$

(xi) $\int(9\,e^x-3\cos x-5\sin x)\,dx$ (xii) $\int(\sec^2 x+\operatorname{cosec}^2 x)\,dx$

2. Evaluate the following indefinite integrals by using standard formulae.

(i) $\int(3x^2+9x+3)^{\frac{1}{2}}(6x+9)dx$ (ii) $\int\sqrt{ax^2+2bx+c}\,(ax+b)dx$

(iii) $\int\frac{(6x+5)dx}{\sqrt{3x^2+5x+2}}dx$ (iv) $\int(x^2+4x+3)^{-9}(2x+4)dx$

(v) $\int(x-2)(x-3)(x-4)dx$ (vi) $\int(2\,x^2-3)^2dx$

(vii) $\int(x^3-3x^2+9)^{\frac{7}{2}}(x^2-2x)dx$ (viii) $\int(x^2-5)^3dx$

(ix) $\int(cos\,x+\sin x)^{\frac{3}{2}}(\cos x-\sin x)dx$

(x) $\int(\tan x+\sin x)\,(\sec^2 x+\cos x)dx$

3. Evaluate by using standard formulae of integration.

(i) $\int\frac{x}{x^2+3}dx$ (ii) $\int\frac{\sec^2 x+\cos x}{\tan x+\sin x}dx$

(iii) $\int\frac{5x^4+4x^3-3x^2+2x}{x^5+x^4-x^3+x^2}dx$ (iv) $\int\frac{\left(e^x+\frac{1}{x}\right)dx}{e^x+\ln x}$

(v) $\int(5\,x^3-3x^2+6x-9)^{-1}(5\,x^2-2x+2)dx$ (vi) $\int\frac{1}{x+\sqrt{x}}dx$

4. Evaluate the following integrals by using standard formulae.

(i) $\int e^x(\sin x+\cos x)dx$ (ii) $\int e^x\left(\sin^{-1}x+\frac{1}{\sqrt{1-x^2}}\right)dx$

(iii) $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx$ (iv) $\int e^x(\sec^2 x+\tan x)dx$

(v) $\int e^x\left(\frac{1}{x}+\ln x\right)dx$

5. Evaluate the following integrals by using appropriate formulae.

(i) $\int \frac{dx}{x^2+9}$ (ii) $\int \frac{dt}{\sqrt{4-t^2}}$ (iii) $\int \frac{dy}{y\sqrt{y^2-9}}$

(iv) $\int \frac{dt}{4t^2-9}$ (v) $\int \frac{dx}{\sqrt{9x^2+16}}$ (vi) $\int \frac{dx}{\sqrt{16x^2-9}}$

(vii) $\int \frac{dx}{9-x^2}$ (viii) $\int \frac{dx}{x\sqrt{4x^2-16}}$ (ix) $\int \sqrt{9-4x^2}\,dx$

(x) $\int \sqrt{25+9x^2}\,dx$ (xi) $\int \frac{dy}{9y^2+81}$ (xii) $\int \frac{dx}{4x^2-16}$

6.3 Integration by substitution

6.3.1 Explain the method of integration by substitution

Sometimes the integrals of the form $\int f(g(x))g'(x)\,dx$ can be converted in standard form or easier by substituting $g(x)$ by introducing a new variable.

To understand the integration by substitution method.

Suppose $u = g(x)$

By the differentials $du = g'(x)dx$

Thus,

$$\int f(g(x))\,g'(x) = \int f(u)\,du$$

Obviously, the integral on the right is much easier to evaluate than that on the left.

6.3.2 Apply method of substitution to evaluate indefinite integrals

Example 1. Evaluate by substitution method.

$$\int (2x^2-3)^{\frac{5}{2}} x^3 dx$$

Splitting x^3, we get

$$\int (2x^2-3)^{\frac{5}{2}} x^3 dx = \int (2x^2-3)^{\frac{5}{2}} x^2 (xdx)$$

Now suppose $u = 2x^2 - 3 \implies x^2 = \frac{u+3}{2}$

$\implies du = 4xdx$

$\implies \frac{1}{4}du = xdx$

By substituting, we get

$$\int (2x^2-3)^{\frac{5}{2}} x^3 dx = \int u^{\frac{5}{2}} \left(\frac{u+3}{2}\right)\left(\frac{1}{4}du\right)$$

$$= \frac{1}{8}\int \left(u^{\frac{7}{2}} + 3u^{\frac{5}{2}}\right) du$$

$$= \frac{1}{8}\left[\frac{u^{\frac{9}{2}}}{\frac{9}{2}} + 3\frac{u^{\frac{7}{2}}}{\frac{7}{2}}\right] + C$$

$$= \frac{1}{4}\left[\frac{u^{\frac{9}{2}}}{9} + 3\frac{u^{\frac{7}{2}}}{7}\right] + C$$

Replacing u by $2x^2 - 3$, we get

$$= \frac{1}{36}(2x^2 - 3)^{\frac{9}{2}} + \frac{3}{28}(2x^2 - 3)^{\frac{7}{2}} + C$$

Example 2. Evaluate $\int \cos^4 3x \sin 3x \, dx$

Solution: Substitute $u = \cos 3x$

$\Longrightarrow$ $du = -3\sin 3x \, dx$

$$-\frac{1}{3} du = \sin 3x \, dx$$

By substituting

$$\int \cos^4 3x \sin 3x \, dx = \int u^4 \left(-\frac{1}{3} du\right)$$

$$= -\frac{1}{3}\left(\frac{u^5}{5}\right) + C$$

$$= -\frac{1}{15}\cos^5 3x + C$$

Example 3. Evaluate $\int \cos^2 x \sin^3 x \, dx$

Solution: $\int \cos^2 x \sin^3 x \, dx = \int \cos^2 x \sin^2 x \sin x \, dx$

$$= \int \cos^2 x \, (1 - cos^2 x) \sin x \, dx$$

Substitute $u = \cos x$

$\Rightarrow$ $du = -\sin x \, dx$

$$\int \cos^2 x \sin^3 x \, dx = -\int u^2(1 - u^2) \, du$$

$$= -\int (u^2 - u^4) \, du$$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

Example 4. Evaluate $\int \sin^2 2x \cos^2 2x\, dx$

Solution: By multiplying and dividing by 2^2.

$$\int \sin^2 2x \cos^2 2x\, dx = \frac{1}{(2)^2}\int (2)^2 \sin^2 2x \cos^2 2x$$

$$= \frac{1}{4}\int (2\sin 2x \cos 2x)^2\, dx$$

$$= \frac{1}{4}\int \sin^2 4x\, dx$$

$$= \frac{1}{4}\int \left(\frac{1-\cos 8x}{2}\right) dx$$

$$= \frac{1}{8}\int (1-\cos 8x)\, dx$$

$$= \frac{1}{8}\left[x - \frac{\sin 8x}{8}\right] + C$$

$$= \frac{1}{8}x - \frac{1}{64}\sin 8x + C$$

Example 5. Evaluate $\int \sin 4x \cos 2x\, dx$

Solution: $\int \sin 4x \cos 2x\, dx = \int \frac{1}{2}[\sin(4x+2x) + \sin(4x-2x)\, dx]$

$$= \frac{1}{2}\int (\sin 6x + \sin 2x)\, dx \quad (\because \text{ Using trigonometric identities})$$

$$= \frac{1}{2}\left(-\frac{\cos 6x}{6} - \frac{\cos 2x}{2}\right) + C$$

$$= -\frac{1}{12}\cos 6x - \frac{1}{4}\cos 2x + C$$

Example 6. Evaluate $\int \tan^5 x\, dx$

Solution: $\int \tan^5 x\, dx = \int \tan^3 x \tan^2 x\, dx$

$$= \int \tan^3 x\,(\sec^2 - 1)dx$$

$$= \int (\tan^3 x \sec^2 x - \tan^3 x)\, dx$$

$$= \int \tan^3 x \sec^2 x\, dx - \int \tan^3 x\, dx$$

$$= \frac{\tan^4 x}{4} - \int \tan x \tan^2 x\, dx$$

$$= \frac{1}{4}\tan^4 x - \int \tan x\,(\sec^2 x - 1)\, dx$$

$$= \frac{1}{4}\tan^4 x - \int \tan x \sec^2 dx + \int \tan x\, dx$$

$$= \frac{1}{4}\tan^4 x - \frac{\tan^2 x}{2} + \ln|\sec x| + C$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C$$

Example 7. Evaluate $\int \sec^6 x\, dx$

Solution:

$$\int \sec^6 x\, dx = \int \sec^4 x \sec^2 x\, dx$$

$$= \int (\sec^2 x)^2 \sec^2 x\, dx$$

$$= \int (1 + \tan^2 x)^2 \sec^2 x\, dx$$

Substitute $u = \tan x$

$du = \sec^2 x\, dx$

By substituting

$$= \int (1 + u^2)^2 du$$

$$= \int (1 + 2u^2 + u^4) du$$

$$= u + \frac{2}{3}u^3 + \frac{u^5}{5} + C$$

$$= \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

Exercise 6.2

1. Evaluate the following integrals by substitution method.

(i) $\int \frac{3x}{\sqrt{x^2+7}}$ (ii) $\int x^{\frac{4}{3}}(a^{\frac{7}{3}} - x^{\frac{7}{3}})^{\frac{5}{2}} dx$

(iii) $\int \frac{1+2x}{\sqrt{1-x}} dx$ (iv) $\int (2x^2 + 4x + 5)^{\frac{3}{2}} (x + 1) dx$

(v) $\int \frac{x}{\sqrt{1-x^2}} dx$ (vi) $\int (x^2 + 2x + 5)^{-1} (x + 1) dx$

(vii) $\int x^3 (9 + x^2)^{\frac{3}{2}} dx$ (viii) $\int (x^3 - 9)^{\frac{5}{2}} x^5 dx$

(ix) $\int x^9 (x^5 + 3)^{\frac{2}{5}} dx$ (x) $\int (x^3 + x^2 + 5x - 1)^{-1} (3x^2 + 2x + 5) dx$

2. Evaluate the following integrals by substitution method.

(i) $\int \frac{\ln x}{x} dx$ (ii) $\int \frac{dx}{x \ln x}$ (iii) $\int \frac{\tan x}{\ln(\cos x)} dx$

(iv) $\int \frac{\tan(\ln x)}{x} dx$ (v) $\int (1 + e^{2x})^{-\frac{1}{2}} e^{2x} dx$

(vi) $\int \frac{5e^{5\tan x}}{\cos^2 x} dx$ (vii) $\int \frac{e^{2x}}{e^x+e^{-x}} dx$

(viii) $\int e^{(cosec\, 2x+1)} \cdot (\text{cosec}\, 2x \cot 2x) dx$

(ix) $\int 3^x \; dx$ (x) $\int e^{(\sin x+\cos x+3)} \cdot (\cos x - \sin x) dx$

3. Evaluate the following integrals by substitution method.

(i) $\int \cos^5 2x \sin 2x \, dx$ (ii) $\int \frac{\cot\sqrt{x}}{\sqrt{x}} dx$

(iii) $\int (2+\sin 3x)^6 \cos 3x \, dx$ (iv) $\int \sin(ax+b)\, dx$

(v) $\int \frac{\sin x+\cos x}{\sin x-\cos x} dx$ (vi) $\int e^{2x} \sec^2 e^{2x} dx$

(vii) $\int \frac{\text{cosec}\ 3x \cot 3x dx}{(a+b\ cosec\ 3x)^2} dx$ (viii) $\int \frac{dx}{(2\cot x+3)\sin^2 x}$

(ix) $\int \frac{\sec x dx}{3\sin x+4\cos x}$ (x) $\int \cos(3x-5)\, dx$

4. Evaluate the following integrals by substitution method.

(i) $\int \cos^2 2y\, dy$ (ii) $\int \sin^3 (3x+5) dx$

(iii) $\int \cos^3 x \sqrt{\sin x}\, dx$ (iv) $\int \sin^4 x \cos^5 x \, dx$

(v) $\int \frac{\sin^3 x}{\sqrt{\cos x}} dx$ (vi) $\int \cos^5 x \sin^7 x \, dx$

(vii) $\int \sin^3 x \cos^3 x \, dx$ (viii) $\int \sin 2x \cos 4x \; dx$

(ix) $\int \cos 3x \cos 5x \, dx$ (x) $\int \sin 3x \cos 7x \, dx$

(xi) $\int \tan^2 x \, dx$ (xii) $\int \cot^4 x \, dx$

(xiii) $\int \tan^7 x \, dx$ (xiv) $\int \sec^4 2x \, dx$

(xv) $\int \tan^5 3x \sec^3 3x \, dx$ (xvi) $\int \text{cosec}^4 3x \, dx$

(xvii) $\int \sec^4 x \sqrt{\tan x}\, dx$ (xviii) $\int \cot 2x\, \text{cosec}^4 2x \, dx$

(xix) $\int \frac{\text{cosec}^4 x}{\sqrt{\cot x}} dx$ (xx) $\int \sqrt{1+\cos x}\, dx$

6.3.3 Apply method of substitution to evaluate integrals of the following types:

$\int \frac{dx}{a^2-x^2}$	$\int \sqrt{a^2-x^2}\, dx$	$\int \frac{dx}{\sqrt{a^2-x^2}}$
$\int \frac{dx}{a^2+x^2}$	$\int \sqrt{a^2+x^2}\, dx$	$\int \frac{dx}{\sqrt{a^2+x^2}}$
$\int \frac{dx}{ax^2+bx+c}$	$\int \frac{dx}{\sqrt{ax^2+bx+c}}$	$\int \frac{px+q}{ax^2+bx+c} dx$
$\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$		

The above types of integrals can be solved easily by using trigonometric substitutions.

If the integrand contains expression of the form $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2}$; we use the following trigonometric substitutions:

S.No.	Integrand	Substitution
1.	$\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
2.	$\sqrt{a^2+x^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
3.	$\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$

If the integral involves a quadratic expression in the denominator or under a radical, we first make the expression perfect square by using $a^2 \pm 2ab + b^2 = (a \pm b)^2$, then use suitable trigonometric substitution.

Example 1. Evaluate $\int \frac{dx}{4-x^2}$

(i) $\int \frac{dx}{4-x^2} = \int \frac{dx}{2^2-x^2}$

Substituting $x = 2\sin\theta$

$\Rightarrow \quad dx = 2\cos\theta\, d\theta$

$$\therefore \int \frac{dx}{4-x^2} = \int \frac{2\cos\theta\, d\theta}{4-4\sin^2\theta}$$

$$= \int \frac{2\cos\theta\, d\theta}{4(1-\sin^2\theta)}$$

$$= \frac{1}{2}\int \frac{\cos\theta\, d\theta}{\cos^2\theta}$$

$$= \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C \qquad \ldots(i)$$

Now,

$x = 2\sin\theta$

$\therefore \quad \sin\theta = \frac{x}{2}$

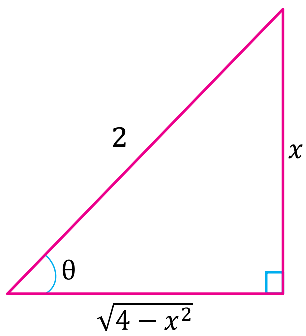

Fig. 6.2

From the figure 6.2, we have

$\sec\theta = \frac{2}{\sqrt{4-x^2}}$

$\boxed{\tan\theta = \frac{x}{\sqrt{4-x^2}}}$

By putting the values in equation (i)

$$\int \frac{dx}{4-x^2} = \frac{1}{2}\ln\left|\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{2+x}{\sqrt{4-x^2}}\right| + c$$

Thus, $\int \frac{dx}{4-x^2} = \frac{1}{2}\ln\left|\sqrt{\frac{2+x}{2-x}}\right| + C$

$$\int \frac{dx}{4-x^2} = \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + C$$

(ii) $\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{dx}{\sqrt{3^2+x^2}}$

Substituting $x = 3\tan\theta$

$\Rightarrow \quad dx = 3\sec^2\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9+9\tan^2\theta}}$$

$$= \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9(1+\tan^2\theta)}}$$

$$= \int \frac{3\sec^2\theta\, d\theta}{3\sec\theta}$$

As $x = 3\tan\theta$ $\quad = \int \sec\theta\, d\theta$

$$= \ln|\sec\theta + \tan\theta| + C \quad \ldots\text{(i)}$$

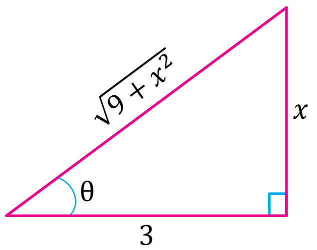

Fig. 6.3

$\therefore \tan\theta = \frac{x}{3}$

From figure 6.3, we have

$$\sec\theta = \frac{\sqrt{9+x^2}}{3}$$

Thus, $\int \frac{dx}{\sqrt{9+x^2}} = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + c_1$

$$= \ln\left|\frac{x+\sqrt{9+x^2}}{3}\right| + c_1$$

$$= \ln\left|x+\sqrt{9+x^2}\right| + (c_1 - \ln 3)$$

$$= \ln\left|x+\sqrt{9+x^2}\right| + C, \quad \text{where } C = c_1 - \ln 3$$

$$= \ln\left|x+\sqrt{9+x^2}\right| + C$$

(iii) $\int \frac{\sqrt{x^2-25}}{x}dx = \int \frac{\sqrt{x^2-5^2}}{x}dx$

Substituting $\quad x = 5\sec\theta$

$\Rightarrow \quad dx = 5\sec\theta\tan\theta\, d\theta$

$$\therefore \quad \int \frac{\sqrt{x^2-25}}{x}dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}(5\sec\theta\tan\theta\, d\theta)$$

$$= \int \sqrt{25(\sec^2\theta - 1)}\tan\theta\, d\theta$$

$$= \int 5\tan\theta\,(\tan\theta) d\theta$$

$$= \int 5\tan^2\theta\, d\theta$$

$$= 5\int (\sec^2\theta - 1)\, d\theta \qquad \ldots(i)$$

From the Fig. 6.4, as $x = 5\sec\theta = 5(\tan\theta - \theta) + C$

$\therefore \quad \sec\theta = \frac{x}{5} \quad \Rightarrow \quad \theta = \sec^{-1}\frac{x}{5}$

$\therefore \quad \tan\theta = \frac{\sqrt{x^2-25}}{5}$

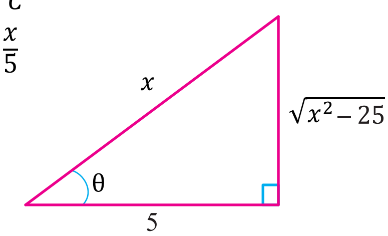

Fig. 6.4

From equation (i)

Thus, $\quad \int \frac{\sqrt{x^2-25}}{5}dx = 5\left(\frac{\sqrt{x^2-25}}{5} - \sec^{-1}\frac{x}{5}\right) + C$

$$= \sqrt{x^2-25} - 5\sec^{-1}\frac{x}{5} + C$$

(iv) $\int \frac{dx}{x^2-4x+8}$

The expression in the denominator is quadratic, so before trigonometric substitution we make it completely square.

i.e., $\quad x^2 - 4x + 8 = (x)^2 - 2(x)(2) + (2)^2 + 4$

$$= (x-2)^2 + 4$$

$$= (x-2)^2 + 2^2$$

$$\therefore \quad \int \frac{dx}{x^2-4x+8} = \int \frac{dx}{(x-2)^2 + (2)^2}$$

Now, substituting $\quad x - 2 = 2\tan\theta$

$\Rightarrow \quad dx = 2\sec^2\theta\, d\theta$

$$\therefore \int \frac{dx}{x^2-4x+8} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta + 4}$$

$$= \int \frac{2\sec^2\theta \, d\theta}{4(\tan^2\theta + 1)}$$

$$= \int \frac{2\sec^2\theta \, d\theta}{4\sec^2\theta}$$

$$= \frac{1}{2}\int d\theta$$

As $x - 2 = 2\tan\theta$ $\quad = \frac{1}{2}\theta + C$

$\therefore \quad \theta = \tan^{-1}\left(\frac{x-2}{2}\right)$

Thus $\quad \int \frac{dx}{x^2-4x+8} = \frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + C$

(v) $\quad \int \frac{(2x+5)dx}{x^2+2x+5} = \int \frac{2x+2+3}{x^2+2x+5}dx$

$$= \int \frac{2x+2}{x^2+2x+5}dx + \int \frac{3}{x^2+2x+5}dx$$

$$= \ln|x^2+2x+5| + 3\int \frac{dx}{x^2+2x+1+4}$$

$$\int \frac{(2x+5)}{x^2+2x+5}dx = \ln|x^2+2x+5| + 3\int \frac{dx}{(x+1)^2+(2)^2} \quad \ldots\text{(i)}$$

Let us find $\int \frac{dx}{(x+1)^2+2^2}$ by trigonometric substituting, we have

$x + 1 = 2\tan\theta$

$\Rightarrow \quad dx = 2\sec^2\theta \, d\theta$

$$\therefore \int \frac{dx}{(x+1)^2+2^2} = \int \frac{2\sec^2\theta \, d\theta}{4\tan^2\theta + 4}$$

$$= \int \frac{2\sec^2\theta \, d\theta}{4(\tan^2\theta + 1)}$$

$$= \int \frac{2\sec^2\theta \, d\theta}{4\sec^2\theta}$$

$$= \frac{1}{2}\int d\theta$$

$$= \frac{1}{2}\theta + c$$

As $x + 1 = 2\tan\theta \quad \Rightarrow \quad \tan\theta = \frac{x+1}{2} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{x+1}{2}\right)$

$$\therefore \int \frac{dx}{(x+1)^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

Now, equation (i), becomes

$$\int \frac{(2x+5)dx}{x^2+2x+5} = \ln|x^2+2x+5| + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

Exercise 6.3

Evaluate by using trigonometric substitution.

1. $\int \frac{x^3 dx}{\sqrt{9-x^2}}$
2. $\int \frac{6dx}{9-x^2}$
3. $\int x^2\sqrt{9-x^2}\,dx$
4. $\int \frac{5dx}{25x^2+9}$
5. $\int \frac{dx}{(4+x^2)^{\frac{3}{2}}}$
6. $\int \frac{dx}{\sqrt{16+4x^2}}$
7. $\int x^3\sqrt{9x^2-36}$
8. $\int \frac{dx}{\sqrt{a^2+x^2}}$
9. $\int \frac{dx}{(16-x^2)^{\frac{5}{2}}}$
10. $\int \frac{x^5 dx}{\sqrt{x^2-9}}$
11. $\int \frac{dx}{x^2+4x+5}$
12. $\int \frac{dx}{\sqrt{5+4x-x^2}}$
13. $\int \frac{dx}{\sqrt{9x-x^2}}$
14. $\int \frac{dx}{(x+1)\sqrt{x^2+2x-15}}$
15. $\int \frac{dx}{(x-4)\sqrt{x^2-8x-9}}$
16. $\int \frac{(2x-5)dx}{\sqrt{8x-x^2}}$
17. $\int \frac{(x+3)dx}{x^2+2x+5}$
18. $\int \frac{(3x+9)dx}{x^2+4x+4}$
19. $\int \frac{(4x+9)dx}{\sqrt{2x^2+8x-10}}$
20. $\int \frac{(2x-5)dx}{\sqrt{5+4x-x^2}}$

6.4 Integration by Parts

6.4.1 Recognize the formula for integration by Parts

The method of integration by parts is used to integrate the product of two functions. Suppose that $f(x)$ and $g(x)$ are two functions and $f'(x)$ and $g'(x)$ are their derivatives respectively which exist in the domain of $f(x)$ and $g(x)$.

According to product rule of differentiation

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

Integrating both sides w.r.t x, we get

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int \left[f(x)\frac{d}{dx}g(x)\right]dx + \int \left[g(x)\frac{d}{dx}f(x)\right]dx$$

$$f(x)g(x) = \int \left[f(x)\frac{d}{dx}g(x)\right]dx + \int \left[g(x)\frac{d}{dx}f(x)\right]dx$$

$$\int \left[f(x)\frac{d}{dx}g(x)\right]dx = f(x)g(x) - \int \left[g(x)\frac{d}{dx}f(x)\right]dx$$

Suppose $u = f(x)$

$v = \frac{d}{dx}g(x) \quad \Rightarrow \quad \int v dx = g(x)$

$$\boxed{\int uv\, dx = u\int vdx - \int (u'\int v\, dx)\, dx}$$

This result is called the formula for integration by parts. It can be stated as integral of the product of two functions equals first function same into integral of second function minus integral of product of derivative of 1st function and integral of 2nd function.

6.4.2 Apply method of integration by parts to evaluate integrals of the following types

- $\int \sqrt{a^2 - x^2}\, dx$, $\int \sqrt{a^2 + x^2}\, dx$, $\int \sqrt{x^2 - a^2}\, dx$

Example 1. Evaluate

(i) $\int \sqrt{a^2 - x^2}\, dx$

Let $\quad \mathrm{I} = \int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - x^2}\,(1)dx$

Integration by parts

$$\mathrm{I} = \sqrt{a^2 - x^2}\int (1)\, dx - \int \left[\frac{d}{dx}(a^2 - x^2)^{\frac{1}{2}} \int (1)\, dx\right] dx$$

$$\mathrm{I} = \sqrt{a^2 - x^2}(x) - \int \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)(x)dx$$

$$\mathrm{I} = x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}}dx$$

$$\mathrm{I} = x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}}dx$$

$$\mathrm{I} = x\sqrt{a^2 - x^2} - \int \left(\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} - \frac{a^2}{\sqrt{a^2 - x^2}}\right)dx$$

$$\mathrm{I} = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$\mathrm{I} = x\sqrt{a^2 - x^2} - \mathrm{I} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + C_1$$

$$2\text{I} = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + C_1$$

$$\text{I} = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\therefore \quad \int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + C$$

Example 2. Evaluate $\int \sqrt{a^2 + x^2}\, dx$.

Solution: Let $\text{I} = \int \sqrt{a^2 + x^2}\, dx$

Integration by parts

$$\text{I} = \sqrt{a^2 + x^2} \int (1) dx - \int \left[\frac{d}{dx}\sqrt{a^2 + x^2} \int (1) dx\right] dx$$

$$\text{I} = x\sqrt{a^2 + x^2} - \int \frac{2x^2}{2\sqrt{a^2 + x^2}}\, dx$$

$$\text{I} = x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}} dx$$

$$\text{I} = x\sqrt{a^2 + x^2} - \int \frac{x^2 + a^2 - a^2}{\sqrt{a^2 + x^2}} dx$$

$$\text{I} = x\sqrt{a^2 + x^2} - \int \frac{x^2 + a^2}{\sqrt{a^2 + x^2}} + \int \frac{a^2}{\sqrt{a^2 + x^2}} dx$$

$$\text{I} = x\sqrt{a^2 + x^2} - \int \sqrt{x^2 + a^2} + a^2 \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

$$2\text{I} = x\sqrt{a^2 + x^2} + a^2 \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

$$\text{I} = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

Example 3. Evaluate $\int \sqrt{x^2 - a^2}\, dx$.

Solution: Let $\text{I} = \int \sqrt{x^2 - a^2}\, dx$

Integration by parts

$$\text{I} = \sqrt{x^2 - a^2} \int (1) dx - \int \left[\frac{d}{dx}\sqrt{x^2 - a^2} \int (1) dx\right] dx$$

$$\text{I} = \sqrt{x^2 - a^2}(x) - \int \frac{2x^2}{2\sqrt{x^2 - a^2}}\, dx$$

$$\text{I} = x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} dx$$

$$\text{I} = x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} dx$$

$$I = x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{x^2 - a^2}} - \int \frac{a^2}{\sqrt{x^2 - a^2}} dx$$

$$I = x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} dx - a^2 \ln\left(x + \sqrt{x^2 - a^2}\right) + C$$

$$2I = x\sqrt{x^2 - a^2} - a^2 \ln\left(x + \sqrt{x^2 - a^2}\right) + C$$

$$I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln\left(x + \sqrt{x^2 - a^2}\right) + C$$

6.4.3 Evaluate integrals using integration by parts

Example 1.

(i) $\int x \sin x \, dx$

Solution: Let $u = x$, $v = \sin x$

Integrating by parts, we have

$$\int x \sin x \, dx = x \int \sin x \, dx - \int \left(\frac{d}{dx}(x) \int \sin x\right) dx$$

$$= x \int (-\cos x) - \int (1)\,(-\cos x) dx$$

$$= -x \cos x + \sin x + C$$

or $$= \sin x - x \cos x + C$$

(ii) $\int x^3 \ln x \, dx$

Solution:

Let $u = \ln x$ and $v = x^3$

Integrating by parts, we have

$$\int uv \, dx = u \int v dx - \int \left(u' \int v dx\right) dx$$

$$\therefore \quad \int x^3 \ln x \, dx = \ln x \int x^3 \, dx - \int \left(\frac{d}{dx} \ln x \int x^3\right) dx$$

$$= \frac{x^4}{4} \ln x - \int \frac{1}{x}\left(\frac{x^4}{4}\right) dx$$

$$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{1}{4} x^4 \ln x - \frac{1}{4}\left(\frac{x^4}{4}\right) + c$$

$$= \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + c$$

(iii) $\int x \cot^{-1} x \, dx$

Solution:

Let $u = \cot^{-1} x$ and $v = x$

$$\int x \cot^{-1} x \, dx = \int (u\, v) \, dx$$

Integrating by parts, we get

$$\int x \cot^{-1} x \, dx = u \int v dx - \int \left(\frac{du}{dx} \int v dx \right) dx$$

$$= \cot^{-1} x \int x \, dx - \int \left(\frac{d}{dx} \cot^{-1} x \int x \, dx \right) dx$$

$$= \frac{x^2}{2} \cot^{-1} x - \int \frac{-1}{1 - x^2} \left(\frac{x^2}{2} \right) dx$$

$$= \frac{1}{2} x^2 \cot^{-1} x + \frac{1}{2} \int \frac{1 + x^2 - 1}{1 + x^2} dx$$

$$= \frac{1}{2} x^2 \cot^{-1} x + \frac{1}{2} \int \left(\frac{1 + x^2}{1 + x^2} - \frac{1}{1 + x^2} \right) dx$$

$$= \frac{1}{2} x^2 \cot^{-1} x + \frac{1}{2} \int dx - \frac{1}{2} \int \frac{dx}{1 + x^2}$$

$$\int x \cot^{-1} x \, dx = \frac{1}{2} x^2 \cot^{-1} x + \frac{1}{2} x - \frac{1}{2} \tan^{-1} x + C$$

(iv) $\int e^x \sin x \, dx$

Solution: Let $\mathrm{I} = \int e^x \sin x \, dx$

selecting $u = \sin x$ and $v = e^x$

Integration by parts, we have

$$\int uv \, dx = u \int v \, dx - \int \left(u' \int v \, dx \right) dx$$

$$\mathrm{I} = \sin x \int e^x \, dx - \int \left(\frac{d}{dx} \sin x \int e^x \, dx \right) dx$$

$$\mathrm{I} = e^x \sin x - \int e^x \cos x \, dx$$

Re-integrating by parts, we get

$$\mathrm{I} = e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) \, dx \right)$$

$$\Rightarrow \quad \mathrm{I} = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$\Rightarrow \quad I + I = e^x(\sin x - \cos x) + C_1$

$\Rightarrow \quad 2I = e^x(\sin x - \cos x) + C_1$

$\Rightarrow \quad \int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + C\,, \quad \text{where } C = \frac{C_1}{2}$

Exercise 6.4

1. Integrate by parts the following:

(i) $\int x^2 e^x dx$ (ii) $\int x^3 e^x dx$ (iii) $\int x \cos x\, dx$

(iv) $\int \ln x\, dx$ (v) $\int x^2 \sin x\, dx$ (vi) $\int x\, \text{cosec}^2 x\, dx$

(vii) $\int x \sec^2 x\, dx$ (viii) $\int (\ln x)^2\, dx$

2. Integrate by parts the following:

(i) $\int (x+1)\ln(x+1)\, dx$ (ii) $\int x \ln x\, dx$

(iii) $\int x^{-3} \ln x\, dx$ (iv) $\int x^{-4} \ln x^2\, dx$

(v) $\int \sin x \cos x \ln(\sin x)\, dx$ (vi) $\int \frac{\tan x}{\cos^2 x}\ln(\tan x)\, dx$

(vii) $\int \cot x\, \text{cosec}^2 x \ln(\cot x)\, dx$ (viii) $\int \sec^3 x \tan x \ln(\sec x)\, dx$

(ix) $\int \text{cosec}\, x \cot x \ln(\text{cosec}\, x)\, dx$ (x) $\int \frac{\ln x^2}{x^2} dx$

(xi) $\int \sec^3 x\, dx$ (xii) $\int \text{cosec}^3 x\, dx$

3. Integrate by parts the following:

(i) $\int 3x \cos(3x)\, dx$ (ii) $\int x^2 \sin x\, dx$

(iii) $\int \frac{x}{\cot^2 x} dx$ (iv) $\int x \sec^2 x\, dx$

(v) $\int \frac{5x}{\sin^2 2x} dx$ (vi) $\int \cos\sqrt{x}\, dx$

(vii) $\int e^{2x} \sin 2x\, dx$ (viii) $\int e^{-x} \cos 2x\, dx$

(ix) $\int \cos(\ln x)\, dx$ (x) $\int e^{ax} \sin bx\, dx$

4. Integrate by parts the following:

(i) $\int \sin^{-1} 3x\, dx$ (ii) $\int x^4 \tan^{-1} x\, dx$

(iii) $\int \tan^{-1}(2x)\, dx$ (iv) $\int x \cos^{-1} x\, dx$

(v) $\int 3x^2 \sin^{-1}(3x)\, dx$ (vi) $\int 2x \sec^{-1} x\, dx$

(vii) $\int 6x \operatorname{cosec}^{-1}(2x) dx$ (viii) $\int x^2 \cot^{-1} x\, dx$

5. Integrate by parts the following:

(i) $\int \sqrt{9-x^2}\, dx$ (ii) $\int \sqrt{16+4x^2}\, dx$ (iii) $\int \sqrt{x^2-25}\, dx$

6.5 Integration using partial fractions

6.5.1 Use partial fraction to find $\int \frac{f(x)}{g(x)} dx$, where $f(x)$ and $g(x)$ are polynomial functions such that $g(x) \neq 0$.

Example 1. Evaluate by using partial fraction

(i) $\int \frac{2x-5}{x^2-5x+6} dx$

Solution: $\int \frac{2x-5}{x^2-5x+6} dx$

By factorization $= \frac{2x-5}{x^2-5x+6} = \frac{2x-5}{(x-2)(x-3)}$

Let $\frac{2x-5}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$ …(i)

$\Rightarrow \frac{2x-5}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$

$\Rightarrow 2x-5 = A(x-3) + B(x-2)$ …(ii)

As (ii) is an identity, so putting $x = 3$, we get

$\boxed{B = 1}$

Similarly, putting $x = 2$ in (ii), we get

$\boxed{A = 1}$

Identity (i), becomes

$$\frac{2x-5}{(x-2)(x-3)} = \frac{1}{x-2} + \frac{1}{x-3}$$

Integrating on both sides w.r.t x, we get

$$\int \frac{2x-5}{x^2-5x+6} dx = \int \frac{1}{x-2} dx + \int \frac{1}{x-3} dx$$

$= \ln|x-2| + \ln|x-3| + C_1$

$= \ln|x-2| + \ln|x-3| + \ln|C|$ where $C_1 = \ln|C|$

$= \ln|C\,(x-2)(x-3)|$

(ii) $\int \frac{\cos x dx}{\sin x(2+\sin x)}$

Solution: To express the integrand in polynomial.

Suppose $u = \sin x$

$du = \cos x\, dx$

$$\int \frac{\cos x\, dx}{\sin x\,(2+\sin x)} = \int \frac{du}{u(2+u)} \quad \ldots(\text{i})$$

Let

$$\frac{1}{u(2+u)} \equiv \frac{A}{u} + \frac{B}{2+u}$$

$$1 = A(2+u) + Bu$$

$$\Rightarrow \quad A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

$$\frac{1}{u(2+u)} = \frac{1}{2u} - \frac{1}{2(2+u)}$$

By putting in equation (i)

$$\int \frac{\cos x\, dx}{\sin x\,(2+\sin x)} = \int \left\{\frac{1}{2u} - \frac{1}{2(2+u)}\right\} du$$

$$= \frac{1}{2}\ln|u| - \frac{1}{2}\ln|2+u| + \ln|C|$$

$$= \ln\left|u^{\frac{1}{2}}\right| - \ln\left|2+u^{\frac{1}{2}}\right| + \ln|C|$$

$$= \ln\left|C\frac{\sqrt{u}}{\sqrt{2+u}}\right|$$

$$= \ln\left|C\frac{\sqrt{\sin x}}{\sqrt{2+\sin x}}\right|$$

(iii) $\int \frac{(5x^2+1)dx}{(x-1)(x+2)^2} dx$

Solution: Partial fraction

$$\frac{5x^2+1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$5x^2 + 1 \equiv A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

To solve it we get $A = \frac{2}{3}, B = \frac{19}{21}, c = -\frac{1}{7}$

$$\frac{5x^2+1}{(x-1)(x+2)^2} = \frac{2}{3(x-1)} + \frac{19}{20(x+2)} - \frac{1}{7(x+2)^2}$$

Thus,

$$\int \frac{(5x^2+1)dx}{(x-1)(x+2)^2} = \frac{2}{3}\int \frac{dx}{x-1} + \frac{19}{20}\int \frac{dx}{x+2} - \frac{1}{7}\int \frac{dx}{(x+2)^2}$$

$$= \frac{2}{3}\ln|x-1| + \frac{19}{20}\ln|x+2| + \frac{1}{7(x+2)} + c$$

(iv) $\int \frac{x+1}{x(x^2+2)} dx$

Solution: Let $\frac{x+1}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$ ...(i)

or $\frac{x+1}{x(x^2+2)} = \frac{A(x^2+2)+x(Bx+C)}{x(x^2+2)}$

$\Rightarrow$ $x+1 = A(x^2+2) + x(Bx+C)$ ...(ii)

As (ii) is an identity, so putting $x = 0$, we get

$$A = \frac{1}{2}$$

$$B - C = -\frac{3}{2}$$

$$B + C = \frac{1}{2}$$

By solving, we get

$B = -\frac{1}{2}$ and $C = 1$

Now, identity (i), becomes

$$\frac{x+1}{x(x^2+2)} = \frac{\frac{1}{2}}{x} + \frac{-\frac{1}{2}x+1}{x^2+2}$$

Integrating both sides w.r.t x, we get

$$\int \frac{x+1}{x(x^2+2)} dx = \frac{1}{2}\int \frac{1}{x} dx - \frac{1}{2}\int \frac{x}{x^2+2} dx + \int \frac{1}{x^2+2} dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{4}\ln|x^2+2| + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

Exercise 6.5

Evaluate the following integrates by using partial fraction.

1. $\int \frac{(5x-2)dx}{(x-3)(x+7)}$

2. $\int \frac{(7x-25)dx}{(x-3)(x-4)}$

3. $\int \frac{dx}{a^2-x^2}$
4. $\int \frac{dx}{x^2-a^2}$
5. $\int \frac{(x^2+2x+3)dx}{x^3-x}$
6. $\int \frac{5dx}{x^2-2x-15}$
7. $\int \frac{(2x+7)dx}{(x-1)(x-5)(x+3)}$
8. $\int \frac{(5x+6)dx}{(x+3)(x-2)^2}$
9. $\int \frac{(7x^2-2x+5)dx}{(x-6)(x-3)^3}$
10. $\int \frac{(2x+1)dx}{(x-3)(x^2+1)}$
11. $\int \frac{sec^2 x dx}{(1+tan\, x)(2+tan\, x)}$
12. $\int \frac{cosec^2 x dx}{cot\, x(2+cot\, x)}$
13. $\int \frac{(3x+7)dx}{(2x-1)(x-4)^2}$
14. $\int \frac{(7x-4)dx}{(x-3)(x^2+2)}$
15. $\int \frac{(2x^2+5x+1)}{x^2+5x+6} dx$
16. $\int \frac{(x^3+3x+1)dx}{x^2+5x-14}$

6.6 Definite integrals

6.6.1 Define definite integral as the limit of a sum.

Suppose that $f(x)$ is a continuous function on the interval $[a,b]$, divide the interval $[a,b]$ into n infinitesimal sub intervals as

$$a = x_0 \le x_1 \le x_2 \le x, \dots \dots \le x_{n-1} \le x_n = b$$

If Δx be the width of each subinterval, then

$\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, 3, \dots \dots, n$ as $\Delta x_i \to 0$ and $n \to \infty$

Select a point c_i on each interval such that

$$x_{i-1} \le c_i \le x_i$$

The limit of the sum

$$= \lim_{n\to\infty} \{f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + \cdots + f(x_n)\Delta x_n\}$$

By using summation notation it can be written a

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

This summation of $f(x)$ on infinitesimal sub intervals is defined as the definite integral of $f(x)$ from a to b denoted by

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Where a and b are called lower and upper limits of the integral respectively.

6.2.2 Describe fundamental theorem of integral calculus and recognize the following basic properties:

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a < c < b$
- $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{when } f(-x) = f(x) \\ 0 & \text{when } f(-x) = -f(x) \end{cases}$

Fundamental theorem of integral calculus:

If $f(x)$ is a continuous function on $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$

i.e., $\frac{d}{dx}F(x) = f(x)$ then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

is called fundamental theorem of calculus.

Example: $\int_0^3 (x^2 + 5)dx$.

Solution: Let $f(x) = x^2 + 5$ then its antiderivative $F(x) = \frac{x^3}{3} + 5x$

Now, by using fundamental theorem of calculus

$$\int_0^3 (x^2 + 5)\,dx = F(3) - F(0)$$

$$= \frac{(3)^3}{3} + 5(3) - \frac{(0)^3}{3} - 5(0)$$

$$= 9 + 15$$

$$= 24$$

Basic properties of definite integrals

(i) $\int_a^a f(x)\,dx = 0$

(ii) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

(iii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iv) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a < c < b$

(v) $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{when } f(-x) = f(x) \\ 0 & \text{when } f(-x) = -f(x) \end{cases}$

(i) $\int_a^a f(x)\,dx = 0$

Proof: By the fundamental theorem of integral calculus

$$\int_a^a f(x)\,dx = F(a) - F(a)$$

$$\boxed{\int_a^a f(x)\,dx = 0}$$

(ii) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

Proof: By the fundamental theorem of integral calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

and

$$\int_a^b f(y)\,dy = F(b) - F(a)$$

Hence

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$$

(iii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Proof: By the fundamental theorem of integral calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f(x)\,dx = -\{F(a) - F(b)\}$$

Thus,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(iv) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \qquad a < c < b$

Proof: By the fundamental theorem of integral calculus

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a)$$

$$= \int_a^b f(x)\,dx$$

Thus,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(v) $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{when } f(-x) = f(x) \\ 0 & \text{when } f(-x) = -f(x) \end{cases}$

Proof: By using the property (iv), we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\,,\ \text{ where } -a < 0 < a \qquad \ldots\text{(i)}$$

Suppose $\quad x = -t \quad \Rightarrow \quad dx = -dt$

When $\ x = -a \ \Rightarrow t = a$

When $\ x = 0 \ \Rightarrow t = 0$

By substituting in (i), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{a}^{0} f(-t)\,(-dt) + \int_{0}^{a} f(x)dx$$

$$\int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt + \int_{0}^{a} f(x)dx$$

By using property (iii), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)dx \qquad \text{... (ii)}$$

When $f(-x) = f(x)$

From equation (ii)

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)dx$$

$$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$$

When $f(-x) = -f(x)$

From equation (ii), we have

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)dx$$

$$\boxed{\int_{-a}^{a} f(x)\,dx = 0}$$

6.6.3 Extend techniques of integration using properties to evaluate definite integrals

Example 1. Evaluate $\int_2^2 5x^4 dx$.

Solution: As the upper and lower limits are equal, by using property of $\int_a^a f(x)\,dx = 0$.

Hence

$$\int_{2}^{2} 5x^4\,dx = 0$$

Example 2. Shown that $\int_1^2 x^2\,dx = \int_1^2 y^2\,dy$.

Solution:

$$\int_{1}^{2} x^2\,dx = \left[\frac{x^3}{3}\right]^2$$

$$= \frac{1}{3}(8-1)$$

$$= \frac{7}{3}$$

By the property $\int_a^b f(x)\,dy = \int_a^b f(y)\,dy$

Thus,

$$\int_{1}^{2} y^3\,dx = \frac{7}{3}$$

Hence

$$\int_{1}^{2} x^2\,dx = \int_{1}^{2} y^2\,dy$$

Example 3. Verify that $\int_0^{\frac{\pi}{2}} \cos x\,dx = -\int_{\frac{\pi}{2}}^{0} \cos x\,dx$.

Solution:

$$\int_{0}^{\frac{\pi}{2}} \cos x\,dx = [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$= 1$$

Now $-\int_{\frac{\pi}{2}}^{0} \cos x\,dx = -[\sin x]_{\frac{\pi}{2}}^{0}$

$$= -\left(\sin 0 - \sin\frac{\pi}{2}\right)$$

$$= 1$$

Hence

$$\boxed{\int_{0}^{\frac{\pi}{2}} \cos x\,dx = -\int_{\frac{\pi}{2}}^{0} \cos x\,dx}$$ verified.

Example 4. Given that $\int_0^1 f(x)\,dx = 5$ and $\int_1^4 f(x)\,dx = 3$, then evaluate $\int_4^0 f(x)\,dx$.

Solution:

As $\int_4^0 f(x)\,dx = -\int_0^4 f(x)\,dx$ (By using property iii)

$= -\left[\int_0^1 f(x)\,dx + \int_1^4 f(x)\,dx\right]$ (By using property iv)

$= -(5+3)$

$$\int_4^0 f(x)\,dx = -8$$

Example 5. Evaluate $\int_{-2}^{1} |x|\,dx$

Solution: By property (iv), we have

$$\int_{-2}^{1} |x|\,dx = \int_{-2}^{0} |x|\,dx + \int_{0}^{1} |x|\,dx \quad \because \begin{cases} |x| = +x, x > 0 \\ |x| = -x, x < 0 \end{cases}$$

$$= \int_{-2}^{0} (-x)\,dx + \int_0^1 x\,dx$$

$$= -\left[\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_0^1$$

$$= -\frac{1}{2}[0-4] + \frac{1}{2}[1-0]$$

$$= 2 + \frac{1}{2} = \frac{5}{2}$$

Example 6. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$

Solution: Here $f(x) = \cos x$

$f(-x) = \cos(-x)$

$f(-x) = \cos x$

$f(-x) = f(x)$

By using property $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = 2\int_0^{\frac{\pi}{2}} \cos x\,dx$$

$$= 2[\sin x]_0^{\frac{\pi}{2}}$$
$$= 2\left(\sin\frac{\pi}{2} - \sin 0\right)$$
$$= 2(1-0)$$
$$= 2$$

Example 7. Evaluate $\int_{-1}^{1} \sin^{-1} x\, dx$

Solution: Here $f(x) = \sin^{-1} x$

$$f(-x) = \sin^{-1}(-x)$$
$$f(-x) = -\sin^{-1} x$$
$$f(-x) = -f(x)$$

$\therefore$ f is an odd function.

Furthermore, limits are additive inverses of each other.

$\therefore$ By property (v), we have

$$\int_{-1}^{1} \sin^{-1} x\, dx = 0$$

Example 8. Evaluate $\int_{-6}^{-2\sqrt{3}} \frac{dx}{x\sqrt{x^2-9}}$ by using trigonometric substitution.

Solution: Let $x = 3\sec\theta \implies dx = 3\sec\theta\tan\theta\ d\theta$

When $x = -2\sqrt{3} \implies -2\sqrt{3} = 3\sec\theta \implies \sec\theta = -\frac{2}{\sqrt{3}} \implies \theta = \frac{5\pi}{6}$

When $x = -6 \implies -6 = 3\sec\theta \implies \sec\theta = -2 \implies \theta = \frac{2\pi}{3}$

$$\therefore \int_{-6}^{-2\sqrt{3}} \frac{dx}{x\sqrt{x^2-9}} = \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \frac{3\sec\theta\tan\theta\ d\theta}{3\sec\theta\sqrt{(3\sec\theta)^2-9}}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \frac{\tan\theta\ d\theta}{\sqrt{9(\sec^2\theta-1)}}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \frac{\tan\theta\ d\theta}{3\sqrt{\tan^2\theta}} \qquad \left(\because \sqrt{x^2} = |x|\right)$$

$$\therefore \qquad = \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \frac{\tan\theta \; d\theta}{3|\tan\theta|}$$

As $\theta \in \left[\frac{2\pi}{3}, \frac{5\pi}{6}\right] \quad \Rightarrow \tan\theta < 0 \quad \Rightarrow \quad |\tan\theta| = -\tan\theta$

$$\therefore \qquad = \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \frac{\tan\theta \; d\theta}{3(-\tan\theta)}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} -\frac{1}{3} d\theta = \left[-\frac{1}{3}\theta\right]_{\frac{2\pi}{3}}^{\frac{5\pi}{6}}$$

$$= -\frac{1}{3}\left[\frac{5\pi}{6} - \frac{2\pi}{3}\right]$$

$$= -\frac{\pi}{18}$$

Example 9. Evaluate $\int_0^{\frac{\pi}{4}} x \sin x \, dx$.

Solution: Integrating by parts

$$\int_0^{\frac{\pi}{4}} x \sin x \, dx = \left[x \int \sin x \, dx\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \left(\frac{dx}{dx} \int \sin x \, dx\right)$$

$$\int_0^{\frac{\pi}{4}} x \sin x \, dx = [-x\cos x]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \cos x \, dx$$

$$= -\left[\frac{\pi}{4}\cos\frac{\pi}{4} - 0\right] + \left[\sin\frac{\pi}{4} - \sin 0\right]$$

$$= -\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - 0\right)$$

$$= \frac{-\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{-\pi + 4}{4\sqrt{2}}$$

$$= \frac{\sqrt{2}}{8}(4 - \pi)$$

Exercise 6.6

1. Evaluate the following definite integrals.

(i) $\int_0^2 \left(4x^3 + 3x^2 + 5\right) dx$ (ii) $\int_{-1}^2 (x^2 + 1)^2 \, dx$

(iii) $\int_1^2 \left(\theta + \sqrt{\theta}\right)^3 d\theta$ (iv) $\int_1^3 \left(y + \frac{1}{\sqrt{y}}\right)^2 dy$

(v) $\int_0^4 \frac{dx}{\sqrt{2+x}+\sqrt{x}}$

2. Evaluate the following definite integrals or by formula.

(i) $\int_1^4 x\left(x^2 + 9\right)^{\frac{3}{2}} dx$ (ii) $\int_2^5 \frac{x\,dx}{7x^2+2}$

(iii) $\int_0^3 \frac{(2x+3)dx}{\sqrt{2x^2+6x+5}}$ (iv) $\int_1^2 (x^3 + 2x)^{-\frac{1}{2}} \left(3x^2 + 2\right)dx$

(v) $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$ (vi) $\int_{\frac{\pi^2}{4}}^{\frac{\pi^2}{36}} \frac{\sin\sqrt{x}}{\sqrt{x}} \, dx$

(vii) $\int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sec^2 x \, dx$ (viii) $\int_0^2 e^{5x-2} \, dx$

(ix) $\int_0^{\frac{\pi}{3}} \cos^3 3x \sin^2 3x \, dx$ (x) $\int_0^{\frac{\pi}{3}} \tan^2 x \sec^4 x \, dx$

3. Compute the following definite integrals by using basic properties.

(i) $\int_2^2 (x^4 + 2x + 3)^{\frac{5}{2}} \left(2x^3 + 1\right)dx$

(ii) $\int_{-50}^{50} (10x^9 - 8x^7 + 6x^5 - 4x^3 + 2x) \, dx$

(iii) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^9 x \cos^6 x \, dx$ (iv) $\int_{-\pi}^{\pi} \sec^8 x \tan x \, dx$

(v) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx$ (vi) $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \, dx$

(vii) $\int_{-2}^2 (x^4 + 2x^2) \, dx$

4. Given that $\frac{d}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right) = \frac{1}{(1+x^2)^{\frac{3}{2}}}$ then evaluate $\int_0^3 \frac{1}{(1+x^2)^{\frac{3}{2}}} dx$

5. Given that $\frac{d}{dx}\left[F(x)\right] = \frac{2+x^2}{1+x^2}$, evaluate $F(\sqrt{3}) - F(1)$. If $F(1) = \pi$, find $F(x)$.

6. Given that $\int_{-2}^{3} f(x)\,dx = 4$ and $\int_{5}^{3} f(x)\,dx = 7$ then evaluate by using suitable properties.

(i) $\int_{3}^{-2} f(x)\,dx$ (ii) $\int_{-2}^{3} f(y)\,dy$ (iii) $\int_{3}^{5} f(y)\,dy$

(iv) $\int_{2}^{2} f(x)\,dx$ (v) $\int_{-2}^{5} f(x)\,dx$ (vi) $\int_{5}^{-2} f(y)\,dy$

7. Evaluate the following integrals by using trigonometric substitutions.

(i) $\int_0^2 \frac{dx}{\sqrt{16-x^2}}$ (ii) $\int_{-1}^{1} \frac{dx}{4-x^2}$

(iii) $\int_3^{2\sqrt{3}} \frac{x^3\,dx}{\sqrt{x^2+4}}$ (iv) $\int_0^{\sqrt{3}} x^2\sqrt{3-x^2}\,dx$

8. Compute the definite integrals by using integration by parts.

(i) $\int_5^9 xe^{4x}\,dx$ (ii) $\int_1^4 x^2 \ln x\,dx$

(iii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\sin 2x\,dx$ (iv) $\int_0^1 \tan^{-1} x\,dx$

6.6.4 Represent definite integral as the area under the curve

Let $y = f(x)$ is the equation of the curve as shown in the figure 6.5. Suppose $x = a$ and $x = b$ be two vertical lines on x-axis. To determine the area under the curve and above the x-axis between $x = a$ and $x = b$, we divide the region into n small rectangular strips each of with Δx.

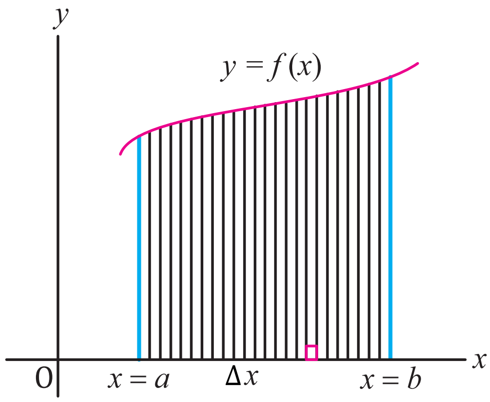

Fig. 6.5

The area of a small rectangular strip of width Δx

$$\Delta A = f(x)\Delta x$$

The total area A bounded by the curve, above x-axis will be equal to the sum of the areas of each rectangular strip from $x = a$ to $x = b$.

$$A = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$

If $\Delta x \to 0$ and $n \to \infty$ by using summation notation.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \text{ where } i = 1, 2, 3, \ldots, n$$

By definition of definite integral it can be written as

$$A = \int_a^b f(x)\,dx$$

or
$$\boxed{A = \int_a^b y\,dx}$$

Thus, the definite integral represents the area under the curve.

Notes: If $y = f(x) \geq 0$, then A is above the x-axis.
If $y = f(x) \leq 0$, then A is below the x-axis.

6.6.5 Apply definite integrals to calculate area under the curve

Example 1. Find the area, above the x-axis under the following curves between the given ordinates:

(i) $y = x^2 + 1\ ,\ x = 2\ ,\ x = 4$

Solution: By using formula

$$A = \int_a^b y\,dx$$

$$A = \int_2^4 (x^2 + 1)\,dx$$

$$A = \left[\frac{x^3}{3} + x\right]_2^4$$

$$A = \frac{1}{3}(64 - 8) + (4 - 2)$$

$$A = \frac{56}{3} + 2$$

$$A = \frac{62}{3}$$

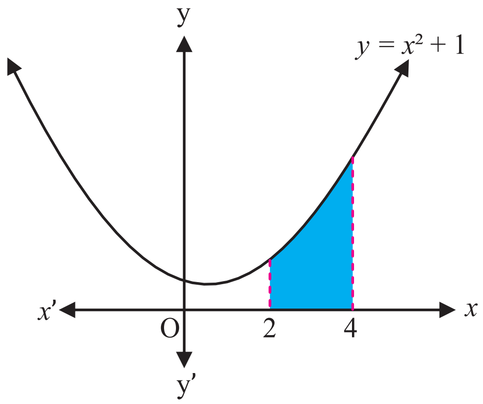

Fig. 6.6

(ii) $y = \cos 3x\ ,\ x = 0\ ,\ x = \frac{\pi}{6}$

Solution: By using formula

$$A = \int_a^b y\,dx$$

$$A = \int_0^{\frac{\pi}{6}} \cos 3x\,dx$$

$$A = \left[\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{6}}$$

$$A = \frac{1}{3}\left[\sin 3\left(\frac{\pi}{6}\right) - \sin 0\right]$$

$$A = \frac{1}{3}\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$A = \frac{1}{3}(1 - 0)$$

$$A = \frac{1}{3}$$

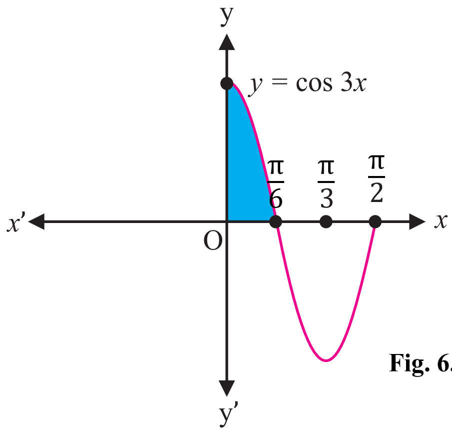

Fig. 6.7`

(iii) $x^2 + y^2 = 16$, $x = 1$, $x = 3$

Solution:

$$x^2 + y^2 = 16$$
$$y^2 = 16 - x^2$$
$$y = \pm\sqrt{16 - x^2}$$

We need area above the x-axis, so we take positive branch of the relation $y = \pm\sqrt{16 - x^2}$

i.e., $y = \sqrt{16 - x^2}$

By using formula

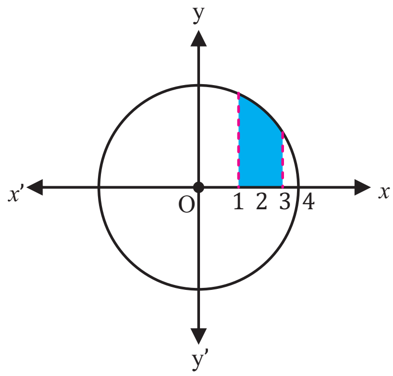

Fig. 6.8

$$A = \int_a^b y\,dx$$

$$A = \int_1^3 \sqrt{16 - x^2}\,dx$$

$$A = \frac{1}{2}\left[x\sqrt{16 - x^2}\right]_1^3 + \left[\frac{1}{2}(16)\sin^{-1}\left(\frac{x}{4}\right)\right]_1^3$$

$$A = \frac{1}{2}\left[3\sqrt{16 - 9} - \sqrt{16 - 1}\right]$$
$$+ 8\left[\sin^{-1}\left(\frac{3}{4}\right) - \sin^{-1}\left(\frac{1}{4}\right)\right]$$

$$A = \frac{1}{2}\left[3\sqrt{7} - \sqrt{15}\right] + 8(0.848 - 0.252)$$

$$A = 2.03 + 4.76$$

$A = 6.8$ approx.

6.6.6 Use MAPLE command int to evaluate definite and indefinite integrals.

The format of int command in MAPLE is as under:

```
> Int(f,x=a..b)
```

where,

f is the function whose definite integral is required x = a..b is the definite integral with lower limit ‘a’ and upper limit ‘b’

In order to compute the integral of a function under definite interval, following examples are given:

$> int(x^3 + 1, x = 1..2)$

$$\frac{19}{4}$$

$> int(x^3 + 1, x = 1..3)$

$$22$$

$> int(x^3 + x^2 + 3x + 4x, x = 1..2)$

$$\frac{175}{12}$$

$> int(e^{3x+1}, x)$

$$\frac{1}{3}e^{3x+1}$$

$> int(e^{3x+1}, x = 0..2)$

$$-\frac{1}{3}e + \frac{1}{3}e^7$$

$> int\left(\sin(x), x = 0..\frac{\pi}{2}\right)$

$$1$$

$> int(\sin(x), x = 0..\pi)$

$$2$$

$> int\left(\cos(x), x = -\frac{\pi}{2}..\frac{\pi}{2}\right)$

$$2$$

$> int(\ln(x + 1), x = 0..1)$

$$-1 + 2\ln(2)$$

$> int\left(\frac{1}{\sqrt{1 - x^2}}, x\right)$

$$\arcsin(x)$$

$> int(e^{\cos x}\sin x, x)$

$$\frac{(\cos x - 1)e^{\cos x}\sin}{\cos^2}$$

Exercise 6.7

Find the area, above the x-axis under the following curves, between the given ordinates.

1.	$y = 3x^2 + 2$	$x = 1\ ,\ x = 2$
2.	$y = \frac{1}{\sqrt{4-x^2}}$	$x = \frac{1}{2}\ ,\ x = \frac{\sqrt{3}}{2}$
3.	$y = \ln x$	$x = 1\ ,\ x = 3$
4.	$y = x\sin x$	$x = \frac{\pi}{3}\ ,\ x = \frac{\pi}{2}$
5.	$y = \frac{1}{9+x^2}$	$x = -\sqrt{3}\ ,\ x = \sqrt{3}$
6.	$y = 4x^3 + 3x^2 + 2x + 1$	$x = 0\ ,\ x = 2$
7.	$y = 3\sec^2 x$	$x = \frac{\pi}{6}\ ,\ x = \frac{\pi}{3}$
8.	$y = 6\sin^2 x$	$x = 0\ ,\ x = \frac{\pi}{3}$
9.	$y = 5e^{5x}$	$x = -2\ , x = 3$
10.	$y = \cos^4 x$	$x = 0\ ,\ x = \frac{\pi}{2}$
11.	$y = \frac{4}{\sin^2 x}$	$x = \frac{\pi}{6},\ x = \frac{\pi}{4}$

12. $x^2+y^2=36$ $\quad x=-1\ ,\ x=1$

13. $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ $\quad x=-1\ ,\ x=1$

14. $y^2=2x+5$ $\quad x=1\ ,\ x=2$

15. $y^2=\tan^4 x$ $\quad x=\dfrac{\pi}{6}\ ,\ x=\dfrac{\pi}{4}$

16. Write MAPLE Command to find integration of the following functions:

(i) $f(x)=e^{2x}$ (ii) $f(x)=\sin x$

(iii) $f(x)=\cos 2x$ (iv) (iv) $f(x)=\ln(1+x)$ (v) $f(x)=\dfrac{1}{x}$

Review Exercise 6

1. Choose the correct option.

(i) $\int f^n(x)f'(x)dx$, where $n\neq -1$, is

(a) $\dfrac{f^{n+1}(x)}{n+1}$ (b) $\dfrac{f^{n-1}(x)}{n-1}+c$

(c) $nf^{n-1}(x)+c$ (d) $\dfrac{f^{n+1}(x)}{n+1}+c$

(ii) $\int f^n(x)f'(x)dx$, where $n=-1$, is

(a) $\dfrac{f^{n+1}(x)}{n+1}+c$ (b) $\ln|f^n(x)|+c$

(c) $\ln|f(x)|+c$ (d) $nf^{n-1}(x)+c$

(iii) $\int x^n dx$, where $n=-1$ is

(a) $\dfrac{x^{n+1}}{n+1}+c$ (b) $nx^{n-1}+c$

(c) $\dfrac{x^{n-1}}{n-1}-c$ (d) $\ln x+c$

(iv) $\int \sin x\cos x\,dx=$

(a) $\sin x+c$ (b) $\cos x+c$

(c) $\dfrac{1}{4}\cos 2x+c$ (d) $-\dfrac{1}{4}\cos 2x+c$

(v) $\int x^2\ln e^{x^2}dx=$

(a) $\dfrac{x^2}{4}+c$ (b) $\dfrac{x^5}{5}+c$

(c) $\ln e^{x^2}+c$ (d) $\ln x^x+c$

(vi) $\int e^{\ln x^3}\,dx =$

(a) $e^{x^3} + c$ (b) $\frac{x^3}{3} + c$

(c) $\frac{x^4}{4} + c$ (d) $\ln x^3 + c$

(vii) $\int (1 + \tan^2 x)\,dx =$

(a) $\tan x + c$ (b) $\sin^2 x + c$

(c) $\frac{\tan^2 x}{2} + c$ (d) $\ln \sec x + c$

8. $\int \frac{2e^x}{1+e^x}\,dx =$

(a) $\ln(1 + e^x) + c$ (b) $\ln(1 + e^x)^2 + c$

(c) $(1 + e^x)^{-2} + c$ (d) $\frac{e^x}{2} + c$

9. $\int \frac{e^{x+3\ln x}}{x^3}\,dx =$

(a) $\frac{1}{3}e^{x+3\ln x} + c$ (b) $e^x + c$

(c) $e^{x+3\ln x} + c$ (d) $3\ln x + c$

10. $\int \ln\left(e^x \cdot e^{\sin x}\right)dx =$

(a) $\frac{1}{e^{x+\sin x}} + c$ (b) $\ln \sin x + c$

(c) $\frac{x^2}{2} - \cos x + c$ (d) $x \ln \sin x + c$

11. $\int \frac{\frac{1}{x\sqrt{x^2-1}}}{cosec^{-1} x}\,dx =$

(a) $\ln(\mathrm{cosec}^{-1} x)^{-1} + c$ (b) $(\mathrm{cosec}^{-1} x)^2 + c$

(c) $\mathrm{cosec}^{-1} x + c$ (d) $\ln(\mathrm{cosec}^{-1} x) + c$

12. If $F(x)$ is an antiderivative of $f(x)$ then

$\int_a^b f(x)\,dx =$

(a) $F(a) - F(b)$ (b) $F(b) - F(a)$

(c) $f(b) - f(a)$ (d) $\frac{f(b)}{f(a)}$

13.

$$\int_{-50}^{+50} (x^3 + x)\,dx$$

(a) 0 (b) 1000 (c) 2000 (d) 3000

14.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^9 x \cos^{11} x\,dx$$

(a) 1 (b) 3 (c) 0 (d) $2\int_0^{\frac{\pi}{2}} \sin^9 x \cos^{11} x\,dx$

15.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{10} x \cos^{11} x\,dx$$

(a) 0 (b) 1 (c) 3 (d) $2\int_0^{\frac{\pi}{2}} \sin^{10} x \cos^{11} x\,dx$

16. Area bounded by the curve $y = \ln e^{x^2}$ from $x = -1$ to $x = 1$ is

(a) $\frac{2}{3}$ (b) 1 (c) ln 2 (d) ln 3

17. $\int_a^b f(x)\,dx =$

(a) $-\int_a^b f(x)\,dx$ (b) $\int_b^a f(x)\,dx$

(c) $-\int_b^a f(x)\,dx$ (d) 0

18. $\int_2^2 \left(x^3 + 3x^2 - 5x^{-\frac{1}{2}}\right) dx =$

(a) 0 (b) $12 - 10\sqrt{2}$

(c) $24 - 20\sqrt{2}$ (d) $20\sqrt{2}$

19. $\int_{-2}^{2} (x^5 - x^3 + x)^5 \left(5x^4 - 3x^2 + 1\right) dx =$

(a) $\frac{1}{3}(26)^6$ (b) 0

(c) $2(26)^6$ (d) $\frac{(26)^6}{6}$

20. $\int \frac{dx}{a^2+x^2} =$

(a) $tan^{-1}\frac{x}{a}$ (b) $\frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(c) $\frac{1}{a}\tan^{-1}\frac{x}{a} + c$ (d) $\sin^{-1}\frac{x}{a} + c$

21. Evaluate the following integrals.

(i) $\int 3x^5 dx$ (ii) $\int x \ln x^n \, dx$

(iii) $\int \sec^5 x \, dx$ (iv) $\int \frac{y^2}{\sqrt{1-y^2}} dy$

(v) $\int \sqrt{1 - \sin 2x} \, dx$ (vi) $\int \tan^5 x \sec^{\frac{5}{2}} x \, dx$

(vii) $\int \frac{\cos x dx}{(2+\sin x)(3+\sin x)}$ (viii) $\int x^2 \sin x \, dx$

(ix) $\int \frac{d\theta}{\sqrt{1+\cos\frac{5\theta}{2}}}$ (x) $\int \frac{dx}{x^2-81}$

(xi) $\int \cot^5 x \operatorname{cosec}^{\frac{3}{2}} x \, dx$ (xii) $\int \cos^5 x \sin^5 x \, dx$

(xiii) $\int_0^2 6(x^2 + 3x + 2)^5 (2x + 3) dx$

(xiv) $\int_0^{\frac{\pi}{2}} \cos^3 x \sqrt{\sin x} \, dx$ (xv) $\int_{-a}^{a} \frac{dx}{x\sqrt{x^2-a^2}}$

(xvi) $\int_0^a \frac{dx}{x^2+a^2}$