Plane Analytic Geometry: Straight Line

Unit 7

Analytic geometry utilizes the concepts of algebra to locate the position of a point on the plane using an ordered pair of numbers. It can be understood as a combination of geometry and algebra. In analytic geometry different algebraic equations are used to describe the dimension and position of different geometric figures.

7.1 Division of a Line Segment

7.1.1 Recall distance formula to calculate distance between two points given in Cartesian plane

We know that, the distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ can be found by

$$d = |\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

known as distance formula (Fig. 7.1).

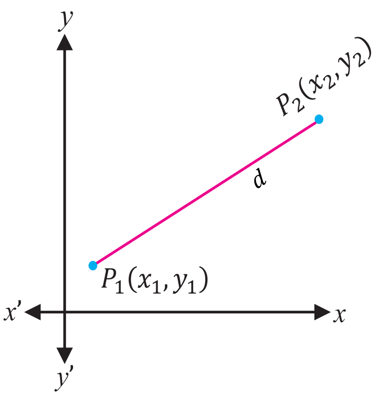

Fig 7.1

Example: Find the distance between two points $P(1, 7)$ and $Q(-2,\ 3)$.

Solution: The distance between the given points P and Q having coordinates $(1, 7)$ and $(-2, 3)$, by using the distance formula;

$$d = |\overline{PQ}| = \sqrt{(-2-1)^2 + (3-7)^2}$$
$$= \sqrt{(-3)^2 + (-4)^2}$$
$$= \sqrt{9 + 16}$$
$$= 5 \text{ units}$$

7.1.2 Recall Mid-point formula

In previous class, we have learnt the mid-point formula to find the mid-point of a line segment when its end points are given. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ are the end points of a line segment then the mid-point $C(x, y)$ of $\overline{AB}$ is found by

$$C(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

As shown in the figure 7.2.

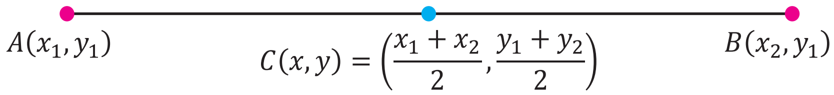

Fig. 7.2

Mid-point $C(x, y)$ divides the $\overline{AB}$ into two equal parts.

For example, the mid-point of line segment $\overline{AB}$ whose endpoints are $(-1, 3)$ and $(5, -3)$ will be

$$\left(\frac{-1+5}{2}, \frac{3+(-3)}{2}\right) = (2, 0)$$

7.1.3 Find coordinates of a point that divides the line segment in given ratio (internally and externally)

Let $\overline{AB}$ is the line segment and A, B and C are the collinear points then C will divide the line segment AB internally or externally in the ratio of m and n.

Let us find the coordinates of division point for both cases.

Internal division

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in the xy-plane. Let $C(x, y)$ be the point which divides line segment AB internally in the ratio $m\!:\!n$. In Fig. 7.3 $\overline{AP}$, $\overline{CN}$ and $\overline{BR}$ are drawn perpendiculars to x-axis. AS and $\overline{CT}$ are drawn parallel to x-axis.

$$m\angle CAS = m\angle BCT \qquad \text{(corresponding angles)}$$

$$m\angle CSA = m\angle BTC = 90°$$

By similarity criterion of triangle

Similarly,

$$\frac{m}{n} = \frac{y - y_1}{y_2 - y} = \frac{x - x_1}{x_2 - x}$$

Take, $$\frac{m}{n} = \frac{y-y_1}{y_2-y}$$

$$\Rightarrow y = \frac{my_2 + ny_1}{m+n}$$

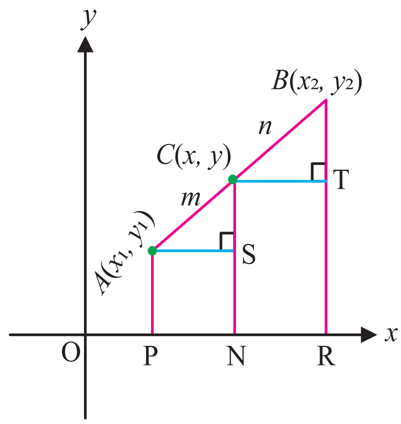

Fig 7.3

$$\Delta CAS \sim \Delta BCT$$

$$\Rightarrow \frac{|\overline{AC}|}{|\overline{CB}|} = \frac{|\overline{AS}|}{|\overline{CT}|} = \frac{|\overline{CS}|}{|\overline{BT}|} = \frac{m}{n} \qquad \text{... (i)}$$

Now,

$$|\overline{AS}| = |\overline{PN}| = |\overline{ON}| - |\overline{OP}| = x - x_1$$

$$|\overline{CT}| = |\overline{NR}| = |\overline{OR}| - |\overline{ON}| = x_2 - x$$

$$|\overline{CS}| = |\overline{CN}| - |\overline{SN}| = y - y_1$$

$$|\overline{BT}| = |\overline{BR}| - |\overline{RT}| = y_2 - y$$

From equation (i),

$$\frac{m}{n} = \frac{y - y_1}{y_2 - y} = \frac{x - x_1}{x_2 - x}$$

Take, $\frac{m}{n} = \frac{x-x_1}{x_2-x}$

$\Rightarrow$ $$x = \frac{mx_2 + nx_1}{m + n}$$

Similarly,

$$\frac{m}{n} = \frac{y - y_1}{y_2 - y} = \frac{x - x_1}{x_2 - x}$$

Take, $\frac{m}{n} = \frac{y-y_1}{y_2-y}$

$\Rightarrow$ $$y = \frac{my_2 + ny_1}{m + n}$$

So, the coordinates of the point $C(x, y)$ which divides the line segment joining points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m:n$ are $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ that is known as section formula.

External Division:

Let $D(x, y)$ divides $\overline{AB}$ externally in the ratio $m:n$, where $m:n = \frac{|\overline{AD}|}{|\overline{BD}|}$. Now draw perpendiculars from $A(x_1, y_1)$, $B(x_2, y_2)$ and $D(x, y)$, along coordinate axes, which meet at $S(x, y_2)$ and $R(x, y_1)$. Therefore, there exist two similar right triangles ADR and BDS, then by similar triangles as shown in Fig. 7.4.

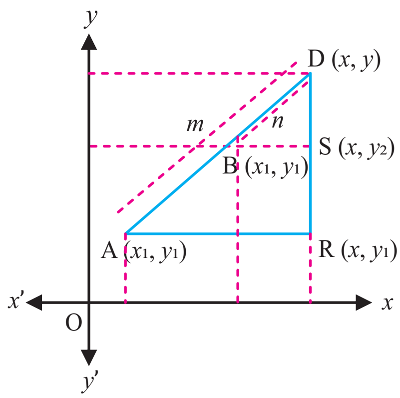

Fig 7.4

Now,

$$\frac{m}{n} = \frac{|\overline{AD}|}{|\overline{BD}|} = \frac{|\overline{AR}|}{|\overline{BS}|} = \frac{x - x_1}{x - x_2}$$

$\therefore$ $$\frac{m}{n} = \frac{x - x_1}{x - x_2}$$

$\Rightarrow$ $$x = \frac{mx_2 - nx_1}{m - n}$$

Similarly,

$$\frac{m}{n} = \frac{|\overline{AD}|}{|\overline{BD}|} = \frac{|\overline{DR}|}{|\overline{DS}|} = \frac{y - y_1}{y - y_2}$$

$\therefore$ $$\frac{m}{n} = \frac{y - y_1}{y - y_2}$$

$\Rightarrow$ $$y = \frac{my_2 - ny_1}{m - n}$$

Hence, coordinate of D which divides externally in the ratio $m_1 : m_2$ are

$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right)$$

Where $m - n \neq 0$.

Note: When point P divides the line segment AB internally, the given ratio $m:n$ will be positive and for external division the ratio will be negative.

Example 1. Find the coordinates of the point, which divides the line segment joining the points $(3,2)$ and $(4,-5)$ internally in the ratio $3:2$.

Solution: Here $(x_1, y_1) = (3,2)$ and $(x_2, y_2) = (4,-5)$. Also $m:n = 3:2$.

By using the section formula;

Point of division is

$$= \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

$$= \left(\frac{3\times 4 + 2\times 3}{3+2}, \frac{3\times -5 + 2\times 3}{3+2}\right)$$

$$= \left(\frac{18}{5}, \frac{-11}{5}\right)$$

Example 2. Find the point of division of the line segment joining $(1,-2)$ to $(-3,4)$ externally in the ratio $3:5$.

Solution: Here $(x_1, y_1) = (1,-2)$ and $(x_2, y_2) = (-3,4)$. Also $m:n = 3:2$

$$P(x,y) = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

$$= \left(\frac{(3)(-3) - 5(-2)}{3-5}, \frac{3(4) - 5(-2)}{3-5}\right)$$

$$= (7, -11)$$

Example 3. If $A(2,4), B(4,5), C(p,q)$ and $D((1,3)$ are the vertices of parallelogram then find the values of p and q.

Solution: We know that the diagonals of a parallelogram bisect each other. Let O be the point at which diagonals intersect. Coordinates of the midpoints (x,y) of both line segments AC and BD will be same. Thus, using midpoint formula;

Point of division

Mid-point of $\overline{BD}$ = mid-point of $\overline{AC}$

$$\left(\frac{4+1}{2}, \frac{5+3}{2}\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\left(\frac{5}{2}, \frac{8}{2}\right) = \left(\frac{2+p}{2}, \frac{4+q}{2}\right)$$

$$\left(\frac{5}{2}, 4\right) = \left(\frac{2+p}{2}, \frac{4+q}{2}\right)$$

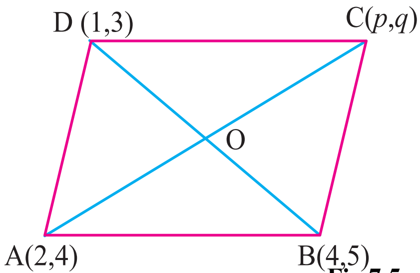

Fig 7.5

Thus,

$$\frac{5}{2} = \frac{2+p}{2}$$

$$\Rightarrow p = 3$$

and

$$4 = \frac{4+q}{2}$$

$\Rightarrow q = 4$

Let us recall the definitions of some important terms which will help us to show many useful results.

Point of Concurrency: A point where three or more lines or rays intersect with each other is known as the point of concurrency.

Perpendicular bisector: A line segment which bisects another line segment at 90° is called perpendicular bisector.

Angle Bisector: An angle bisector is a straight-line drawn from the vertex of a triangle to its opposite side in such a way, that it divides the angle into two equal or congruent angles.

Median: Line segment joining a vertex to the mid-point of the side opposite to that vertex is called the median of a triangle.

Altitude: The altitude of a triangle is the perpendicular line segment drawn from the vertex to the opposite side of the triangle.

As four different types of line segments can be drawn to a triangle, therefore there will be four different points of concurrency in a triangle. Such concurrent points are referred to as different centers according to the lines meeting at that point. The four different points of concurrency in a triangle are:

Circumcentre: The point where three perpendicular bisectors of the triangle meet is called circumcentre of the triangle.

Incentre: The point where three angle bisectors of the triangle meet is called incenter of the triangle.

Centroid: The point where three medians of the triangle meet is called centroid of the triangle.

Orthocentre: The point where three altitudes of the triangle meet is called orthocentre of the triangle.

7.1.4 Show that the medians and angle bisectors of a triangle are concurrent

Show that the Medians of a Triangle are concurrent

Proof: Let ABC is a triangle (Fig. 7.6) with medians $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$ respectively where F is the midpoint of line segment BC, D of $\overline{AB}$ and E of $\overline{AC}$ respectively.

The midpoint of side $\overline{BC}$ is

$$F = \frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}$$

The midpoint of side $\overline{AB}$ is

$$D = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The midpoint of side $\overline{AC}$ is

$$E = \left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right)$$

In triangle ABC, say P is the point of intersection. The coordinates of point P that divides the $\overline{AF}$ in the ratio 2: 1 are as under:

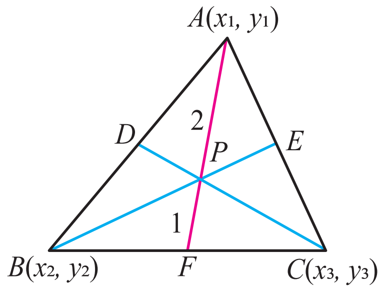

Fig 7.6

Let $(x_1, y_1) = (x_1, y_1)$ and

$(x_2, y_2) = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$ because F is midpoint of $\overline{BC}$.

$$P(x, y) = \left(\frac{(1)x_1 + (2)\left(\frac{x_2 + x_3}{2}\right)}{1 + 2}, \frac{(1)y_1 + (2)\left(\frac{y_2 + y_3}{2}\right)}{1 + 2}\right)$$

$$P(x, y) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Again, the coordinates of point $P(x, y)$ that divides the $\overline{BE}$ in the ratio 2: 1 are as under:

Let $(x_1, y_1) = (x_2, y_2)$ and $(x_2, y_2) = \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$ because E is midpoint of $\overline{AC}$.

Thus,

$$P(x, y) = \left(\frac{(1)x_1 + (2)\left(\frac{x_2 + x_3}{2}\right)}{1 + 2}, \frac{(1)y_1 + (2)\left(\frac{y_2 + y_3}{2}\right)}{1 + 2}\right)$$

$$P(x, y) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Similarly, the coordinates of point $P(x, y)$ that divides the $\overline{CD}$ in the ratio 2: 1 are as under:

Let $(x_1, y_1) = (x_3, y_3)$ and $(x_2, y_2) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ because D is midpoint of AB.

Thus,

$$P(x, y) = \left(\frac{(1)x_3 + (2)\left(\frac{x_1 + x_2}{2}\right)}{1 + 2}, \frac{(1)y_3 + (2)\left(\frac{y_1 + y_2}{2}\right)}{1 + 2}\right)$$

$$P = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Hence, the medians of the triangle are concurrent. It is called centroid of the triangle.

Example: Find the point of concurrency of medians of triangle ABC where coordinates of A, B and C are $(4,10),(8,2)$ and $(-8,4)$.

Solution: Here, $(x_1,y_1)=(4,10),(x_2,y_2)=(8,2)$ and $(x_3,y_3)=(-8,4)$

The point of concurrency of the points $(4,10),(8,2)$ and $(-8,4)$ is

$$=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$$

$$=\left(\frac{4+8-8}{3},\frac{10+2+4}{3}\right)=\left(\frac{4}{3},\frac{16}{3}\right)$$

Show that angle bisectors of triangle are concurrent

Let $A(x_1,y_1),B(x_2,y_2)$ and $C(x_3,y_3)$ be the vertices of any ΔABC. Let a,b and c be the measures of the sides $\overline{BC},\overline{AC}$ and $\overline{AB}$ respectively as shown in the figure 7.7.

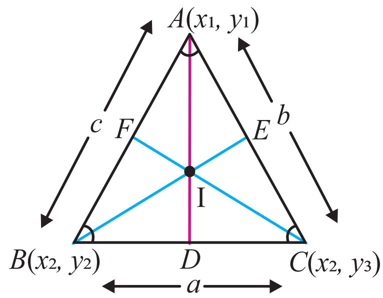

Fig 7.7

Let $\overline{AD}$ be the angle bisector of $\angle A$ which divides $\overline{BC}$ at D internally in the ratio of the sides containing the angle.

i.e., $$\frac{|\overline{BD}|}{|\overline{DC}|}=\frac{c}{b} \quad \ldots\text{(i)}$$

$$\therefore \quad D(x,y)=\left(\frac{bx_2+cx_3}{b+c},\frac{by_2+cy_3}{b+c}\right)$$

Also angle bisector of $\angle$B divides $\overline{AD}$ at I in $|\overline{BD}|:|\overline{AB}|$

$$\therefore \quad \frac{|\overline{AB}|}{|\overline{BC}|}=\frac{c}{|\overline{BC}|} \quad \ldots\text{(ii)}$$

From (i), we have

$$\frac{|\overline{BD}|}{|\overline{DC}|}=\frac{c}{b}$$

By componendo property

$$\frac{|\overline{BD}|}{|\overline{BD}|+|\overline{DC}|}=\frac{c}{c+b}$$

$$\frac{|\overline{BD}|}{|\overline{BC}|}=\frac{c}{c+b}$$

$$|\overline{BD}|=\frac{ac}{b+c} \qquad [\because\ |\overline{BC}|=a]$$

From (ii), we have

$$\frac{|\overline{AB}|}{|\overline{BD}|}=\frac{c}{|\overline{BD}|}=\frac{c}{\frac{ac}{b+c}}=\frac{b+c}{a}=(b+c):a$$

i.e., I divides $|\overline{AD}|$ internally in the ratio $(b+c):a$

$$\therefore \quad I(x,y)=\left(\frac{ax_1+(b+c)\cdot\left(\frac{bx_2+cx_3}{b+c}\right)}{a+(b+c)},\frac{ay_1+(b+c)\cdot\left(\frac{by_2+cy_3}{b+c}\right)}{a+(b+c)}\right)$$

$$\Rightarrow \quad I(x,y)=\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right) \qquad \ldots\text{(i)}$$

Similarly,

$$E(x,y)=\left(\frac{ax_1+cx_3}{a+c},\frac{ay_1+cy_3}{a+c}\right)$$

Point I divides $\overline{BE}$ internally in the ratio $(a+c):b$

$$\therefore \quad I(x,y)=\left(\frac{bx_2+(a+c)\cdot\left(\frac{ax_1+cx_3}{a+c}\right)}{b+(a+c)},\frac{by_2+(a+c)\cdot\left(\frac{ay_1+cy_3}{a+c}\right)}{b+(a+c)}\right)$$

$$\Rightarrow \quad I(x,y)=\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right) \qquad \ldots\text{(ii)}$$

Again,

$$F(x,y)=\left(\frac{ax_1+bx_2}{a+b},\frac{ay_1+by_2}{a+b}\right)$$

Point I divides $\overline{CF}$ internally in the ratio $(a+b):c$

$$\therefore \quad I(x,y)=\left(\frac{cx_3+(a+b)\cdot\left(\frac{ax_1+bx_2}{a+b}\right)}{c+(a+b)},\frac{cy_3+(a+b)\cdot\left(\frac{ay_1+by_2}{a+b}\right)}{c+(a+b)}\right)$$

$$\Rightarrow \quad I(x,y)=\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right) \qquad \ldots\text{(iii)}$$

From equation (i), (ii) and (iii), the coordinates of point I on each angle bisector are found to be same. This means all angle bisectors passes through I. Thus, angle bisectors of a triangle are concurrent.

Exercise 7.1

1. Find the distance between the following pairs of points:
   (i) $A(-1,3)$ and $B(5,-5)$ (ii) $C(-1,0)$ and $D(0,-1)$
   (iii) $E(1,-1)$ and $F(2,7)$ (iv) $G(-1,-4)$ and $H(5,-4)$
2. Find the point on the y-axis which is $5\sqrt{2}$ units away from $(5,2)$.
3. Find the point on the x-axis which is $\sqrt{41}$ units away form $(-7,5)$.
4. If ABC is triangle whose vertices are $A(-3,3), B(2,6)$ and $C(3,0)$. Give the most specific name for ΔABC.

5. If the point $P(2,1)$ lies on the line segment joining the points $A(4,2)$ and $B(8,4)$, then show that $|\overline{AP}| = \frac{1}{2}|\overline{AB}|$.

6. Find the coordinates of the midpoint of following points:
   (i) $M(-4,2)$ and $N(-4,-2)$ (ii) $P(-1,-4)$ and $Q(5,-4)$
   (iii) $S(1,-2)$ and $T(2,-4)$ (iv) $X(6,2)$ and $Y(-2,-7)$

7. Find the point which divides the line segment joining $(4,-1)$ and $(4,3)$ in the ratio $3:1$ internally.

8. The points $P(-2,2), Q(2,-1)$ and $R(-1,4)$ are the mid-points of the sides of the triangle. Find the vertices.

9. $Z(4,5)$ and $X(7,-1)$ are two given points and the point Y divides the line-segment ZX externally in the ratio $4:3$. Find the coordinates of Y.

10. If a point $P(k,7)$ divides the line segment joining $A(8,9)$ and $B(1,2)$ in a ratio $m:n$ then find ratio $m:n$ also find k.

11. $A(2,7)$ and $B(-4,-8)$ are coordinates of the line segment AB. There are two points that trisect the segment AB. Find the points of trisection.

12. The vertices P, Q and R of a triangle are $(2,1)$, $(5,2)$ and $(3,4)$ respectively. Find the coordinates of the circum-centre and also the radius of the circum-circle of the triangle.

13. $\overline{AB}$ is divided into 20 equal parts by $P_1, P_2, P_3, \ldots P_{10}, \ldots P_{19}$. If A and B are (2,3) and (10,11) respectively, find the coordinates of P_{13}.

14. If A, B and C are three collinear point and the coordinates of A and B are $(3,4)$ and $(7,7)$ respectively. Find the coordinates of C if $|\overline{AC}| = 10$ units.

15. Find the coordinates of the incentre of triangle whose angular points are respectively;
   (i) $L(2,8), M(8,2)$ and $N(9,9)$ (ii) $P(-36,7), Q(20,7)$ and $R(0,-8)$

16. The line segment joining $P(-8,10)$ and $Q(6,-4)$ is cut by x and y axes at A and B respectively. Find the ratios in which A and B divide $\overline{PQ}$.

17. Find the coordinates of the centroid of a triangle whose angular points are
   (i) $A(1,3), (2,7)$ and $5,6)$
   (ii) $P(-2,5) Q\ (-7,1)$ and $R(-8,-4)$.

18. A straight line passes through the points $(7,9)$ and $(-1,1)$. Find a point in the line whose ordinate is 4.

7.2 Slope (Gradient) of a Straight line

Slope or gradient of a line is a number that describes both the direction and the steepness of the line. The concept of slope has many applications in the real world.

In construction, the pitch of a roof, the slant of the plumbing pipes and the steepness of the stairs are few applications of slope.

7.2.1 Define the slope of a line

To understand the definition of the slope, first we understand the inclination of a line.

Inclination of a line:

Inclination of a line is the smallest positive angle between the line and the positive direction of x-axis. In Fig. 7.8, θ is the inclination of l, where θ is $0 < \theta < \pi$.

Note: The inclination of x-axis is taken as 0.

Slope of a line:

Slope of a line is the tangent of its inclination. It is denoted by m.

i.e., $m = \tan\theta$

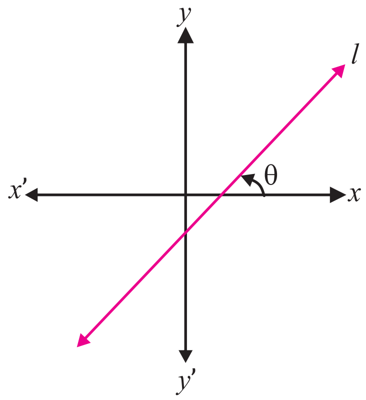

Fig 7.8

7.2.2 Derive the formula to find the slope of a line passing through two points

Let l is a line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$ as shown in the figure 7.9.

Here θ is the inclination of the line and m is the slope of the line,

i.e., $m = \tan\theta$ ...(i)

The changes in abscissa and ordinates are $x_2 - x_1$ and $y_2 - y_1$ respectively.

Consider the right triangle ABC

$$\tan\theta = \frac{|\overline{BC}|}{|\overline{AC}|} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{...(ii)}$$

By comparing equations (i) and (ii)

We get,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

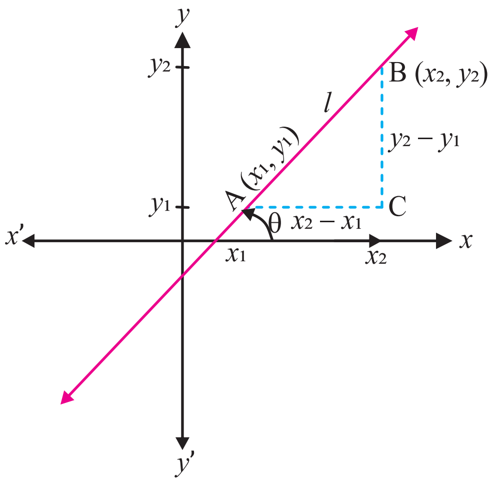

Fig 7.9

The formula states that the slope of a line is equal to the rise over run.

Example 1. Find the slope of a line whose coordinates are (1, 5) and (4, 7).

Solution: Here $(x_1, y_1) = (3, 7)$ and $(x_2, y_2) = (5, 8)$.

We get using Slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$\Rightarrow m = \frac{8-7}{5-3} = \frac{1}{2}$$

Example 2. Find the value of b, if the slope of a line passing through the points $(-2, b)$ and $(3, 4)$ is 5.

Solution: Here, $(x_1, y_1) = (-2, b)$ and $(x_2, y_2) = (3, 4)$.

Using Slope formula $= m = \frac{y_2 - y_1}{x_2 - x_1}$

$$5 = \frac{4 - b}{3 - (-2)}$$

$$5 = \frac{4 - b}{5}$$

$\Rightarrow \quad 25 = 4 - b$

$\Rightarrow \quad b = -21$

Thus, the value of b is -21.

7.2.3 Find the condition that two straight lines with given slopes may be:

- **parallel to each other,**
- **perpendicular to each other.**

Find the condition that two straight lines with given slopes may be:

- **Parallel to each other**

Let l_1 and l_2 are two straight lines with slopes m_1 and m_2 respectively as shown in the figure 7.10.

Since both lines are parallel to each other, therefore they have same inclination as θ is in our case.

Now, slope of first line

$m_1 = \tan\theta \quad \ldots\text{(i)}$

Slope of second line

$m_2 = \tan\theta \quad \ldots\text{(ii)}$

From (i) and (ii)

$m_1 = m_2$

$\Rightarrow l_1 || l_2$ iff $m_1 = m_2$

Hence two straight lines are parallel to each other iff they have same slopes.

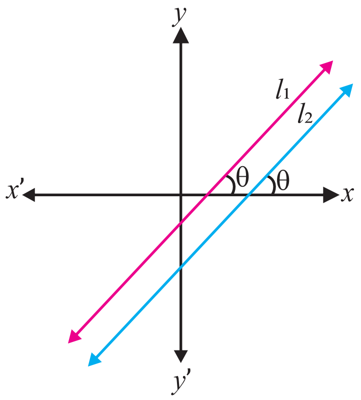

Fig 7.10

- **Perpendicular to each other**

Let l_1 and l_2 are two straight lines perpendicular to each other with slopes m_1 and m_2 respectively. If θ is the inclination of l_1 then $90° + \theta$ will be the inclination of l_2 as shown in the figure 7.11.

Now,

$m_1 = \tan\theta \quad \ldots\text{(i)}$

$m_2 = \tan(90° + \theta)$

$$m_2 = -\cot\theta$$

or $$m_2 = \frac{-1}{\tan\theta}$$

$\Rightarrow$ $$\tan\theta = \frac{-1}{m_2} \quad \ldots\text{(ii)}$$

From equation (ii) and (iii),

we get

$$m_1 = \frac{-1}{m_2}$$

$$\Longrightarrow \quad m_1 m_2 = -1$$

$$\Longrightarrow l_1 \perp l_2 \text{ iff } m_1 m_2 = -1$$

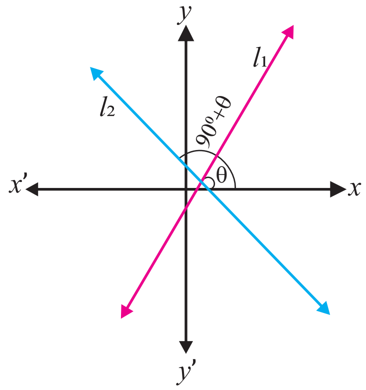

Fig 7.11

Hence two non-vertical straight lines are perpendicular to each other iff product of their slopes is -1.

Example 1. If a line l_1 passes through two given points $(1,3)$ and $(3,7)$ and l_2 passes through $(2,9)$ and $(3,11)$. Check whether both lines are parallel or not.

Solution: Line l_1 passes through two given points $(1,3)$ and $(3,7)$.

The slope of $l_1 = m_1 = \frac{7-3}{3-1} = \frac{4}{2} = 2$

Line l_2 passes through two points $(2,9)$ and $(3,11)$.

The slope of $l_2 = m_2 = \frac{11-9}{3-2} = \frac{2}{1} = 2$

Since, $m_1 = m_2$

Therefore, both lines are parallel to each other.

Example 2. If a line l_1 passes through two given points $(0,1)$ and $(1,-1)$, a line l_2 passes through $(2,2)$ and $(4,3)$. Check whether the following lines are perpendicular or not.

Solution:

The slope of line $l_1 = m_1 = \frac{-1-1}{1-0} = -2$

The slope of line $l_2 = m_2 = \frac{3-2}{4-2} = \frac{1}{2}$

Since $m_1 m_2 = (-2)\left(\frac{1}{2}\right) = -1$

Therefore, both lines are perpendicular to each other.

Exercise 7.2

1. Find the slope of the line passing through given pair of points.

(i) $A(3,7)$ and $B(2,9)$ (ii) $C(5,-2)$ and $D(3,6)$

(iii) $E(5,3)$ and $F(-2,3)$ (iv) $G(0,0)$ and $H(a,b)$

2. Find the slope of the perpendicular line when the given line passes through the following pair of points.

   (i) $A(2,1)$ and $B(4,5)$ (ii) $C(-1,0)$ and $D(3,5)$

   (iii) $E(2,1)$ and $F(-3,1)$ (iv) $G(-1,2)$ and $H(-1,-5)$

3. In each of the following the slope of the line is given. What is the slope of a line (a) parallel (b) perpendicular, to it?

   (i) $\frac{2}{3}$ (ii) $\frac{-7}{2}$ (iii) -1 (iv) 4

4. Are the lines l_1 and l_2 passing through the given pairs of points parallel, perpendicular or neither?

   (i) l_1: $(1,2),(3,1)$ and l_2: $(0,-1),(2,0)$

   (ii) l_1: $(0,3),(3,1)$ and l_2: $(-1,4),(-7,-5)$

   (iii) l_1: $(2,-1),(5,-7)$ and l_2: $(0,0),(-1,2)$

   (iv) l_1: $(1,0),(2,0)$ and l_2: $(5,-5),(-10,-5)$

5. The line through $(6,-4)$ and $(-3,2)$ is parallel to the line through $(2,1)$ and $(0,y)$. Find y.

6. The line through $(2,5)$ and $(-3,-2)$ is perpendicular to the line through $(4,-1)$ and $(x,3)$. Find x.

7. Using slopes prove that $(-1,4),(-3,-6)$ and $(3,-2)$ are the vertices of right triangle.

8. Using slopes, find the fourth vertex of a rectangle if $(0,-1),(4,-3)$ and $(12,3)$ are its three consecutive vertices.

7.3 Equation of a Straight Line Parallel to Co-ordinate Axes

7.3.1 Find the equation of a straight line parallel to

- y-axis at a distance a from it,
- *x*-axis at a distance b from it.

- **Line parallel to y-axis at a distance a from it,**

Let l be a line parallel to y-axis at a distance 'a' from it and cutting the axis of x at A, such that $|\overline{OA}| = a$, as shown in the figure 7.12.

Let $P(x,y)$ be any point on l, draw $\overline{PN}$ perpendicular to y-axis.

Then $|\overline{NP}| = |\overline{OA}| = a$

i.e., $x = a$.

which is the equation of line parallel to y-axis.

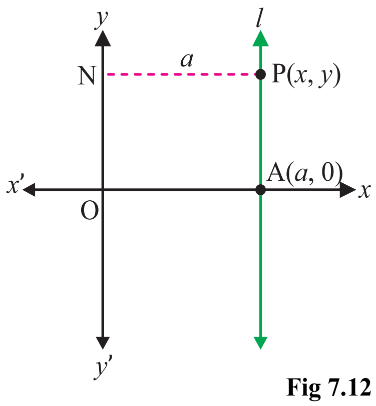

Fig 7.12

If a is positive then line l is to the right of y-axis and if a is negative then line l is to the left of y-axis.

- **Line parallel to x-axis at a distance b from it**

Let l be a line parallel to x-axis is at a distance b from it and cutting the axis of y at B so that $|\overline{OB}| = b$, as shown in the figure 7.13.

Let $P(x, y)$ be any point on l. Draw $\overline{PM}$ perpendicular to x-axis.

Then, $|\overline{MP}| = |\overline{OB}| = b$

i.e., $y = b$

which is equation of line parallel to y-axis.

If b is positive then line is above the x-axis and if b is negative then line is below the x-axis.

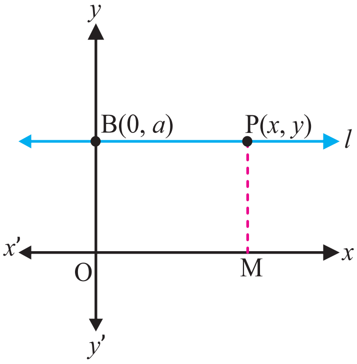

Fig 7.13

Corollary:

Since the axis of x is parallel to itself and at a distance zero from it, the equation of the x-axis is $y = 0$.

Since the axis of y is parallel to itself and at a distance 0 from it, the equation of the y-axis is $x = 0$.

Example 1. Find the equation of straight line parallel to the axis of x at a distance.

(i) 3 unit above it (ii) 5 unit below it

Solution:

(i) Since the line is parallel to x-axis and 3 unit above it, its equation $y = 3$.

(ii) Since the line is parallel to x-axis and 5 unit below it, its equation is $y = -5$.

Example 2. Find the equation of straight line parallel to the axis of y and a distance of

(i) 2 units to its right (ii) 7 units to its left

Solution:

(i) Since the line is parallel to the y-axis and 2 units to its right is $x = 2$.

(ii) Since the line is parallel to the x-axis and 7 units to its left is $x = -7$.

7.4 Standard Form of Equation of a Straight Line

7.4.1 Define intercepts of a straight line. Derive equation of a straight line in

- slope-intercept form
- point-slope form
- two-point forms
- intercepts form
- symmetric form
- normal form

- **x-intercept of a straight line**

When a straight line cuts the x-axis at a point $A(a, 0)$, then a is called x-intercept of the line, as shown in the figure 7.14.

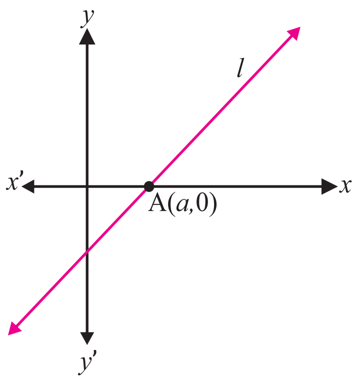

Fig 7.14

y-intercept of a straight line

When a straight line cuts the y-axis at a point $B(0, b)$ then b is called y-intercept of the line, as shown in the figure 7.15.

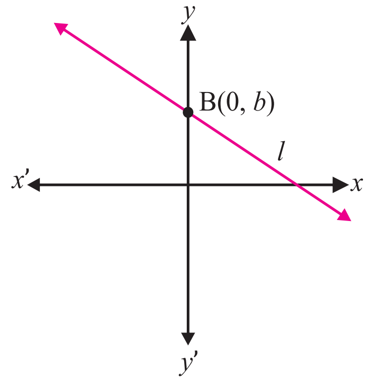

Fig 7.15

Example: If a line cuts the coordinates axes at $(3, 0)$ and $(0, -8)$ respectively. Find the x and y-intercept of the line.

Solution: Here the line cuts the x-axis at $(3, 0)$ then x-intercept of the line is 3. Similarly, the line cuts the y-axis at $(0, -8)$, then y-intercept of the line is -8.

Slope intercept form of straight line

Let the line l intersects the y-axis at P and θ is its inclination and $P(0, c)$ and $Q(x, y)$ be any two points on the line $\overline{AB}$ as shown in Fig. 7.16. Then the slope of the line as discussed in section 7.2 (i) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Here,

$$m = \frac{y - c}{x - 0}$$

$$mx = y - c$$

$$y = mx + c$$

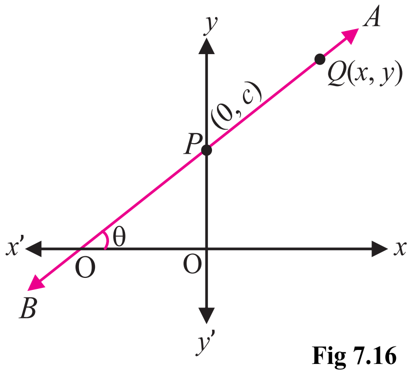

Fig 7.16

which is the required equation of the straight line with slope and intercept form, where m is the slope of the line and c is the y-intercept.

Example: Find the equation of a straight line whose slope is 3 which intersects the y-axis at $(0, 5)$.

Solution: We have, $m = 3$ and $c = 5$

The equation of a line in slope-intercept form is: $y = mx + c$.

So, the required equation is: $y = 3x + 5$

Note:

(i) If the slope or gradient i.e., $m = 0$ and y-intercept i.e., $c \neq 0$, then equation $y = mx + c \Rightarrow y = 0x + b \Rightarrow y = b$, which represents the equation of a line parallel to x-axis.

(ii) When slope and y-intercept is zero (i.e., $m = 0$ and $c = 0$) then equation $y = mx + c \Rightarrow y = 0x + 0 \Rightarrow y = 0$, which represents the equation of x-axis.

- **Point-slope form**

Consider a straight line l in the cartesian plane with slope m and a fixed point $Q(x_1, y_1)$ that lies on the line. Let $P(x, y)$ be another point on the line (Fig. 7.17).

Since the two points lie on the same line with slope m then;

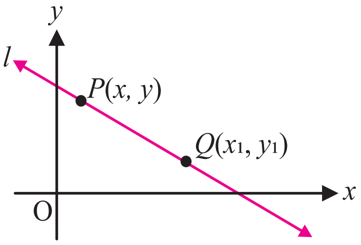

Fig 7.17

$$m = \frac{y - y_1}{x - x_1}$$

$$\Rightarrow \quad (y - y_1) = m(x - x_1)$$

It is called point-slope form of equation of straight line that contains a fixed point $Q(x_1, y_1)$ and slope m.

Corollary: If the line passes through the origin, i.e., if $x_1 = 0$ and $y_1 = 0$, then the equation of line is $y = mx$.

Example: Find the equation of straight line with slope -2 and passing through (2,6).

Solution: Here $m = -2$, $(x_1, y_1) = (2, 6)$. Using the point-slope formula we get;

$$(y - 6) = -2(x - 2)$$
$$y - 6 = -2x + 4$$
$$y + 2x - 10 = 0$$

which is required equation of straight line.

- **two-point form,**

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points on line l_1. Let $P(x, y)$ be any any point on the line l.

From the (Fig. 7.18), we can say that the three points A, P and B are collinear. It shows that the slope of $\overline{AP}$ = slope of $\overline{AB}$.

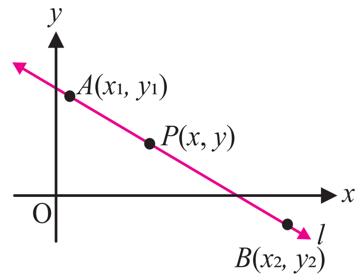

Fig 7.18

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

This is the equation of a line in two-point form.

Corollary 1: The above equation can also be written in the form $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Corollary 2: Another way of writing the two-point form is;

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

Corollary 3: If the line passes through one point and origin, i.e. if $x_2 = 0, y_2 = 0$ then the equation of the line is;

$$\frac{y}{y_1} = \frac{x}{x_1}$$

Example 1. Find the equation of a line passing through the points $(-1, 2)$ and $(3, 5)$.

Solution: Let the given points be: $(x_1, y_1) = (-1, 2)$ and $(x_2, y_2) = (3, 5)$. Then using;

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

We get

$$y - 2 = \frac{(5 - 2)}{(3 + 1)}(x + 1)$$

$$3x - 4y + 11 = 0$$

This is the required equation of line passing through two points, $(-1, 2)$ and $(3, 5)$

Example 2. Find the equation of the line passing through the points $(1, -3)$ and $(5,7)$.

Solution: Using the Corollary (1), the required equation is;

$$\begin{vmatrix} x & y & 1 \\ 1 & -3 & 1 \\ 5 & 7 & 1 \end{vmatrix} = 0$$

After simplifying the determinant, we get;

$$(-3 - 7)x - (1 - 5)y + (7 + 15) = 0$$

$$-10x + 4y + 22 = 0$$

or $$5x - 2y - 11 = 0$$

- **Intercepts form**

The intercepts form of the equation of the line can be derived from the two-point form of line. Let l is a line which passes through two points $(a, 0)$ and $(0, b)$ where a and b are the x and y intercepts of l respectively as shown in Fig. 7.19.

Thus, by two-point form of equation

$$\frac{(y - y_1)}{(x - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

We will get;

$$\frac{(y - 0)}{(x - a)} = \frac{(b - 0)}{(0 - a)}$$

$$y = \frac{-b}{a}(x - a)$$

$$y = \frac{-bx}{a} + b$$

$$y = b\left(\frac{-bx}{a} + 1\right)$$

$$\frac{y}{b} = \frac{-x}{a} + 1$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

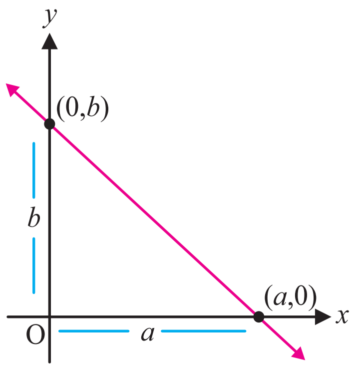

Fig 7.19

which is the required equation of straight line in intercepts form. Where a and b are x and y intercept respectively.

Corollary 1: The equation $\frac{x}{a}+\frac{y}{b}=1$ may be written in the form $lx+my=1$, where $l=\frac{1}{a}$ and $m=\frac{1}{b}$ and l, m are reciprocals of the intercepts on the axes.

Corollary 2: The equation of the straight-line which has equal intercept (say a) is $x+y=a$.

Example 1. Find the equation of the straight line which makes intercepts $\frac{1}{5}$ and $\frac{1}{7}$ on the axes respectively.

Solution: Here $a=\frac{1}{5}$ and $b=\frac{1}{7}$. Thus, the required equation is;

$$\frac{x}{\frac{1}{5}}+\frac{y}{\frac{1}{7}}=1$$

i.e., $5x+7x=1$

or $5x+7y-1=0$

- **Symmetric form,**

Let θ is the inclination of a straight-line l passing through the point $P(x,y)$. Consider another point $Q(x_1,y_1)$. Now using the slope formula;

$$m=\tan\theta=\frac{y-y_1}{x-x_1}$$

As we know that $\tan\theta=\frac{sin\,\theta}{cos\,\theta}$, then the above formula becomes;

$$\tan\theta=\frac{sin\,\theta}{cos\,\theta}=\frac{y-y_1}{x-x_1}$$

$$\frac{sin\,\theta}{cos\,\theta}=\frac{y-y_1}{x-x_1}$$

or

$$\frac{x-x_1}{cos\,\theta}=\frac{y-y_1}{sin\,\theta}$$

This is called the symmetric form of an equation of a straight line with inclination θ and passing through (x_1,y_1).

Example 7. Find the symmetric form equation of a straight line with inclination 45° and passing through the point $(2,\sqrt{2})$.

Solution: Here an inclination is $\theta=45°$ and point $(x_1,y_1)=(2,\sqrt{2})$. Using the equation of line in its symmetric form;

$$\frac{x-x_1}{cos\,\theta}=\frac{y-y_1}{sin\,\theta}$$

Substitute the above values in the formula to get the symmetric form equation of a straight line;

$$\frac{x-2}{cos\,45°} = \frac{y-\sqrt{2}}{sin\,45°}$$

$$\frac{x-2}{\left(\frac{1}{\sqrt{2}}\right)} = \frac{y-\sqrt{2}}{\left(\frac{1}{\sqrt{2}}\right)}$$

$\Rightarrow$ $y - x + 2 - \sqrt{2} = 0$ is __________ equation.

- **Normal form or perpendicular form**

Let $P(x, y)$ be any point on the straight-line l. The line intersects the coordinate axes at points A and B respectively. The $|\overline{OA}|$ and $|\overline{OB}|$ become its x-intercept and y-intercept respectively as shown in the Fig. 7.20.

Now using intercepts form of equation of straight line, we have

$$\frac{x}{|\overline{OA}|} + \frac{y}{|\overline{OB}|} = 1 \qquad \text{... (i)}$$

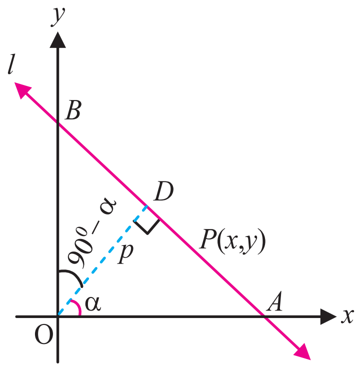

Fig 7.20

Let p be the length of the normal drawn from the origin to the line, which subtends an angle α with the positive direction of x-axis. If D is foot of perpendicular drawn from O then consider the triangle ODA as given in Fig. 7.20. Using the trigonometric ratios; we get;

$$\frac{|\overline{OD}|}{|\overline{OA}|} = \cos\alpha$$

$$\Rightarrow \quad \frac{p}{|\overline{OA}|} = \cos\alpha$$

$$\Rightarrow \quad |\overline{OA}| = \frac{p}{cos\,\alpha} \qquad (\because |\overline{OD}| = p)$$

Similarly, ODB is a right-angle triangle, then;

$$\frac{|\overline{OD}|}{|\overline{OB}|} = \cos(90° - \alpha)$$

$$\Rightarrow \quad \frac{p}{|\overline{OB}|} = \sin\alpha$$

$$\Rightarrow \quad |\overline{OB}| = \frac{p}{\sin\alpha}$$

Now substituting the values of $|\overline{OA}|$ and $|\overline{OB}|$ in equation (i) we get;

$$\frac{x}{\frac{p}{cos\,\alpha}} + \frac{y}{\frac{p}{\sin\alpha}} = 1$$

$$\Rightarrow \quad \frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1$$

$$\Rightarrow \quad x\cos\alpha + y\sin\alpha = p$$

where p is always kept positive and α is measured in counter-clockwise direction $(0 < \alpha < 2\pi)$.

This equation is called normal or perpendicular form of equation of line.

Example 8. Find the equation of the straight line which is at a of distance 9 units from the origin and the perpendicular from the origin to the line makes an angle 30° with the positive direction of x-axis.

Solution: Here $p = 9$ and $\alpha = 30°$, Using the Normal form of equation of straight line, we have

$$x\cos 30° + y\sin 30° = 9$$

$$x\left(\frac{\sqrt{3}}{2}\right) + y\left(\frac{1}{2}\right) = 9$$

$$\frac{x\sqrt{3}}{2} + \frac{y}{2} = 9$$

$$x\sqrt{3} + y = 18$$

which is the required equation of line.

(ii) Show that a linear equation in two variables represents a straight line.

A linear equation in two variables x and y is the equation of the form

$$ax + by + c = 0 \qquad \ldots(i)$$

where a, b and c are real numbers (constants). Also, a and b are not both zero.

Theorem: Every linear equation in two variables represents a straight line.

Linear equation into two variables is $ax + by + c = 0$, both a and b are not 0.

Proof: Case-I when $b = 0$.

In this case $a \neq 0$. Thus, the equation (i) reduces to

$$ax + c = 0$$

$$x = \frac{-c}{a} \qquad \ldots(ii)$$

which is the equation of straight line and parallel to y-axis.

Case-II when $a = 0$.

In this case $b \neq 0$. Thus, the equation (i) reduces to

$$by + c = 0$$

$$y = \frac{-c}{b} \qquad \ldots(iii)$$

which is the equation of straight line and parallel to x-axis.

Case-III When $a \neq 0$ and $b \neq 0$. In this the equation (i) reduces to

$$ax + by + c = 0$$

$$y = \frac{-a}{b}x + \frac{-c}{b}$$

$$y = mx + c' \quad \text{... (iv)}$$

where $m = \frac{-a}{b}$ and $c' = \frac{-c}{b}$. Thus (iv) is also equation of straight line in slope intercept form. Hence in all cases a linear equation in two variables represents a straight line.

(iii) Reduce the general form of the equation of a straight line to the other standard forms.

- **Reduce the general equation $ax + by + c = 0$ into slope intercept form**

The general equation of straight line is

$$ax + by + c = 0 \quad \text{...(i)}$$

Now adding $-ax - c$ on both sides of equation (i) we get;

$$ax + by + c - ax - c = -ax - c$$

$$by = -ax - c$$

$$y = \left(\frac{-a}{b}\right)x + \left(\frac{-c}{b}\right)$$

which is the slope intercept form of line, where $\frac{-a}{b}$ is the slope and $\frac{-c}{b}$ is the y-intercept from the line.

- **Reduce the general equation $ax + by + c = 0$ into intercept form.**

The general equation of straight line is

$$ax + by + c = 0 \quad \text{...(i)}$$

If $a \neq 0, b \neq 0, c \neq 0$ then from the equation (i) we get,

$$ax + by = -c$$

$$\frac{ax}{c} + \frac{by}{c} = \frac{-c}{c} \quad \text{(Dividing both sides by } -c\text{)}$$

$$\frac{x}{\frac{c}{a}} + \frac{y}{\frac{c}{b}} = -1$$

$$\Rightarrow \quad \frac{x}{\left(\frac{-c}{a}\right)} + \frac{y}{\left(\frac{-c}{b}\right)} = 1$$

which is the required intercept form of equation of line, where $\frac{-c}{a}$ is the x-intercept and $\frac{-c}{b}$ is the y-intercept.

- **Reduce the general equation $ax + by + c = 0$ into Normal form.**

The general equation of straight line is

$$ax + by + c = 0 \quad \text{...(i)}$$

Let the normal equation of line is

$$x\cos\alpha + y\sin\alpha - p = 0 \qquad \text{where } p > 0 \qquad \text{... (ii)}$$

By comparing equation (i) and (ii) as both the equations are identical, we get;

$$\Rightarrow \frac{\cos\alpha}{a} = \frac{\sin\alpha}{b} = \frac{-p}{c} = k$$

Thus $\cos\alpha = ak$ and $\sin\alpha = bk$, $p = -ck$.

Squaring and adding we get;

$$\cos\alpha^2 + \sin\alpha^2 = a^2k^2 + b^2k^2$$

$$\Rightarrow \quad a^2k^2 + b^2k^2 = 1$$

$$\Rightarrow \quad k = \pm\frac{1}{\sqrt{a^2+b^2}}$$

Since p is always positive therefore select k as positive. Thus, the equation (ii) becomes;

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y = -\frac{c}{\sqrt{a^2+b^2}} \qquad \text{... (iii)}$$

which is the required normal form of equation of line.

Exercise 7.3

1. Find the equations of the straight lines parallel to the coordinate axes and passing through the point $(3, -4)$.
2. Find the equations of the straight lines parallel to the coordinate axes and passing through the point $(5, 2)$.
3. Write the equation of the straight lines parallel to x-axis which is at a distance of 5 units from above the x-axis.
4. Find the equation of a line parallel to y-axis which is at a distance of 6 units on its left.
5. Find the equations of the straight line determined by each of the following set of conditions.
   (i) through $(5, -2)$ with the slope 4
   (ii) through $(-1, -4)$ with the slope $\frac{-2}{3}$
   (iii) through $\left(\frac{-1}{4}, \frac{3}{4}\right)$ with slope $\frac{2}{5}$
   (iv) through $(0, b)$ with the slope m.
   (v) through the points $(7, -3)$ and $(-4, 1)$
   (vi) through the points $(5, -5)$ and $(-3, 1)$
   (vii) through $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$

(viii) y-intercept $= 3$; slope $= 2$

(ix) y-intercept $= -2$; slope $= \frac{-2}{3}$

(x) y-intercept $= -5$; slope $= \frac{1}{2}$

(xi) y-intercept $= 0$; slope $= 0$

(xii) x-intercept $= 4$; y-intercept $= 3$

(xiii) x-intercept $= -2$; y-intercept $= 5$

(xiv) x-intercept $= -5$; y-intercept $= -1$

(xv) the perpendicular from the origin to the line, $p = 3$ units and it makes an angle $\alpha = 60°$ with x-axis.

(xvi) $p = \frac{3}{2}, \alpha = 150°$

6. Reduce the equation $3x + 4y - 12 = 0$ to the

(i) slope-intercept form (ii) two-intercept form

(iii) Normal or Perpendicular form

7. Find the equations of the sides of the triangle whose vertices are $(1,4), (2,-3)$ and $(-1,-2)$.

8. Find the equation of the perpendicular bisector of the segment joining $(-1,2)$ and $(9,12)$.

9. The x-intercept of a line is the reciprocal of its y-intercept and line passes through $(2,-1)$. Find its equation.

10. Find the equation of the line which passes through $(-2,-4)$ and the sum of its intercepts equal to 3.

11. Find the equation of the line which passes through $(5,6)$ and the y-intercept is twice that of the x-intercept.

12. Find an equation of the line through $(11,-5)$ and parallel to a line having slope $\frac{3}{2}$.

13. Find an equation of the line through $(-4,-6)$ and perpendicular to the line having slope $\frac{-3}{2}$.

7.5 Distance of a Point from a Line

We know that using the distance formula we can find the distance between the two points that could be a distance between two objects, a distance between two houses, etc. Similarly, the distance from a point can be measured from a line as well. Thus, the perpendicular distance of a point from a line is the shortest distance between the point and the line.

7.5.1 Recognize a point with respect to position of a line

As we know that a line divides a plane into two regions such that every point of the plane not on the line lies in one of the regions. Thus, if a line is not parallel to y-axis, every point of the plane not on the line is either above the line or below the line.

Theorem 1: Let l denote the line $ax + by + c = 0$ with $b > 0$.

If $P(x_1, y_1)$ is a point above the line l, then $ax_1 + by_1 + c > 0$.

If $P(x_2, y_2)$ is a point below the line l, then $ax_1 + by_1 + c < 0$.

Proof: Through P_1 draw a line l' parallel to y-axis, intersecting the line l at a point M (Fig. 7.21).

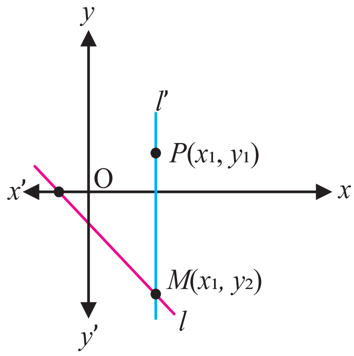

Fig 7.21

Now the abscissa of the M is same as the abscissa of P_1, viz, x_1. Let us denote the ordinate of M by y_2, so that M is the point (x_1, y_2).

If P_1 is a point lies above the line then $y_1 > y_2$. Since $b > 0$ we have $by_1 > by_2$. Adding $ax_1 + c$ to both sides of this inequality we have

$$ax_1 + by_1 + c > ax_1 + by_2 + c$$

Since $M(x_1, y_2)$ lies on the line l, $ax_1 + by_2 + c = 0$.

Thus, $ax_1 + by_1 + c > 0$

If $P_1(x_1, y_1)$ is a point lies below the line then $y_1 < y_2$ and in a similar way we can show that

$$ax_1 + by_1 + c < 0$$

The following is the converse of the theorem 1.

Theorem 2: Let l denote the line $ax + by + c = 0$ with $b > 0$.

(i) If x_1 and y_1 are real numbers such that $ax_1 + by_1 + c > 0$ then the point $P_1(x_1, y_1)$ lies above the line l.

(ii) If x_1 and y_1 are real numbers such that $ax_1 + by_1 + c < 0$ then the point $P_1(x_1, y_1)$ lies below the line l.

Note: The above two theorems should be used after the equation of the given line has been put in the general form; i.e., $ax_1 + by_1 + c = 0$ with $b > 0$

In case $b = 0$ the line would be parallel to y-axis then the question of a point below or above does not arise.

Example: Find whether each of the points $(-8, -3)$, $(10, -5)$, $(-35, 9)$ is above or below the line $2x - 3y + 4 = 0$.

Solution: First of all, we write the equation of straight line in the form in which coefficient of y is positive. Thus,

$$-2x + 3y - 4 = 0$$

For the point $(-8,-3)$, we have,

$$-2(-8)+3(-3)-4=3>0$$

Hence $(-8,-3)$ lies above the given line. Again, for the point $(10,-5)$, we have,

$$-2(10)+3(-5)-4=-39<0$$

Hence $(10,-5)$ lies below the given line. Again, for the point $(-35,9)$, we have,

$$-2(-35)+3(9)-4=93>0$$

Hence $(-35,9)$ lies above the given line.

7.5.2 Find the perpendicular distance from a point to the given straight line.

Consider a line $l: ax+by+c=0$ and a point $P(x_1,y_1)$ not on l.

To find perpendicular distance of P from l. Through P, draw a line l' perpendicular to l cutting l at Q. (Fig. 7.22)

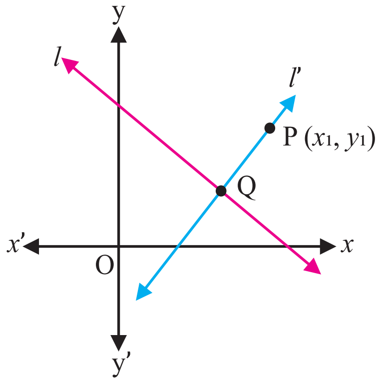

Fig 7.22

Slope of $l=-\frac{a}{b}$ $\Rightarrow$ slope of $l'=\frac{b}{a}$

Equation of l' is $y-y_1=\frac{b}{a}(x-x_1)$

$\Rightarrow \quad bx-ay+ay_1-bx_1=0$

To find Q, we solve equation of l and l', we get

$$\frac{x}{b(ay_1-bx_1)+ac}=\frac{y}{bc-a(-bx_1+ay_1)}=\frac{1}{-a^2-b^2}$$

$$\Rightarrow \quad \frac{x}{aby_1-b^2x_1+ac}=\frac{1}{-a^2-b^2} \text{ and } \frac{y}{bc+abx_1-a^2y_1}=\frac{1}{-a^2-b^2}$$

$$\Rightarrow \quad x=\frac{aby_1-b^2x_1+ac}{-a^2-b^2} \qquad \Rightarrow \quad y=\frac{bc+abx_1-a^2y_1}{-a^2-b^2}$$

So,

$$x=\frac{b^2x_1-aby_1-ac}{a^2+b^2} \qquad y=\frac{a^2y_1-abx_1-bc}{a^2+b^2}$$

Thus, the coordinates of Q are $\left(\frac{b^2x_1-aby_1-ac}{a^2+b^2},\frac{a^2y_1-abx_1-bc}{a^2+b^2}\right)$

By using distance formula,

$$d=|\overline{PQ}|=\sqrt{\left(\frac{b^2x_1-aby_1-ac}{a^2+b^2}-x_1\right)^2+\left(\frac{a^2y_1-abx_1-bc}{a^2+b^2}-y_1\right)^2}$$

$$=\sqrt{\frac{(a^2x_1+aby_1+ac)^2}{(a^2+b^2)^2}+\frac{(abx_1+b^2y_1+bc)^2}{(a^2+b^2)^2}}$$

$$= \sqrt{\frac{a^2(ax_1 + by_1 + c)^2}{(a^2 + b^2)^2} + \frac{b^2(ax_1 + by_1 + c)^2}{(a^2 + b^2)^2}}$$

$$= \sqrt{\frac{(ax_1 + by_1 + c)^2(a^2 + b^2)}{(a^2 + b^2)^2}}$$

$$d = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

$\because$ d is always $+ve$

$\therefore$ $d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

In case $b = 0$, the same formula holds.

Example: Find the distance of the point $(-3, 5)$ from the line $4x - 3y - 26 = 0$.

Solution: Given equation of a line is: $4x - 3y - 26 = 0$ and the point $(x_1, y_1) = (-3, 5)$

Comparing these with the general forms,

$$a = 4, \ \ b = -3, \ \ c = -26$$

We know that the perpendicular distance (d) of a line $ax + by + c = 0$ from a point (x_1, y_1) is given by

$$d = \left[\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right]$$

After substituting the values, we will get;

$$d = \left|\frac{(4)(-3) + (-3)(5) - 26}{\sqrt{(4)^2 + (-3)^2}}\right|$$

$$d = \frac{53}{5}$$

7.5.3 Find the distance between two parallel lines

The distance between two parallel lines is equal to the perpendicular distance between the two lines. We know that the slopes of two parallel lines are the same; therefore, the equation of two parallel lines can be given as:

$$y = mx + c_1 \quad \text{...(i)}$$

$$y = mx + c_2 \quad \text{...(ii)}$$

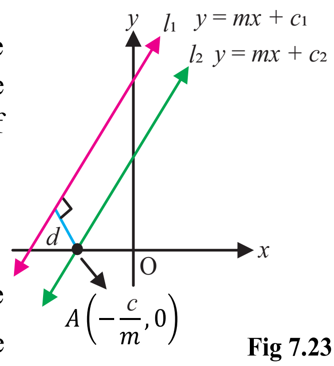

Fig 7.23

The point $A\left(-\frac{c}{m}, 0\right)$ is the intersection point of the second line and the x-axis (Fig. 7.23). The perpendicular distance from A to l_1 will be the required perpendicular distance between two parallel lines. The distance between the point A and the line $y = mx + c_2$ can be given by using the formula:

$$d = \left[\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right]$$

$$d = \left[\frac{(-m)\left(\frac{-c_1}{m}\right) - c_2}{\sqrt{(1 + m^2)}}\right]$$

$$d = \frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$$

Thus, we can conclude that the perpendicular distance between two parallel lines is given by:

$$d = \frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$$

Example: Find the distance between two parallel lines $3x + 4y = 9$ and $6x + 8y = 15$.

Solution: Given equations of lines are:

$$3x + 4y = 9 \qquad \ldots\text{(i)}$$

$$6x + 8y = 15$$

or $$3x + 4y = \frac{15}{2} \qquad \ldots\text{(ii)}$$

Now, by comparing with the general equations of straight lines we get;

$$a = 3, b = 4, c_1 = -9 \text{ and } c_2 = \frac{-15}{2}$$

Thus, the required distance will be;

$$d = \frac{\left|(-9) - \left(\frac{-15}{2}\right)\right|}{\sqrt{(3)^2 + (4)^2}}$$

$$d = \frac{3}{5}$$

That is the required distance between two lines.

Exercise 7.4

1. Determine whether each of the specified points is above or below the given straight line:
   - (i) $3x + 11y - 44 = 0$, $(10, 1)$, $(-4, 6)$ and $(5, 3)$
   - (ii) $10x - 12y + 17 = 0$, $(-20, -15)$, $(5, 5)$ and $(100, 84)$
   - (iii) $29x - 17y + 31 = 0$, $(0, 2)$, $(-3, -3)$ and $(20, 30)$
2. In each of the following, find the perpendicular distance from the point to the line;
   - (i) $15x - 8y - 5 = 0$, $(2, 1)$
   - (ii) $3x - 4y + 5 = 0$, $(4, -3)$

(iii) $3x + 4y + 10 = 0, (3, -2)$

(iv) $2x - 7y + 1 = 0, (7, 4)$

(v) $5x + 12y - 16 = 0, (3, -1)$

3. Find the distance between the parallel lines;

(i) $5x - 12y + 10 = 0, 5x - 12y - 16 = 0$

(ii) $x + y - 2 = 0, 2x + 2y - 1 = 0$

(iii) $4x - 3y + 12 = 0, 4x - 3y - 12 = 0$

7.6 Angle Between Lines

7.6.1 Find the angle between two coplanar intersecting straight lines

Let the equations of the straight lines l_1 and l_2 are $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively intersect at a point P and make angles θ_1 and θ_2 respectively with the positive direction of x-axis as shown in Fig. 7.24.

Let $\angle APC = \theta$ is positive angle from l_1 to l_2.

Clearly, the slope of the line l_1 and l_2 are m_1 and m_2 respectively.

Then, $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

Now, from the elementary geometry

$$\theta_2 = \theta + \theta_1 \qquad \Rightarrow \theta = \theta_2 - \theta_1$$

Now, $\tan\theta = \tan(\theta_2 - \theta_1) = \dfrac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2}$

Thus, substituting the values of $\tan\theta_1$ and $\tan\theta_2$ for m_1 and m_2 respectively, we have;

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1m_2}$$

It should be noted that the value of $\tan\theta$ in this equation will be positive if θ is acute and negative if θ is obtuse.

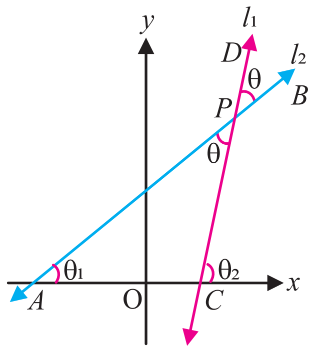

Fig 7.24

- The angle between two lines having equations $l_1: a_1x + b_1y + c_1 = 0$ and $l_2: a_2x + b_2y + c_2 = 0$ is $\theta = \tan^{-1}\dfrac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}$.
- If the lines l_1 and l_2 are perpendicular, then $\theta = 90°$ or $\dfrac{\pi}{2}$. Thus, we have,

$$1 + m_1m_2 = \frac{m_2 - m_1}{\tan\theta}$$

$$\Rightarrow \quad 1 + m_1m_2 = \frac{m_2 - m_1}{\tan 90°}$$

$$\Rightarrow \quad 1 + m_1m_2 = \frac{m_2 - m_1}{\infty}$$

$\Rightarrow \quad 1 + m_1 m_2 = 0$

$\Rightarrow \quad m_1 m_2 = -1$ or $a_1 a_2 + b_1 b_2 = 0$

This is the condition for two lines to be perpendicular.

- If the lines l_1 and l_2 are parallel, then $\theta = 0°$ or π radian. Thus, we have,

$$a_2 b_1 - a_1 b_2 = 0$$

- If two linear equations have the same x and y coefficients, the lines represented by them are parallel.
- If the coefficients of the later of the two linear equations are those of the former reversed in order and with the sign of one coefficient changed, the lines represented by them are perpendicular.

For example, line $ax + by + c = 0$ is respectively parallel to $ax + by + c_1 = 0$ and perpendicular to $ax - by + c_2 = 0$.

Example 1. If $A(-2,1), B\ (2,3)$ and $C\ (-2,-4)$ are three points, find the acute angle between the straight lines AB and BC.

Solution: Let the slope of the line AB and BC are m_1 and m_2 respectively.

Then, $m_1 = \frac{3-1}{2-(-2)} = \frac{1}{2}$ and $m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$

Let θ be the angle between AB and BC. Then,

$$\tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2} = \pm\frac{2}{3}$$

$\theta = \tan^{-1}\left(\frac{2}{3}\right)$ is the required acute angle.

Example 2. Find the acute angle between the lines $7x - 4y = 0$ and $3x - 11y + 5 = 0$.

Solution:

Method 1: First we need to find the slope of both the lines.

Thus, $7x - 4y = 0$ the slope of the line is $\frac{7}{4}$.

Again, $3x - 11y + 5 = 0$, the slope of the line is $\frac{3}{11}$.

Using the formula, we have; $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$

$$\theta = \tan^{-1}\left(\frac{\frac{7}{4} - \frac{3}{11}}{1 + \left(\frac{3}{11}\right)\left(\frac{7}{4}\right)}\right) = \tan^{-1}(1)$$

$\because\ \theta$ is acute $\therefore\ \theta = 45°$.

Method 2: The given two equations of the lines are $7x - 4y = 0$ and $3x - 11y + 5 = 0$.

Here we have $a_1 = 7, b_1 = -4, a_2 = 3$ and $b_2 = -11$.

The angle between the two lines can be calculated using the formula

$$\tan\theta = \frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}$$

$$= \frac{3(-4) - (7)(-11)}{(3)(7) + (-4)(-11)}$$

$$= \frac{-12 + 77}{21 + 44} = \frac{65}{65} = 1$$

$$\tan\theta = 1 \quad \Longrightarrow \quad \theta = 45°$$

7.6.2 Find the equation of family of lines passing through the point of intersection of two given lines.

A family of lines is a set of lines having one or two factors in common with each other. Straight lines can belong to two types of families: one where the slope is the same and one where the y-intercept is the same.

Consider the two straight lines;

$$a_1 x + b_1 y + c_1 = 0 \quad \ldots(i)$$

$$a_2 x + b_2 y + c_2 = 0 \quad \ldots(ii)$$

For any nonzero constant k, the equation of the form

$$a_1 x + b_1 y + c_1 + k(a_2 x + b_2 y + c_2) = 0 \quad \ldots(iii)$$

being linear in x and y is an equation of a straight line.

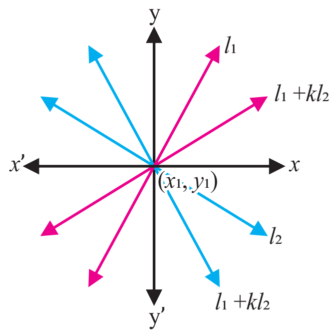

Fig 7.25

If (x_1, y_1) is the point of intersection of line (i) and (ii) then it must satisfy the both equation (i) and (ii);

$$a_1 x_1 + b_1 y_1 + c_1 = 0 \quad \ldots(iv)$$

$$a_2 x_1 + b_2 y_1 + c_2 = 0 \quad \ldots(v)$$

Next, we check whether the point (x_1, y_1) lies on (iii) or not. For this, we replace x by x_1 and y by y_1 in equation (iii), we will get,

$$a_1 x_1 + b_1 y_1 + c_1 + k(a_2 x_1 + b_2 y_1 + c_2) = 0 \quad \ldots(vi)$$

Using equation (iv) and (v) in Equation (vi), we will get;

$$0 + k(0) = 0$$

This shows that equation (vi) is true for all k and for $x = x_1,\ y = y_1$. Thus, the point (x_1, y_1) lies on (vi) for all k. Equation (vi) represents the equation of the line through the point of intersection of lines (i) and (ii). Since k is any real number, equation (vi) shows that there will be an infinite number of lines (Family of Lines) through the point of intersection of lines (i) and (iii).

Example: Find the equation of a line through the point (1,3) and the point of intersection of lines $2x-3y+4=0$ and $4x+y-1=0$.

Solution: The family of the equation of a straight line through the point of intersection of lines $2x-3y+4=0$ and $4x+y-1=0$ is given as

$$2x-3y+4+k(4x+y-1)=0 \quad \ldots\text{(i)}$$

Since the required line passes through the point (1,3), this point must satisfies the equation (i) i.e.

$$2(1)-3(3)+4+k(4(1)+3-1)=0-3+6k=0$$

$$k=12$$

Substituting the value of k in the required equation of a straight line, we have

$$2x-3y+4+12(4x+y-1)=0$$

$8x-5y+7=0$ is the required equation of straight line.

7.6.3 Calculate angles of the triangle when the slopes of the sides are given

The angles of triangle will be calculated using the formula as discussed in section 7.6(i)

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}$$

where m_1 and m_2 are the slopes of two lines.

Example: Find the angles of the given triangle, the slopes of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are $-3, 2$ and $\frac{1}{3}$ respectively.

Solution:

Let m_1 is the slope of $\overline{AB}$ i.e., $m_1=-3$, m_2 is the slope of $\overline{BC}$ i.e., $m_2=2$ and m_3 is the slope of $\overline{AC}$ i.e., $m_3=\frac{1}{3}$

Assume θ_1 is the positive angle from $\overline{AB}$ to $\overline{AC}$, θ_2 is the positive angle from $\overline{BC}$ to $\overline{AB}$ and θ_3 is the positive angle from $\overline{AC}$ to $\overline{BC}$.

Now,

$$\tan\theta_1=\frac{m_1-m_3}{1+m_1m_3}$$

$$\tan\theta_1=\frac{(-3)-\left(\frac{1}{3}\right)}{1+(-3)\left(\frac{1}{3}\right)}$$

$$\tan\theta_1=\frac{-8}{0} \quad \text{(undefined)}$$

$$\tan\theta_1=-\infty$$

$$\boxed{\theta_1=90^\circ}$$

Similarly,

$$\tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan\theta_2 = \frac{(-3)-(2)}{1+(-3)(2)}$$

$$\tan\theta_2 = \frac{-5}{-5}$$

$$\boxed{\theta_2 = 45°}$$

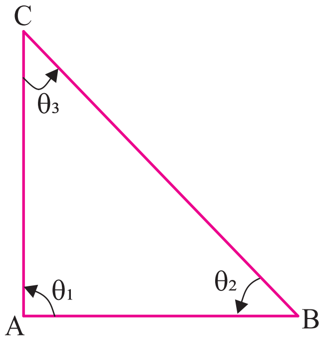

Fig 7.26

and

$$\theta_3 = 180° - \theta_1 - \theta_2$$

$$\theta_3 = 180° - 90° - 45°$$

$$\boxed{\theta_3 = 45°}$$

These interior angles of the triangles are 90°, 45° and 45°.

Exercise 7.5

1. What is the angle between two lines when they intersect at origin and one of the line passes through $(2, 3)$ and the other line passes through $(-3, 6)$?
2. Find the angle between the following two lines. $l_1: 4x - 3y = 8$ and $l_1: 2x + 5y = 4$.
3. Find the acute angle between $l_1: y = 3x + 1$ and $l_2: y = -4x + 3$.
4. Find the angle between two lines, one of which is the x-axis and the other line is $x - y + 4 = 0$, is?
5. Find the angle between the lines $2x - 3y + 7 = 0$ and $7x + 4y - 9 = 0$.
6. Find the equation of line through point $(3, 2)$ and making angle $45°$ with the line $x - 2y = 3$.
7. Determine the measure of the acute angle between the straight-line $x - y + 4 = 0$ and the straight line passing through the points $(3, 2)$ and $(2, 4)$.
8. Find the equation of family of lines that pass through the point of intersection of $2x + 3y - 8 = 0$ and $x - y + 1 = 0$. Also find the point of intersection.
9. Find the equation of a line through the intersection of the lines;
   (i) $2x + 3y + 1 = 0, 3x - 4y = 5$ and passing through the point (2, 1).
   (ii) $x - 4y = 3, x + 2y = 9$ and passing through the origin.
   (iii) $3x + 2y = 8, 5x - 11y + 1 = 0$ and parallel to $6x + 13y = 25$.
   (iv) $2x - 3y + 4 = 0, 3x + 3y - 5 = 0$ and parallel to y-axis.
   (v) $5x - 6y = 1, 3x + 2y + 5 = 0$ and perpendicular to $5y - 3x = 11$.
   (vi) $3x - 4y + 1 = 0, 5x + y - 1 = 0$ and cutting off equal intercepts

from the axes.

(vii) $43x + 29y + 43 = 0, 23x + 8y + 6 = 0$ and having y-intercept -2.

(viii) $2x + 7y - 8 = 0, 3x + 2y + 5 = 0$ and making an angle of $45°$ with the line $2x + 3y - 7 = 0$.

10. Find the angles of the triangle with the given vertices $(1, 2), (3, 4)$ and $(2, 5)$.

11. What are the angles of the triangle with vertices $A(3, 2), B(4, 5)$ and $C(-1, -1)$?

12. Find the angles of triangle where the slopes of its sides, are $3, \frac{1}{2}, -2$.

7.7 Concurrency of Straight Lines

7.7.1 Find the condition of concurrency of three straight lines

Three or more distinct lines are said to be concurrent, if they pass through the same point. The point of intersection of any two lines, which lie on the third line is called the point of concurrence.

Let the equations of the three concurrent straight lines be;

$$a_1x + b_1y + c_1 = 0 \quad \ldots(i)$$

$$a_2x + b_2y + c_2 = 0 \quad \ldots(ii)$$

$$a_3x + b_3y + c_3 = 0 \quad \ldots(iii)$$

Suppose the equations (i) and (ii) of two intersecting lines intersect at $P(x_1, y_1)$. Then (x_1, y_1) will satisfy both the equations (i) and (ii). Therefore,

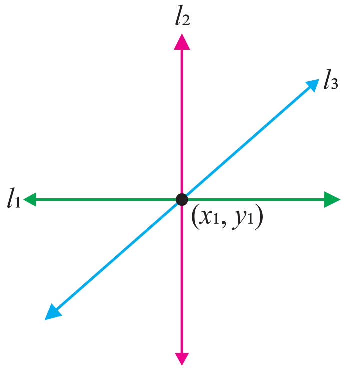

Fig 7.27

$a_1x_1 + b_1y_1 + c_1 = 0$ and

$a_2x_1 + b_2y_1 + c_2 = 0$

Solving the above two equations, we get,

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

Therefore,

$$x_1 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$y_1 = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

where $a_1b_2 - a_2b_1 \neq 0$

Therefore, the required co-ordinates of the point of intersection of the lines (i) and (ii) are;

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

where $a_1b_2 - a_2b_1 \neq 0$

Since the straight lines (i), (ii) and (iii) are concurrent, hence (x_1, y_1) must satisfy the equation (iii).

Therefore,

$$a_3x_1 + b_3y_1 + c_3 = 0$$

$$\Rightarrow \quad a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

$$\Rightarrow \quad a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrence of three straight lines.

The above condition is not sufficient to ensure that the three given lines are concurrent. However, it can be shown that, if the above determinant vanishes, then the lines are concurrent.

Example 1. Show that the lines $2x - 3y + 5 = 0,\ 3x + 4y - 7 = 0$ and $9x - 5y + 8 = 0$ are concurrent.

Solution: The given lines are $2x - 3y + 5 = 0,\ 3x + 4y - 7 = 0$ and $9x - 5y + 8 = 0$

We have, $$\begin{vmatrix} 2 & -3 & 5 \\ 3 & 4 & -7 \\ 9 & -5 & 8 \end{vmatrix} = 0$$

$$= 2\,(32 - 35) - (-3)(24 + 63) + 5(-15 - 36)$$

$$= 2(-3) + 3(87) + 5(-51)$$

$$= 0$$

Therefore, the given three straight lines are concurrent.

Example 2. For what value of 'a' the lines $2x + y - 1 = 0, ax + 2y - 2 = 0$ and $2x - 3y - 5 = 0$ are concurrent.

Solution: The given lines are $2x + y - 1 = 0, ax + 2y - 2 = 0$ and $2x - 3y - 5 = 0$, We have;

We have, $$\begin{vmatrix} 2 & 1 & -1 \\ a & 2 & -2 \\ 2 & -3 & -5 \end{vmatrix} = 0$$

$$= 2\,(-10 - 6) - 1(-5a + 4) - 1(-3a - 4) = 0$$

$$= -32 + 5a - 4 + 3a + 4 = 0$$

$$a = 4$$

7.7.2 Find the equation of median, altitude and right bisector of a triangle the equations of altitudes of triangle:

Consider ΔABC be a triangle as shown in Fig. 7.28.

The equation of the altitude through vertex A (Fig. 7.28) can be calculated using the following steps;

- Find the slope of $\overline{BC}$.
- Since $\overline{AD}$ and $\overline{BC}$ are perpendicular to each other so using the coordinates of point B and C we will get; slope of $\overline{AD}$ × slope of $\overline{BC} = -1$.

  slope of $\overline{AD} = \dfrac{-1}{slope\ of\ \overline{BC}}$

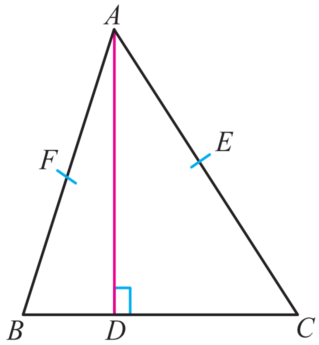

Fig 7.28

- Using the slope intercept form of equation of a line. The equation of altitude of $\overline{AD}$ is $y = mx + c_1$, where m is the slope of $\overline{AD}$. For finding c_1 we will use the coordinates of point A.
- Thus, using the slope of equation of $\overline{AD}$, i.e. m, and y intercept c_1 we will get, the equation of altitude of $\overline{AD}$, as follows; $y = mx + c_1$

Similarly, we can find the equations of altitudes through the vertices B and C.

Example 1. $A(3,2)$ $B(6,-2)$ and $C(-7,3)$ are the vertices of ΔABC. Find the equations of the altitudes through A.

Solution: Here we have $P(x_1,y_1) = A(3,2), B(x_2,y_2) = B(6,-2)$ and $C(x_3,\ y_3) = C(-7,3)$.

For equation of the altitude through A (Fig. 7.28);

we have; slope of $\overline{BC} = \frac{y_2 - y_1}{x_2 - x_1}$

slope of $\overline{BC} = \frac{(3+2)}{(-7-6)}$

slope of $\overline{BC} = \frac{-5}{13}$

and slope of $\overline{AD}$ (where D is the point on the $\overline{BC}$) will be;

slope of $\overline{AD} = \frac{-1}{\frac{-5}{13}}$

slope of $\overline{AD} = \frac{13}{5}$

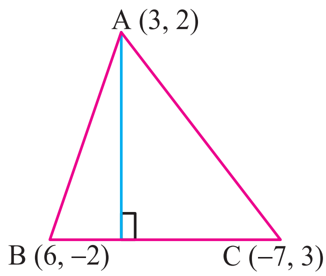

Fig 7.29

Now the equation of altitude of $\overline{AD}$

$$y = mx + c_1$$

or $$y = \frac{13}{5}x + c_1$$

For finding c_1 we will use the coordinates of point A, i.e.,

$$2 = \frac{13}{5}(3) + c_1$$

$$c_1 = \frac{39}{5} - 2 = \frac{39 - 10}{5} = \frac{29}{5}$$

Thus, the equation of the altitude through the point A is

$$y = \frac{13}{5}x + \frac{29}{5}$$

or $$13x + 5y - 29 = 0$$

- **Equation of medians of triangle**

The equation of the median through vertex A (Fig. 7.30) can be calculated using the following steps;

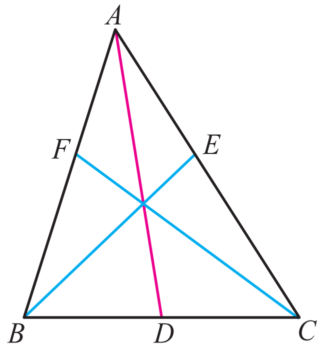

Fig 7.30

- Using midpoint formula, find the midpoint of $\overline{BC}$, which gives the coordinates of point D.
- Find the slope of median $\overline{AD}$ using the points A and D.
- Using point slope form equation $y - y_1 = m(x - x_1)$, find the equation of the median $\overline{AD}$.

Similarly, we can find the equations of medians through the vertices B and C.

Example 2. Find the equations of median of ΔABC with vertices $A(1,2)$ $B(-2,5)$ and $C(-7,4)$ through A.

Solution: Here we have $(x_1, y_1) = A(1,2), (x_2, y_2) = B(-2,5)$ and $(x_3, y_3) = C(-7,4)$.

Let D and F be the midpoints of the sides $\overline{BC}$,

Now, for the equation of Median $\overline{AD}$:

$$\text{Midpoint of } \overline{BC} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$= \left(\frac{(-2)+(-7)}{2}, \frac{5+4}{2}\right)$$

$$= \left(\frac{-9}{2}, \frac{9}{2}\right)$$

Now, find the slope of $\overline{AD}$, i.e.,

$$m = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$= \frac{\frac{9}{2} - 2}{\frac{-9}{2} - 1}$$

$$= -\frac{5}{11}$$

Now, equation of median $\overline{AD}$ is as follows:

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{5}{11}(x - 1)$$

or $\quad 11y - 22 = -5x + 5$

or $\quad 5x + 11y - 27 = 0$

- **Equation of the right bisector**

The equation of the right bisector through $\overline{BC}$ (Fig. 7.31) can be calculated using the following steps;

- Find the slope of $\overline{BC}$.
- Take its negative reciprocal as it is making a perpendicular line.
- Find the midpoint of $\overline{BC}$.
- Use $y = mx + c$ formula for finding the equation of line using the steps 2 and 3.

Similarly, we can find the equations of right bisectors through the vertices $\overline{AC}$ and $\overline{AB}$.

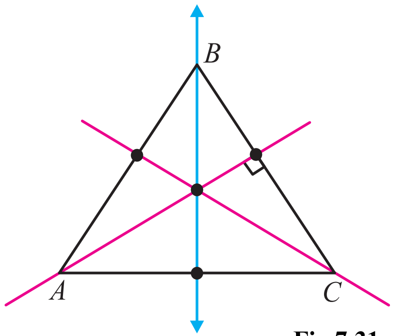

Fig 7.31

Example 3. Find the equations of right bisector of ΔABC with vertices $A(1, 2)$ $B(10, -6)$ and $C(-7, 2)$ through A.

Solution: Here we have $(x_1, y_1) = A(1,1), (x_2, y_2) = B(10, -6)$ and $(x_3, y_3) = C(-7,2)$.

Let D and F be the midpoints of the sides $\overline{BC}$,

Now, for the equation of right bisector through $\overline{BC}$, we have;

$$\text{slope of } \overline{BC} = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$= \left(\frac{2 - (-6)}{-7 - 10}\right)$$

$$= \left(\frac{8}{-17}\right)$$

Then its negative reciprocal is $\left(\frac{17}{8}\right)$.

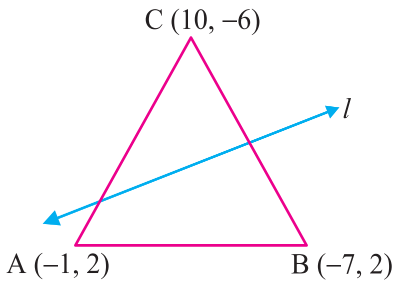

Fig 7.32

Now, find the midpoint of segment $\overline{BC}$;

$$\text{Midpoint of } \overline{BC} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{(-7) + 10}{2}, \frac{2 + (-6)}{2}\right)$$

$$= \left(\frac{3}{2}, \frac{-4}{2}\right)$$

Now, c_1 will be calculated as follows:

$$y = m(x + c_1)$$

$$-2 = \frac{17}{8}\left(\frac{3}{2}\right) + c_1$$

or $\qquad c_1 = \frac{-83}{16}$

$$y = \frac{17}{8}x - \frac{83}{16}$$

or $\qquad 34x - 16y - 83 = 0$

7.7.3 Show that

- **three right bisectors,**
- **three medians,**
- **three altitudes, of a triangle are concurrent.**

- **three right bisectors of the triangle are concurrent**

Let $A(-a, 0)$, $B(a, 0)$ and $C(0, b)$ are the vertices of triangle as shown in Fig. 7.33.

Here l_1, y-axis and l_2 are the right bisectors of sides $\overline{BC}$, $\overline{AB}$ and $\overline{AC}$ respectively. First, we find the equation of each right bisectors.

Equation of right bisector through $\overline{AB}$

Equation of right bisectors through $\overline{AB}$ is

$$x = 0 \qquad \ldots\text{(i)}$$

- **Equation of right bisector through $\overline{BC}$**

Slope of $\overline{BC} = \frac{-b}{a}$

Slope of $l_1 = \frac{a}{b}$

Mid-point of $\overline{BC} = \left(\frac{a}{2}, \frac{b}{2}\right)$

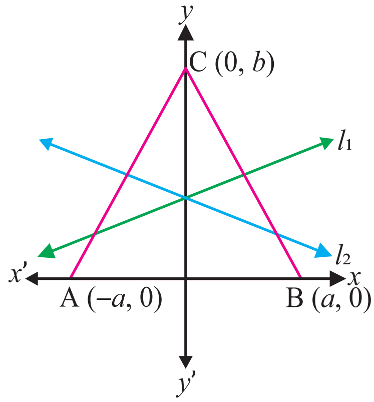

Fig 7.33

Equation of l_1 is

$$y = \left(\frac{a}{b}\right)x + c_1$$

$$\frac{b}{2} = \left(\frac{a}{b}\right)\left(\frac{a}{2}\right) + c_1$$

$\Rightarrow \qquad c_1 = \frac{b^2 - a^2}{2b}$

Equation of right bisector through $\overline{BC}$ is

$$y = \left(\frac{a}{b}\right)x + \left(\frac{b^2 - a^2}{2b}\right) \qquad \ldots\text{(ii)}$$

Similarly, equation of right bisector through $\overline{AC}$

$$y = \left(-\frac{a}{b}\right)x + \left(\frac{b^2-a^2}{2b}\right) \qquad \ldots\text{(iii)}$$

By solving equation (i) and (ii)

We get, $\quad x = 0$ and $y = \dfrac{b^2-a^2}{2b}$

Similarly, by solving equation (i) and (iii)

We get, $\quad x = 0$ and $y = \dfrac{b^2-a^2}{2b}$

Since $x = 0$ and $y = \dfrac{b^2-a^2}{2b}$ satisfy this equation (iii).

Since $\left(0, \dfrac{b^2-a^2}{2b}\right)$ is the intersecting point of three bisectors.

Therefore, right bisectors of triangle are concurrent.

- **Three medians of a triangle are concurrent**

Consider ΔABC be a triangle as shown in Fig. 7.34 with $A(-2a, 0), B(2a, 0)$ and $C(0,2b)$ are its vertices.

The equations of the median through vertex A, B and C (Fig. 7.34) will be calculated as follows:

Midpoint of $\overline{AB} = F = (0, 0)$

Midpoint of $\overline{BC} = D = (a, b)$

Midpoint of $\overline{AC} = E = (-a, b)$

The equation of median $\overline{AD}$ by using two-point form of equation.

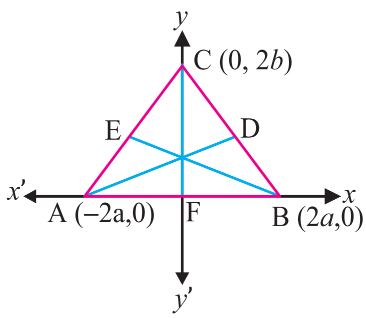

Fig 7.34

$$\frac{b-0}{a+2a} = \frac{y-0}{x+2a}$$

$$\Rightarrow \quad bx + 2ab = ay + 2ay$$

$$\Rightarrow \quad bx - 3ay + 2ab = 0 \qquad \ldots\text{(i)}$$

Similarly, equation of median $\overline{BE}$ is

$$bx - 3ay - 2ab = 0 \qquad \ldots\text{(ii)}$$

and equation of median $\overline{CF}$ is

$$x = 0 \qquad \ldots\text{(iii)}$$

Now the determinant of coefficient of equation (i), (ii) and (iii)

$$\begin{vmatrix} b & -3a & 2ab \\ b & -3a & -2ab \\ 1 & 0 & 0 \end{vmatrix} = 0$$

Hence the medians of the triangle are concurrent.

- **three altitudes of a triangle are concurrent.**

Consider ΔABC be a triangle as shown in Fig. 7.35 with $A(a,0), B(b,0)$ and $C(0,c)$. Let the base $\overline{AB}$ of triangle be taken as axis of x and a line through C perpendicular to base $\overline{AB}$ is taken as the axis of y. The point of intersection is G.

The equations of the altitudes through vertex A, B and C (Fig. 7.35) will be calculated as follows:

Now, the equation of altitude through C is as follows;

$$x = 0 \quad \text{...(i)}$$

Now, we will find the slope of $\overline{BC}$ and $\overline{CA}$ of the triangle, respectively,

Slope of $\overline{BC} = \dfrac{y_2-y_1}{x_2-x_1}$

$$= \frac{c-0}{0-b}$$

Slope of $\overline{BC} = -\dfrac{c}{b}$

and Slope of $\overline{CA} = -\dfrac{c}{a}$

So, the slope of the respective lines perpendicular to them are $\dfrac{b}{c}$ and $\dfrac{a}{c}$.

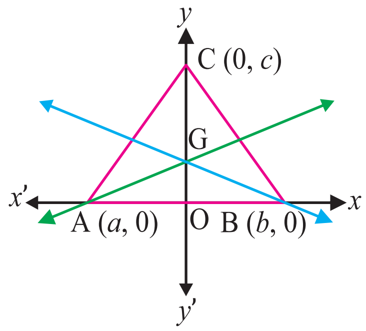

Fig 7.35

Now, the equation of altitude from A is as follows;

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{b}{c}(x - a)$$

or $\quad bx - cy - ab = 0 \quad$ ...(ii)

Now, the equation of altitude from B is as follows;

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{a}{c}(x - b)$$

or $\quad ax - cy - ab = 0 \quad$ ...(iii)

Now, the determinant of the coefficients of equations (i), (ii) and (iii) is

$$\begin{vmatrix} 1 & 0 & 0 \\ b & -c & -ab \\ a & -c & -ab \end{vmatrix}$$

which is zero. Hence the altitudes of a triangle are concurrent.

Exercise 7.6

1. If $A(2,5), B(3,7)$ and $C(0,8)$ are the vertices of a triangle then find the
   (i) Equation of median through A
   (ii) Equation of altitude through B
   (iii) Equation of right bisector of side $\overline{AC}$
2. Show that the following lines are concurrent. Also find their point of concurrency.
   (i) $x - y = 6, 4y + 22 = 3x$ and $6x + 5y + 8 = 0$
   (ii) $\frac{x}{a} + \frac{y}{b} = 1, \frac{x}{b} + \frac{y}{a} = 1$ and $y = x$.
   (iii) $5x + y + 11 = 0, x + 7y + 9 = 0$ and $2x + y + 5 = 0$.
3. If $A(-1,5), B(2,3)$ and $C(7,6)$ the vertices of triangle, then show right bisectors, medians and altitudes the triangle is concurrent.

7.8 Area of a Triangular Region

7.8.1 Find the area of a triangular region whose vertices are given

Consider ΔABC be a triangle as shown in Fig. 7.36 with the coordinates of triangle in anticlockwise direction, i.e.; $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$.

Let $\overline{AD}$ be perpendicular to $\overline{BC}$ (Fig. 7.36). By using the elementary geometry, the area of triangle ABC, i.e.; ▲;

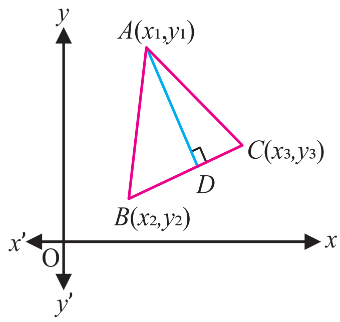

Fig 7.36

$$▲ = \frac{1}{2} \times (\text{base}) \times (\text{Altitude})$$

$$= \frac{1}{2} \times |\overline{BC}| \times |\overline{AD}|$$

where $\overline{BC} = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$

Also, the equation of the line l of which $\overline{BC}$ is a segment can be found, as it passes through $B(x_2,y_2)$ and $C(x_3,y_3)$.

So, the equation of l is;

$$y - y_2 = \frac{y_3 - y_2}{x_3 - x_2}(x - x_2)$$

or $$(y_2 - y_3)x + (x_3 - x_2)y + x_2y_3 - x_3y_2 = 0$$

Now $|\overline{AD}|$ = The perpendicular distance of $A(x_1,y_1)$ from the line l is;

$$= \frac{((y_2 - y_3)\, x_1 + (x_3 - x_2)y_1 + x_2y_3 - x_3y_2}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}}$$

Thus, area of triangle ABC is;

$$\blacktriangle = \frac{1}{2} \times (\text{base}) \times (\text{Altitude})$$

$$= \frac{1}{2}\{(y_2 - y_3)\, x_1 + (x_3 - x_2)y_1 + (x_2y_3 - x_3y_2)\}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Using the properties of determinant this result can also be written as;

$$\Delta = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

Corollary 1: If A, B and C are three collinear points, then the area of triangle ABC = 0, i.e., the condition for collinearity of three points $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ is;

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Corollary 2: The area of polygon whose vertices taken in order in anticlockwise direction are; (x_1, y_1), (x_2, y_2), (x_3, y_3) ... (x_n, y_n)

$$= \frac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)\}$$

Area of a quadrilateral can thus be written as

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}$$

Example: Find the area of the triangle whose vertices are

(i) $(2, 9), (-2, 1)$ and $(6, 3)$ (ii) $(3, 8), (7, 2)$ and $(-1, 1)$

Solution (i): The area of triangle with the vertices $(2, 9), (-2, 1)$ and $(6, 3)$ is given by;

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 9 & 1 \\ -2 & 1 & 1 \\ 6 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{2(1 - 3) - 9(-2 - 6) + 1(-6 - 6)\}$$

$$= \frac{1}{2}(56)$$

$= 28$ square units.

Solution (ii): The area of triangle with the vertices $(3, 8), (7, 2)$ and $(-1, 1)$ is given by;

$$\blacktriangle = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ 7 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{3(2 - 1) - 8(7 + 1) + 1(7 + 2)\}$$

$= \frac{1}{2}(-52)$

$= 26$ square units.

Thus, omitting the negative sign, the magnitude of the area = 26 square units.

Exercise 7.7

1. Find the area of the triangle whose vertices are:
   (i) $(11, -12), (6, 2)$ and $(-5, 10)$ (ii) $(3, 1), (-2, 5)$ and $(-4, -5)$
   (iii) $(-5, -2), (4, -6)$ and $(1, 7)$
   (iv) $(-a, b + c), (a, b - c)$ and $(a, -c)$
   (v) $(a\cos\theta_1, b\sin\theta_1), (a\cos\theta_2, b\sin\theta_2)$ and $(a\cos\theta_3, b\sin\theta_3)$
2. Find the area of a quadrilateral whose consecutive vertices are;
   (i) $(3, -3), (7, 5), (1, 2)$ and $(-3, 4)$
   (ii) $(2, 3), (-1, 2), (-3, 2)$ and $(3, -3)$
3. Prove, by the method of the area of a triangle, that the following points are collinear;
   (i) $(2, 3), (5, 0)$, and $(4, 1)$ (ii) $(2, 1), (4, -1)$, and $(1, 2)$
   (iii) $(-1, -1), (5, 7)$ and $(8, 11)$
4. Find the area of triangle formed by the lines;
   (i) $y = 0, y = 2x$ and $y = 6x + 5$
   (ii) $y - x = 0, y + x = 0$ and $x - c = 0$.
   (iii) $y = 2x + 3, 2y + 3x = 3$ and $x + y + 2 = 0$.

7.9 Homogenous Equation

7.9.1 Recognize homogeneous linear and quadratic equations in two variables

When a straight line passes through the origin, then its equation will become $ax + by = 0$ and is known as homogeneous linear equation in two variables.

Similarly, in general quadratic equation in two variables

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

if $g = f = c = 0$ then it becomes homogeneous quadratic equation in two variables, we write as $ax^2 + 2hxy + by^2 = 0$

7.9.2 Investigate that the 2nd degree homogeneous equation in two variables x and y represents a pair of straight lines through the origin and find acute angle between them.

Let l_1 and l_2 are two straight lines passing through the origin and $y = m_1x$ and $y = m_2x$ are the equations of l_1 and l_2 respectively, as shown in the figure 7.37.

Now, take any point P on the line l_1. The path it travels on these two lines is its locus. To find the equation of the locus we multiply equation of l_1 and l_2

i.e., $(y - m_1x)(y - m_2x) = 0$

$\Rightarrow$ $m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$

is the equation pair of straight lines passing through the origin, which is homogeneous second order quadratic equation.

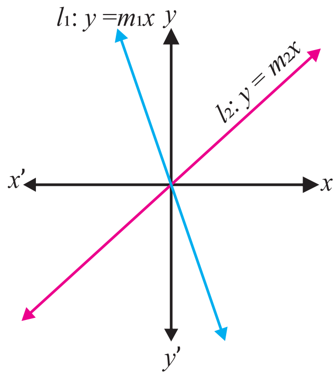

Fig. 7.37

Theorem 1: A homogeneous equation of degree two in x and y, i.e., $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through the origin if $h^2 - ab \geq 0$.

Proof: Let the second-degree homogeneous equation in x and y be $ax^2 + 2hxy + by^2 = 0$ where a, h, b are real numbers and not all zero.

Case I: Let $a = 0$ and $b = 0$, but $h \neq 0$.

$hxy = 0$ as $h \neq 0, xy = 0$

$xy = 0 \Rightarrow x = 0$ or $y = 0$

Separate equations are $x = 0$ and $y = 0$ which are the equations of the coordinate axes.

$ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through origin when $a = 0$ and $b = 0$.

Case II: Let $a = 0$, Given equation becomes $2hxy + by^2 = 0$

$y(2hx + by) = 0$ gives us $y = 0$ and $2hx + by = 0$.

The two factors are linear in x and y and do not contain constant terms.

Hence $ax^2 + 2hxy + by^2 = 0$ represent a pair of lines through origin for $a = 0$. The same thing can be proved by taking $b = 0$.

Case III: Let $a \neq 0$ Now, $ax^2 + 2hxy + by^2 = 0$, multiply both sides of equation by a;

$$a^2x^2 + 2ahxy + aby^2 = 0$$

$$a^2x^2 + 2(ax)(hy) + aby^2 = 0$$

$$\Rightarrow \quad a^2x^2 + 2(ax)(hy) + h^2y^2 + (aby^2 - h^2y^2) = 0$$

$$\Rightarrow \quad (ax + hy)^2 - y^2(h^2 - ab) = 0$$

$$\Rightarrow \quad (ax + hy)^2 - \left(y\sqrt{(h^2 - ab)}\right)^2 = 0$$

$$\left(ax + hy - y\sqrt{(h^2 - ab)}\right)\left(ax + hy + y\sqrt{(h^2 - ab)}\right) = 0$$

The two factors are linear in x and do not contain constant term. and lines will be real if and only if $h^2 - ab \geq 0$. It is a pair of lines. Their separate equations

are $(ax + hy - y\sqrt{(h - ab)} = 0$ and $\left(ax + hy + y\sqrt{(h^2 - ab)}\right) = 0$

which are separately satisfied by the origin. Hence $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through origin, for $a \neq 0$.

Where their slopes are

$$m_1 = \frac{-h + \sqrt{h^2 - ab}}{b}$$

$$m_2 = \frac{-h - \sqrt{h^2 - ab}}{b}$$

Combining the cases (I), (II) and (III) we get that every second-degree homogeneous equation in x and y in general represents a pair of lines through origin.

Theorem 2: If m_1 and m_2 are the slopes of the two lines represented by $ax^2 + 2hxy + by^2 = 0$ show that $m_1 + m_2 = \frac{-2h}{b}$ and $m_1 m_2 = \frac{a}{b}$.

Deduce that lines are perpendicular if $a + b = 0$ and lines are coincident if $h^2 = ab$.

Proof: m_1 and m_2 are the slopes of the two lines represented by $ax^2 + 2hxy + by^2 = 0$, Equation of first line is $y = m_1 x$, i.e., $m_1 x - y = 0$ and equation of second line is $y = m_2 x$, i.e., $m_2 x - y = 0$. Combined equation is:

$$(m_1 x - y)(m_2 x - y) = 0$$

$$m_1 x(m_2 x - y) - y(m_2 x - y) = 0$$

or $$m_1 m_2 x^2 - (m_1 + m_2)y - y^2 = 0$$

Also $ax^2 + 2hxy + by^2 = 0$ is combined equation of the two lines. The equation $ax^2 + 2hxy + by^2 = 0$ and $m_1 m_2 x^2 - (m_1 + m_2)y - y^2 = 0$ are identical.

Hence their corresponding coefficients are proportional.

$$\frac{a}{m_1 m_2} = \frac{2h}{-(m_1 + m_2)} = \frac{b}{1}$$

Now

$$\frac{a}{m_1 m_2} = \frac{b}{1}$$

$$m_1 m_2 = \frac{a}{b}$$

Now

$$\frac{2h}{-(m_1 + m_2)} = \frac{b}{1}$$

$$m_1 + m_2 = \frac{-2h}{b}$$

Case – I: Lines are perpendicular if $m_1 m_2 = -1$

$$\frac{a}{b} = -1$$

$$a = -b$$
$$a + b = 0$$

Case – II: Lines are coincident if $m_1 = m_2$

$$m_1 - m_2 = 0$$
$$(m_1 - m_2)^2 = 0$$

If $\quad (m_1 + m_2)^2 - 4m_1m_2 = 0$

Then substituting the values we will get;

$$\left(\frac{-2h}{b}\right)^2 - 4\frac{a}{b} = 0$$
$$\frac{4h^2}{b^2} - 4\frac{a}{b} = 0$$
$$h^2 - ab = 0 \quad \Longrightarrow \quad h^2 = ab$$

Nature of Roots:

- If $h^2 > ab$ then the roots of the equation are real and distinct.
- If $h^2 = ab$ then the roots of the equation are real and equal.
- If $h^2 < ab$ then the roots of the equation are imaginary.

Pair of Straight Lines Formulas:

We know that angle between two lines is:

$$\tan\theta = \pm\frac{m_2 - m_1}{1 + m_1m_2}$$

and on substituting the value of $m_1 + m_2 = \frac{-2h}{b}$ and $m_1m_2 = \frac{a}{b}$,

we get,

$$\theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{b}\right)$$

which is the angle between the two lines expressed as $ax^2 + 2hxy + by^2 = 0$.

Example: Find the equations of the pair of lines represented jointly by the given equation. State nature of lines and also find the acute angle between lines $5x^2 + 13xy - 6y^2 = 0$.

Solution: Factorizing the given equation, we get;

$$5x^2 + 13xy - 6y^2 = 0 \Longrightarrow (x + 3y)(5x - 2y) = 0$$

So, the given equations represent the lines $x + 3y = 0$ and $5x - 2y = 0$.

Now compare the given equation with $ax^2 + 2hxy + by^2 = 0$, so, $a = 5, 2h = 13$ and $b = -6$.

Put the values in $D = h^2 - ab \Longrightarrow D = \frac{73}{2}$.

Hence $D > 0$ then the lines are real and distinct.

The acute angle between the line

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{b}$$

$$\tan\theta = \frac{2\sqrt{\left(\frac{13}{2}\right)^2 - 5(-6)}}{(-6)} = -\frac{17}{6}$$

$$\theta = \tan^{-1}\left(\frac{17}{6}\right) = 70.55°$$

Exercise 7.8

1. Find the equations of the pair of lines represented jointly by each of the following equations. State nature of lines and also trace the pair of lines;

   (i) $x^2 - 5xy + 6y^2 = 0$ (ii) $4x^2 - xy - 5y^2 = 0$

   (ii) $9x^2 - 6xy + y^2 = 0$ (iii) $10x^2 - 3xy - y^2 = 0$

   (v) $7x^2 - 3xy + 5y^2 = 0$

2. Find the combined equation of the pair of lines through the origin which are perpendicular to the lines represented by

   (i) $2x^2 - 5xy + y^2 = 0$ (ii) $6x^2 - 13xy + 6y^2 = 0$

3. Trace the pair of lines given by the following equations;

   (i) $x^2 - 3xy + 2y^2 = 0$ (ii) $x^2 - 6xy + 9y^2 = 0$

   (iii) $6x^2 - xy - y^2 = 0$ (vi) $8x^2 - 3xy - y^2 = 0$

4. Find the angle between the lines represented by;

   (i) $x^2 - 5xy + 6y^2 = 0$ (ii) $3x^2 + 7xy + 2y^2 = 0$

   (iii) $x^2 + 2xy - 3y^2 = 0$ (iv) $x^2 + xy - 2y^2 = 0$

5. The gradient of one of the lines of $ax^2 + 2hxy + by^2 = 0$ is twice that of the other. Show that $8h^2 = 9ab$.

Review Exercise 7

1. Select correct option.

   (i) The slope of the line that passes through $(-3, -4)$ and $(7, 6)$ is

   (a) 0 (b) undefined (c) -1 (d) 1

   (ii) A line with a slope of $\frac{1}{2}$ and a y-intercept 7 is

   (a) $2y = x - 7$ (b) $y = \frac{1}{2}x + 7$

   (c) $y = \frac{1}{2}x - 7$ (d) $x - 2y = 14$

(iii) Two lines are said to be parallel if and only if their slopes are

(a) Equal (b) Unequal

(c) Does not exit (d) negative reciprocals of each other

(iv) Two lines l_1 and l_2 are said to be perpendicular if and only if

(a) $m_1 m_2 = -1$ (b) $m_1 m_2 = 1$ (c) $m_1 = -m_2$ (d) $m_1 = \frac{1}{m_2}$

(v) The slope of a line that is perpendicular to a vertical line is

(a) 0 (b) 1 (c) 90° (d) undefined

(vi) The slope of the line which makes an angle 45° with the line $3x - y = -5$ is

(a) 2 (b) $\frac{1}{2}$ (c) $\frac{1}{2}, -2$ (d) -2

(vii) The point on the line $2x - 3y = 5$ is equidistant from $(1, 2)$ and $(3, 4)$ is

(a) $(-2, 2)$ (b) $(4, 1)$ (c) $(1, -1)$ (d) $(4, 6)$

(viii) In a plane three or more points are said to be collinear if

(a) they lie on a circle (b) they form a closed loop together

(c) they lie on a straight line (d) they do not make any defined shape

(ix) If the line coincides with x-axis then its equation is

(a) $y = b$ (b) $-b$ (c) $y = 0$ (d) ∞

(x) The general equation of line also known as standard equation of line is,

(a) $ax + by + c = 0$ (b) $y = ax + c$

(c) $y - y_1 = m(x - x_1)$ (d) $\frac{x}{a} + \frac{y}{b} = 1$

2. If the distance between the points $(5, -2)$ and $(1, a)$ is 5, find the values of a.
3. $M(3, 8)$ is the midpoint of the line AB. A has the coordinates $(-2, 3)$, find the coordinates of B.
4. The diameter of a circle has endpoints: $(2, -3)$ and $(-6, 5)$. Find the coordinates of the center of this circle?
5. Find the coordinates of the points which divides the join of $P(-1, 7)$ and $Q(4, -3)$ in the ratio $2 : 3$.
6. Find the points of trisection of the line segment AB, where $A(-6, 11)$ and $B(10, -3)$.
7. Two vertices of a triangle are $(1, 4)$ and $(3, 1)$. If the centroid of the triangle is the origin, find the third vertex.
8. Find the slope of the line which is perpendicular to the given line whose equation is $-2y = -8x + 9$.
9. If a straight line intercepts the x-axis at (6,0) and intercepts the y-axis at $(0, 5)$, write the equation ofthe straight line in two intercept form.

10. Determine the slope (gradient) and y-intercept of each line.
    (a) $y = 12x - 6$ (b) $y = 5 - 2x$
    (c) $4x - y + 13 = 0$ (d) $y = 4x$
11. Find the x and y-intercept of the equation $4x + 8y + 2 = 0$.
12. What is the value of k so that the line through $(3, k)$ and $(2, 7)$ is parallel to the line through $(-1, 4)$ and $(0, 6)$?
13. Tell whether the following pair of lines is parallel, perpendicular or neither?
    (a) $y = 2x + 7$ and $y = 7x - 2$ (b) $x = 1$ and $x = 5$.
    (c) $x = -3$, and $y = -3$
14. For what value of k will the three lines $y = 3x - 1,\ 2y = x + 3$ and $3y = kx + 4$ be concurrent.
15. Find the value of a and b for which the lines given by $ax - 2y - 1 = 0$ and $6x - 4y - b = 0$
    (i) are parallel (ii) are perpendicular
    (iii) coincided (iv) have no common point.
16. Find the equation of the line passing through the intersection of the lines $x + 2y - 5 = 0$ and $3x - 2y + 1 = 0$ and
    (i) parallel to the line $4x + 3y - 5 = 0$
    (ii) perpendicular to the line $2x - 3y + 7 = 0$.
17. Find the distance of the point from the line
    (i) $15x - 8y - 5 = 0,\ (2, 1)$ (ii) $2x - 7y + 1 = 0, (7, 4)$