Unit 8

Circle

8.1 Conics

Introduction to conics

According to the Greek mathematicians, conics or conic sections are the curves that can be obtained as intersections of a cone and a plane, the most important of which are circles, ellipses, parabolas and hyperbolas.

With the advent of analytic geometry and calculus, conics got great importance in the physical sciences. In 1609 Johannes Kepler presented his landmark discovery that the path of each planet about the sun is an ellipse. Galileo and Newton showed that objects under the gravitational forces can also move along paths that are parabolas and hyperbolas.

Nowadays, properties of conics are used in the construction of telescopes, radar antennas, medical equipment, navigational systems and in the determination of satellite orbits.

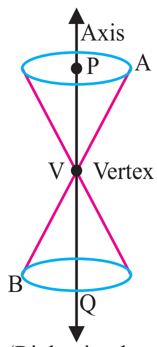
8.1.1 Define conics and demonstrate members of its family, i.e., circle, parabola, ellipse and hyperbola

As discussed earlier, the Greek mathematicians studied conics and defined them as sections of a right circular cone by planes.

In analytic geometry, a conic is the path or locus of a point moving so that the ratio of its distance from a fixed point to the distance from a fixed line is constant.

A double right circular cone or simply a cone is the surface in three-dimensional space which is generated by all the lines through a fixed point, called "vertex" and the circumference of a circle as shown in Fig. 8.1.

All such lines are called generators. The line through the centre of the circle and perpendicular to its plane is called "axis" of the cone. It also passes through vertex of the cone.



(Right circular cone)

Fig. 8.1

In Fig. 8.1, a double right circular cone or simply a cone is shown which has two parts called nappes. $\overleftrightarrow{PQ}$ is the axis and $\overline{AB}$ is a generator of the cone, whereas V is the vertex.

A circle is a conic or curve which is obtained by cutting a cone with a plane that is perpendicular to the axis and does not contain the vertex as shown in Fig. 8.2.

An ellipse is a conic or curve which is obtained if the intersecting plane is slightly tilted and cuts only one nappe of the cone as shown in Fig. 8.3.

A parabola is a conic or curve which is obtained if the cutting plane is parallel to a generator of the cone but intersects its one nappe only as shown in Fig. 8.4.

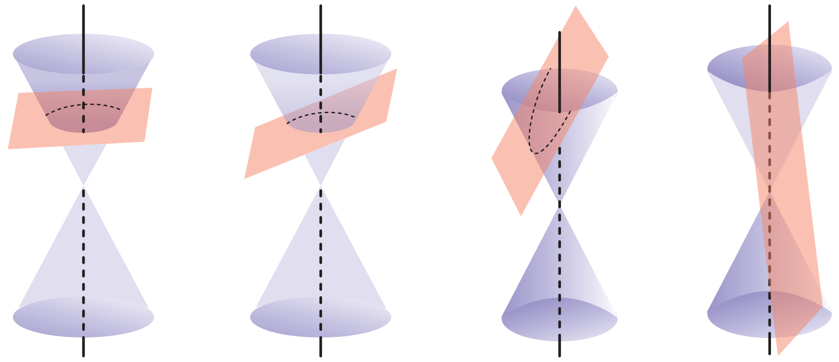


Fig. 8.2 **Fig. 8.3** **Fig. 8.4** **Fig. 8.5**

If the plane intersects both nappes but does not contain the vertex, the resulting intersection is a hyperbola as shown in Fig. 8.5.

If the cutting plane passes through the vertex then it is possible to obtain a point, a line or a pair of lines. These are called degenerate conics as shown in Fig. 8.6.

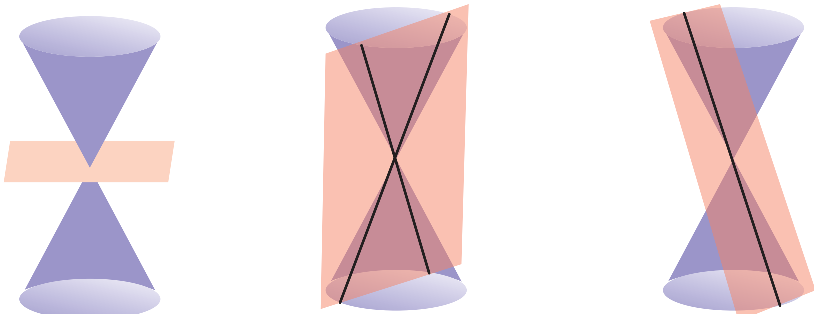


Fig. 8.6

8.2 Circle and its standard form of Equation

We know that curves in plane are described by their equations. Likewise, circle has its own equation which has different forms. First, we will discuss standard form of the equation of circle.

8.2.1 Define circle and derive its equation in standard form

i.e., $(x-h)^2+(y-k)^2=r^2$

We are already familiar with the concept of circle and its related terms. Let us recall the definition of circle.

A circle is a set of the points in plane which are equidistant from a given fixed point.

The fixed point is called the centre of the circle and the constant distance of each point of circle from the centre is called radius of the circle.

- **Standard form of equation of circle:**

Let $C(h,k)$ be the centre and r the radius of a circle as shown in the figure 8.7.

If $P(x,y)$ is any point on the circle then using distance formula.

We have $\quad |\overline{CP}| = \sqrt{(x-h)^2 + (y-k)^2}$

or $\quad r = \sqrt{(x-h)^2+(y-k)^2} \qquad (\because |\overline{CP}| = r)$

squaring both sides

we get $\quad (x-h)^2 + (y-k)^2 = r^2 \quad \ldots$(i)

Equation (i) represents the circle with centre (h,k) and radius r. This equation is called standard form of equation of circle or standard equation of circle.

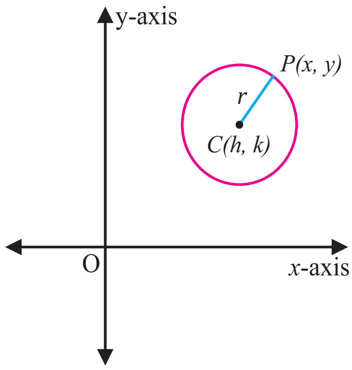


Fig. 8.7

If $(h,k) = 0$ then equation (i) becomes

$$x^2 + y^2 = r^2 \qquad \ldots\text{(ii)}$$

Equation (ii) represents the circle with centre at origin and radius r

Example 1. Find equation of the circle whose centre is at $(-3,5)$ and radius $\sqrt{2}$ units.

Solution:

Here $\quad (h,k) = (-3,5)$

and $\quad r = \sqrt{2}$ units

Using standard form of equation of circle.

We get $\quad (x+3)^2 + (y-5)^2 = \left(\sqrt{2}\right)^2$

$\Rightarrow x^2 + 6x + 9 + y^2 - 10y + 25 = 2$

$\Rightarrow x^2 + y^2 + 6x - 10y + 32 = 0$

Example 2. $P(3,4)$ is the point of a circle with centre at origin. Find the radius of the circle.

Solution: Let r be the radius of the circle. We know that equation of circle with centre at origin and radius r is

$$x^2 + y^2 = r^2 \qquad \ldots\text{(i)}$$

$\because P(3,4)$ lies on this circle

$\therefore$ equation (i) becomes

$3^2 + 4^2 = r^2$

$\Rightarrow r^2 = 25$

$\Rightarrow r = 5$ units.

So, the radius of the circle is 5 units.

8.3 General Form of an Equation of a Circle

General form of an equation of a circle is infact the simplified form of standard equation of circle. Here, we will recognize its equation and find its centre and radius.

8.3.1 Recognize general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and find its centre and radius

Let us consider the standard equation of circle with centre (h, k) and radius r

$$(x-h)^2 + (y-k)^2 = r^2$$

On simplification, we get

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$

using $\quad h = -g, k = -f$ and $h^2 + k^2 - r^2 = c$

we get $\quad \boxed{x^2 + y^2 + 2gx + 2fy + c = 0} \quad \ldots$(i)

This equation is called the general form of the equation of circle or simply general equation where

$g, f, c \in \mathbb{R}$ and $g^2 + f^2 - c \geq 0$

In order to find centre and radius of the circle of equation (i)

We convert equation (i) in standard form.

From equation (i), we get

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) - g^2 - f^2 + c = 0$$

$\Rightarrow \quad (x+g)^2 + (y+f)^2 = g^2 + f^2 - c \quad \ldots$ (ii)

Comparing equation (ii) with standard equation of circle

We get centre $= (-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

If we compare general equation of second degree i.e.,

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0 \quad \ldots$(iii)

with general equation of circle represented by equation (i), we notice that equation (iii) represents a circle if $h = 0$ and $a = b = 1$.

In case $a = b = k$ and $h = 0$, equation (iii) becomes

$kx^2 + ky^2 + 2gx + 2fy + c = 0 \quad \ldots$(iv)

which is also equation of circle, equation (iv) will take form of equation (i) if we divide both sides by k.

Example 1. Find centre and radius of the circle $x^2 + y^2 + 14x - 6y - 6 = 0$.

Solution: Comparing given equation of circle with general equation, we get

$\Rightarrow \quad 2g = 14, 2f = -6$ and $c = -6$

i.e., $\quad g = 7,\ f = -3$ and $c = -6$

We know that

$$\text{Centre} = (-g, -f) = (-7, 3)$$

and

$$\text{radius} = \sqrt{g^2 + f^2 - c}$$
$$= \sqrt{49 + 9 + 6}$$
$$= \sqrt{64} = 8 \text{ units.}$$

Thus, centre of circle is $(-7, 3)$ and radius 8 units.

Example 2. Find the value of k, if radius of the circle $3x^2 + 3y^2 - 18x + 12y + k = 0$ is 5 units.

Solution: We have

$$3x^2 + 3y^2 - 18x + 12y + k = 0$$

Dividing both sides, by 3

so, we get

$$x^2 + y^2 - 6x + 4y + \frac{k}{3} = 0$$

Comparing with the general equation of circle

$$2g = -6, 2f = 4 \text{ and } c = \frac{k}{3}$$

we get $\quad g = -3, f = 2 \text{ and } c = \frac{k}{3}$

We know that

$$r = \sqrt{g^2 + f^2 - c}$$

so, $\quad 5 = \sqrt{9 + 4 - \frac{k}{3}} \qquad (\because r = 5 \text{ units})$

$\Rightarrow \quad 25 = 13 - \frac{k}{3}$

$\Rightarrow \quad -12 = \frac{k}{3}$

or $\quad k = -36$

so, the value of k is -36.

Exercise 8.1

1. Describe the condition under which a plane cuts right circular cone to produce.
   (i) circle (ii) parabola (iii) ellipse
   (iv) hyperbola (v) a degenerate conic
2. Find the equation of the circle if:
   (i) Centre is at origin and radius $5\sqrt{2}$ units.
   (ii) Centre is $(-5, 7)$ and radius 6 units.

(iii) $(2,-3)$ and $(-4,7)$ are the ends of its diameter.

(iv) Centre is at origin and contains a point $(5,6)$.

(v) Centre is at $(2,3)$ and contains the point $(5,7)$.

(vi) Centre is at (p,q) and radius of $\sqrt{p^2+q^2}$ units.

3. Find the centre and radius of each of the following circles. Also draw the circles.

(i) $x^2+y^2-25=0$

(ii) $(x+3)^2+(y-5)^2=49$

(iii) $x^2+y^2-6x+8y+10=0$

(iv) $x^2+y^2-8x+9=0$

(v) $5x^2+5y^2+20x-15y+10=0$

4. Find the value of k if the radius of the following circle is 10 units.

$2x^2+2y^2-8x+4y+3k=0$

5. Find the equation of the circle passing through $(-3,-4)$ and is concentric with the circle whose equation is $x^2+y^2-6x+8y-24=0$. Also identify the outer circle.

6. Show that the equation $x=a\cos\theta$ and $y=a\sin\theta$ represent a circle with centre at origin and radius equal to a

7. Prove that the equation of a circle

(i) through the origin has no constant term.

(ii) with centre on x-axis has no term in y.

(iii) with centre on y-axis has no term in x.

(iv) with centre at origin has no term in x and y both.

8.4 Equation of Circle determined by a given condition

In this section, we will find the equation of circle which is determined by the following conditions.

8.4.1 Find the equation of a circle passing through:

- **three non-collinear points,**
- **two points and having its centre on a given line,**
- **two points and equation of tangent at one of these points is known,**
- **two points and touching a given line.**

(a) Equation of circle passing through three non-collinear points

We know that one and only one circle can pass through three non-collinear points. So a unique circle can be determined if three non-collinear points are given. The method of finding equation of circle under this condition is explained in the following example.

Example: Find the equation of circle through the points $(3,0),(0,-2),(-3,4)$.

Solution: Let the equation of given circle be

$x^2+y^2+2gx+2fy+c=0$ ...(i)

$\because$ $(3,0)$ lines on it

$\therefore$ $9+6g+c=0$ ... (ii)

As $(0,-2)$ and $(-3,4)$ also lie on it. So, we have

$4-4f+c=0$ ... (iii)

$25-6g+8f+c=0$ ... (iv)

Subtracting equation (iii) from equation (ii)

we get $5+6g+4f=0$ ...(v)

Subtracting equation (iii) from equation (iv)

we get $21-6g+12f=0$ ...(vi)

Solving equation (v) and (vi) simultaneously,

we get $f=-\frac{13}{8}$ and $g=\frac{1}{4}$

From equation (iii), we get $c=-\frac{21}{2}$

By using these values of g,f and c in equation (i)

We get $x^2+y^2+\frac{x}{2}-\frac{13y}{4}-\frac{21}{2}=0$

or $4x^2+4y^2+2x-13y-42=0$

This is the required equation of the circle.

(b) Equation of a circle passing through two points and having its centre on a given line.

We know that infinitely many circles can be drawn from two points but particular circle or circles can be obtained under certain condition. In the following example we find the equation of a circle passing through two points with the condition that its centre lies on a given line.

Example: Find the equation of a circle passing through two points $(1,4)$ and $(3,2)$ and having its centre on the line $2x+y-1=0$.

Solution: Let equation of the circle be

$x^2+y^2+2gx+2fy+c=0$ ...(i)

The circle passes through the points $(1,4)$ and $(3,2)$.

$\therefore$ we have

$17+2g+8f+c=0$ ...(ii)

and $13+6g+4f+c=0$ ...(iii)

As centre $(-g,-f)$ of the circle lies on the line

$2x + y - 1 = 0$

So, we have $\quad -2g - f - 1 = 0$

or $\quad 2g + f + 1 = 0 \quad \ldots$(iv)

Subtracting equation (iii) from equation (ii)

We get $\quad 4 - 4g + 4f = 0$

or $\quad 1 - g + f = 0 \quad \ldots$(v)

Solving equation (iv) and (v) simultaneously,

We get $\quad g = 0$ and $f = -1$

By using $\quad g = 0$ and $f = -1$, equation (ii) becomes

$17 - 8 + c = 0$

$\Rightarrow \quad c = -9$

By using these values of g, f and c in equation (i)

we get $\quad x^2 + y^2 - 2y - 9 = 0$

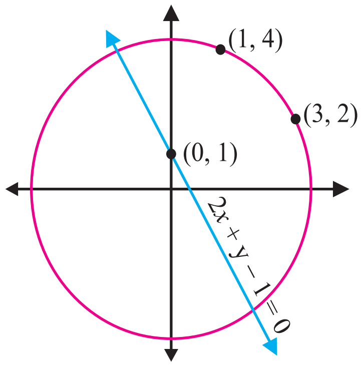


Fig. 8.8

(c) Equation of circle passing through two points and equation of tangent at one of these points is known

In the following example, the method of finding equation of circle is explained when the circle passes through two points and equation of tangent at one of these points is known.

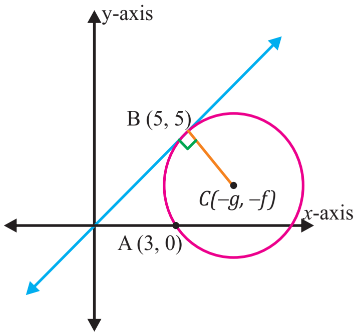


Fig. 8.9

Example: Find the equation of circle passing through $A(3, 0)$ and $B(5, 5)$, whereas the line $x - y = 0$ is tangent to the circle at B.

Solution: Let the general equation of circle with centre $C(-g, -f)$ be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots\text{(i)}$$

$\because \quad$ The circle passes through $(5, 5)$ and $(3, 0)$

$\therefore \quad$ we have $\quad 50 + 10g + 10f + c = 0 \quad \ldots$(ii)

and $\quad 9 + 6g + c = 0 \quad \ldots$(iii)

Subtracting equation (iii) from equation (ii)

we get $\quad 41 + 4g + 10f = 0 \quad \ldots$(iv)

$\because \quad$ radial segment BC and the given tangent are perpendicular

$\therefore \quad$ product of their slopes is -1

i.e., $\quad \dfrac{5+f}{5+g} = -1 \quad$ (where slope of $\overline{BC} = \dfrac{5+f}{5+g}$ and slope of tangent $= 1$)

$\Rightarrow \quad 5 + f = -5 - g$

$\Rightarrow \qquad f + g + 10 = 0 \qquad \ldots(v)$

Solving equation (iv) and equation (v) simultaneously,

we get $\qquad g = -\frac{59}{6}$ and $f = -\frac{1}{6}$

By using $\qquad g = -\frac{59}{6}$ in equation (iii)

we get $\qquad c = 50$

By subtracting values of g, f and c in equation (i), we get

$$x^2 + y^2 - \frac{59x}{3} - \frac{y}{3} + 50 = 0$$

or $\qquad 3x^2 + 3y^2 - 59x - y + 150 = 0$

(d) Equation of circle passing through two points and touching a given line.

In this case the given tangent does not pass through any of the two given points of the circle. The method is explained in the following example.

Example 1. Find the equation of circle passing through two points $(1, 0)$ and $(0, 1)$ and touches the line $x + y = 0$.

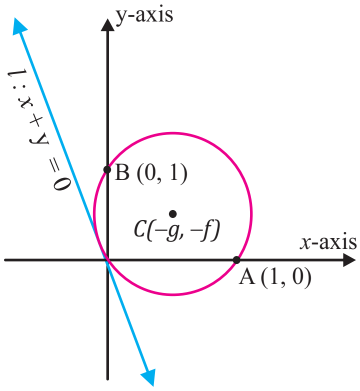


Fig. 8.10

Solution: Since line is touches the circle S at origin $(0, 0)$

$\therefore \qquad c = 0$

Let $x^2 + y^2 + 2gx + 2fy = 0$ be the desire equation of a circle.

Since points $(1, 0)$ and $(0, 1)$ are the points in the circle.

$\therefore \qquad 1 + 0 + 2g + 0 = 0 \;\Rightarrow\; g = -\frac{1}{2}$

and $\qquad 0 + 1 + 2f + 0 = 0 \;\Rightarrow\; g = -\frac{1}{2}$

Thus, required equation of a circle is

$$x^2 + y^2 + 2\left(-\frac{1}{2}\right)x + 2\left(-\frac{1}{2}\right)y = 0$$

$\Rightarrow \qquad \boxed{x^2 + y^2 - x - y = 0}$

Example 2. Find the equation of circle containing the points $(-2, 1)$ and $(-4, 3)$ and touching y-axis.

Solution: Let equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots(i)$$

$\because \qquad$ circle passes through $(-2, 1)$ and $(-4, 3)$

$\therefore \qquad$ we have

$$5 - 4g + 2f + c = 0 \qquad \ldots(ii)$$

and $25 - 8g + 6f + c = 0$ ...(iii)

$\because$ circle touches y-axis.

$\therefore$ radius of circle = modulus of abscissa of centre

i.e., $\sqrt{g^2 + f^2 - c} = |-g|$

squaring both sides

we get $c = f^2$

Multiplying equation (ii) by 2

and using $c = f^2$ in the resultant equation and equation (iii)

we get $10 - 8g + 4f + 2f^2 = 0$ ...(iv)

$25 - 8g + 6f + f^2 = 0$ ...(v)

Subtracting equation (v) from equation (iv)

we get $-15 - 2f + f^2 = 0$

or $f^2 - 2f - 15 = 0$

$\Rightarrow$ $(f - 5)(f + 3) = 0$

$\Rightarrow$ $f = 5$ and $f = -3$

So, $c = 25$ and $c = 9$

If $f = 5$ and $c = 25$ then equation (ii)

becomes $5 - 4g + 10 + 25 = 0$

$\Rightarrow$ $4g = 40$

or $g = 10$

If $f = -3$ and $c = 9$ then equation (ii) becomes

$5 - 4g - 6 + 9 = 0$

$\Rightarrow$ $4g = 8$

$\Rightarrow$ $g = 2$

Now, if $f = 5, c = 25$ and $g = 10$ then equation (i), becomes

$x^2 + y^2 + 20x + 10y + 25 = 0$

and if $f = -3, c = 9$ and $g = 2$

then equation (i) becomes

$x^2 + y^2 + 4x - 6y + 9 = 0$

Exercise 8.2

1. Find the equation of the circle through the given points.

(i) $(0,0), (0,3), (-4,0)$ (ii) $(0,10), (-10,0), (8,6)$

(iii) $(0,3), (2,-1), (1,0)$ (iv) $(7,-3), (-7,5), (11,5)$

(v) $(1,1), (2,-1), (3,-2)$

2. Find the equation of circle through the points $(1, 2), (2, 3)$ and having centre on
   (i) x-axis (ii) y-axis.
3. Find the equation of circle through the points $(3, 1), (2, 2)$ and having centre on the line $x + y - 3 = 0$.
4. Find the equation of the circle through the points $(0, -1)$ and $(3, 0)$ and the line $3x + y - 9 = 0$ is tangent to it at $(3, 0)$.
5. Find the equation of the circle through origin with x-intercept 2 and is tangent to the line $y - 1 = 0$.
6. Find the equation of a circle containing the points $(1, 2), (2, 3)$ and having centre on $x - y + 1 = 0$.
7. Find the equation of the circum-circle of the triangle with vertices $(1, -2), (-5, 2)$ and $(3, 4)$.
8. Find the equation of circle containing the points $(1, -2)$ and $(3, -4)$ and touching x-axis.
9. Find the equation of circle containing the points $(6, 0)$ and touching the line $x = y$ at $(4, 4)$.
10. Show that the equation of circle with centre $(-g, -f)$ and;
    (i) touching x-axis is of the form $x^2 + y^2 + 2gx + 2fy + g^2 = 0$
    (ii) touching y-axis is of the form $x^2 + y^2 + 2gx + 2fy + f^2 = 0$
11. Find the equation of circle passing through origin and having intercepts 6 and 8.
12. Find the equation of the circle which passes through the two points $(b, 0)$ and $(-b, 0)$ and whose radius is a unit.
13. Find the equation of the circle which passes through the point $(5, 0)$ and $(0, -5)$ and whose radius is 5 unit.

8.5 Tangent and Normal

In this section, we will discuss about the conditions of tangency and normality of a line to a circle along with their equations. But, first let us revise the concepts of secant, tangent and normal.

In geometry, a line which touches a curve at a single point is tangent to the curve and the point called point of tangency or point of contact.

Any line which is perpendicular to the tangent at the point of tangency is called normal to the curve whereas, secant is the line which intersects a curve at a minimum of two distinct points as shown in Fig. 8.11.

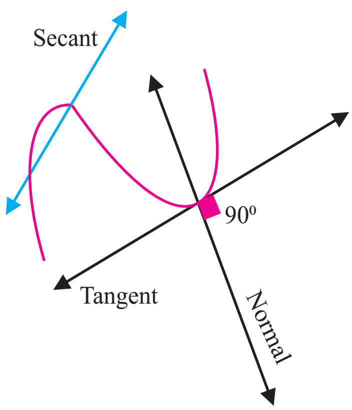


Fig. 8.11

In case of circle, secant intersects the circle at exactly two points. In the figure 8.12 line l is secant to the circle through two points P and Q. If P gets closer to Q and ultimately becomes coincident with Q, the secant l becomes tangent to the circle at point Q. In circle, normal always passes through the centre of the circle.

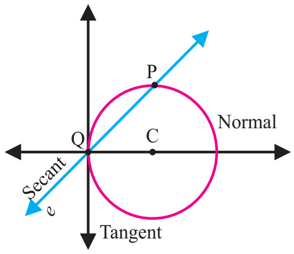


Fig. 8.12

8.5.1 Find the condition when a line intersects the circle

We are aware of the fact that a line can cut or touch a circle and sometimes, it neither cuts nor touches the circle at a point. In this section we will discuss these conditions in detail.

Consider a line $y = mx + c$ and the circle $x^2 + y^2 = r^2$.

Solving both equations simultaneously,

we get

$$x^2 + (mx + c)^2 = r^2$$

$$\Rightarrow \quad x^2 + m^2x^2 + 2mcx + c^2 = r^2$$

$$\Rightarrow \quad (1 + m^2)x^2 + 2mcx + c^2 - r^2 = 0 \quad \ldots(\text{i})$$

Since roots of equation (i) represent abscissas of the points of intersection A and B of the given circle and line as shown in the figure 8.13.

Therefore, nature of roots of the quadratic equation (i) will represent the nature of parallel lines l_1, l_2 and l_3, each of the slope m, with respect to the given circle.

Here discriminant of equation (i) is:

$$\Delta = (2mc)^2 - 4(1 + m^2)(c^2 - r^2)$$

or $$\Delta = 4m^2c^2 - 4c^2 + 4r^2 - 4m^2c^2 + 4m^2r^2$$

$$\Rightarrow \quad \Delta = 4\{r^2(1 + m^2) - c^2\}$$

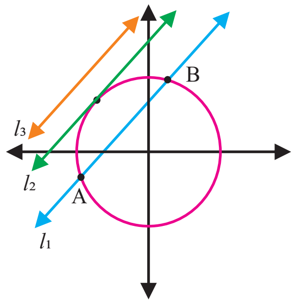


Fig. 8.13

We know that the roots of equation (i) will be real and unequal if $\Delta > 0$.

i.e., $$r^2(1 + m^2) - c^2 > 0$$

$$\Rightarrow \quad r^2(1 + m^2) > c^2 \quad \ldots(\text{ii})$$

This is the condition when points of intersection are real and distinct. This value of c^2corresponds to l_1 which intersects the circle at two real and distinct points. Condition (ii) is the condition when line intersects the circle i.e., condition of secancy.

We know that

the roots of equation (i) will be real and equal.

If $\Delta = 0$

i.e., $r^2(1 + m^2) - c^2 = 0$

$\Rightarrow$ $\boxed{r^2(1+m^2)=c^2}$ ...(iii)

This is the condition when points of intersection are real and coincident. This value of c^2 corresponds to l_2 which touches the given circle at a single point.

Hence condition (iii) is the condition of tangency of the line. We also know that the roots of equation (i) will be non-real if $\Delta< 0$.

i.e., $r^2(1+m^2)-c^2<0$

$\Rightarrow$ $\boxed{r^2(1+m^2)<c^2}$ ...(iv)

This is the condition when points of intersection are imaginary. This value of c^2 corresponds to l_3 which neither cuts nor touches the given circle. Hence condition (iv) is the condition when the line is neither secant nor tangent.

8.5.2 Find the condition when a line touches the circle

As discussed in section 8.5.1 a line $y=mx+c$ will be tangent to the circle $x^2+y^2=r^2$ if $c^2=r^2(1+m^2)$.

In general, a line l will be tangent to any given circle if distance of the line from centre is always equal to the radius of the circle. So, in order to find the condition of tangency of line to the given circle, we equate the distance of the line from the centre of the circle and radius of the given circle. (Fig. 8.14)

Alternatively, we take discriminant of the quadratic equation as zero which is obtained by solving equations of given circle and line simultaneously as we did in section 8.5.1.

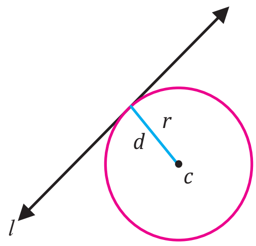


Fig. 8.14

Condition of tangency of a line $lx+my+n=0$ to the circle $x^2+y^2=r^2$.

$\because$ centre of the circle $x^2+y^2=r^2$ is origin. (Fig. 8.15)

$\therefore$ distance of line: $lx+my+n=0$ from centre is:

$$d=\left|\frac{n}{\sqrt{l^2+m^2}}\right|$$

Now, given line will be tangent to the given circle.

if d = radius of circle

i.e., $\left|\frac{n}{\sqrt{l^2+m^2}}\right|=r$

Squaring both sides

$$\frac{n^2}{l^2+m^2}=r^2$$

$\Rightarrow$ $\boxed{n^2=r^2(l^2+m^2)}$

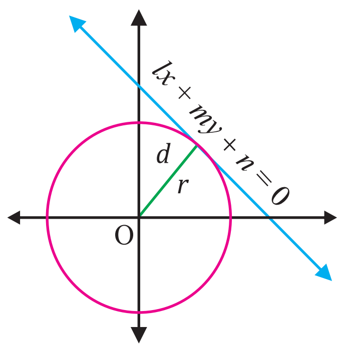


Fig. 8.15

This is the condition when line $lx+my+n=0$ will touch the circle $x^2+y^2=r^2$.

Alternative Method

We have, equation of line

$$lx + my + n = 0 \quad \ldots\text{(i)}$$

and equation of circle.

$$x^2 + y^2 = r^2 \quad \ldots\text{(ii)}$$

Solving (i) and (ii), simultaneously, we get

$$x^2 + \left(\frac{-lx - n}{m}\right)^2 = r^2$$

$$\Rightarrow \quad m^2x^2 + l^2x^2 + 2nlx + n^2 = m^2r^2$$

$$\Rightarrow \quad (l^2 + m^2)x^2 + 2nlx + n^2 - m^2r^2 = 0 \quad \ldots\text{(iii)}$$

Given line will be tangent to the given circle

if Discriminant of equation (iii) vanishes

i.e., $4n^2l^2 - 4(l^2 + m^2)(n^2 - m^2r^2) = 0$

$$\Rightarrow \quad 4n^2l^2 - 4n^2l^2 + 4l^2m^2r^2 - 4m^2n^2 + 4m^4r^2 = 0$$

$$\Rightarrow \quad 4m^2(l^2r^2 - n^2 + m^2r^2) = 0$$

$$\Rightarrow \quad l^2r^2 - n^2 + m^2r^2 = 0 \qquad (\text{Let } m \neq 0)$$

$$\Rightarrow \quad n^2 = r^2(l^2 + m^2)$$

This is the condition of tangency of given line $lx + my + n = 0$ to the given circle $x^2 + y^2 = r^2$.

Example 1. Find the condition of tangency and secancy of the line $y = mx + k$ with the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Solution: We have the line $mx - y + k = 0$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

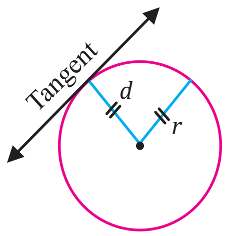


Fig. 8.16

Given line will be tangent to the circle, if distance of line from the centre of circle is equal to the radius of the circle, i.e., $d = r$ as shown in Fig. 8.16.

or $$\left|\frac{-mg+f+k}{\sqrt{m^2+1}}\right| = \sqrt{g^2 + f^2 - c}$$

squaring both sides

$$(f + k - mg)^2 = (g^2 + f^2 - c)(m^2 + 1)$$

This is the required condition of tangency.

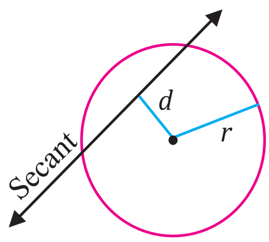


Fig. 8.17

We know that given line will be secant to the circle if, distance of line from the centre is less than the radius of the circle as shown in Fig. 8.17.

i.e., $d < r$

or $\left|\frac{-mg+f+k}{\sqrt{m^2+1}}\right| < \sqrt{g^2 + f^2 - c}$

This is the required condition of secancy.

Example 2. Find the value of r when the line $x = 2y + 4$ should be:

(i) a tangent to the circle $x^2 + y^2 = r^2$

(ii) a secant to the circle $x^2 + y^2 = r^2$

Solution: (i) value of r when line is tangent

We have, line: $x = 2y + 4$ …(i)

and circle: $x^2 + y^2 = r^2$ …(ii)

solving equations (i) and (ii)

we get $(2y + 4)^2 + y^2 = r^2$

$\Rightarrow$ $5y^2 + 16y + 16 - r^2 = 0$ …(iii)

Given line will be tangent to the circle,

if $\Delta = 0$

i.e., $(16)^2 - 4(5)(16 - r^2) = 0$

$\Rightarrow$ $256 - 320 + 20r^2 = 0$

$\Rightarrow$ $20r^2 = 64$

$\Rightarrow$ $r = \pm\frac{4}{\sqrt{5}}$

(ii) Value of r when given line is secant

From equation (iii)

$$\Delta = 256 - 20(16 - r^2)$$
$$= 20r^2 - 64$$

We know that given line will be secant to the circle,

if $\Delta > 0$

i.e., $20r^2 - 64 > 0$

$\Rightarrow$ $r^2 > \frac{16}{5}$

$\Rightarrow$ $r > \frac{16}{5}$ or $r < -\frac{16}{5}$

8.5.3 Find the equation of a tangent to a circle in slope form:

Let m be the slope of a line which is tangent to the circle $x^2 + y^2 = r^2$ then the equation of tangent will be

$y = mx + c$ …(i)

where c is the y-intercept of the tangent.

According to the condition of tangency

$$c^2 = r^2(1 + m^2)$$

$$\Rightarrow \quad c = \pm r\sqrt{1 + m^2}$$

By using value of c in equation (i)

we get $\boxed{y = mx \pm r\sqrt{1 + m^2}}$

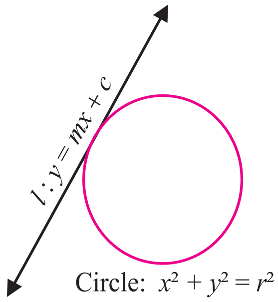


Fig. 8.18

This is the equation of tangent to the circle $x^2 + y^2 = r^2$ in slope form.

Example: Find the equation of tangent to $x^2 + y^2 = 25$ with the slope 2.

Solution:

Here, slope of tangent $= m = 2$

and radius $= r = 5$

We know that the equation of tangent to the given circle will be

$$y = mx \pm r\sqrt{1 + m^2}$$

By using values of m and r

we get $y = 2x \pm 5\sqrt{5}$

So, there will be two tangents to the given circle with slope 2 which are

$$y = 2x + 5\sqrt{5} \text{ and } y = 2x - 5\sqrt{5}$$

8.5.4 Find the equations of a tangent and a normal to a circle at a point

Equation of tangent and normal to a circle at a given point

Let the line l be the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the given point (x_1, y_1) as shown in the figure 8.19.

$\because$ tangent to the circle is perpendicular to the radial segment at the point of contact i.e., $l \perp \overline{CP}$

$$\therefore \quad \text{slope of tangent} = m = -\frac{1}{\text{slope of } \overline{CP}}$$

$$= -\frac{1}{\frac{y_1 + f}{x_1 + g}}$$

$$= -\frac{x_1 + g}{y_1 + f}$$

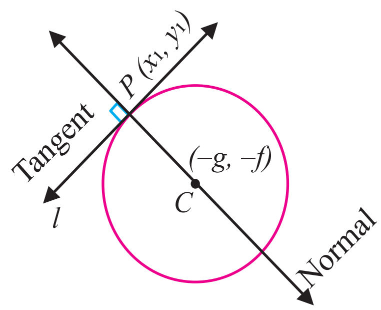


Fig. 8.19

By point-slope form the equation of tangent will be

$$y - y_1 = m(x - x_1)$$

i.e., $y - y_1 = -\frac{x_1+g}{y_1+f}(x - x_1)$ ...(i)

$\Rightarrow$ $(y - y_1)(y_1 + f) + (x - x_1)(x_1 + g) = 0$

or $yy_1 + fy - y_1^2 - fy_1 + xx_1 + gx - x_1^2 - gx_1 = 0$

$\Rightarrow$ $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1$ ...(ii)

$\because$ (x_1, y_1) lies on the given circle

$\therefore$ $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

$\Rightarrow$ $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 = -c$

So, equation (ii) becomes

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ ...(iii)

This is the equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1).

In case centre of circle is at origin then $g = f = 0$

So, equation (iii) becomes

$xx_1 + yy_1 + c = 0$ ...(iv)

We know that

radius of circle $= r = \sqrt{g^2 + f^2 - c}$

i.e., $c = g^2 + f^2 - r^2$

$\Rightarrow$ $c = -r^2$ $(\because \; g = f = 0)$

So, equation (iv) becomes

$xx_1 + yy_1 - r^2 = 0$

i.e., $xx_1 + yy_1 = r^2$ ...(v)

This is the equation of tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1).

We know that normal is perpendicular to the tangent at the point of contact.

So, slope of normal $= m' = -\frac{1}{m}$

$= \frac{y_1 + f}{x_1 + g}$

Now, by point-slope form, the equation of normal will be

$$y - y_1 = m'(x - x_1)$$

i.e., $y - y_1 = \frac{y_1+f}{x_1+g}(x - x_1)$ ...(vi)

This is the equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1).

In case centre is at origin then $g = f = 0$.

So, equation (vi) becomes

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

$\Rightarrow \quad x_1y - x_1y_1 = xy_1 - x_1y_1$

$\Rightarrow \quad x_1y - xy_1 = 0$

This is the equation of normal to the circle $x^2 + y^2 = r^2$ at a given point (x_1, y_1).

- **Equation of a tangent and a normal to a circle at a given point using derivative.**

We know that derivative $\left(\frac{dy}{dx}\right)$ at a point (x_1, y_1) of a curve $y = f(x)$ is the slope of the tangent to the curve at that point.

Thus, the equation of tangent to any circle at the point (x_1, y_1) is

$y - y_1 = m(x - x_1)$

where $m = \frac{dy}{dx}$ at (x_1, y_1)

Since normal is perpendicular to the tangent at the point (x_1, y_1)

Therefore, equation of the normal to any circle will be

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

Example 1. Find the tangents and normals to the following circles without using derivatives.

(i) $x^2 + y^2 = 25$ at $(3, 4)$

(ii) $x^2 + y^2 + 6x + 4y = 132$ at $(6, 6)$

Solution: (i) $\boldsymbol{x^2 + y^2 = 25}$ **at** $\boldsymbol{(3, 4)}$

Here $(x_1, y_1) = (3, 4)$

and $r = 5$

Now, equation of tangent to the given circle will be

$xx_1 + yy_1 = r^2$

i.e., $3x + 4y = 25$

Also, the equation of normal to the given circle will be

$x_1y - xy_1 = 0$

i.e., $3y - 4x = 0$

(ii) $\boldsymbol{x^2 + y^2 + 6x + 4y = 132}$ **at** $\boldsymbol{(6, 6)}$

Given circle is $x^2 + y^2 + 6x + 4y - 132 = 0$

Comparing with general equation of circle

we get $2g = 6 \quad \Rightarrow \quad g = 3$;

$2f = 4 \quad \Rightarrow \quad f = 2$

and $\quad c = -132$

We also have $(x_1, y_1) = (6, 6)$

Now, equation of tangent to the given circle will be

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Using values, we get

$$6x + 6y + 3(x + 6) + 2(y + 6) - 132 = 0$$

Also, the equation of normal to the given circle will be

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

i.e., $\quad y - 6 = \frac{6+2}{6+3}(x - 6)$

$\Longrightarrow \quad y - 6 = \frac{8}{9}(x - 6)$

$\Longrightarrow \quad 9y - 54 = 8x - 48$

$\Longrightarrow \quad 9x - 9y = 6 = 0$

Example 2. Find the equation of tangent and normal to $x^2 + y^2 = 100$ at $(6, 8)$.

Solution: We have circle,

$$x^2 + y^2 = 100$$

Differentiating w.r.t x

$$2x + 2y\frac{dy}{dx} = 0$$

$\Longrightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$

So, slope of tangent to the given circle at $(6, 8) = \left(\frac{dy}{dx}\right)_{(6,8)}$

$$= -\frac{6}{8}$$

$$= -\frac{3}{4} = m$$

We also have $(x_1, y_1) = (6, 8)$

By point-slope form the equation of tangent will be

$$y - y_1 = m(x - x_1)$$

i.e., $\quad y - 8 = -\frac{3}{4}(x - 6)$

$\Longrightarrow \quad 4y - 32 = -3x + 18$

$\Longrightarrow \quad 3x + 4y - 50 = 0$

$\because$ Normal is perpendicular to the tangent at $(6, 8)$

$\therefore$ its slope $= m' = -\frac{1}{m}$

$= \frac{4}{3}$

By point-slope form the equation of normal will be

$$y - y_1 = m'(x - x_1)$$

i.e., $y - 8 = \frac{4}{3}(x - 6)$

$\Rightarrow \quad 3y - 24 = 4x - 24$

$\Rightarrow \quad 4x - 3y = 0$

8.5.5 Find the length of tangent to a circle from a given external point

Let $P(x, y)$ be the point of contact of the tangent from the external point $E(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ whose centre is $C(-g, -f)$. $\overline{EP}$ is called tangent segment and its length is called length of tangent.

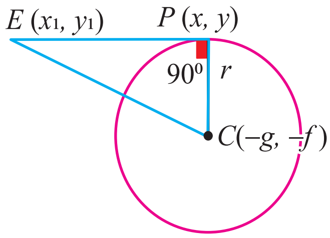


Fig. 8.20

$\because \quad \overline{EP} \perp \overline{CP}$

$\therefore$ CEP is a right angled triangle as shown in the figure 8.20.

In ΔCEP, by Pythagoras theorem

$$|\overline{CE}|^2 = |\overline{CP}|^2 + |\overline{EP}|^2$$

i.e., $(x_1 + g)^2 + (y_1 + f)^2 = r^2 + |\overline{EP}|^2$

$\Rightarrow \quad |\overline{EP}|^2 = x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - (g^2 + f^2 - c)$

$\left(\because \quad r = \sqrt{g^2 + f^2 - c}\right)$

So, $\quad |\overline{EP}| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} \qquad \ldots$(i)

So, the length of tangent from (x_1, y_1) to the circle in general form is

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Example: Find the length of tangent from $(-2, 3)$ to the circle $x^2 + y^2 - 5x - 2y + 1 = 0$.

Solution: Given circle is $x^2 + y^2 - 5x - 2y + 1 = 0$

Here $\quad 2g = -5$,

$2f = -2$,

$c = 1$ and $(x_1, y_1) = (-2, 3)$

Now,

length of tangent $= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$= \sqrt{(-2)^2 + (3)^2 - 5(-2) - 2(3) + 1}$

$= \sqrt{4 + 9 + 10 - 6 + 1}$

$= \sqrt{18}$ units.

8.5.6 Prove that two tangents drawn to a circle from an external point are equal in length

Let $\overrightarrow{PA}$ and $\overrightarrow{PB}$ be two tangents to the given circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $C(-g, -f)$from an external point $P(x_1, y_1)$ as shown in the figure 8.21.

We know that a tangent to the circle is perpendicular to its radial segment at the point of contact.

So, we have two right triangles PAC and PBC with right angles at A and B respectively.

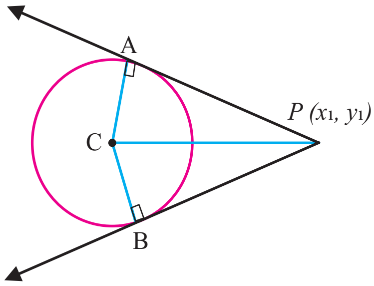


Fig. 8.21

In right ΔPAC, by using Pythagoras theorem

$$|\overline{CP}|^2 = |\overline{AC}|^2 + |\overline{AP}|^2$$

$$\Rightarrow \quad (x_1 + g)^2 + (y_1 + f)^2 = r^2 + |\overline{AP}|^2$$

$$\Rightarrow \quad |\overline{AP}|^2 = x_1^2 + 2gx_1 + g^2 + y_1^2 + 2gy_1 + f^2 - r^2$$

$$\Rightarrow \quad |\overline{AP}|^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + g^2 + f^2 - g^2 - f^2 + c$$

$$\left(\because \quad r = \sqrt{g^2 + f^2 - c}\right)$$

$$\Rightarrow \quad |\overline{AP}|^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$\Rightarrow \quad |\overline{AP}|^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

So, $\quad |\overline{AP}| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} \qquad \ldots$(i)

Similarly, in right ΔPBC

$$|\overline{PB}| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$|\overline{AP}| = |\overline{PB}|$$

Hence two tangents drawn to a circle from an external point are equal in length.

Exercise 8.3

1. Check whether the following lines are tangent, secant or neither to the circle $x^2 + y^2 = 25$.
   (i) $y = x + 3$
   (ii) $y = \sqrt{3}x + 10$
   (iii) $y = 2x + 15$
2. Find the condition of tangency and secancy of the line $y = 2x + k$ with the circle $x^2 + y^2 + 10x + 20y + c = 0$.
3. Find the equation of tangent to $x^2 + y^2 = 36$ with the slope $\sqrt{3}$.

4. Find the equation of tangent and normal:
   (i) at $(1,-4)$ to the circle $x^2+y^2=17$
   (ii) at $(4,1)$ to the circle $x^2+y^2-4x+2y=3$
5. Find the length of tangent:
   (i) from $(6,1)$ to the circle $x^2+y^2=4$
   (ii) from $(2,5)$ to the circle $x^2+y^2+8x-5y=7$.
6. Find the condition that the line $y=mx+c$ may be tangent to the circle $(x-h)^2+(y-k)^2=r^2$.
7. Show that circles $x^2+y^2-6x-6y+10=0$ and $x^2+y^2=2$ touch each other and find the point of contact.
8. Find the equation of tangent (s) to the circle $x^2+y^2=25$.
   (i) at the point whose abscissa is 3.
   (ii) at the point whose ordinate is -4.
   (iii) which is parallel to $3x+4y+1=0$
   (iv) which is perpendicular to $3x+4y+1=0$
9. Find the equations of tangents to $x^2+y^2-6x-2y+9=0$ through origin. Find also their respective points of contact.
10. Show that the line $ax+by+al+bm=0$ is normal to the circle $x^2+y^2+2lx+2my+c=0$ for all values of *a* and *b*.
11. Find (i) the product of abscissa (ii) the product of ordinates of the points, where the line $y=mx$ meets the circle $x^2+y^2+2gx+2fy+c=0$.
12. Prove that the line $y=x+k\sqrt{2}$ touches the circle $x^2+y^2=k^2$ and find its point of contact.
13. Find the condition that the line $3x+4y=c$ may touch the circle $x^2+y^2=8x$.
14. Find whether the line $x+y=2+\sqrt{2}$ touches the circle $x^2+y^2-2x-2y-1=0$.
15. Prove that the two circles $x^2+y^2+2gx+c=0$ and $x^2+y^2+2fy+c=0$ touch each other, if $\frac{1}{f^2}+\frac{1}{g^2}=\frac{1}{c}$.

8.6 Properties of Circle

In this section we will prove some theorem of Euclidean geometry analytically which are related to the circle.

Prove analytically the following properties of a circle.

- **Perpendicular from the centre of a circle on a chord bisects the chord.**

Let $\overline{AB}$ be a chord of circle $x^2+y^2+2gx+2fy+c=0$ with centre $C(-g,-f)$. where the end points of the chord are $A(x_1,y_1)$ and $B(x_2,y_2)$ as shown in the Fig. 8.22.

Furthermore $\overline{CD}$ is perpendicular from centre $\overline{CD}$ to the chord $\overline{AB}$.

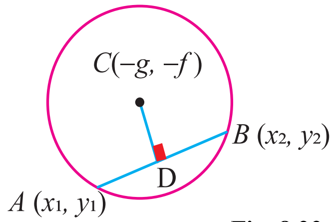


Fig. 8.22

∵ $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on the circle

∴ we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \ldots\text{(i)}$$

and $$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \ldots\text{(ii)}$$

Subtracting equation (ii) from equation (i), we get

$$(x_1^2 - x_2^2) + (y_1^2 - y_2^2) + 2g(x_1 - x_2) + 2f(y_1 - y_2) = 0$$

$$\Rightarrow \boxed{2g(x_1 - x_2) + 2f(y_1 - y_2) = -x_1^2 + x_2^2 - y_1^2 + y_2^2} \quad \ldots\text{(iii)}$$

Now, slope of $\overline{CD} = -\dfrac{1}{\frac{y_2 - y_1}{x_2 - x_1}}$

or $m = -\dfrac{(x_1 - x_2)}{y_1 - y_2}$

Now, equation of $\overline{CD}$, by point-slope form will be

$$y + f = m(x + g)$$

i.e., $$y + f = -\frac{(x_1 - x_2)}{y_1 - y_2}(x + g)$$

$$\Rightarrow (y_1 - y_2)(y + f) = -(x_1 - x_2)(x + g)$$

$$\Rightarrow y(y_1 - y_2) + f(y_1 - y_2) = -(x_1 - x_2)x - g(x_1 - x_2)$$

$$\Rightarrow g(x_1 - x_2) + f(y_1 - y_2) = -(x_1 - x_2)x - y(y_1 - y_2)$$

Using equation (iii), we get

$$2x(x_1 - x_2) + 2y(y_1 - y_2) = x_1^2 - x_2^2 + y_1^2 - y_2^2 \quad \ldots\text{(iv)}$$

This is the equation of perpendicular $\overline{CD}$ from centre of circle to the chord $\overline{AB}$.

Now, midpoint of $\overline{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

By substituting midpoint in equation (iv), we get

$$2\left(\frac{x_1 + x_2}{2}\right)(x_1 - x_2) + 2\left(\frac{y_1 + y_2}{2}\right)(y_1 - y_2) = x_1^2 - x_2^2 + y_1^2 - y_2^2$$

$$\Rightarrow x_1^2 - x_2^2 + y_1^2 - y_2^2 = x_1^2 - x_2^2 + y_1^2 - y_2^2$$

∵ Midpoint of $\overline{AB}$ satisfies equation of perpendicular $\overline{CD}$

∴ $\overline{CD}$ bisects the chord $\overline{AB}$.

- **Perpendicular bisector of any chord of a circle passes through the centre of the circle.**

Let $\overline{AB}$ be a chord of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $C(-g, -f)$.

Let l be the perpendicular bisector of $\overline{AB}$ which cuts $\overline{AB}$ at midpoint D whereas the coordinates of A and B are (x_1, y_1) and (x_2, y_2) respectively as shown in Fig. 8.23.

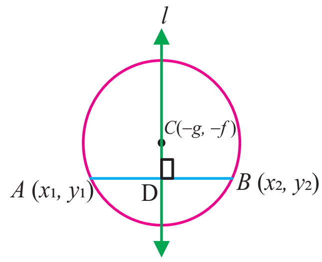


Fig. 8.23

Here midpoint $D = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

and slope of $\overline{AB} = \frac{y_2-y_1}{x_2-x_1}$

$\because$ l is perpendicular on $\overline{AB}$

$\therefore$ slope of $l = -\frac{(x_2-x_1)}{y_2-y_1}$

Now, equation of l will be

$$\left\{y - \left(\frac{y_1+y_2}{2}\right)\right\} = -\frac{(x_2-x_1)}{y_2-y_1}\left\{x - \left(\frac{x_1+x_2}{2}\right)\right\} \quad \text{... (i)}$$

We know that

$$|\overline{AC}| = |\overline{BC}|$$

$$\Rightarrow \quad \sqrt{(x_1+g)^2 + (y_1+f)^2} = \sqrt{(x_2+g)^2 + (y_2+f)^2}$$

Squaring both sides

$$(x_1+g)^2 + (y_1+f)^2 = (x_2+g)^2 + (y_2+f)^2$$

$$\Rightarrow \quad x_1^2 + 2gx_1 + y_1^2 + 2fy_1 = x_2^2 + 2gx_2 + y_2^2 + 2fy_2$$

$$\Rightarrow \quad x_1^2 - x_2^2 + 2g(x_1 - x_2) = -(y_1^2 - y_2^2) - 2f(y_1 - y_2)$$

$$\Rightarrow \quad (x_1 - x_2)(x_1 + x_2 + 2g) = -(y_1 - y_2)(y_1 + y_2 + 2f)$$

$$\Rightarrow \quad -\frac{(x_2-x_1)}{y_2-y_1} = \frac{(y_1+y_2+2f)}{x_1+x_2+2g}$$

Using this in equation (i), we get

$$2y - y_1 - y_2 = \frac{(y_1+y_2+2f)}{x_1+x_2+2g}(2x - x_1 - x_2)$$

$$\Rightarrow \quad (2y - y_1 - y_2)(x_1 + x_2 + 2g) - (y_1 + y_2 + 2f)(2x - x_1 - x_2) = 0 \quad \text{...(ii)}$$

This is equation of perpendicular bisector of chord $\overline{AB}$.

Now, we substitute centre $(-g, -f)$ in equation (ii), we get

$$(-2f - y_1 - y_2)(x_1 + x_2 + 2g) - (y_1 + y_2 + 2f)(-2g - x_1 - x_2) = 0$$

$$\Rightarrow \quad -(2f + y_1 + y_2)(x_1 + x_2 + 2g) + (y_1 + y_2 + 2f)(2g + x_1 + x_2) = 0$$

$$\Rightarrow \quad 0 = 0$$

$\because$ centre $C(-g, -f)$ satisfies equation (ii)

$\therefore$ Perpendicular bisector of the chord $\overline{AB}$ passes through the centre.

- **Line joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.**

Let l be the line joining the centre $C(-g,-f)$ of the circle $x^2+y^2+2gx+2fy+c=0$ to the mid-point of chord. $\overline{AB}$ whose end points are $A(x_1,y_1)$ and $B(x_2,y_2)$ as shown in the figure 8.24.

Mid-point $D=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$\because \quad |\overline{AC}|=|\overline{BC}|$

$\therefore \quad \sqrt{(x_1+g)^2+(y_1+f)^2}=\sqrt{(x_2+g)^2+(y_2+f)^2}$

Squaring both sides

$$(x_1+g)^2+(y_1+f)^2=(x_2+g)^2+(y_2+f)^2$$

$$\Rightarrow \quad x_1^2+2gx_1+y_1^2+2fy_1=x_2^2+2gx_2+y_2^2+2fy_2$$

$$\Rightarrow \quad x_1^2-x_2^2+2g(x_1-x_2)=-(y_1^2-y_2^2)-2f(y_1-y_2)$$

$$\Rightarrow \quad (x_1-x_2)(x_1+x_2+2g)=-(y_1-y_2)(y_1+y_2+2f)$$

$$\Rightarrow \quad -\frac{(x_1+x_2+2g)}{y_1+y_2+2f}=\frac{y_2-y_1}{x_2-x_1} \qquad \ldots(\text{i})$$

Now, slope of chord $\overline{AB}=\frac{y_2-y_1}{x_2-x_1}$

$$=-\frac{(x_1+x_2+2g)}{y_1+y_2+2f} \qquad (\text{Using equation (i)})$$

and slope of l = slope of $\overline{CD}$.

$$=\frac{\frac{y_1+y_2}{2}+f}{\frac{x_1+x_2}{2}+g}$$

$$=\frac{y_1+y_2+2f}{x_1+x_2+2g}$$

Now, (slope of l) × (slope of $\overline{AB}$)

$$=\frac{y_1+y_2+2+f}{x_1+x_2+2g}\times\left\{-\frac{(x_1+x_2+2g)}{(y_1+y_2+2f)}\right\}$$

$$=-1$$

$\because \quad$ product of slopes of l and chord $\overline{AB}$ is -1.

$\therefore \quad$ The line l through the centre of circle and the mid-point of chord is perpendicular to the chord.

- **Congruent chords of a circle are equidistant from its centre and its converse**

Let $\overline{AB}$ and $\overline{CD}$ be two congruent chords of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $O(-g, -f)$ whereas the end of chords are $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and $D(x_4, y_4)$ as shown in the figure 8.25.

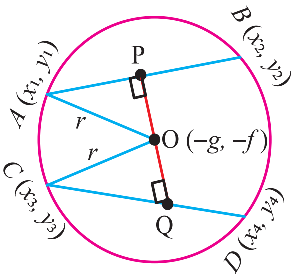


Fig. 8.25

Let $\overline{OP}$ is perpendicular distance from centre to chord $\overline{AB}$ then P is mid-point of chord $\overline{AB}$ according to property 1.

Also let $\overline{OQ}$ is perpendicular distance of centre to chord $\overline{CD}$ then Q is mid-point of chord $\overline{CD}$ according to property 1.

According to the condition

$$|\overline{AB}| = |\overline{CD}|$$

i.e., $$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2}$$

Squaring both sides

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_4 - x_3)^2 + (y_4 - y_3)^2 \qquad \ldots\text{(i)}$$

In right angled ΔAOP

$$|\overline{OP}|^2 = |\overline{AO}|^2 - |\overline{AP}|^2$$

$$= r^2 - \left(\frac{1}{2}|\overline{AB}|\right)^2 \qquad (\because \; |\overline{OA}| = r)$$

$$= r^2 - \frac{1}{4}\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}$$

$$\Rightarrow \qquad |\overline{OP}| = \frac{\sqrt{4r^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2}}{2} \qquad \ldots\text{(ii)}$$

In right angled ΔCOQ

$$|\overline{OQ}|^2 = r^2 - |\overline{CQ}|^2$$

$$\Rightarrow \qquad |\overline{OQ}|^2 = r^2 - \left(\frac{1}{2}|\overline{CD}|\right)^2$$

$$\Rightarrow \qquad |\overline{OQ}|^2 = r^2 - \frac{1}{4}\{(x_4 - x_3)^2 + (y_4 - y_3)^2\}$$

$$\Rightarrow \qquad |\overline{OQ}|^2 = r^2 - \frac{1}{4}\{(x_2 - x_1)^2 + (y_2 - y_1)^2\} \text{ (using equation (i))}$$

$$\Rightarrow \qquad |\overline{OQ}| = \frac{\sqrt{4r^2 - (x_2 - x_1)^2 + (y_2 - y_1)^2}}{2} \qquad \ldots\text{(iii)}$$

From (ii) and (iii), we get

$$|\overline{OP}| = |\overline{OQ}|$$

Hence chords are equidistant from the centre.

- **Converse of Theorem 4a**

If the perpendicular distances from the centre of a circle to its two chords are equal, then the chords are congruent.

Let $\overline{AB}$ and $\overline{CD}$ be two chords of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $O(-g, -f)$ where the ends of chords are $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and $D(x_4, y_4)$ as shown in the Fig. 8.26.

Let $\overline{OP}$ is perpendicular distance of centre to the chord $\overline{AB}$, So P is the mid-point of chord $\overline{AB}$.

So, mid-point $P = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Also let $\overline{OQ}$ is perpendicular distance of centre to the chord $\overline{CD}$, So Q is mid-point of chord $\overline{CD}$.

So, mid-point $Q = \left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}\right)$

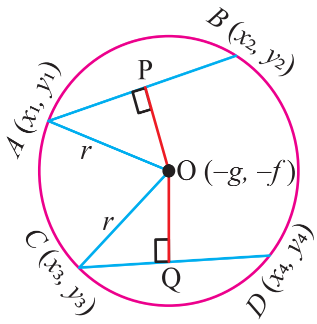


Fig. 8.26

According to the condition

$$|\overline{OP}| = |\overline{OQ}|$$

i.e., $$\sqrt{\left(\frac{x_1+x_2}{2} + g\right)^2 + \left(\frac{y_1+y_2}{2} + f\right)^2} = \sqrt{\left(\frac{x_3+x_3}{2} + g\right)^2 + \left(\frac{y_3+y_3}{2} + f\right)^2}$$

Squaring both sides

$$\frac{(x_1 + x_2 + 2g)^2}{4} + \frac{(y_1 + y_2 + f)^2}{4} = \frac{(x_3 + x_4 + 2g)^2}{4} + \frac{(y_3 + y_4 + 2f)^2}{4} \quad \ldots\text{(i)}$$

In right angled ΔAOP

$$|\overline{AP}|^2 = |\overline{AO}|^2 - |\overline{OP}|^2$$

$$\Rightarrow |\overline{AP}|^2 = r^2 - \frac{(x_1 + x_2 + 2g)^2}{4} - \frac{(y_3 + y_4 + 2f)^2}{4} \qquad (\because \ |\overline{OA}| = r)$$

$$\Rightarrow \qquad |\overline{AP}| = \frac{\sqrt{4r^2 - (x_1+x_2+2g)^2 - (y_3+y_4+2f)^2}}{2} \qquad \ldots\text{(ii)}$$

In right angled ΔCOQ

$$|\overline{CQ}|^2 = |\overline{OC}|^2 - |\overline{OQ}|^2$$

$$\Rightarrow \qquad |\overline{CQ}|^2 = r^2 - \left\{\frac{(x_3+x_4+2g)^2}{4} + \frac{(y_3+y_4+2f)^2}{4}\right\} \qquad (\because \ r = |\overline{OC}|)$$

$$\Rightarrow \qquad |\overline{CQ}|^2 = r^2 - \left\{\frac{(x_1+x_2+2g)^2}{4} + \frac{(y_1+y_2+2f)^2}{4}\right\} \text{ (using equation (i))}$$

$$\Rightarrow \qquad |\overline{CQ}| = \frac{\sqrt{4r^2 - (x_1+x_2+2g)^2 - (y_1+y_2+2f)^2}}{2} \qquad \ldots\text{(iii)}$$

From (ii) and (iii), we get

$$|\overline{AP}| = |\overline{CQ}|$$

$\Rightarrow \quad 2|\overline{AP}| = 2|\overline{CQ}|$

i.e., $\quad |\overline{AB}| = |\overline{CD}|$

Hence two chords are congruent it they are equidistant from centre.

- **Measure of the central angle of a minor arc is double the measure of the angle subtended by the corresponding major arc.**

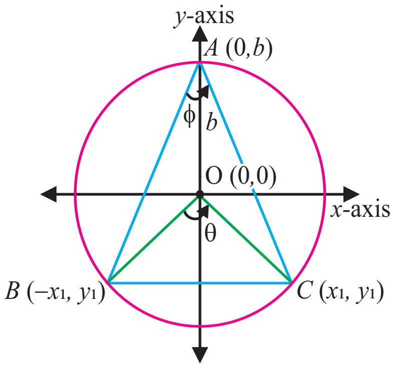


Fig. 8.27

Let $\overline{BC}$ be a minor arc of circle $x^2 + y^2 = r^2$ such that its ends are $B(-x_1, y_1)$ and $C(x_1, y_1)$ whereas $A(0, b)$ is any point of corresponding major arc on the given circle as shown in Fig. 8.27.

Now, $\angle BOC$ is the central angle of minor arc BC. and $\angle BAC$ is the angle subtended by the corresponding major arc BAC.

Here $\quad$ slope of $\overline{BO} = m_1 = \dfrac{y_1}{-x_1}$

Slope of $\overline{CO} = m_2 = \dfrac{y_1}{x_1}$

Now $\quad \tan m\angle BOC = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

i.e, $\quad \tan\theta = \dfrac{\frac{y_1}{x_1} + \frac{y_1}{x_1}}{1 + \left(-\frac{y_1}{x_1}\right)\left(\frac{y_1}{x_1}\right)} \qquad$ (Let $m\angle BOC = \theta$)

$$= \frac{\dfrac{2y_1}{x_1}}{\dfrac{x_1^2 - y_1^2}{x_1^2}}$$

$\tan\theta = \dfrac{2x_1 y_1}{x_1^2 - y_1^2} \qquad \ldots$(i)

we have

slope of $\overline{AB} = m_3 = \dfrac{b - y_1}{x_1}$

and $\quad$ slope of $\overline{AC} = \dfrac{b - y_1}{-x_1} = m_4$

Furthermore,

$$x^2 + y^2 = r^2$$

i.e., $\quad x_1^2 + y_1^2 = b^2 \qquad (\because \ r = b)$

$\Rightarrow \quad x_1^2 = b^2 - y_1^2 \qquad \ldots$(ii)

Now $\tan(m\angle BAC) = \frac{m_4 - m_3}{1 + m_4 m_3}$

i.e., $\tan\emptyset = \frac{\frac{-(b-y_1)}{x_1} - \frac{(b-y_1)}{x_1}}{1 - \frac{(b-y_1)}{x_1}\frac{(b-y_1)}{x_1}}$ (Let $m\angle BAC = \emptyset$)

$$= \frac{\frac{-2(b-y_1)}{x_1}}{\frac{x_1^2 - b^2 + 2by_1 - y_1^2}{x_1^2}}$$

$$= \frac{-2(b-y_1)x_1}{b^2 - y_1^2 - b^2 + 2by_1 - y_1^2}$$ (Using equation (ii))

$$= \frac{-2(b-y_1)x_1}{-2y_1^2 + 2by_1}$$

$$= \frac{-2(b-y_1)x_1}{-2y_1(y_1 - b)}$$

$$= \frac{-2(y_1-b)x_1}{2y_1(y_1 - b)}$$

$$\tan\emptyset = -\frac{x_1}{y_1} \quad \text{... (iii)}$$

We know that

$$\tan 2\emptyset = \frac{2\tan\emptyset}{1 - \tan^2\emptyset}$$

$$= \frac{2\left(-\frac{x_1}{y_1}\right)}{1 - \frac{x_1^2}{y_1^2}}$$

$$= \frac{-2x_1y_1}{y_1^2 - x_1^2}$$

$$= \frac{2x_1y_1}{x_1^2 - y_1^2} \quad \text{... (iv)}$$

From equation (i) and equation (iv)

$$\tan\emptyset = \tan 2\emptyset$$

$$\Rightarrow \quad \theta = 2\emptyset$$

Hence central angle of minor arc is double than the angle subtended by the corresponding major arc.

- **An angle in a semi-cricle is a right angle**

Let $P(x, y)$ be any point on the circle $x^2 + y^2 = r^2$ with radius r and centre at origin, whereas $A(r, 0)$ and $B(-r, 0)$ are two points of its diameter AB as shown in Fig. 8.28.

Now $\angle APB$ is the angle in semi-circle

Now,

$$\text{Slope of } \overline{PB} = \frac{y}{x+r} = m_1$$

and $$\text{Slope of } \overline{AP} = \frac{y}{x-r} = m_2$$

Product of slopes $= m_1 m_2$

$$= \left(\frac{y}{x+r}\right)\left(\frac{y}{x-r}\right)$$

$$= \frac{y^2}{x^2 - r^2}$$

$$= \frac{y^2}{x^2 - (x^2 + y^2)} (\because x^2 + y^2 = r^2)$$

$$= \frac{y^2}{-y^2} = -1$$

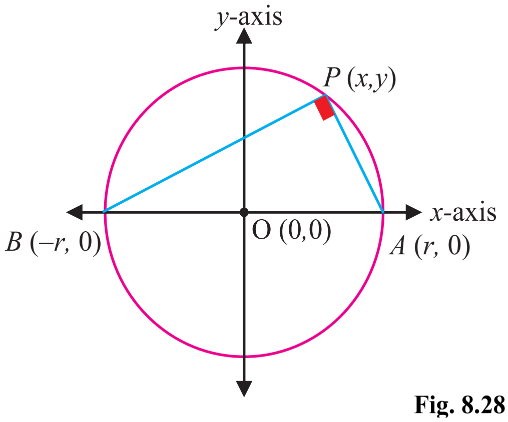


Fig. 8.28

$\because$ product of slopes $= -1$.

$\therefore$ $\overline{AP} \perp \overline{PB}$

Hence $\angle APB$ is right angle.

So, angle in a semi-circle is right angle.

- **The perpendicular at the outer end of radial segment is tangent to the circle**

Let l be the line perpendicular to the radial segment $\overline{CP}$ of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the outer end $P(x_1, y_1)$ whereas $C(-g, -f)$ is the centre of the circle as shown in Fig. 8.29.

Now,

$$\text{Slope of radial segment } \overline{CP} = \frac{y_1+f}{x_1+g} = m$$

So, $$\text{slope of } l = -\frac{1}{m}$$

$$= -\frac{(x_1 + g)}{y_1 + f} \ldots \text{(i)}$$

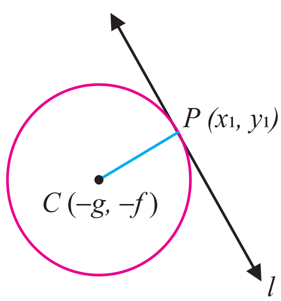


Fig. 8.29

We have

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Differentiating w.r.t x

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (2y + 2f)\frac{dy}{dx} = -2x - 2g$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2(x+g)}{2(y+f)} = \frac{-(x+g)}{y+f}$$

Now,

Slope of tangent to the circle at $(x_1, y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$

$$= -\frac{(x_1 + g)}{y_1 + f} \ldots \text{(ii)}$$

From equation (i) and equation (ii)

Slope of l = slope of tangent at (x_1, y_1)

So, line l is tangent to the circle at (x_1, y_1).

Hence the perpendicular at the outer end of a radial segment is tangent to the circle.

- **The tangent to a circle at any point of the circle is perpendicular to the radial segment at that point.**

Let t be the tangent to the circle at any point $P(x, y)$ of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $C(-g, -f)$ whereas $\overline{CP}$ is the radial segment of the circle at the point P as shown in Fig. 8.30.

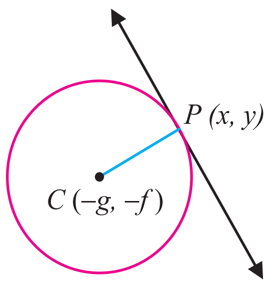


Fig. 8.30

Now,

Slope of radial segment $\overline{CP} = \frac{y+f}{x+g} = m_1$

We have the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Differentiating w.r.t x

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$2(y + f)\frac{dy}{dx} = -2(x + g)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{x+g}{y+f}\right)$$

i.e., Slope of tangent to the circle at any point of the circle

$$= -\frac{(x + g)}{y + f} = m_2$$

Here $$m_1 m_2 = \left(\frac{y+f}{x+g}\right)\left[-\left(\frac{x+g}{y+f}\right)\right]$$

$$= -1$$

∵ $m_1m_2 = -1$

∴ Tangent to the circle is perpendicular to the radial segment at the point of contact.

Exercise 8.4

Prove the following analytically

1. The tangents drawn at the ends of a diameter of a circle are parallel.
2. A normal to a circle passes through the centre of circle.
3. The mid-point of hypotenuse of a right triangle is the centre of the circle circumscribing the triangle.
4. Measure of the central angle of a major arc is double the measure of the inscribed angle of corresponding minor arc.
5. The parallelogram circumscribing a circle is a rhombus.

Review Exercise 8

1. Tick the correct option.

(i) If a plane cuts one nappe of a right circular cone perpendicularly then conic is -----

(a) parabola (b) circle (c) ellipse (d) hyperbola

(ii) The centre of circle with equation $(x+3)^2 + (y-5)^2 = 36$ is --------------

(a) $(3,-5)$ (b) $(-3,-5)$ (c) $(-3,5)$ (d) $(3,5)$

(iii) The centre of circle with equation $x^2 + y^2 + 10x - 8y + 1 = 0$ is --------------

(a) $(-5,\ 8)$ (b) $(-10,\ 8)$ (c) $(5,-4)$ (d) $(-5,\ 4)$

(iv) The radius of circle with equation $x^2 + y^2 + 4x + 6y + 1 = 0$ is --------------

(a) $\sqrt{13}$ (b) $\sqrt{12}$ (c) $\sqrt{10}$ (d) $\sqrt{71}$

(v) Which of the following is a degenerate conic

(a) circle (b) ellipse (c) line (d) parabola

(vi) The equation of circle with centre at origin and diameter of 10 units is ---------

(a) $x^2 + y^2 = 100$ (b) $x^2 + y^2 + 100 = 0$

(c) $x^2 + y^2 = 50$ (d) $x^2 + y^2 = 25$

(vii) For what value of k the radius of circle $x^2 + y^2 + 6x - 4y + k = 0$ is 5

(a) 11 (b) -12 (c) 10 (d) 12

(viii) The centre of the circle $x^2 + y^2 + 6x + 8 = 0$ is:

(a) on x-axis (b) on y-axis (c) in 1st quadrant (d) at origin

(ix) The circle $x^2 + y^2 + 6x + 10y + 9 = 0$

(a) touches x-axis (b) touches y-axis

(c) passes through origin (d) cuts x-axis

(x) The circle $x^2 + y^2 + 20x - 8y + 16 = 0$
(a) touches x-axis (b) touches y-axis
(c) passes through origin (d) cuts y-axis

(xi) The line $y = 2x + c$ will be tangent to $x^2 + y^2 = 25$ if
(a) $c^2 = 25$ (b) $c^2 = 625$ (c) $c^2 = 50$ (d) $c^2 = 125$

(xii) For what value of k, the line $y = 2x + 3$ is tangent to $x^2 + y^2 = k^2$
(a) $\pm\frac{5}{\sqrt{3}}$ (b) $\pm\frac{3}{\sqrt{5}}$ (c) $\pm\sqrt{\frac{3}{5}}$ (d) $\pm\frac{\sqrt{5}}{3}$

(xiii) For what value of k, the line $2x + 3y + k = 0$ is normal to the circle $x^2 + y^2 + 2x + 9 = 0$
(a) -2 (b) 2 (c) 3 (d) -3

(xiv) Equation of tangent to the circle $x^2 + y^2 = 25$ at $(3, 4)$ is:
(a) $3x + 4y = 0$ (b) $4x + 3y = 25$
(c) $3x + 4y = 25$ (d) $3x + 4y = 5$

(xv) Equation of normal to the circle $x^2 + y^2 = 36$ at $(2, 4\sqrt{2})$ is:
(a) $2x + 4\sqrt{2}y = 0$ (b) $4\sqrt{2}x + 2y = 0$
(c) $2x - 4\sqrt{2}y = 0$ (d) $4\sqrt{2}x - 2y = 0$

(xvi) The equation of tangent to $x^2 + y^2 = 100$ is ________ if slope of tangent is $\sqrt{15}$
(a) $y = \sqrt{15}x \pm 40$ (b) $y = -\sqrt{15x} \pm 40$
(c) $y = \sqrt{15x} \pm 40y$ (d) $y = -\sqrt{15x} \pm 40y$

(xvii) The length of tangent to the circle $x^2 + y^2 + 2y - 1 = 0$ from $(5, 2)$ is:
(a) $\sqrt{24}$ units (b) $\sqrt{33}$ units (c) $\sqrt{32}$ units (d) $\sqrt{31}$ units

(xviii) Congruent chords of a circle are equidistant from its
(a) diameter (b) centre (c) arc (d) segment

(xix) Angle in a semi-circle is --------------
(a) acute angle (b) obtuse angle (c) right angle (d) straight angle

(xx) The point $(3, 3)$ is ______________ the circle $x^2 + y^2 = 64$
(a) outside (b) inside
(c) on (d) cannot be determined

2. Find the equation of circle passing through $(2, 3), (4, 6)$ and
(i) centre on x-axis (ii) centre on y-axis

3. $y = \sqrt{3}x + 10$ is the equation of tangent to the circle with centre at origin. Find the equation of normal to the circle at the point of tangent.

4. Find the condition of tangency, secancy and normality of line $x + y + k = 0$ to the circle $x^2 + y^2 + 2x - 3 = 0$.