Parabola, Ellipse and Hyperbola

Unit 9

Introduction

As a matter of fact, parabola, ellipse and hyperbola are the types of conic. In previous chapter, the Greek concept of conic was discussed in detail whereas the analytic concept was given in short.

In this chapter we will study conics analytically. Recall that in analytic geometry, a conic is the locus of a moving point or the set of all points whose distance from a fixed point (in the plane) bears a constant ratio to its distance from a fixed line in the same plane.

The fixed point, the fixed line and the constant ratio are called focus, directrix and eccentricity of the conic respectively, whereas eccentricity is denoted by e.

The line through the focus and perpendicular to the directrix is called the axis of the conic. The distance of a point on the conic from its focus is called the focal distance. The chord through the focus of a conic is called focal chord of the conic and the focal chord which is perpendicular to its axis is called the latus rectum.

Different conics are identified on the basis of the value of eccentricity as mentioned below.

If $e = 1$, the conic is called a parabola.

If $e < 1$, the conic is called an ellipse.

If $e > 1$, the conic is called a hyperbola.

whereas

$$e = \frac{\text{The distance from the focus to any point on the conic}}{\text{The distance from the directrix to that point on the conic}}$$

9.1 Parabola

9.1.1 Define parabola and its elements (i.e., focus, directrix, eccentricity, vertex, focal chord and latus rectum).

Parabola and its elements:

A parabola is the set of all points in the plane which are equidistant from a fixed line and a fixed point not on the line.

The fixed line is called the directrix of the parabola and the fixed point is called its focus.

The straight line through the focus and perpendicular to the directrix is called the axis of the parabola. The point where the parabola meets its axis is called the vertex of the parabola. A chord which passes through the focus is called focal chord and the focal chord which is perpendicular to the axis of parabola is called its latus rectum. Focus is the mid-point of latus rectum. In figure 9.1, P, Q and R are three points on the parabola whose focus is F and the vertex is V. The focal chord QR is its latus rectum. By definition of eccentricity,

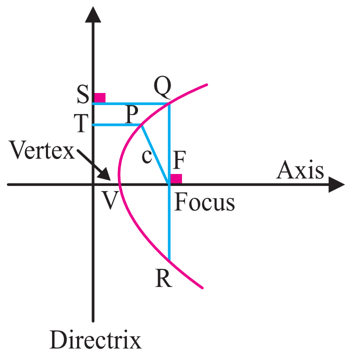


Fig 9.1

$$e = \frac{|\overline{PF}|}{|\overline{PT}|} \quad \text{or} \quad e = \frac{|\overline{QF}|}{|\overline{QS}|}$$

$$= 1 \quad \text{or} \quad e = 1 \qquad (\because \ |\overline{PF}| = |\overline{PT}| \text{ and } |\overline{QF}| = |\overline{QS}|)$$

So, the eccentricity of parabola is 1.

Note: The axis of parabola is also called the axis of symmetry of the parabola as the parabola is symmetric about its axis.

9.2 General Form of Equation of a Parabola

General form of equation of parabola means the equation which can be used to find parabola for any focus and any directrix.

9.2.1 Derive the general form of an equation of a parabola

Consider a parabola whose focus is $F(h,k)$ and equation of its directrix is $lx+my+n=0$.

Let $P(x,y)$ be any point on the parabola. By definition of parabola

$$|\overline{PF}| = |\overline{PT}|$$

i.e., $$\sqrt{(x-h)^2+(y-k)^2} = \left|\frac{lx+my+n}{\sqrt{l^2+m^2}}\right|$$

$\Rightarrow \quad (l^2+m^2)[(x-h)^2+(y-k)^2] = (lx+my+n)^2$

$\Rightarrow \quad l^2x^2-2hl^2x+l^2h^2+m^2x^2-2hm^2x+h^2m^2+l^2y^2-2kl^2y+k^2l^2$
$+m^2y^2-2km^2y+k^2m^2 = l^2x^2+m^2y^2+n^2+2lmxy+2mny+2lnx$

$\Rightarrow \quad m^2x^2-2lmxy+l^2y^2-2hl^2x-2hm^2x-2lnx-2kl^2y-2km^2y$
$-2mny+l^2h^2+h^2m^2+k^2l^2+k^2m^2-n^2=0$

$\Rightarrow \quad (mx-ly)^2-2(hl^2+hm^2+ln)x-2(kl^2+km^2+mn)y+l^2h^2+h^2m^2$
$+k^2l^2+k^2m^2-n^2=0$

$\Rightarrow \quad (mx-ly)^2+2gx+2fy+c=0 \qquad \ldots(i)$

where

$$g = -(hl + hm + ln)$$
$$f = -(kl + km + mn)$$

and $\quad c = l^2h^2 + h^2m^2 + k^2l^2 + k^2m^2 - n^2$

Equation (i) is the general equation of parabola.

It is evident from the equation that second degree terms in the equation of parabola form a perfect square.

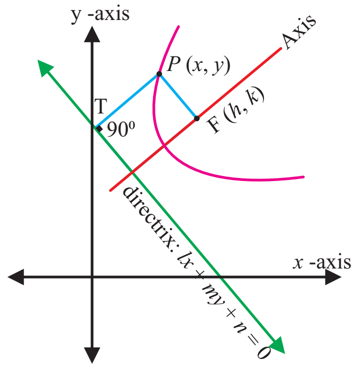


Fig 9.2

In case directrix is parallel to x-axis then $l = 0$. So, by adjusting values of g, f, c accordingly equation (i) is reduced to

$$m^2x^2 + 2gx + 2fy + c = 0 \qquad \ldots\text{(ii)}$$

In case directrix is parallel to y-axis then $m = 0$. So, by adjusting values of g, f, c accordingly equation (i) is reduced to

$$l^2y^2 + 2gx + 2fy + c = 0 \qquad \ldots\text{(iii)}$$

Let $a = m^2$ and $b = l^2$

then

$$ax^2 + by^2 + 2gx + 2fy + c = 0 \qquad \ldots\text{(iv)}$$

represents the parabola whose directrix is parallel to either of axes if either $a = 0$ or $b = 0$.

Example: Find the equation of parabola whose focus is $F(3, 4)$ and directrix $l: 2x - 3 = 0$.

Solution: Let $P(x, y)$ be any point on the parabola.

According to the definition of parabola

$|\overline{PF}|$ = distance of P from l

i.e., $\quad \sqrt{(x-3)^2 + (y-4)^2} = \left|\frac{2x-3}{2}\right|$

$\Rightarrow \quad 4\{(x-3)^2 + (y-4)^2\} = (2x-3)^2$

$\Rightarrow \quad 4(x^2 - 6x + 9 + y^2 - 8y + 16) = 4x^2 - 12x + 9$

$\Rightarrow \quad 4x^2 + 4y^2 - 24x - 32y + 100 = 4x^2 - 12x + 9$

$\Rightarrow \quad 4y^2 - 12x - 32y + 91 = 0$

This is the required equation of parabola.

9.3 Standard Form of Equation of Parabola

The four possible orientations of parabola such that vertex is at origin and the axis of parabola is along the x-axis or y-axis, are called the standard positions of a parabola and the resulting equations are called the standard equations of a parabola.

9.3.1 Derive the standard equations of parabola, sketch their graphs and find their elements:

(a) Standard equations of parabola

- **Standard equation of parabola when axis of parabola is along x-axis and vertex is at origin.**

Consider a parabola whose vertex is at origin and axis of symmetry is along x-axis as shown in the figure 9.3.

Let $F(a,0)$ be the focus on x-axis then the equation of directrix l will be $x=-a$ or $x+a=0$.

Let $P(x,y)$ be any point on the parabola, where $a \neq 0$. (Fig. 9.3)

According to the definition of parabola

$|\overline{PF}|$ = distance of P from l

i.e., $\sqrt{(x-a)^2+y^2} = |x+a|$

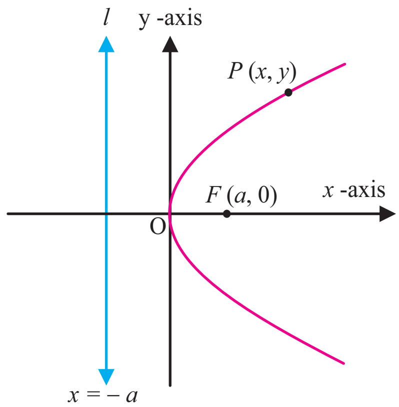


Fig 9.3

Squaring both sides

$$(x-a)^2+y^2=(x+a)^2$$

$$\Rightarrow \quad y^2=4ax$$

This is the required standard equation of parabola.

If $a>0$ then it is cup-right parabola.

If $a<0$ then it is cup-left parabola as shown in fig. 9.4.

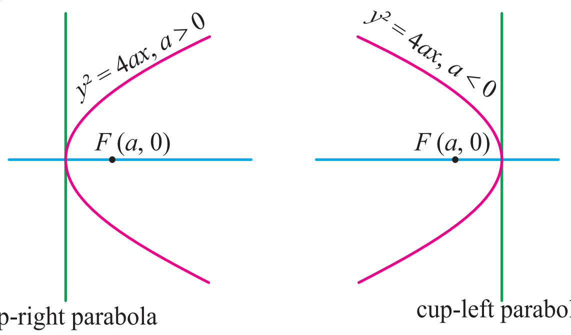


Fig 9.4

- **Standard equation of parabola when axis of parabola is along y-axis and vertex is at origin.**

By using definition of parabola, we can derive standard equation of parabola, when axis of parabola is along y-axis and vertex is at origin which is

$$x^2=4ay$$

where $F(0,a)$ is focus and equations of directrix is $y=-a$ as shown in fig. 9.5

If $a>0$ then it is cup-up parabola.

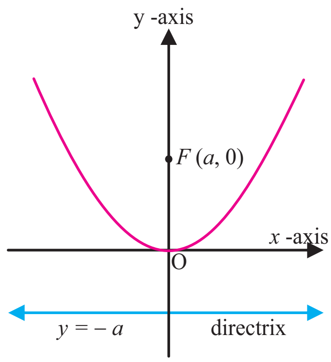
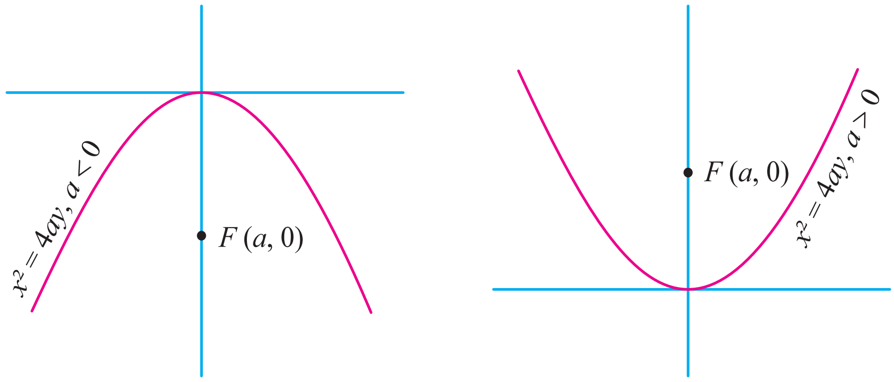


Fig 9.5

If $a < 0$ then it is cup-down parabola as shown in fig. 9.6.



cup-down parabola　　　　cup-up parabola

Fig. 9.6

Latus Rectum:

We know that the chord through the focus of a parabola and perpendicular to its axis is called the latus rectum of the parabola. In the Fig. 9.7 $\overline{AB}$ is latus rectum.

Here $\quad |\overline{AB}| = 2|\overline{AF}| = 2|\overline{AC}|$

or $\quad |\overline{AB}| = 2|\overline{EF}| = 2(2a) = 4a$

Thus, the length of the latus rectum is $|4a|$.

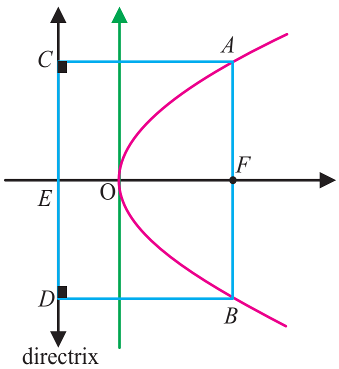


Fig 9.7

(b) Sketching the graph of parabolas from their standard equations

Graphs of parabolas from their standard equations can be sketched using the following steps.

1. Determine the axis of parabola from the given standard equations. If equation contains x^2-term, then its axis of symmetry is along y-axis. If equation contains y^2-term then its axis of symmetry is along x-axis.
2. Determine, in which way, the parabola opens. If parabola is along x-axis then it is cup-right and cup-left if $a > 0$ and $a < 0$ respectively. If parabola is along y-axis then it is cup-up and cup-down if $a > 0$ and $a < 0$ respectively.
3. Locate focus and draw the latus rectum of length $|4a|$.
4. Sketch parabola joining the ends of latus rectum with its vertex.

Example: Sketch the graphs of the following parabolas.

(i) $y^2 = 12x$ $\qquad$ (ii) $x^2 = -10y$

Solution:

(i) $y^2 = 12x$

Comparing with $y^2 = 4ax$

We get $4a = 12 \quad \Longrightarrow \quad a = 3$

$\because$ equation has y^2 −term

$\therefore$ parabola is along x-axis and it is cup-right because $a > 0$.

Here, latus rectum $= |4a| = |4(3)| = 12$

Now, we draw latus rectum $\overline{AB}$ through focus $F(3, 0)$ and sketch the parabola as shown in figure 9.8.

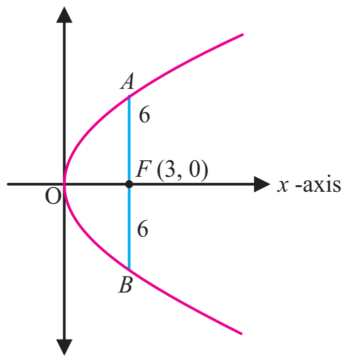


Fig 9.8

(ii) $x^2 = -10y$

Comparing with $x^2 = 4ay$

We get $4a = -10 \quad \Longrightarrow \quad a = -\frac{5}{2}$

$\because$ equation has x^2 −term

$\therefore$ parabola is along y-axis and it is cup-down as $a < 0$.

Here, latus rectum $= |4a| = \left|4\left(-\frac{5}{2}\right)\right| = 10$

Now, we draw latus rectum $\overline{AB}$ through focus $F\left(0, -\frac{5}{2}\right)$ and sketch the parabola as shown in figure 9.9.

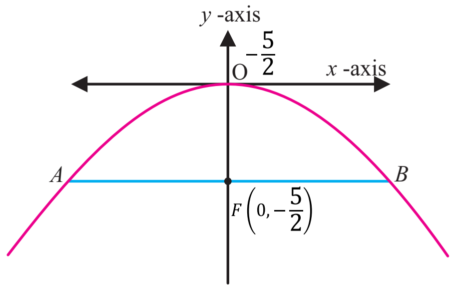


Fig 9.9

Standard forms of translated equations of parabola

The topic of translation and rotation of axes will be discussed in detail in section 9.12. At this stage we should know that if standard parabola is translated h units horizontally and k units vertically then its vertex will be (h, k) and the resulting equations will be

(1) $(y - k)^2 = 4a(x - h)$ in case axis of symmetry is parallel to x-axis as shown in the Fig. 9.10.

Here focus and directrix will be $(h + a, k)$ and $x = h - a$ respectively.

(2) $(x - h)^2 = 4a(y - k)$ in case axis of symmetry is parallel to y-axis as shown in the Fig. 9.11.

Here focus and directrix are $(h, k + a)$ and $y = k - a$ respectively.

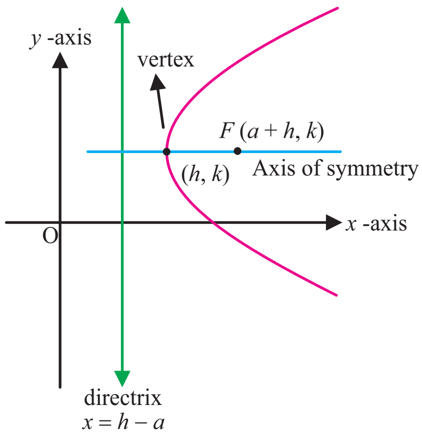


Fig 9.10

In order to find elements of parabola from its equation, we summarize the equations along with its elements as under:

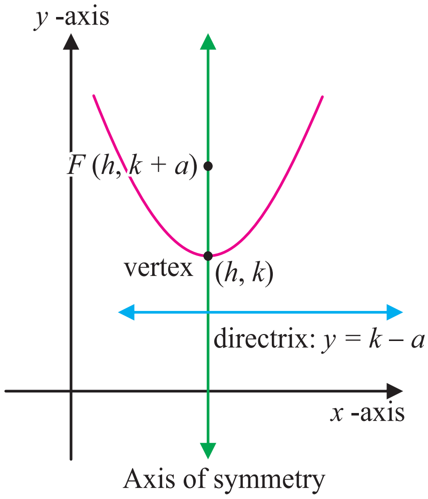


Fig 9.11

Equation of Parabola	Related information
(1) $y^2 = 4ax$	• Axis of symmetry is along x-axis with vertex at origin. Axis of symmetry: $y = 0$ • If $a > 0$ then it is cup-right. • If $a < 0$ then it is cup-left. • Focus is $(a, 0)$ • Latus rectum $= \|4a\|$ • Directrix is $x = -a$ • End points of latus rectum $= (a, \pm 2a)$
(2) $x^2 = 4ay$	• Axis of symmetry is along y-axis with vertex at origin. Axis of symmetry: $x = 0$. • If $a > 0$ then it is cup-up. • If $a < 0$ then it is cup-down. • Focus is $(0, a)$ • Latus rectum $= \|4a\|$ • Directrix is $y = -a$ • Ends points of latus rectum $= (\pm 2a, a)$
(3) $(y-k)^2 = 4a(x-h)$	• Axis of symmetry is parallel to x-axis with vertex $(h, k) y - k = 0$. • If $a > 0$ then it is cup-right. • If $a < 0$ then it is cup-left.

Equation of Parabola	Related information
	• Focus is $(h+a, k)$ • Latus rectum $= \|4a\|$ • Directrix is: $x-h=-a$ • End points of latus rectum $=(h+a, k\pm a)$
(4) $(x-h)^2=4a(y-k)$	• Axis of symmetry is parallel to y-axis i.e., $x-h=0$ • If $a>0$ then it is cup-up. • If $a<0$ then it is cup-down. • Focus is $(h, k+a)$ • Latus rectum $= \|4a\|$ • Directrix is: $y-k=-a$ • End points of latus rectum $=(h\pm 2a, k+a)$

(c) Finding elements of parabola

We find different elements of parabola with the help of the following examples.

Example 1. Find focus, latus rectum and equation of directrix of the parabola with equation $y^2=12x$. Also sketch its graph.

Solution: Given parabola is: $y^2=12x$

comparing with $y^2=4ax$

We get, $4a=12$

$\Rightarrow \quad a=3$

$\because$ Parabola is along x-axis with vertex $(0,0)$.

$\therefore$ Its focus $=(a,0)$

$=(3,0)$

Now, latus rectum $=|4a|$

$=|4(3)|$

$=|12|=12$

Equation of directrix will be

$x=-a$

i.e., $x=-3$

or $x+3=0$

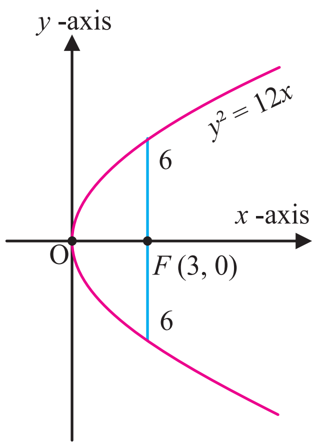


Fig 9.12

Graph of Parabola

Here,

Axis of symmetry is along x-axis with vertex at origin.

$\because \quad a>0$

$\therefore$ Its is cup-right parabola and latus rectum is 12 units.

This graph is shown in Fig. 9.12.

Example 2. Find vertex, focus, latus rectum, equation of axis and directrix of parabola $(x+2)^2 = -8(y-3)$. Also sketch its graph.

Solution: Given parabola is $(x+2)^2 = -8(y-3)$

comparing with $(x-h)^2 = 4a(y-k)$

We get, $h = -2, k = 3$ and $4a = -8$

$\Rightarrow$ $a = -2$

$\because$ Axis of symmetry is parallel to y-axis.

$\therefore$ Its focus $= (h, k+a)$

$= (-2, 3-2)$

$= (-2, 1)$

Its vertex $= (h, k) = (-2, 3)$

Now, latus rectum $= |4a|$

$= |-8| = 8$ units

Equation of axis will be

$x = h$

i.e., $x = -2$

or $x + 2 = 0$

Equation of directrix will be

$y - k = -a$

$\Rightarrow$ $y - 3 = 2$

$\Rightarrow$ $y - 5 = 0$

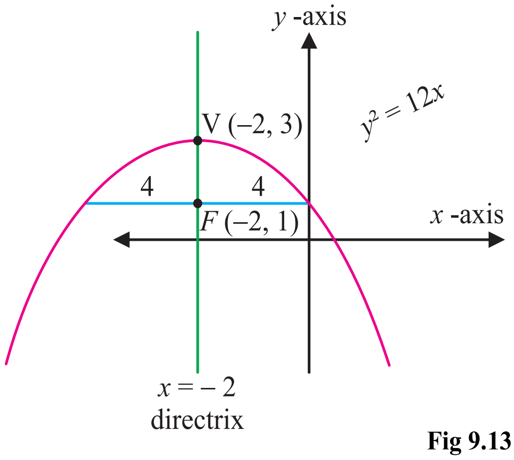


Fig 9.13

Graph of Parabola

Here, axis of symmetry is $x = -2$, which is parallel to y-axis. Vertex, focus and latus rectum are $(-2, 3), (-2, 1)$ and 8 respectively.

$\because$ $a < 0$

$\therefore$ Its is cup-down parabola. The graph is shown in Fig. 9.13.

9.3.2 Find the equation of a parabola with the following given elements:

- **focus and vertex,**
- **focus and directrix,**
- **vertex and directrix,**
- **vertex and points.**

(i) Equation of parabola when focus and vertex are given.

The method of finding equation of parabola when focus and vertex are given is explained with the help of the following examples.

Example 1. Find the equation of parabola when focus is $(5, 0)$ and vertex is $(0, 0)$.

Solution: Here focus $= (5, 0) = (a, 0)$

So $a = 5$

$\because$ Focus is on x-axis and vertex is at origin

$\therefore$ Its equation will be

$$y^2 = 4ax$$

i.e., $y^2 = 4(5)x$

or $y^2 = 20x$

Example 2. Find equation of parabola whose vertex is $(2, 3)$ and focus is $(2, 7)$.

Solution: Here, vertex $= (2, 3) = (h, k)$ and focus $= (2, 7) = (h, k + a)$

So, $k + a = 7$

$\Rightarrow a = 4$

According to the condition axis of symmetry is parallel to y-axis with vertex (h, k).

So, its equation will be

$$(x - h)^2 = 4a(y - k)$$

By using values of a, h, k

We get, $(x - 2)^2 = 4(4)(y - 3)$

$\Rightarrow x^2 - 4x + 4 = 16y - 48$

$\Rightarrow x^2 - 4x - 16y - 52 = 0$

(ii) Equation of parabola when focus and directrix are given

Method of finding equation of parabola when focus and directrix are given is explained with the following examples.

Example: Find the equation of parabola whose focus is $(2, 4)$ and equation of directrix is $x + 3 = 0$.

Solution:

Here, focus $= (2, 4)$ and directrix is $x + 3 = 0$.

Let $P(x, y)$ be any point of parabola.

So, $|\overline{PF}|$ = distance of P from directrix

i.e., $\sqrt{(x-2)^2 + (y-4)^2} = |x + 3|$

Squaring both sides

$$(x - 2)^2 + (y - 4)^2 = (x + 3)^2$$

$\Rightarrow x^2 - 4x + 4 + y^2 - 8y + 16 = x^2 + 6x + 9$

$\Rightarrow y^2 - 8y - 10x + 11 = 0$

This is the required equation of parabola.

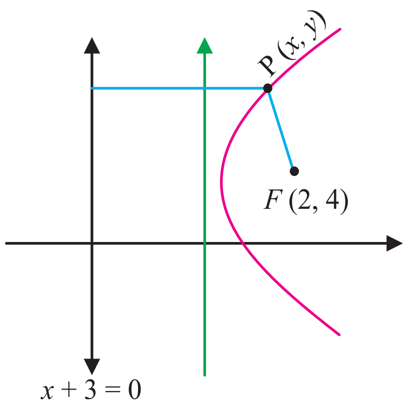


Fig 9.14

(iii) Equation of parabola when vertex and directrix are given

Method of finding equation of parabola when vertex and directrix are given is explained with the help of the following examples.

Example 1. Find the equation of parabola whose directrix is $x = 5$ and vertex is at origin.

Solution:

$\because$ directrix is parallel to y-axis and vertex is origin.

$\therefore$ axis of symmetry is along x-axis and its equation will be

$$y^2 = 4ax \quad \ldots(i)$$

with directrix $x = -a$ ...(ii)

whereas given directrix is: $x = 5$ ...(iii)

comparing equations (ii) and (iii)

we get $a = -5$

By using $a = -5$ in equation (i)

we get, $y^2 = -20x$

This is the required equations of parabola.

Example 2. Find the equation of parabola whose vertex is $(1, 2)$ and directrix is $y = 4$.

Solution:

$\because$ directrix is parallel to x-axis and vertex is not at origin.

$\therefore$ Axis of symmetry will be parallel to y-axis and its equation will be

$(x - h)^2 = 4a(y - k)$ ...(i)

with vertex (h, k) and directrix $y = k - a$ ...(ii)

Given directrix is $y = 4$ ...(iii)

Here vertex $= (h, k) = (1, 2)$

comparing equation (ii) and (iii)

we get, $k - a = 4$

i.e., $2 - a = 4$

$\Rightarrow$ $a = -2$

Using $a = -2, h = 1$ and $k = 2$ in equation (i)

we get, $(x - 1)^2 = -8(y - 2)$

This is the required equation of parabola.

(iv) Equation of parabola when vertex and point are given:

The method of finding equation of parabola when vertex and point are given is explained by the following example.

Example: Find the equation of parabola whose vertex is $(0, 0)$ and passes through $(1, 2)$.

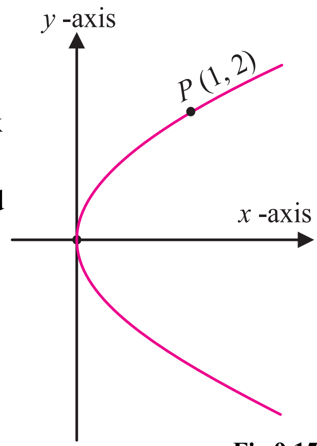


Fig 9.15

Solution:

$\because$ Vertex is at origin.

$\therefore$ Axis of symmetry may be along x-axis or y-axis.

Case I: When axis of symmetry is along x-axis

Let $y^2 = 4ax$ ...(i)

be the equation of parabola

∵ point $P(1,2)$ lies on the parabola

∴ equation (i) becomes

$$4 = 4a \implies a = 1$$

By using $a = 1$ in equation (i)

we get,

$y^2 = 4x$, which is the required equation of parabola.

Case II: When axis of symmetry is along y-axis

Let $x^2 = 4ay$ ...(i)

be the equation of parabola

∵ point $P(1,2)$ lies on the parabola

∴ equation (i) becomes

$$1 = 8a \implies a = \frac{1}{8}$$

By using $a = \frac{1}{8}$ in equation (i)

we get

$$x^2 = \frac{1}{2}y$$

This is the required equation.

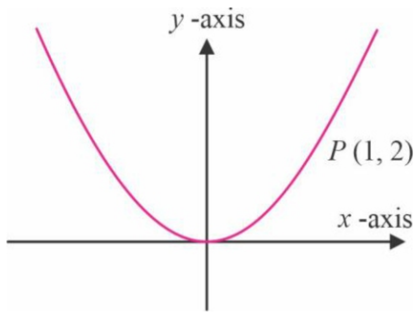


Fig 9.16

Exercise 9.1

1. Draw the following parabolas:
   (i) $y^2 = 10x$ (ii) $x^2 = -12y$
   (iii) $y^2 - x - 2y - 1 = 0$ (iv) $x^2 - 6x - 2y + 5 = 0$
2. Determine vertex, focus, latus rectum and equation of directrix of the following. Also find the equation of the axis of symmetry.
   (i) $y^2 = -8x$ (ii) $x^2 = -16y$
   (iii) $(y+3)^2 = 12(x-2)$ (iv) $(x+5)^2 = 8(y-3)$
   (v) $x^2 + 4x - y + 5 = 0$ (vi) $y^2 - 6y + 8x - 23 = 0$
3. Find the equation of parabola whose focus is $F(1,-2)$ and directrix is $3x - 5 = 0$.
4. Find the equation of the parabola whose focus is $(3,4)$ and the directrix is the line $x + y - 1 = 0$.
5. Find the equation of the parabolas whose focus and vertex are as under:
   (i) Vertex $(0,0)$; focus $(5,0)$ (ii) Vertex $(0,0)$; focus $(0,-2)$
   (iii) Vertex $(1,-3)$; focus $(1,2)$ (iv) Vertex $(2,4)$; focus $(3,4)$

6. Find the equation of parabola whose focus and directrix are given:
   (i) focus $(3, 0)$ and directrix $x - 5 = 0$
   (ii) focus $(0, 4)$ and directrix $y + 6 = 0$
   (iii) focus $(-4, 3)$ and directrix $y = 6$
7. Find equation of the parabola whose vertex and directrix are as under:
   (i) vertex $(0, 0)$; directrix $x = -6$
   (ii) vertex $(0, 0)$; directrix $y = 5$
   (iii) vertex $(3, 4)$; directrix $x = 5$
8. Find the equation of parabola whose vertex and point are given:
   (i) vertex $(0, 0)$; point $(3, 4)$
   (ii) vertex $(5, 0)$; point $(4, 6)$
9. Find the standard equation of parabola whose latus rectum and vertex are the diameter and centre of the circle respectively $x^2 + y^2 - 4x$ and$-8y - 5 = 0$.
10. Find the equation of circle and its circle is at the focus, whose diameter is the latus rectum of the parabola $x^2 = 12y$ and its centre is at the focus.
11. For what point of the parabola $y^2 = 10x$, the abscissa is equal to three times its ordinate.

9.4 Equation of Tangent and Normal of Parabola

In this section we will study about tangent and normal to a parabola along with their equations and conditions.

9.4.1 Recognize tangent and normal to a parabola

We know that a line which touches a parabola at a single point is called tangent and the line perpendicular to the point of tangency is called normal. In the figure 9.17, the line l is tangent to the parabola at point P whereas the line m is normal.

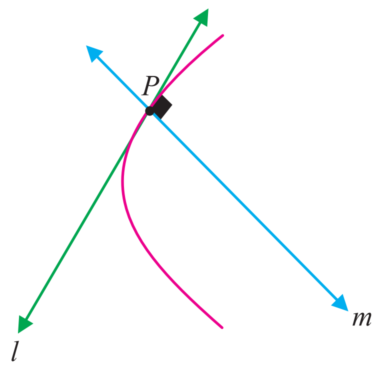


Fig 9.17

9.4.2 Find the condition when a line is tangent to a parabola at a point and hence write the equation of a tangent line in slope form

Consider a line

$l: y = mx + c$ ...(i)

and parabola $y^2 = 4ax$ ...(ii)

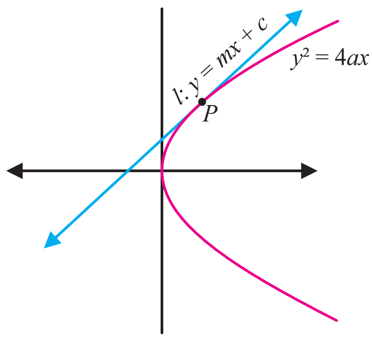


Fig 9.18

By using $y = mx + c$ in equation (i),

we get $(mx + c)^2 = 4ax$

$\Rightarrow \quad m^2x^2 + 2cmx + c^2 = 4ax$

$\Rightarrow \quad m^2x^2 + 2(cm - 2a)x + c^2 = 0$ ...(iii)

Given line will be tangent to the parabola

if $\Delta = 0$ (where Δ is discriminant of equation (iii))

i.e., $\quad 4(cm - 2a)^2 - 4c^2m^2 = 0$

$\Rightarrow \quad c^2m^2 - 4acm + 4a^2 - c^2m^2 = 0$

$\Rightarrow \quad 4a^2 = 4acm$

$\Rightarrow \quad a = cm$

$\Rightarrow \quad c = \frac{a}{m}$

This is the condition of tangency of $y = mx + c$ to the parabola $y^2 = 4ax$.

By using $c = \frac{a}{m}$ in equation (i) (since $a \neq 0$)

we get,

$$y = mx + \frac{a}{m}$$

This is the equation of tangent to parabola $y^2 = 4ax$ in slope form.

From equation (iii), by quadratic formula

we have $x = -\frac{2(cm-2a)}{2m^2}$

$= -\frac{(a - 2a)}{m^2}\left(\because \; c = \frac{a}{m}\right)$

$= \frac{a}{m^2}$

By using $x = \frac{a}{m^2}$ in equation (i)

we get,

$y = m\left(\frac{a}{m^2}\right) + c$

$\Rightarrow \quad y = \frac{2a}{m}\left(\because \; c = \frac{a}{m}\right)$

So, the point of tangency is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

Example: Find the condition when the line $2x + 3y = p$ is tangent to the parabola $y^2 = 12x$. Also find equation of tangent and point of tangency.

Solution: We have parabola $y^2 = 12x$ in which $a = 3$ and the line $l: 2x + 3y = p$

i.e., $\quad y = -\frac{2}{3}x + \frac{p}{3}$

Comparing with $y = mx + c$

we get, $m = -\frac{2}{3}$ and $c = \frac{p}{3}$

Now condition of tangency is:

$$c = \frac{a}{m}$$

i.e., $$\frac{p}{3} = \frac{3}{-\frac{2}{3}}$$

$$\Rightarrow \quad p = -\frac{27}{2}$$

Equation of tangent will be $y = mx + \frac{a}{m}$

i.e., $$y = -\frac{2}{3}x + \frac{3}{-\frac{2}{3}}$$

$$\Rightarrow \quad y = -\frac{2x}{3} - \frac{9}{2}$$

This is the required equation of tangent.

Now the point of tangency $= \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$$= \left(\frac{3}{\frac{4}{9}}, \frac{6}{-\frac{2}{3}}\right)$$

$$= \left(\frac{27}{4}, -9\right)$$

9.4.3 Find the equation of a tangent and a normal to a parabola at a point

Let $P(x_1, y_1)$ be a point of parabola $y^2 = 4ax$,

So $y_1^2 = 4ax_1$

Differentiating w.r.t x

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{2a}{y}}$$

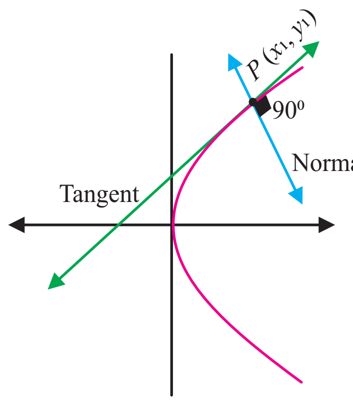


Fig 9.19

Now slope of tangent at $P(x_1, y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

i.e., $$m = \frac{2a}{y_1}$$

By point slope form, the equation of tangent will be

$$y - y_1 = m(x - x_1)$$

i.e., $y - y_1 = \frac{2a}{y_1}(x - x_1)$

$\implies$ $yy_1 - y_1^2 = 2ax - 2ax_1$

$\implies$ $yy_1 - 4ax_1 = 2ax - 2ax_1$ $\quad (\because y_1^2 = 4ax_1)$

$\implies$ $yy_1 = 2a(x + x_1)$

This is the equation of tangent to $y^2 = 4ax$ at (x_1, y_1).

$\because$ Normal is perpendicular to the tangent at point of contact $P(x_1, y_1)$

$\therefore$ slope of normal $= -\frac{1}{m}$

$$= -\frac{y_1}{2a}$$

Now equation of normal, by point slope form is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

$\implies$ $y_1(x - x_1) + 2a(y - y_1) = 0$

This is the equation of normal to the parabola $y^2 = 4ax$ at (x_1, y_1).

Example: Find the equation of tangent and normal to $x^2 = 8y$ at $(4, 2)$.

Solution: We have

$$x^2 = 8y$$

Differentiating w.r.t x

$$2x = 8\frac{dy}{dx}$$

$\implies$ $\boxed{\frac{dy}{dx} = \frac{x}{4}}$

Now slope of tangent at $(4, 2) = \left(\frac{dy}{dx}\right)_{(4,2)}$

i.e., $\boxed{m = 1}$

By point slope form the equation of tangent will be

$$y - y_1 = m(x - x_1)$$

$y - 2 = 1(x - 4)$ $\quad [\because (x_1, y_1) = (4, 2)]$

$\implies$ $x - y - 2 = 0$

$\because$ Normal is perpendicular to tangent

$\therefore$ Slope of normal $= m' = -1$

By point-slope form, equation of normal will be

$$y - 4 = -1(x - 2)$$

$\implies$ $x + y - 6 = 0$

9.5 Application of Parabola

Parabolas have important applications in suspension bridges, design of telescopes, radar antennas and lighting systems.

This is because of the important geometrical property of parabola which is stated as under:

Theorem: The tangent at a point P of parabola makes equal angles with the line through P parallel to the axis of parabola and the line through P and the focus.

Proof: Let line *l* be a tangent to parabola $y^2 = 4ax$ at point $P(x_1, y_1)$ as shown in Figure.

Let α be the angle between tangent and the line PQ parallel to the axis of the parabola and β be the angle between the tangent and the line through P and focus $F(a, 0)$.

∵ $P(x_1, y_1)$ lies on the parabola

∴ $y_1^2 = 4ax_1$

$\Rightarrow \quad a = \dfrac{y_1^2}{4x_1}$

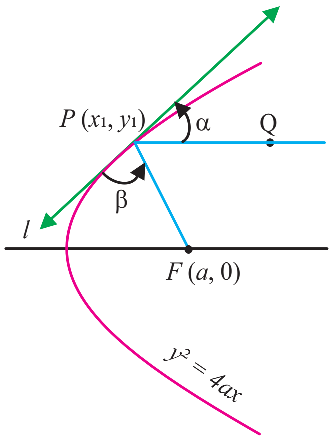


Fig 9.20

We have $y^2 = 4ax$

differentiating w.r.t *x*

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

Now, the slope of tangent to the parabola at P will be

$$m_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{2a}{y_1}$$

$$= \frac{2}{y_1}\left(\frac{y_1^2}{4x_1}\right)$$

$$= \frac{y_1}{2x_1}$$

Slope of $\overleftrightarrow{PQ} = m_2 = 0$ and slope of $\overline{PF} = m_3 = \dfrac{y_1}{x_1 - a}$

$$= \frac{y_1}{x_1 - \dfrac{y_1^2}{4x_1}}$$

$$= \frac{4x_1y_1}{4x_1^2 - y_1^2}$$

Angle from $\overleftrightarrow{PQ}$ to the tangent

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$= \frac{\frac{y_1}{2x_1} - 0}{1 + 0}$$

$$\tan\alpha = \frac{y_1}{2x_1} \quad \text{... (i)}$$

Angle from tangent to $\overline{PF}$

$$\tan\beta = \frac{m_3 - m_1}{1 + m_1 m_3}$$

$$= \frac{\frac{4x_1y_1}{4x_1^2 - y_1^2} - \frac{y_1}{2x_1}}{1 + \frac{4x_1y_1}{4x_1^2 - y_1^2} \times \frac{y_1}{2x_1}}$$

$$= \frac{8x_1^2y_1 - 4x_1^2y_1 + y_1^3}{8x_1^3 - 2x_1y_1^2 + 4x_1y_1^2}$$

$$= \frac{4x_1^2y_1 + y_1^3}{8x_1^3 + 2x_1y_1^2}$$

$$= \frac{y_1(4x_1^2 + y_1^2)}{2x_1(4x_1^2 + y_1^2)}$$

$$\tan\beta = \frac{y_1}{2x_1} \quad \text{... (ii)}$$

∵ α and β are acute angles and tan α = tan β (from (i) and (ii))

∴ α = β Hence proved.

9.5.1 Solve suspension and reflection problems related to parabola

In physics, according to the law of reflection of light, the angle of incidence is equal to the angle of reflection at point P of the surface as shown in the Fig. 9.21.

i.e., $\theta_1 = \theta_2$

So, $\alpha = \beta$ (complements of congruent angles)

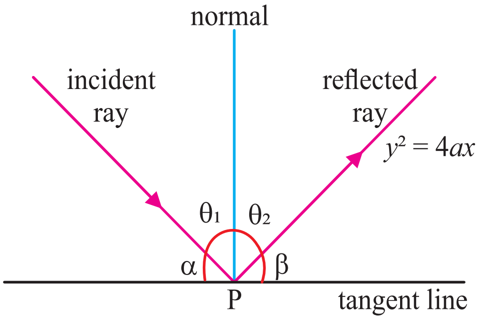


Fig 9.21

It means the angle between incident ray and the tangent line at P is equal to the angle between the reflected ray and the tangent line at P. Therefore, if the reflecting surface has parabolic cross sections with a common focus, then all light rays entering parallel to the axis of parabola will be reflected through the focus as shown in the figure 9.22.

In reflecting telescopes, this rule is used to reflect the parallel rays of light from the stars or planets off a parabolic mirror to an eye piece at the focus of the parabola.

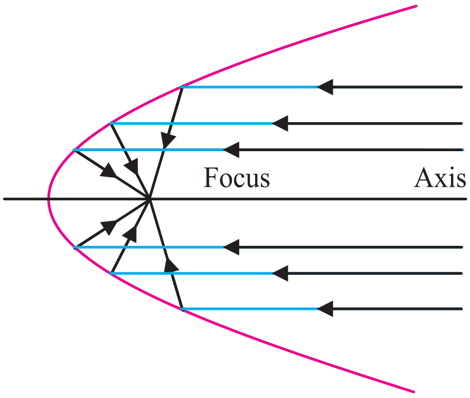


Fig. 9.22

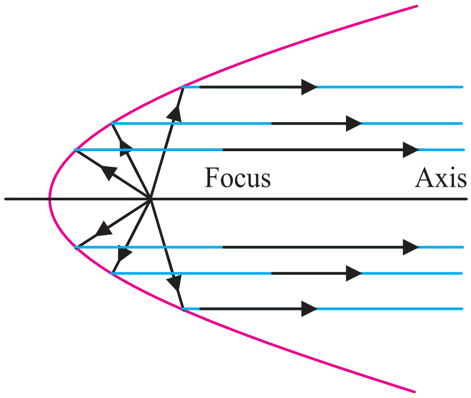


Fig. 9.23

Conversely, if a light source is located at the focus of a parabolic reflector, then the reflected rays will form a beam of parallel rays parallel to the axis of parabola as shown in the figure 9.23. The parabolic reflectors in automobile headlights and flash light use this rule. The optical principles which have been discussed above are also valid for radar signals, sound waves, radio waves etc.

Parabolas are also used in suspension problems related to suspension bridges and structures.

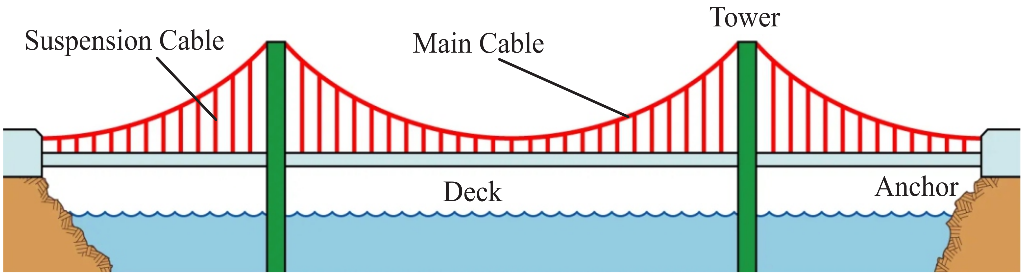


Fig. 9.24

We know that the cables of suspension bridges are mostly parabolic in shape. This shape provides the stability of bridges. The weight of the bridge and other physical forces (tensions, compressions) acting on the cable are transferred by the parabolic cables to the towers to which the cables are attached. This transfer of physical forces helps the bridges to remain operational for a long period of time.

Let us solve few examples related to suspension and reflection.

Example 1. How far from vertex should a light source be placed on the axis of parabolic reflector so that it produces a beam of parallel rays, whereas the depth and length of chord perpendicular on axis of parabolic reflector are 10 cm and 12 cm respectively and the parabola is cup-right.

Solution: Let the vertex of parabolic reflector is at origin as shown in the figure 9.25.

According to the condition $|\overline{AB}| = 12\ cm$ and $|\overline{OC}| = 10\ cm$.

$\because$ point C is the mid-point of $\overline{AB}$

$\therefore$ $|\overline{AC}| = |\overline{BC}| = 6\ cm$

Hence coordinates of A and B are $(10, 6)$ and $(10, -6)$ respectively.

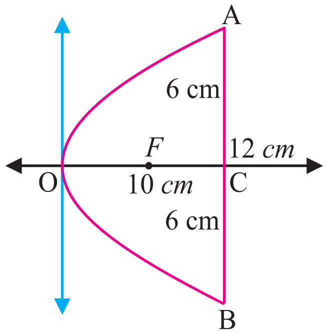


Fig 9.25

Le the equation of parabola be

$y^2 = 4ax$ ...(i)

$\because$ $A(10, 6)$ lies on the parabola

$\therefore$ we have $36 = 40a$

$\Rightarrow$ $a = \frac{9}{10}$

Now, focus $= (a, 0) = \left(\frac{9}{10}, 0\right)$

So, the light source should be placed at $F\left(\frac{9}{10}, 0\right)$ which is at a distance of $\frac{9}{10}$ cm from vertex.

Example 2. A main cable of a suspension bridge is suspended in the shape of parabola between two towers that are 600 ft apart and 90 ft above the roadway. If cable is at the height of 10 ft from the roadway at the centre of bridge then find:

(i) equation of parabola

(ii) height of suspender cable which is 150 ft away from the centre of bridge.

Solution: Let $\overline{AC}$ and $\overline{BD}$ represent towers of suspension bridge as shown in the Fig. 9.26.

According to the condition $|\overline{QE}| = 10, |\overline{QP}| = 80, |\overline{BD}| = 90$ and $|\overline{DE}| = 300$.

Let vertex Q is on y-axis then equation of parabola will be

$x^2 = 4a\,(y - 10)$ ...(i)

According to the condition, point $B(300, 90)$ lies on parabola.

So from equation (i), we get

$90000 = 4a(80)$

$\Rightarrow$ $a = \frac{9000}{32}$

$\Rightarrow$ $a = \frac{1125}{4}$

So, equation (i) becomes

$x^2 = 1125\,(y - 10)$ ...(ii)

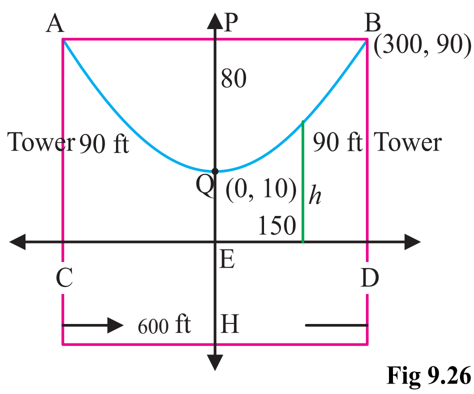


Fig 9.26

Let h be the height of suspender cable at 150 ft away from the centre of bridge.

$\because$ $T(150, h)$ lies on the parabola

$\therefore$ equation (ii) becomes

$$(150)^2 = 1125\,(h - 10)$$

$$\frac{22500}{1125} = h - 10$$

$\Rightarrow$ $h - 10 = 20$

$\Rightarrow$ $h = 30$

So, the required height is 30 ft.

Exercise 9.2

1. Find the condition when the line $y = mx + c$ is tangent to the parabola $x^2 = 4ay$. Also find point of contact and the equation of tangent.
2. Find condition of tangency and the point of tangency for the following lines and parabolas. Also find equation of tangent in each case:
   (i) $2x + y = c$; $y^2 = 10x$
   (ii) $3x + 4y = p$; $x^2 = 12y$
   (iii) $y = cx$; $y^2 = 8(x - 1)$
3. Find the equation of tangent and normal to the following parabolas at the given points:
   (i) $y^2 = 8x$; $(2, 4)$
   (ii) $x^2 = 4y$; $(-6, 9)$
   (iii) $(y - 1)^2 = 9(x - 2)$; $(3, 4)$
4. Find the equation of tangent and normal at $P(x_1, y_1)$ to the parabola $x^2 = 4ay$.
5. A light house uses a parabolic reflector that is 1 m in diameter. How deep should the reflector be if light source is placed halfway between the vertex and the plane of rim to produce parallel beam of light to the axis of parabola.
6. There is a parabolic reflector of 12 cm in diameter is used in a vehicle where should the light source be placed to produce parallel beam of light whereas the reflector is 8 cm deep.
7. The main cable of suspension bridge is suspended in the shape of parabola between two towers that are 100 m apart and 30 m high from the roadway. If the cable is at the height of 5 m from the roadway at the centre of the bridge then find the equation of parabola and the distance of 10 m high suspended cable from the centre of the bridge.
8. The main cable of a suspension bridge is in the shape of a parabola. The towers are 600 feet apart and 60 feet high from the roadway. If the cable touches at the roadway at the midway between the towers. What is height of the suspender cable 150 feet from the centre of the bridge.

9.6 Ellipse

We have already studied about ellipse that ellipse is an special type of conic. Here we will discuss its definition and elements in detail.

9.6.1 Define ellipse and its elements (i.e., centre, foci, vertices, covertices, directrices, major and minor axes, eccentricity, focal chord and latera recta)

An ellipse is defined on the basis of two geometrical properties, one is called focus-directrix property and the other one is related to the distances of a point of ellipse to two fixed points.

Definition 1: An ellipse is a set of all the points in plane whose distance from a fixed point bears a constant ratio to its distance from a fixed line. The fixed point is called focus, the fixed line is called directrix and the constant ratio is called eccentricity. We denote eccentricity by e whereas $0 < e < 1$. By symmetry ellipse has two foci and two directrices as shown in the figure 9.27. In the figure F_1 and F_2 are two foci whereas l_1 and l_2 are two directrices. The mid-point of the foci is called centre of the ellipse. The chord through two foci and the centre is called major axis whereas a chord through centre and perpendicular to the major axis is called minor axis. In the figure point C, $\overline{A_1A_2}$ and $\overline{B_1B_2}$ are the centre, major axis and minor axis of the ellipse respectively. The end points of major axis and minor axis are called vertices and covertices respectively.

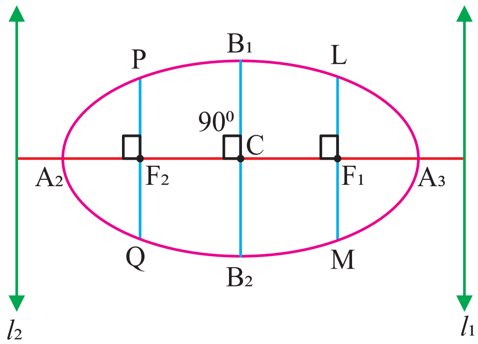


Fig 9.27

In the figure, A_1 and A_2 are vertices whereas B_1 and B_2 are covertices. Any chord through a focus is called focal chord of the ellipse whereas the focal chord which is perpendicular to the major axis is called latus rectum of the ellipse. In the figure $\overline{LM}$ and $\overline{PQ}$ are latera recta (plural of latus rectum) of the ellipse.

> **Note:** The major and minor axes together are called principal axes and their halves are called semi-axes.

Definition 2: An ellipse is the set of all points in the plane, the sum of whose distances from two fixed points is a positive constant that is greater than the distance between the fixed points and equal to the length of major axis. The fixed points are called foci as shown in the figure 9.28.

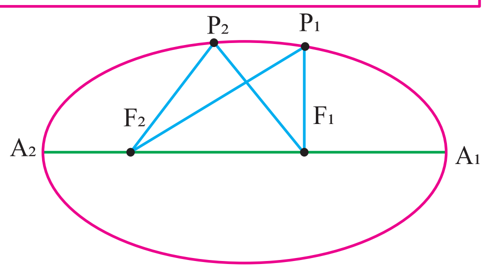


Fig 9.28

Let P_1 and P_2 be any two points of ellipse whereas F_1 and F_2 are foci as shown in the figure then by definition.

$$|\overline{P_1F_1}| + |\overline{P_1F_2}| = |\overline{P_2F_1}| + |\overline{P_2F_2}| = k$$

Where $k > 0$, $k > |\overline{F_1F_2}|$ and $k = |\overline{A_1A_2}|$

Basic relation of distances of focus, vertex and covertex from the centre of ellipse.

Let a, b, c are respectively the distance of vertex, covertex and focus from centre of the ellipse as shown in the figure 9.29. The basic relation of a, b, c is $a^2 = b^2 + c^2$. Let us prove it. First of all, we take two points P and Q such that P is at vertex and Q is at covertex.

According to the definition 2 of ellipse

$$|\overline{PF_1}| + |\overline{PF_2}| = |\overline{QF_1}| + |\overline{QF_2}|$$

i.e., $(a - c) + (a + c) = \sqrt{b^2 + c^2} + \sqrt{b^2 + c^2}$

$\Rightarrow \quad 2a = 2\sqrt{b^2 + c^2}$

Squaring both sides

we get $a^2 = b^2 + c^2$.

Hence proved.

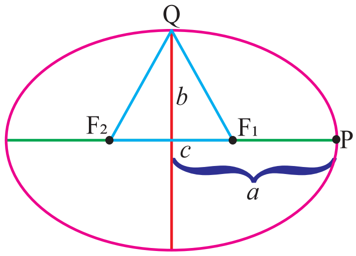


Fig 9.29

Relation of a, c and e where e is eccentricity of the ellipse.

Consider an ellipse whose centre is at C and directrices l_1 and l_2 as shown in the figure 9.30. A_1 and A_2 are two vertices and F_1 and F_2 are the foci. By definition of eccentricity

$$e = \frac{m\overline{A_1F_1}}{m\overline{A_1D_1}}$$

$\Rightarrow \quad m\overline{A_1F_1} = e(m\overline{A_1D_1}) \quad \ldots(i)$

Similarly, $m\overline{A_2F_1} = e(m\overline{A_2D_1}) \quad \ldots(ii)$

From Fig. 9.30

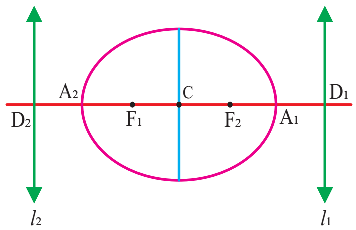


Fig 9.30

$m\overline{A_1A_2} = m\overline{A_2F_1} + m\overline{A_1F_1}$

or $\quad 2a = e(m\overline{A_2D_1} + m\overline{A_1D_1})$

$\Rightarrow \quad 2a = e(m\overline{A_2C} + m\overline{CD_1} + m\overline{CD_1} - m\overline{A_2C})$

$\Rightarrow \quad 2a = e(2m\overline{CD_1})$

$\Rightarrow \quad m\overline{CD_1} = \frac{a}{e} \quad \ldots(iii)$

Now, $\quad m\overline{CF_1} = m\overline{CA_1} - m\overline{A_1F_1}$

i.e., $\quad c = a - e\, m\overline{A_1D_1} \quad$ (using (i))

$c = a - e(m\overline{CD_1} - m\overline{CA_1})$

$\Rightarrow \quad c = a - e\left(\frac{a}{e} - a\right) \quad$ (using (iii))

$\Rightarrow \quad c = a - e\left(\frac{a - ae}{e}\right)$

$\Rightarrow \quad c = ae$

or $\boxed{e = \dfrac{c}{a}}$

i.e., eccentricity of ellipse is also the ratio of distances of focus and vertex from centre.

9.6.2 Explain that circle is a special case of an ellipse

We know that in ellipse, the eccentricity "*e*" is given by

$$e = \frac{c}{a} \quad \text{where } 0 \le e < 1$$

and it is the measure of the flatness of ellipse.

If we keep major axis constant then the closer the eccentricity is to 1, the flatter will be the ellipse. Conversely if e gets closer to zero the ellipse will become circle, as shown in the figure 9.31.

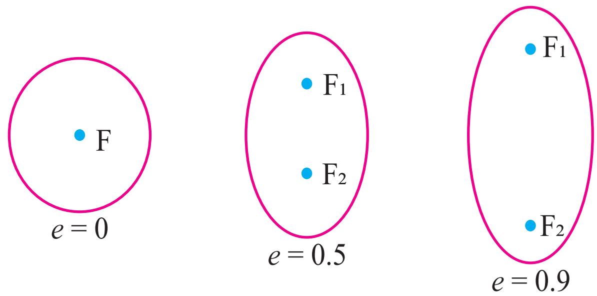


Fig 9.31

We have

$$e = \frac{c}{a}$$

If c approaches to zero then eccentricity will be zero and two foci will coincide and the resulting ellipse will be a circle.

We also have

$$a^2 = b^2 + c^2$$

If $\quad c = 0$

then $\quad a^2 = b^2$

i.e., $\quad a = b$

Hence circle is an special case of circle when eccentricity is zero, foci coincide and $a = b$.

9.7 Standard Form of Equation of an Ellipse

The simplest equations of ellipse are obtained when coordinate axes are positioned in such a way that the centre of ellipse is at the origin and the foci are on either x-axis or y-axis. The two possible such orientations are shown in the figure 9.32 and 9.33. These are called the standard positions of ellipse and their equations are called standard equations of ellipse.

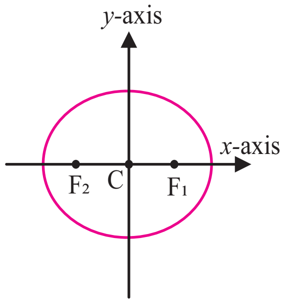


Fig 9.32

9.7.1 Derive the standard form of equation of an ellipse and identify its elements

There are two standard forms of equation of ellipse: one when major axis is along x-axis and the other when major axis is along y-axis. We derive both standard forms.

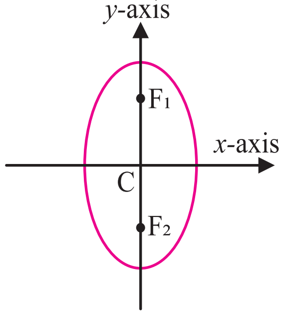


Fig 9.33

(a) Standard form of equation of ellipse when major axis is along x-axis

Let $P(x,y)$ be any point of ellipse with major axis along x-axis and centre at origin as shown in the figure 9.34. Let a, b, c be the distances of vertex, covertex and focus from the centre respectively.

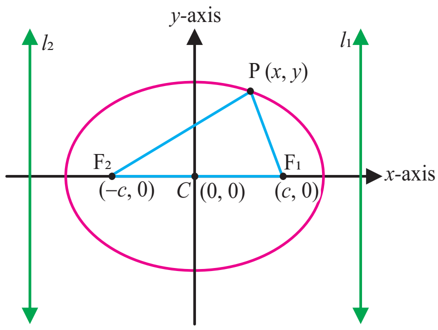


Fig 9.34

$\because$ foci are on x-axis

$\therefore$ foci are $F_1(c,0)$ and $F_2(-c,0)$

By the definition 2 of ellipse

$|\overline{PF_1}| + |\overline{PF_2}| = 2a$ where $a > b$

$\Rightarrow$ $\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$

or $\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$

Squaring both sides

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2$$

or $a\sqrt{(x-c)^2+y^2} = a^2 - cx$

Again, squaring both sides

$$a^2\{(x-c)^2 + y^2\} = a^4 - 2a^2cx + c^2x^2$$

$\Rightarrow$ $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

Dividing both sides by $a^2(a^2-c^2)$

we get $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2-c^2} = 1$

i.e., $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $\quad (\because\ a^2 = b^2 + c^2$ and $a > b)$

This is the standard equation of ellipse when major axis is along x-axis where coordinates of vertices and covertices are $(\pm a, 0)$ and $(0, \pm b)$ respectively.

We have already proved in section 9.6.1 that directrix is at distance of $\dfrac{a}{e}$ from the centre. So, equations of directrices are: $x = \pm\dfrac{a}{e}$.

(b) Standard form of equation of ellipse when major axis is along y-axis

Let $P(x, y)$ be any point of ellipse with major axis along y-axis and centre at origin as shown in the figure 9.35. Let a, b, c be the distances of vertex, covertex and focus from the centre respectively.

$\because$ foci are on y-axis

$\therefore$ foci are $F_1(0, c)$ and $F_2(0, -c)$

By the definition 2 of ellipse

$$|\overline{PF_1}| + |\overline{PF_2}| = 2a \quad \text{where } a > b$$

$$\Rightarrow \quad \sqrt{x^2 + (y-c)^2} + \sqrt{x^2 + (y+c)^2} = 2a$$

or $$\sqrt{x^2 + (y+c)} = 2a - \sqrt{x^2 + (y-c)^2}$$

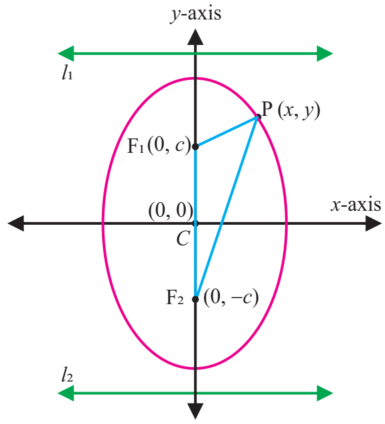


Fig 9.35

Squaring both sides

$$(y+c)^2 + x^2 = 4a^2 - 4a\sqrt{x^2 + (y-c)^2} + x^2 + (y-c)^2$$

or $$a\sqrt{x^2 + (y-c)^2} = a^2 - cy$$

Again, squaring both sides

$$a^2\{x^2 + (y-c)^2\} = a^4 - 2a^2cy + c^2y^2$$

$$\Rightarrow \quad a^2x^2 + (a^2 - c^2)y^2 = a^2(a^2 - c^2)$$

Dividing both sides by $a^2(a^2 - c^2)$

We get $$\frac{x^2}{a^2 - c^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad (\because \; a^2 = b^2 + c^2 \text{ and } a > b)$$

This is the standard equation of ellipse when major axis is along y-axis where coordinates of vertices and covertices are $(0, \pm a)$ and $(\pm b, 0)$ respectively.

We have already proved in section 9.6.1 that directrix is at distance of $\frac{a}{e}$ from the centre. So, equations of directrices will be: $y = \pm\frac{a}{e}$.

Length of latus rectum:

Let $\overline{AB}$ be the latus rectum of ellipse with major axis along x-axis having equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{... (i)}$$

$\because$ Focus is on x-axis

$\therefore$ Coordinates of one focus are $(c, 0)$

Now equation of line containing latus rectum is

$x = c$ or $x = ae$ $(\because c = ae)$

By using $x = ae$ in equation (i)

We get, $\dfrac{a^2e^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\Rightarrow \quad y^2 = b^2(1 - e^2)$

$\Rightarrow \quad y = \pm b\sqrt{1 - e^2}$

So, the end points of latus rectum are

$A\left(c, b\sqrt{1 - e^2}\right)$ and $B\left(c, -b\sqrt{1 - e^2}\right)$

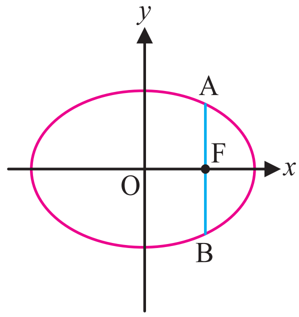


Fig 9.36

Now, $m\overline{AB} = 2b\sqrt{1 - e^2}$

$$= 2b\sqrt{\frac{a^2 - c^2}{a^2}} \qquad \left(\because\ e = \frac{c}{a}\right)$$

$$= 2b\left(\frac{b}{a}\right) \qquad (\because\ a^2 - c^2 = b^2)$$

$$= \frac{2b^2}{a}$$

So, the length of latus rectum is $\dfrac{2b^2}{a}$.

Standard equations of translated ellipses:

If the axes of an ellipse are parallel to the coordinate axes and centre is not at the origin then, by the translation the equations of ellipse may be determined which are as under:

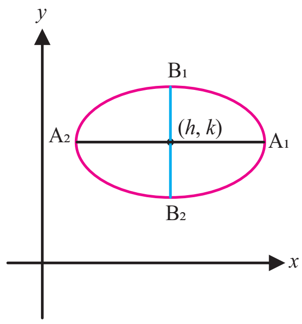


Fig 9.37

(i) Ellipse with centre (h, k) and major axis parallel to x-axis.

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \ (a > b)$$

In this case foci are $(h \pm c, k)$, vertices $(h \pm a, k)$, covertices are $(h, k \pm b)$ and directrices are $x - h = \pm\dfrac{a}{e}$.

(ii) Ellipse with centre (h, k) and major axis parallel to y-axis.

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1 \ (a > b)$$

In this case foci are $(h, k \pm c)$, vertices are $(h, k \pm a)$ covertices are $(h \pm b, k)$ and directrices are $y - k = \pm\dfrac{a}{e}$.

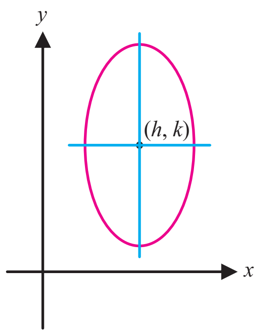


Fig 9.38

General equation of ellipse when axes of ellipse are parallel to coordinate axes

Consider a translated ellipse

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$\Rightarrow \quad b^2(x^2-2hx+h^2)+a^2(y^2-2ky+k^2)=a^2b^2$

$\Rightarrow \quad b^2x^2+a^2y^2-2hb^2x-2ka^2y+b^2h^2+a^2k^2-a^2b^2=0 \quad$ ...(i)

Let $\quad A=b^2, B=a^2$

$\quad G=-2hb^2, H=-2ka^2$ and $C=b^2h^2+a^2k^2-a^2b^2$

then equation (i) becomes

$\quad Ax^2+By^2+Gx+Hy+C=0$, where A and B are non-zero with same sign.

This is general equation of ellipse when axes of ellipse are parallel to coordinate axes.

Summary of equations of ellipse

Equation	Related terms and conditions
(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad , \quad a>b$	Major axis is along *x*-axis and centre is origin Foci: $(\pm c, 0)$, Vertices: $(\pm a, 0)$ Covertices: $(0, \pm b)$ and Directrices: $x=\pm\frac{a}{e}$ Length of latus rectum: $\frac{2b^2}{a}$
(ii) $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \quad , \quad a>b$	Major axis is along *y*-axis and centre is at origin. Foci: $(0, \pm c)$, Vertices: $(0, \pm a)$ Covertices: $(\pm b, 0)$ and Directrices: $y=\pm\frac{a}{e}$ Length of latus rectum: $\frac{2b^2}{a}$
(iii) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1(a>b)$	Major axis is parallel to *x*-axis with centre at (h, k) Foci: $(h\pm c, k)$, Vertices: $(h\pm a, k)$ Covertices: $(h, k\pm b)$ and Directrices: $x-h=\pm\frac{a}{e}$ Length of latus rectum: $\frac{2b^2}{a}$

Equation	Related terms and conditions
(iv) $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$	Major axis is parallel to y-axis with centre at (h,k) Foci: $(h,k\pm c)$, Vertices: $(h,k\ \pm a)$ Covertices: $(h\pm b,k)$ and Directrices: $y-k=\pm\frac{a}{e}$ Length of latus rectum: $\frac{2b^2}{a}$
General equation $Ax^2+By^2+Gx+Hy+C=0$	A and B are non-zero with same signs. All related elements can be found by converting into a standard form.

Note: For all types of ellipses mentioned above, we have

$$a^2=b^2+c^2,\text{ latus rectum }=\frac{2b^2}{a}\text{ and }c=ae.$$

Auxiliary Circle: Auxiliary circle of ellipse is the circle whose diameter is the major axis of ellipse. For ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, auxiliary circle is: $x^2+y^2=a^2$.

Example 1. Find foci, vertices, covertices, latus rectum and equations of directrices, of the ellipse: $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Solution: This is the ellipse with centre at origin and major axis along x-axis.

Here, $a^2=25$ and $b^2=9$

So, $a=5$ and $b=3$ are the semi-axes of the ellipse,

we know that

$$a^2=b^2=c^2$$

$\Rightarrow\quad 25=9+c^2$

$\Rightarrow\quad c^2=16$

$\Rightarrow\quad c=4$

We know that $c=ae$

$\Rightarrow\quad 4=5e\qquad\Rightarrow\qquad e=\frac{4}{5}$

So, the eccentricity is $\frac{4}{5}$.

Now,

Foci $=(\pm c,0)=(\pm 4,0)$,

$$\text{Vertices} = (\pm a, 0) = (\pm 5, 0),$$

$$\text{Covertices} = (0, \pm b) = (0, \pm 3),$$

$$\text{Latus rectum} = \frac{2b^2}{a} = \frac{2\times 9}{5} = \frac{18}{5}$$

and equation of directrices are

$$x = \pm\frac{a}{e}$$

i.e., $\quad x = \pm\left(\frac{5}{\frac{4}{5}}\right)$

or $\quad x = \pm\frac{25}{4}.$

Example 2. Find semi-axes, centre, foci, vertices, covertices, latus rectum and equations of directrices of the ellipse $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{25} = 1.$

Solution: Comparing given ellipse with $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

we get $a^2 = 25, b^2 = 16, h = 2, k = -3$ and major axis is parallel to y-axis.

We know that

$$a^2 = b^2 + c^2$$

$\Rightarrow \quad 25 = 16 + c^2$

$\Rightarrow \quad c^2 = 9$

$\Rightarrow \quad c = 3$

We know that $\quad c = ae$

$\Rightarrow \quad 3 = 5e \qquad \Rightarrow \qquad e = \frac{3}{5}$

So, the eccentricity is $\frac{3}{5}$ and $a = 5, b = 4$ are the semi-axes of the ellipse.

Now,

$$\text{Centre} = (h, k) = (2, -3)$$

$$\text{Foci} = (h, k \pm c) = (2, -3 \pm 3)$$

So, $\quad$ Foci are $(2, 0)$ and $(2, -6)$

$$\text{Vertices} = (h, k \pm a) = (2, -3 \pm 5)$$

So, $\quad$ Vertices are $(2, 2)$ and $(2, -8)$

Covertices are $(h \pm b, k) = (2 \pm 4, -3)$,

So, $\quad$ Covertices are $(2, 1)$ and $(2, -7)$

$$\text{Latus rectum} = \frac{2b^2}{a} = \frac{2\times 16}{5} = \frac{32}{5}$$

and equation of directrices are

$$y - k = \pm\frac{a}{e}$$

$$y + 3 = \pm\left(\frac{5}{\frac{3}{5}}\right)$$

or $$y + 3 = \pm\frac{25}{3}.$$

9.7.2 Find the equation of an ellipse with the following given elements:

- **major and minor axes,**
- **two points,**
- **foci, vertices or lengths of a latera recta,**
- **foci, minor axes or length of a latus rectum.**

- **Equation of ellipse whose major and minor axes are given**

The method is explained with the help of the following example.

Example: Find the equation of ellipse with centre at origin where major and minor axes are 10 and 8 units respectively and major axis is along x-axis.

Solution: Here length of major axis $= 10$

i.e., $2a = 10$

$\Rightarrow$ $a = 5$

and length of minor axis $= 8$

i.e., $2b = 8$

$\Rightarrow$ $b = 4$

According to the condition, equation of ellipse will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{... (i)} \quad (\because \text{ major axis is along } x\text{-axis})$$

By using values of a and b in equation (i)

we get,

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

This is the required equation of ellipse.

- **Equation of ellipse when two points are given**

The method is explained with the help of the following example.

Example: Find the equation of ellipse passing through $\left(1, \sqrt{2}\right)$ and $\left(\frac{\sqrt{6}}{2}, 1\right)$ whereas the centre is at origin and major axis is along y-axis.

Solution:

$\because$ major axis is along y-axis and centre at origin

$\therefore$ equation of ellipse will be

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \ldots\text{(i)}$$

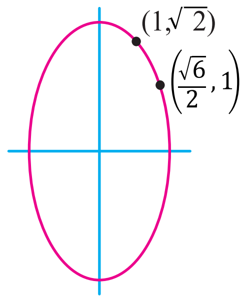


Fig 9.39

$\because$ $\left(1, \sqrt{2}\right)$ lies on the ellipse

$\therefore$ we have from equation (i)

$$\frac{1}{b^2} + \frac{2}{a^2} = 1 \quad \ldots\text{(ii)}$$

$\because$ $\left(\frac{\sqrt{6}}{2}, 1\right)$ lies on the ellipse

$\therefore$ we have from equation (i)

$$\frac{3}{2b^2} + \frac{1}{a^2} = 1 \quad \ldots\text{(iii)}$$

Multiplying equation (ii) by $\frac{3}{2}$ and subtracting equation (iii) from the resultant equation

$$\frac{3}{2b^2} + \frac{3}{a^2} = \frac{3}{2}$$

and $\quad \pm\frac{3}{2b^2} \pm \frac{1}{a^2} = -1$

or $\quad \frac{2}{a^2} = \frac{1}{2}$

$\Longrightarrow \quad a^2 = 4 \quad \Longrightarrow \quad a = 2$

By using $a^2 = 4$ in (ii)

We get, $\frac{1}{b^2} + \frac{2}{4} = 1$

$\Longrightarrow \quad \frac{1}{b^2} = \frac{1}{2} \quad \Longrightarrow \quad b^2 = 2$

By using values of *a* and *b* in equation (i)

we get, $\frac{x^2}{2} + \frac{y^2}{4} = 1$

$\Longrightarrow \quad 4x^2 + 2y^2 = 8$

This is the required equation of ellipse.

- **Equation of ellipse when foci, vertices or length of latera recta are given**

The method is explained with the help of the following examples.

Example 1. Find the equation of ellipse with centre at origin whose focus and vertex are $(8, 0)$ and $(10, 0)$ respectively.

Solution:

According to the condition, focus and vertex lie on x-axis.

So, major axis is along x-axis.

Here $c = 8$

$a = 10$

We know that

$$a^2 = b^2 + c^2$$

i.e., $100 = b^2 + 64$

$\Rightarrow \quad b^2 = 36 \quad \Rightarrow \quad b = 6$

Equation of ellipse will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

i.e., $$\frac{x^2}{100} + \frac{y^2}{36} = 1$$

This is the required equation of ellipse.

Example 2. Find the equation of ellipse with centre at origin such that its focus is $(0, 3)$ and latus rectum is of $\frac{32}{5}$ units.

Solution:

According to the condition, major axis is along y-axis.

We have,

$c = 3$

and length of latus rectum $= \frac{32}{5}$

i.e., $$\frac{2b^2}{a} = \frac{32}{5}$$

$\Rightarrow \quad b^2 = \frac{16a}{5}$

We know that

$$a^2 = b^2 + c^2$$

i.e., $a^2 = \frac{16a}{5} + 9$

$\Rightarrow \quad 5a^2 = 16a + 45$

$\Rightarrow \quad 5a^2 - 16a - 45 = 0$

$\Rightarrow \quad 5a^2 - 25a + 9a - 45 = 0$

$\Rightarrow \quad 5a(a-5) + 9(a-5) = 0$

$\Rightarrow \quad (a-5)(5a+9) = 0$

$\Rightarrow \quad a = 5 \text{ or } a = -\frac{9}{5}$

$\because \quad a$ cannot be $-$ve

$\therefore \quad$ we neglect $a = -\frac{9}{5}$

Hence $a = 5$

We knot that

$$a^2 = b^2 + c^2$$

i.e., $\quad 25 = b^2 + 9$

$\Rightarrow \quad b^2 = 16 \quad \Rightarrow \quad b = 4$

Now equation of ellipse will be

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

i.e., $$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

This is the required equation of ellipse.

- **Equation of ellipse when foci, minor axis or length of latus rectum are given**

The method is explained with the help of the following examples.

Example 1. Find the equation of an ellipse whose focus is $(5, 0)$ and minor axis is 12 units long and along y-axis where centre is at origin.

Solution:

According to the given conditions, equation of ellipse will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{... (i)}$$

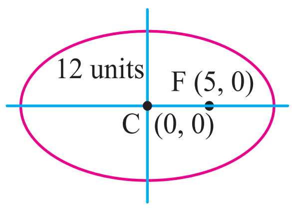


Fig 9.40

Here $\quad c = 5$

and length of minor axis $= 10$

i.e., $\quad 2b = 12$

$\Rightarrow \quad b = 6$

We know that

$$a^2 = b^2 + c^2$$

i.e., $\quad a^2 = 36 + 25$

$a^2 = 61 \quad \Rightarrow \quad a = \sqrt{61}$

By using values of a and b in equation (i)

we get

$$\frac{x^2}{61} + \frac{y^2}{36} = 1$$

This is the required equation of ellipse.

Example 2. Find the equation of an ellipse whose minor axis is 10 units long and along x-axis whereas latus rectum is 8 units long and centre is at origin.

Solution:

According to the condition, the equation of ellipse will be

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Here length of minor axis $= 10$

i.e., $2b = 10 \quad \Rightarrow \quad b = 5$

and length of latus rectum $= 8$

i.e., $\frac{2b^2}{a} = 8$

$\Rightarrow \quad b^2 = 4a$

i.e., $a = \frac{25}{4}$

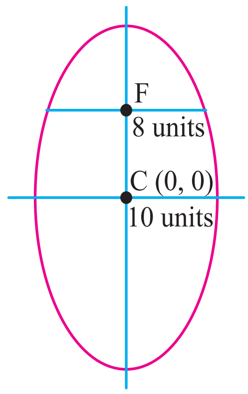


Fig 9.41

By using values of a and b in equation (i)

We get

$$\frac{x^2}{25} + \frac{y^2}{\frac{25}{4}} = 1$$

$$\Rightarrow \quad \frac{x^2}{25} + \frac{4y^2}{25} = 1$$

This is the required equation of ellipse.

9.7.3 Convert a given equation to the standard form of equation of an ellipse, find its elements and draw the graph

In section 9.7.1, we have already studied the general equation of ellipse when major and minor axes of ellipses are parallel to the coordinate axes. The equation is as under:

$$Ax^2 + By^2 + Gx + Hy + C = 0$$

where A and B are non-zero and having same sign.

Equation (i) can be converted into standard form of equation of an ellipse. The method will be explained in the following examples.

Technique for drawing graph of an ellipse

Graph of an ellipse can be drawn from their standard equations using the following three steps:

1. Determine whether the major axis is along x-axis or y-axis or parallel to one of them. If denominator of x^2 or $(x-h)^2$ is larger then major axis is along x-axis or parallel to x-axis respectively. If denominator of y^2 or $(y-k)^2$ is larger then major axis is along y-axis or parallel to y-axis respectively. If both denominators are equal then it is a circle.
2. Determine the values of a and b and draw rectangle extending a units on each side of the centre along major axis and b units on each side of the centre along the minor axis.
3. Using the rectangle as guide, sketch the ellipse so that it touches the sides of the rectangle where the sides intersect the axes of the ellipse.

Example: Find foci, eccentricity, vertices, covertices, latus rectum and equations of directrices of ellipse $9x^2 + 16y^2 - 144 = 0$. Also draw its graph.

Solution: We have

$$9x^2 + 16y^2 - 144 = 0$$

or $$9x^2 + 16y^2 = 144$$

Dividing both sides by 144

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Here $a^2 = 16$ and $b^2 = 9$, so $a = 4$ and $b = 3$ and major axis is along x-axis and centre at origin.

We know that

$$a^2 = b^2 + c^2$$

$$16 = 9 + c^2 \quad \Longrightarrow \quad c^2 = 7 \quad \text{or} \quad c = \sqrt{7}$$

Also, we know that

$$c = ae$$

So, $$\frac{\sqrt{7}}{4} = e$$

Now,

$$\text{Foci} = (\pm c, 0) = \left(\pm\sqrt{7}, 0\right)$$

So, Foci are $\left(\sqrt{7}, 0\right)$ and $\left(-\sqrt{7}, 0\right)$

$$\text{Vertices} = (\pm a, 0) \text{ and } (\pm 4, 0)$$

So, Vertices are $(4, 0)$ and $(-4, 0)$

Covertices $= (0, \pm b) = (0, \pm 3)$

So, Covertices are $(0, 3)$ and $(0, -3)$

Latus rectum $= \frac{2b^2}{a} = \frac{2\times 9}{4} = \frac{9}{2}$

equations of directrices will be

$$x = \pm \frac{a}{e}$$

$$x = \pm \frac{4}{\frac{\sqrt{7}}{4}}$$

$$\Rightarrow \quad x = \pm \frac{16}{\sqrt{7}}.$$

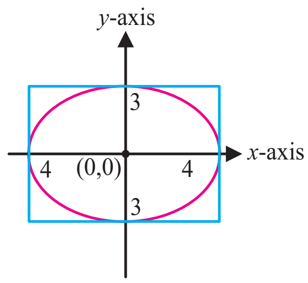


Fig 9.42

Graph of ellipse

Here centre $= (0, 0)$

Semi axes are $a = 4$ and $b = 3$

Major axis is along x-axis

Graph is shown in Fig. 9.42.

Example 2. Find centre, foci, vertices, and latus rectum of ellipse $9x^2 - 18x + 4y^2 + 16y - 11 = 0$. Also draw its graph.

Solution: We have

$$9x^2 - 18x + 4y^2 + 16y - 11 = 0$$

$$\Rightarrow \quad 9(x^2 - 2x) + 4(y^2 + 4y) = 11$$

$$\Rightarrow \quad 9(x^2 - 2x + 1) + 4\,(y^2 + 4y + 4) = 11 + 9 + 16$$

$$\Rightarrow \quad 9(x-1)^2 + 4(y+2)^2 = 36$$

Dividing both sides by 36

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$

Comparing with $\frac{(x-h)^2}{b^2} + \frac{(y+k)^2}{a^2} = 1$

We get centre $= (h, k) = (1, -2)$

$$a^2 = 9 \text{ and } b^2 = 4$$

$$\Rightarrow \quad a = 3 \text{ and } b = 2$$

Also, major axis is parallel to y-axis.

We know that

$$a^2 = b^2 + c^2$$

i.e., $9 = 4 + c^2$

$\Rightarrow$ $c^2 = 5$ $\Rightarrow$ $c = \sqrt{5}$

Now, foci $= (h, k \pm c) = (1, -2 \pm \sqrt{5})$

So, foci are $(1, -2 + \sqrt{5})$ and $(1, -2, -\sqrt{5})$

Vertices $= (h, k \pm a) = (1, -2, \pm 3)$

So, Vertices are $(1, 1)$ and $(1, -5)$

Latus rectum $= \dfrac{2b^2}{a}$

$= \dfrac{2(4)}{9} = \dfrac{8}{9}$

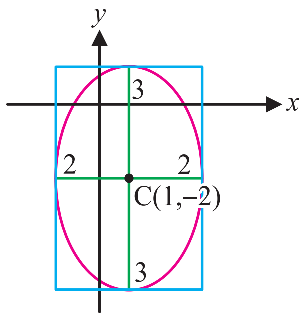


Fig 9.43

Graph of ellipse

Here centre $= (1, -2)$

Semi-axes are $a = 3$ and $b = 2$

Major axis is parallel to y-axis.

Graph of the ellipse is shown in the Fig. 9.43.

Exercise 9.3

1. Find semi-axes, eccentricity, foci, vertices, covertices, latus rectum and equations of directrices of the following ellipses. Also draw their graphs.

(i) $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$ (ii) $\dfrac{x^2}{16} + \dfrac{y^2}{10} = 1$

(iii) $\dfrac{(x-3)^2}{25} + \dfrac{(y+4)^2}{16} = 1$ (iv) $\dfrac{(x+1)^2}{9} + \dfrac{(y-2)^2}{16} = 1$

(v) $9x^2 + 25y^2 = 225$ (vi) $4x^2 - 16x + 25y^2 + 200y - 316 = 0$

2. Find the equations of the following ellipse whose centres are at origin and their axes are along coordinate axes. Also satisfy the given conditions:

(i) Major and minor axes are 12 and 8 respectively with minor axis is along y-axis.

(ii) Ellipse passes through $\left(1, \sqrt{\dfrac{3}{2}}\right)$ and $\left(\dfrac{2}{\sqrt{3}}, 1\right)$ with major axis is along y-axis.

(iii) Foci at $(\pm 3, 0)$ and vertices at $(\pm 5, 0)$

(iv) Foci at $(0, \pm 4)$ and latus rectum $\dfrac{18}{5}$

(v) Foci at $(\pm 5, 0)$ and minor axis is 12 units long and along y-axis.

(vi) Minor axis along x-axis with length is 8 units and latus rectum are 6 units long and along y-axis.

(vii) Covertices at $(0, \pm 3)$ and distance between foci $= 10$ units.

(viii) Directrix $y = \frac{25}{3}$ and latus rectum $\frac{32}{5}$

3. Find equation of auxiliary circle to $5x^2 + 7y^2 = 11$.
4. Is the point $(4, 5)$ inside on or outside the ellipse $2x^2 + 3y^2 = 6$.
5. Find equation of ellipse with centre at $(5, -3)$, one vertex at $(10, -3)$ and one focus at $(9, -3)$.
6. If ellipse is $9x^2 + 13y^2 = 117$ then find:
   (i) distance between foci;
   (ii) distance between vertices;
   (iii) distance between covertices.
7. Find eccentricity of ellipse if:
   (i) axes are 32 and 24;
   (ii) latus rectum is equal to half of its major axis.
8. Find equation of the circle passing through focus of parabola $y^2 + 8x = 0$ and foci of ellipse $25x^2 + 16y^2 = 400$.
9. Find the length of, and the equations to, the focal radii drawn to a point $(4\sqrt{3}, 5)$ of the ellipse $25x^2 + 16y^2 = 1600$.
10. Find equation of ellipse with centre at $(0, 1)$ and major axis parallel to y-axis. Also, it passes through $(2, 1)$ and $(0, 4)$.

9.8 Equations of Tangent and Normal of an Ellipse

As tangent and normal are very important to solve many physical problems, so we will discuss concept, conditions and equations of tangents and normals to an ellipse in this section.

9.8.1 Recognize tangent and normal to an ellipse

Like any other curve, tangent to an ellipse is the line which touches the ellipse at a certain point P. The point P is called point of tangency. Normal to the ellipse is a line which is perpendicular to the tangent at the point of tangency.

In the figure 9.44, l_1 is tangent to the ellipse at P and l_2 is the normal.

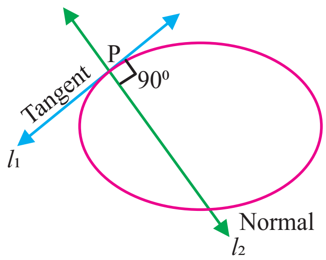


Fig 9.44

9.8.2 Find points of intersection of an ellipse with a line including the condition of tangency

As a matter of fact, a line can cut or touch an ellipse and sometimes it neither cuts nor touches the ellipse as shown in the figure 9.45. We will discuss the method of finding points of intersection of an ellipse with a line along with condition of tangency in this section.

Consider a line $y = mx + c$ ...(i)

and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(ii)

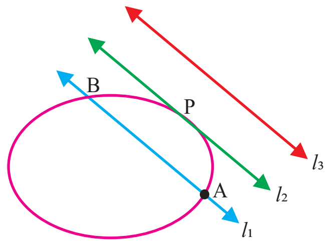


Fig 9.45

Solving both equations simultaneously,

we get $b^2x^2 + a^2(mx + c)^2 = a^2b^2$

$\Rightarrow$ $(b^2 + a^2m^2)x^2 + 2mca^2x + a^2(c^2 - b^2) = 0$ ...(iii)

$\because$ Roots of quadratic equation (iii)

represent abscissas of the points of intersection.

$\therefore$ Abscissas of points of intersection will be the roots of (iii) and the corresponding ordinates of points of intersection will be obtained by

Substituting the values of x in (i)

Moreover, the nature of roots of quadratic equation (iii) will represent the nature of parallel lines, l_1, l_2 and l_3 each having slope m with respect to given ellipse.

Here, discriminant of equation (iii) is

$$\Delta = 4m^2c^2a^4 - 4a^2(b^2 + a^2m^2)(c^2 - b^2)$$
$$= 4m^2c^2a^4 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4m^2c^2 + 4a^4m^2b^2$$
$$= 4a^2b^2(b^2 - c^2 + a^2m^2)$$

If $\Delta > 0$ then line will cut the ellipse because there will be two distinct real roots.

If $\Delta < 0$ then the line will neither cut nor touch the ellipse because there will be no real root.

If $\Delta = 0$ then line will be tangent to the ellipse because there will be only one real root.

i.e., $4a^2b^2(b^2 - c^2 + a^2m^2) = 0$

$\Rightarrow$ $b^2 - c^2 + a^2m^2 = 0$

$\Rightarrow$ $c^2 = b^2 + a^2m^2$

This is the condition of tangency of line $y = mx + c$ with the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Example: Show that the line $y = 2x + 4$ is tangent to the ellipse $4x^2 + 3y^2 = 12$. Also find point of contact.

Solution: We have

Line $y = 2x + 4$ ...(i)

and ellipse: $4x^2 + 3y^2 = 1$ ...(ii)

Solving equation (i) and (ii) simultaneously, we get $4x^2 + 3(2x+4)^2 - 12 = 0$

$\Rightarrow$ $4x^2 + 3(4x^2 + 16x + 16) - 12 = 0$

$\Rightarrow$ $16x^2 + 48x + 36 = 0$

$\Rightarrow$ $4x^2 + 12x + 9 = 0$ ...(iii)

Here $\Delta = 144 - 4(4)(9)$

$= 144 - 144 = 0$

$\because$ $\Delta = 0$

$\therefore$ Given line is tangent to the given ellipse.

From equation (iii), we have $x = -\dfrac{b}{2a}$

i.e., $x = -\dfrac{12}{8} = -\dfrac{3}{2}$

By using $x = -\dfrac{3}{2}$ in equation (i)

we get $y = 2\left(-\dfrac{3}{2}\right) + 4$

$= 1$

So, the point of intersection is $\left(-\dfrac{3}{2}, 1\right)$.

9.8.3 Find the equation of a tangent in slope form

Let $y = mx + c$ ...(i)

be the equation of tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

By the condition of tangency

$c^2 = b^2 + a^2m^2$

or $c = \sqrt{b^2 + a^2m^2}$

By using this value of c in equation (i)

we get $y = mx + \sqrt{b^2 + a^2m^2}$

This is the required equation of tangent in slope form to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Example: Find the equation of tangent to $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ whose slope $= 2$.

Solution: Here slope $= m = 2, a^2 = 16$ and $b^2 = 9$.

We know that the equation of tangent in slope form is

$$y = mx + \sqrt{b^2 + a^2m^2}$$

i.e., $y = 2x + \sqrt{9 + 16 \times 4}$

$y = 2x + \sqrt{73}$

This is the required tangent.

9.8.4 Find the equation of a tangent and a normal to an ellipse at a point

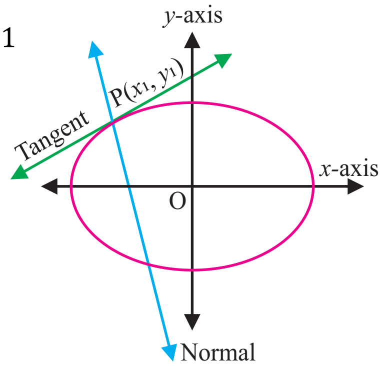


Fig 9.46

Let $P(x_1, y_1)$ be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Differentiating w.r.t x

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\Rightarrow \boxed{\frac{dy}{dx}=-\frac{b^2x}{a^2y}}$$

Now, slope of tangent at $(x_1, y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$

$$=-\frac{b^2x_1}{a^2y_1}$$

By point-slope form, the equation of tangent will be

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$\Rightarrow \frac{x_1(x-x_1)}{a^2}+\frac{y_1(y-y_1)}{b^2}=0$$

$$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)=0$$

$$\Rightarrow \boxed{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$$

This is the required equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1).

$\because$ Normal is perpendicular to the tangent at $P(x_1, y_1)$

$\therefore$ slope of normal $=\frac{a^2y_1}{b^2x_1}$ at (x_1, y_1).

By point-slope form, the equation of normal will be

$$y-y_1=-\frac{a^2y_1}{b^2x_1}(x-x_1)$$

$$\Rightarrow \frac{b^2(y-y_1)}{y_1}=\frac{a^2(x-x_1)}{x_1}$$

$$\Rightarrow \frac{a^2x}{x_1}-\frac{b^2y}{y_1}-a^2+b^2=0$$

$$\text{or } \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

This is the equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1).

Example: Find equation of tangent and normal to $\frac{x^2}{5} + \frac{y^2}{3} = 1$ at $\left(\sqrt{\frac{5}{2}}, \sqrt{\frac{3}{2}}\right)$.

Solution: We have ellipse: $\frac{x^2}{5} + \frac{y^2}{3} = 1$

differentiating w.r.t x

$$\frac{2x}{5} + \frac{2y}{3}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{5} \times \frac{3}{2y}$$

$$\frac{dy}{dx} = -\frac{3x}{5y}$$

Now, slope of tangent at $\left(\sqrt{\frac{5}{2}}, \sqrt{\frac{3}{2}}\right) = m = \left(\frac{dy}{dx}\right)_{\left(\sqrt{\frac{5}{2}}, \sqrt{\frac{3}{2}}\right)}$

$$= -\sqrt{\frac{3}{5}}$$

So, by point-slope form, the equation of tangent will be

$$y - y_1 = m(x - x_1)$$

i.e., $\quad y - \sqrt{\frac{3}{2}} = -\sqrt{\frac{3}{5}}\left(x - \sqrt{\frac{5}{2}}\right)$

$\because$ Normal is perpendicular to the tangent at $\left(\sqrt{\frac{5}{2}}, \sqrt{\frac{3}{2}}\right)$

$\therefore$ slope of normal $= \sqrt{\frac{5}{3}} = m'$

By point-slope form, the equation of normal will be

$$y - y_1 = m'(x - x_1)$$

i.e., $y - \sqrt{\frac{3}{2}} = \sqrt{\frac{5}{3}}\left(x - \sqrt{\frac{5}{2}}\right)$

Exercise 9.4

1. Find the condition when line $y = \sqrt{5}x + c$ is tangent to the ellipse $4x^2 + 9y^2 = 36$.
2. Show that the line $x = 2y + 4$ touches the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$. Also find point of contact.

3. Find the condition of tangency of line $y = mx + c$ to the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

4. Find the condition of tangency of given line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   (i) $\frac{x}{p} + \frac{y}{q} = 1$ (ii) $x\cos\alpha + y\sin\alpha = p$

   (iii) $lx + my + n = 0$

5. Find the equation of tangent to $\frac{x^2}{5} + \frac{y^2}{4} = 1$ with slope 3.

6. Find the equation of tangent and normal to

   (i) $9x^2 + 25y^2 = 225$ at $\left(3, \frac{12}{5}\right)$

   (ii) $49x^2 + 64y^2 = 64 \times 49$ at $(8\cos\alpha, 7\sin\alpha)$

7. Find the equation of tangent and normal at the ends of the latus rectum with positive abscissa of the ellipse $\frac{x^2}{3} + \frac{y^2}{2} = 1$.

8. Find the equation of tangent to the ellipse $\frac{x^2}{5} + \frac{9y^2}{20} = 1$ at the points where abscissa is 1.

9.9 Hyperbola

In previous chapter we defined hyperbola as conic section of right circular cone but in this section, we will discuss hyperbola in detail on the basis of eccentricity and directrix.

9.9.1 Define hyperbola

Hyperbola is defined on the basis of two geometrical properties, one is directrix-focus property whereas the other one is based on the distances of a point from two fixed points.

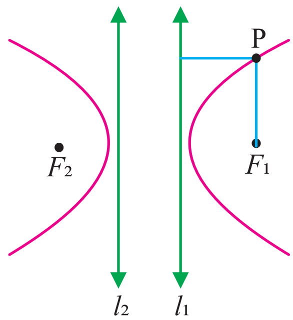


Fig 9.47

Definition 1: A hyperbola is the set of all the points in a plane whose distance from a fixed point bears a constant ratio to its distance from a fixed line such that the ratio is greater than 1.

The fixed point, fixed line and ratio are called focus, directrix and eccentricity respectively. Hyperbola has two foci and two directrices as shown in the figure 9.47.

Definition 2: The locus of a point, the difference of whose distances from two fixed points is constant, is called hyperbola. The fixed points are foci.

Let $P(x, y)$ be any point of hyperbola whereas F_1 and F_2 are two foci as shown in the figure 9.48 then

$|\overline{PF_2}| - |\overline{PF_1}| = $ constant (say k).

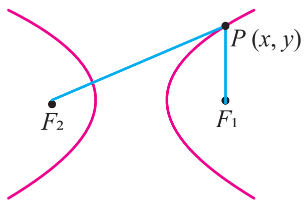


Fig 9.48

9.9.2 Define elements of hyperbola (i.e., centre, foci, vertices, directrices, transverse and conjugate axes, eccentricity, focal chord and latera recta)

As discussed above hyperbola has two fixed points and two fixed lines which are called foci and directrices respectively. In the figure 9.49, F_1, F_2 are foci whereas l_1, l_2 are directrices of hyperbola.

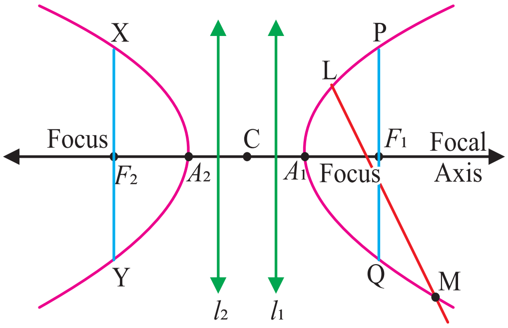


Fig 9.49

The mid-point of the line segment joining the foci is called centre. In the figure point C is the centre of hyperbola. The line through foci is called focal axis.

Hyperbola intersects focal axis at two points called vertices. In the Fig. 9.49 A_1 and A_2 are two vertices of the hyperbola. The two parts of hyperbola are called its branches. Any chord of hyperbola through any one of its foci is called focal chord. In the figure $\overline{LM}$ is a focal chord. The focal chord perpendicular to the focal axis is called latus rectum. In the figure $\overline{PQ}$ and $\overline{XY}$ are the latera recta (plural of latus rectum) of the hyperbola. The ratio of distances of any point from focus and directrix is called eccentricity and is denoted by e where $e > 1$.

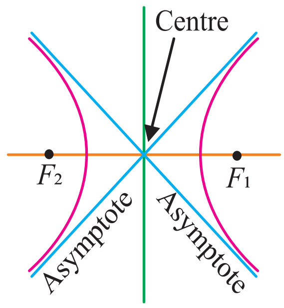


Fig 9.50

A line passing through centre and gets closer and closer to hyperbola but never touches it is called asymptote. There are two asymptotes of any hyperbola as shown in the figure 9.50.

Let distance of focus and vertex from centre are c and a respectively. Asymptotes are in fact diagonal lines of a rectangle which extends a unit from centre on either side on focal axis and it extends b units from centre on either side on the line perpendicular to focal axis and through centre as shown in the Fig. 9.51 where we define b as $b = \sqrt{c^2 - a^2}$.

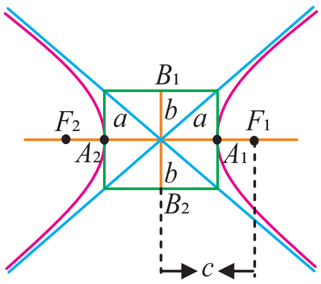


Fig 9.51

The line segments A_1A_2 and B_1B_2 are called transverse and conjugate axes respectively where A_1 and A_2 are the vertices but B_1 and B_2 are the ends of conjugate axis of hyperbola.

The relationship of a, b and c is pictured geometrically in the figure.

Relation is: $c^2 = a^2 + b^2$

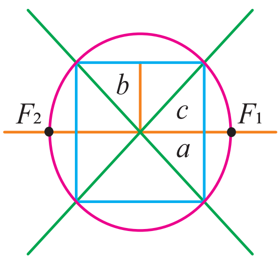


Fig 9.52

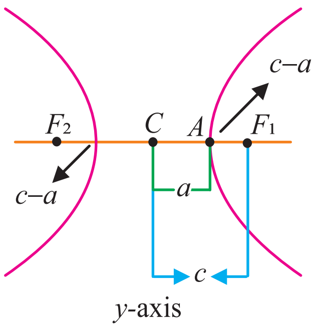


Fig 9.53

Property: The differences of the focal distances of a point on a hyperbola is equal to the length of its transverse axis as explained with the help of the figure 9.53. We take vertex A as a point of hyperbola.

By definition 2 of hyperbola

$m\overline{AF_2} - m\overline{AF_1} = k$ (constant)

i.e., $\{(c-a)+2a\}-(c-a) = k$

$\Rightarrow \quad c - a + 2a - c + a = k$

or $\quad k = 2a.$

This is valid for any point of hyperbola.

Hence difference of focal distances of a point on a hyperbola is equal to the length of its transverse axis.

Distance of directrix from the centre and the relation $c = ae$

Consider a hyperbola in which distance of a focus and vertex from centre is c and a units respectively and l is the directrix cutting focal axis at point D as shown in the figure 9.54.

Now, $\dfrac{m\overline{F_1A_1}}{m\overline{A_1D}} = e > 1 \Rightarrow m\overline{F_1A_1} = e(m\overline{A_1D})$ as point A_1 is nearer to D then F_1 and

$\dfrac{m\overline{F_1A_2}}{m\overline{A_2D}} = -e$ (as $m\overline{F_1A_2} = -m\overline{A_2F_1}$)

$\Rightarrow \quad m\overline{F_1A_2} = -e(m\overline{A_2D})$

$\Rightarrow \quad m\overline{A_2F_1} = e(m\overline{A_2D})$

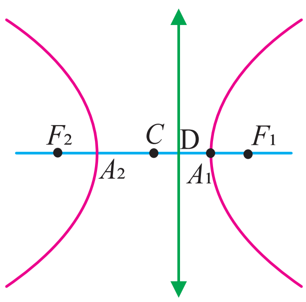


Fig 9.54

The points A_1 and A_2 are on the opposite sides of l. Take C as origin and $\overline{CF_1}$ as positive x-axis. Now, by definition of hyperbola

$$\begin{aligned} m\overline{A_1A_2} &= m\overline{A_2F_1} - m\overline{A_1F_1} \\ &= m\overline{A_2F_1} - m\overline{F_1A_1} \\ &= e(m\overline{A_2D} + m\overline{A_1D}) \\ &= e(m\overline{A_2C} + m\overline{CD}) + e(m\overline{A_1C} + m\overline{CD}) \end{aligned}$$

$$2a = 2e(m\overline{CD}) \qquad (\because m\overline{A_2C} = -m\overline{A_1C})$$

$$\Rightarrow \quad m\overline{CD} = \frac{a}{e} \qquad \ldots\text{(i)}$$

Hence, directrix l is at a distance of $\dfrac{a}{e}$ from the centre.

Similarly, $\quad m\overline{CF_1} = m\overline{CA_1} + m\overline{A_1F_1}$

$\qquad = m\overline{CA_1} + e(m\overline{A_1D})$

$$c = a + e(m\overline{CA_1} - m\overline{CD})$$

$\Rightarrow \qquad c = a + e\left(a - \frac{a}{e}\right)$

$\Rightarrow \qquad c = a + ae - a$

or $\qquad c = ae \qquad \ldots$(ii)

9.10 Standard Form of Equation of Hyperbola

The simplest form of equation of hyperbola is when the coordinate axes are positioned in such a way that centre of hyperbola is at the origin and the transverse axis and conjugate axis are on the coordinate axes. The two possible such orientation are shown in the figure 9.55 and 9.56.

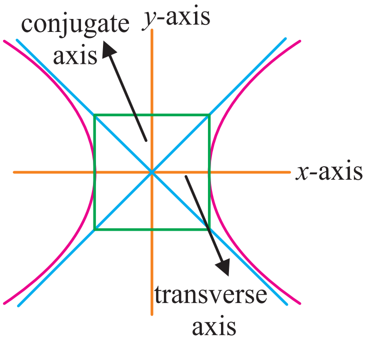


Fig 9.55

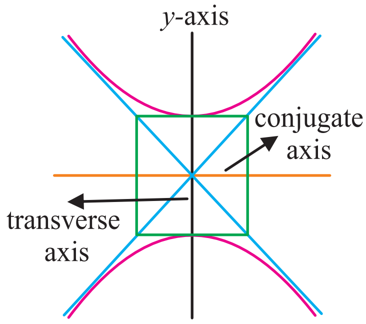


Fig 9.56

These are the standard positions of hyperbola and the resulting equations are called standard equations which will be derived in the next section.

9.10.1 Derive the standard form of equation of a hyperbola and identify its elements

(a) Standard equation of hyperbola when transverse axis is along *x*-axis.

Let $P(x, y)$ be any point on hyperbola with centre at origin, transverse axis on *x*-axis and conjugate axis along y-axis as shown in the figure 9.57 whereas foci are $F_1(c, 0)$ and $F_2(-c, 0)$. The length of transverse axis is $2a$.

By the definition of hyperbola

$$|\overline{PF_2}| - |\overline{PF_1}| = 2a$$

i.e., $\qquad \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$

or $\qquad \sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$

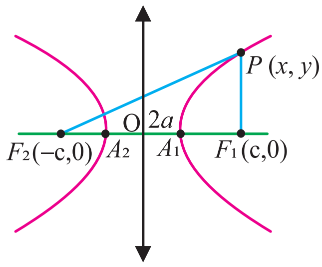


Fig 9.57

Squaring both sides

$$(x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

$$\Rightarrow \quad 4cx - 4a^2 = 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow \quad \frac{cx}{a} - a = \sqrt{(x-c)^2 + y^2}$$

Again, squaring both sides

$$\frac{cx^2}{a^2} - 2cx + a^2 = x^2 - 2cx + c^2 + y^2$$

$$\Rightarrow \quad c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

$$\Rightarrow \quad (c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

Dividing both sides by $(c^2 - a^2)a^2$

we get

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

i.e., $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (\because c^2 = a^2 + b^2)$$

This is the required equation of hyperbola whose transverse axis is along x-axis.

with centre at origin whereas foci are $(0, \pm c)$, vertices are $(0, \pm a)$, ends of conjugate axis are $(\pm bi, 0)$, equations of directrices are $y = \pm\frac{a}{e}$ and latus rectum $= \frac{2b^2}{a}$.

With centre at origin whereas foci are $(\pm c, 0)$, vertices are $(\pm a, 0)$.

In order to find ends of conjugate axis, we find y-intercepts of hyperbola by using $x = 0$ in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

we get $\quad y^2 = -b^2$

$$\Rightarrow \quad y = \pm bi$$

So, the ends of conjugate axis are $(0, \pm bi)$

$\because$ Distance of directrix from centre is $\frac{a}{e}$ as we studied in section 9.9.2.

$\therefore$ Equation of directrices will be $x = \pm\frac{a}{e}$.

Length of latus rectum of hyperbola

Let PQ be the latus rectum of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ …(i)

with a focus $F_1(c, 0)$ as shown in Fig. 9.58

$\because$ latus rectum PQ passes through focus $F_1(c, 0)$ and perpendicular to the focal axis.

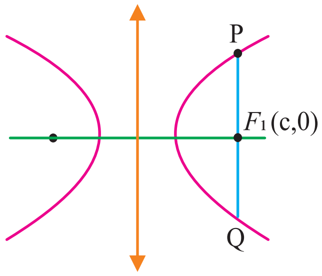


Fig 9.58

$\therefore$ its equation will be $x = c$ ...(ii)

Solving (i) and (ii) simultaneously

We get $\dfrac{c^2}{a^2} - \dfrac{y^2}{b^2} = 1$

i.e., $y = \pm b\dfrac{\sqrt{c^2-a^2}}{a}$

$$y = \pm\frac{b^2}{a} \qquad (\because c^2 - a^2 = b^2)$$

So, the coordinates of P and Q are $\left(c, \dfrac{b^2}{a}\right)$ and $\left(c, \dfrac{-b^2}{a}\right)$

Now, $|\overline{PQ}| = \sqrt{\left(\dfrac{2b^2}{a}\right)^2} = \dfrac{2b^2}{a}$

So, the length of latus rectum is $\dfrac{2b^2}{a}$.

(b) Standard equation of hyperbola when transverse axis is along y-axis

Let $P(x, y)$ be any point on hyperbola with centre at origin, transverse axis along y-axis and conjugate axis along x-axis as shown in the figure 9.59, whereas foci are $F_1(0, c)$ and $F_2(0, -c)$. The length of transverse axis is $2a$.

By the definition of hyperbola

$$|\overline{PF_2}| - |\overline{PF_1}| = 2a$$

i.e., $\sqrt{x^2 + (y + c)^2} - \sqrt{x^2 + (y - c)^2} = 2a$

or $\sqrt{x^2 + (y + c)^2} = 2a + \sqrt{x^2 + (y - c)^2}$

Squaring both sides

$$x^2 + (y + c)^2 = 4a^2 + 4a\sqrt{x^2 + (y - c)^2} + x^2 + (y - c)^2$$

$\Rightarrow$ $4cy - 4a^2 = 4a\sqrt{x^2 + (y - c)^2}$

$\Rightarrow$ $\dfrac{cy}{a} - a = \sqrt{x^2 + (y - c)^2}$

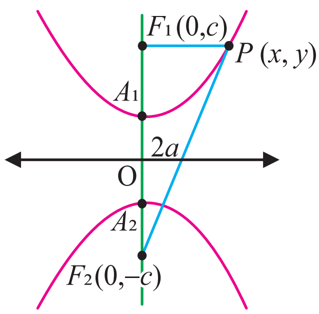


Fig 9.59

Again, squaring both sides

$$\frac{c^2y^2}{a^2} - 2cy + a^2 = x^2 + y^2 - 2cy + c^2$$

$\Rightarrow$ $c^2y^2 + a^4 = a^2x^2 + a^2y^2 + a^2c^2$

$\Rightarrow$ $(c^2 - a^2)y^2 - a^2x^2 = a^2(c^2 - a^2)$

Dividing both sides by $a^2(c^2 - a^2)$

we get

$$\frac{y^2}{a^2}-\frac{x^2}{c^2-a^2}=1$$

i.e., $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ $\qquad (\because c^2=a^2+b^2)$

This is the required equation of hyperbola whose transverse axis is along *y*-axis.

Equilateral or Rectangular hyperbola

A hyperbola, in which transverse axis and conjugate axis are of same length is called rectangular or equilateral hyperbola.

i.e., $2a=2b$

i.e., $b=a$

So, the equation of hyperbola is: $x^2-y^2=a^2$

where $e=\sqrt{2}$

because $c^2=a^2+b^2$

i.e., $a^2e^2=2a^2$ $\qquad (\because c=ae)$

$\Longrightarrow$ $e=\sqrt{2}$

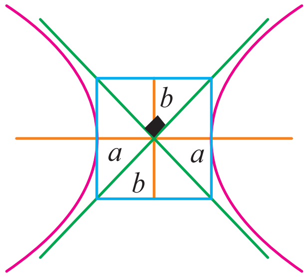


Fig 9.60

In rectangular hyperbola asymptotes are perpendicular to each other as shown in the figure 9.60.

Conjugate Hyperbola

The conjugate hyperbola of a given hyperbola is the hyperbola whose transverse and conjugate axes are respectively conjugate and transverse axes of given hyperbola.

Thus, $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ is conjugate hyperbola of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Standard forms of Translated Hyperbolas

If centre of hyperbola is not at origin but the transverse and conjugate axes are parallel to the coordinate axes then it is standard form of translated hyperbola and its equations along with its elements are given below.

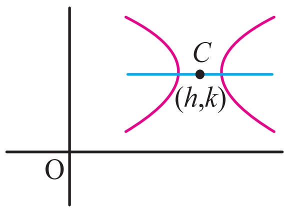


Fig 9.61

(a) Equation of hyperbola when centre is at (h,k) and transverse axis parallel to *x*-axis is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Here foci are $(h\pm c,k)$, vertices are $(h\pm a,k)$, ends of conjugate axes are $(h,k\pm bi)$.

Equations of directrices are: $x-h=\pm\frac{a}{e}$ and latus rectum is $\frac{2b^2}{a}$.

(b) Equation of hyperbola when centre is at (h, k) and transverse axis parallel to y-axis is:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

Here foci are $(h, k \pm c)$, vertices are $(h, k \pm a)$, ends of conjugate axes are $(h \pm bi, k)$. Equations of directrices are $y-k=\pm\frac{a}{e}$ and latus rectum $=\frac{2b^2}{a}$.

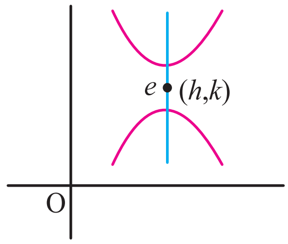


Fig 9.62

(c) General equation of hyperbola when transverse and conjugate axes are parallel to coordinate axes.

Consider a hyperbola

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

On simplification, it becomes

$$b^2(x^2-2hx+h^2)-a^2(y^2-2ky+k^2)=a^2b^2$$

$$\Rightarrow \quad b^2x^2-2hb^2x+b^2h^2-a^2y^2+2ka^2y-a^2k^2-a^2b^2=0$$

$$\Rightarrow \quad b^2x^2-a^2y^2-2hb^2x+2ka^2y+b^2h^2-a^2k^2-a^2b^2=0 \qquad \ldots\text{(i)}$$

Let $A=b^2, B=-a^2, G=-2hb^2, F=2ka^2$ and $C=b^2h^2-a^2k^2-a^2b^2$

So, equation (i) becomes

$$Ax^2+By^2+Gx+Fy+C=0 \qquad \ldots\text{(ii)}$$

Where A and B are non-zero and have different signs.

Equation (ii) is the general equation of hyperbola.

Summary of standard equations of hyperbola and related terms

Equation	Related Terms
(i) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Centre at origin and transverse axis is along x-axis.	Foci are $(\pm c, 0)$ Vertices are $(\pm a, 0)$ Ends of conjugate axis are $(0, \pm bi)$ Directrices: $x=\pm\frac{a}{e}$ Asymptotes: $y=\pm\frac{b}{a}x$
(ii) $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ Centre at origin and transverse axis is along y-axis.	Foci are $(0, \pm c)$ Vertices are $(0, \pm a)$ Ends of conjugate axis are $(\pm bi, 0)$ Directrices: $y=\pm\frac{a}{e}$ Asymptote: $y=\pm\frac{a}{b}x$

Equation	Related Terms
(iii) $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ Centre at (h,k) and transverse axis is parallel to x-axis.	Foci are $(h \pm c, k)$ Vertices are $(h \pm a, k)$ Ends of conjugate axis are $(h, k \pm bi)$ Directrices: $x-h=\pm\frac{a}{e}$ Asymptote: $y-k=\pm\frac{b}{a}(x-h)$
(iv) $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ Centre at (h,k) and transverse axis is parallel to y-axis.	Foci: $(h, k \pm c)$ Vertices: $(h, k \pm a)$ Ends of conjugate axis: $(h, \pm bi, k)$ Directrices: $y-k=\pm\frac{a}{e}$ Asymptote: $y-k=\pm\frac{a}{b}(x-h)$

Note: For all standard equations of hyperbola, we have

Latus rectum $=\frac{2b^2}{a}$

$c=ae$ and $c^2=a^2+b^2$

Similarities and differences between ellipse and hyperbola

- Similarities for standard forms along x-axis

Ellipse	Hyperbola
(i) Two foci: $(\pm c, 0)$	(i) Two foci: $(\pm c, 0)$
(ii) Two vertices: $(\pm a, 0)$	(ii) Two vertices: $(\pm a, 0)$
(iii) Two directrices: $x=\pm\frac{a}{e}$	(iii) Two directrices: $x=\pm\frac{a}{e}$
(iv) Length of latus rectum $=\frac{2b^2}{a}$	(iv) Length of latus rectum $=\frac{2b^2}{a}$
(v) $c=ae$	(v) $c=ae$
(vi) axes are $2a$ and $2b$	(vi) axes are $2a$ and $2b$

- Differences

Ellipse	Hyperbola
(i) $a>c$	(i) $a<c$
(ii) $c^2=a^2-b^2$	(ii) $c^2=a^2+b^2$
(iii) Ellipse is closed curve	(iii) Hyperbola is not closed figure
(iv) If $b=a$ then it is an auxiliary circle with $e=0$ and foci coincide	(iv) If $b=a$ then it is an rectangular hyperbola with $e=\sqrt{2}$ and foci do not coincide

Example 1. Find eccentricity, foci, vertices, ends of conjugate axis and latus rectum of hyperbola: $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

Solution: We have hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Comparing with standard equation.

we get $a^2 = 16, b^2 = 9$, transverse axis is along x-axis and centre at origin.

we know that

$$c^2 = a^2 + b^2$$

i.e., $c^2 = 16 + 9 = 25$

So, $a = 4, b = 3$ and $c = 5$.

We obtain eccentricity by $c = ae$

i.e., $\frac{5}{4} = e$

Foci $= (\pm c, 0) = (\pm 5, 0)$

Vertices $(\pm a, 0) = (\pm 4, 0)$

Ends of conjugate axis $= (0, \pm bi) = (0, \pm 3i)$

and Latus rectum $= \frac{2b^2}{a}$

$$= \frac{18}{4} = \frac{9}{2}$$

Example 2. Find eccentricity, equation of directrices and equations of asymptotes of hyperbola $\frac{y^2}{9} - \frac{x^2}{4} = 1.$

Solution: Comparing given hyperbola with standard equation of hyperbola.

we get $a^2 = 9, b^2 = 4$, transverse axis along y-axis and centre at origin.

We know that

$$c^2 = a^2 + b^2$$

i.e., $c^2 = 9 + 4 = 13$

So, $a = 3, b = 2$ and $c = \sqrt{13}$.

We obtain eccentricity by $c = ae$

i.e., $\sqrt{13} = 3e$

$\Rightarrow$ $e = \frac{\sqrt{13}}{3}$

$\because$ Transverse axis is along y-axis

$\therefore$ Its directrices will be

$$y = \pm\frac{a}{e}$$

i.e., $$y = \pm\frac{3}{\frac{\sqrt{13}}{3}}$$

or $$y = \pm\frac{9}{\sqrt{13}}$$

Also, the equation of asymptotes will be

$$y = \pm\frac{a}{b}x$$

i.e., $$y = \pm\frac{3}{2}x.$$

Example 3. Find centre, foci, vertices, latus rectum and equations of directrices, for the hyperbola $\frac{(x-3)^2}{64} - \frac{(y+4)^2}{36} = 1$

Solution: Comparing given equation of hyperbola with

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

We get $h = 3$ and $k = -4$

Also, $a^2 = 64 \implies a = 8$

and $b^2 = 36 \implies b = 6$

Its transverse axis is parallel to x-axis with centre $= (h, k) = (3, -4)$

We know that

$$c^2 = a^2 + b^2$$

$\implies c^2 = 64 + 36$

$\implies c^2 = 100$

$\implies c = 10$

Now, foci $= (h \pm c, k)$

$= (3 \pm 10, -4)$

So, foci are $(13, -4)$ and $(-7, -4)$

Vertices $= (h \pm a, k)$

$= (3 \pm 8, -4)$

So, vertices are $(11, -4)$ and $(-5, -4)$

Latus rectum $= \frac{2b^2}{a}$

$= \frac{2(36)}{8}$

$= 9$

Here, $c = ae$

i.e., $10 = 8e$

$\Rightarrow \quad e = \frac{5}{4}$

$\because$ Transverse axis is parallel to x-axis

$\therefore$ Equations of directrices will be

$$x - h = \pm\frac{a}{e}$$

i.e., $x - 3 = \pm\frac{8}{\frac{5}{4}}$

$\Rightarrow \quad x - 3 = \pm\frac{32}{5}$

9.10.2 Find the equation of hyperbola with the following given elements

- **transverse and conjugate axes with centre at origin,**
- **two points,**
- **eccentricity, latera recta and transverse axes,**
- **focus, eccentricity and centre,**
- **focus, centre and directrix.**

The equation of hyperbola can be found with different conditions and elements. Here we discuss some of them.

(a) When transverse and conjugate axes are given with centre at origin

The method is explained with the help of the following example.

Example: Find the equation of hyperbola if transverse axis and conjugate axis are 8 and 6 units long respectively, where centre is at origin and transverse axis is along y-axis.

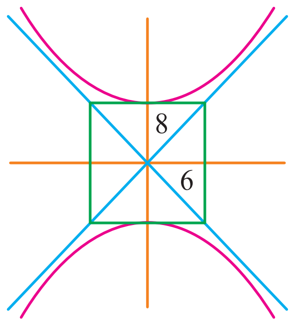


Fig 9.63

Solution: Here

$2a = 8$ and $2b = 6$

$\Rightarrow \quad a = 4 \quad \Rightarrow \quad b = 3$

$\because$ centre is at origin and transverse axis is along y-axis.

$\therefore$ Its equation will be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

i.e., $\frac{y^2}{16} - \frac{x^2}{9} = 1 \quad \Rightarrow \quad 9y^2 - 16x^2 = 144$

(b) When two points of hyperbola are given

The method is explained with the help of the following example.

Example: Find the equation of hyperbola with centre at origin and transverse axis along y-axis, such that the hyperbola passes through the points $\left(\frac{1}{2},\frac{\sqrt{5}}{2}\right)$ and $\left(\frac{1}{\sqrt{8}},\frac{-3}{\sqrt{8}}\right)$.

Solution: Centre is at origin and transverse axis is along y-axis.

∴ its equation will be $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ …(i)

∵ $\left(\frac{1}{2},\frac{\sqrt{5}}{2}\right)$ lies on the hyperbola

∴ we have

$$\frac{5}{4a^2}-\frac{1}{4b^2}=1 \quad \text{…(ii)}$$

∵ $\left(\frac{1}{\sqrt{8}},\frac{-3}{\sqrt{8}}\right)$ is on hyperbola

∴ we have

$$\frac{9}{8a^2}-\frac{1}{8b^2}=1 \quad \text{…(iii)}$$

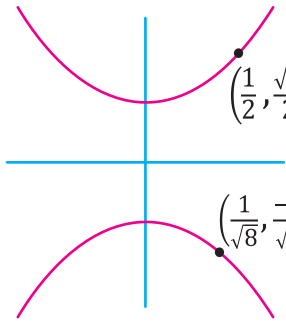


Fig 9.64

Multiplying equation (ii) by $\frac{1}{2}$ and subtracting resultant equation from (iii)

we get

$$\frac{4}{8a^2}=\frac{1}{2}$$

⇒ $a^2=1$

By using $a^2=1$ in equation (ii)

We get $\frac{5}{4}-\frac{1}{4b^2}=1$

⟹ $\frac{1}{4}=\frac{1}{4b^2}$ ⟹ $b^2=1$

By using values of a^2 and b^2 in (i)

We get $y^2-x^2=1$

(c) When eccentricity, latera recta or transverse axis are given

The method is explained with the help of following examples.

Example 1. Find the equation of hyperbola when centre is at origin and transverse axis is along x-axis with the length 10 units, whereas eccentricity is $\sqrt{3}$.

Solution: ∵ Centre is at origin and transverse axis is along x-axis

∴ Its equation will be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots\text{(i)}$$

Here, $e = \sqrt{3}$

and $2a = 10$

i.e., $a = 5 \implies a^2 = 25$

Now, $c = ae$

$\implies c = 5\sqrt{3}$

We know that

$c^2 = a^2 + b^2$

i.e., $75 = 25 + b^2$

$\implies b^2 = 50$

By using values of a^2 and b^2 in equation (i)

We get $\dfrac{x^2}{25} - \dfrac{y^2}{50} = 1$

$\implies 2x^2 - y^2 = 50$

Example 2. Find the equation of hyperbola with centre at origin and transverse axis is along y-axis, such that latus rectum is 12 units long and eccentricity is 2.

Solution: $\because$ Centre is at origin and transverse axis along y-axis

$\therefore$ its equation will be

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \ldots\text{(i)}$$

We have $e = 2$ and latus rectum $= 12$

i.e., $\dfrac{2b^2}{a} = 12$

$\implies b^2 = 6a \qquad \ldots\text{(ii)}$

We know that

$c^2 = a^2 + b^2$

i.e., $a^2e^2 = a^2 + 6a \qquad (\because c = ae)$

$\implies 4a^2 = a^2 + 6a$

$\implies 3a^2 - 6a = 0$

$\implies 3a(a - 2) = 0$

$\implies a = 0$ or $a = 2$

Neglecting $a = 0$

We have $a = 2$ or $a^2 = 4$

So, from equation (ii)

we get $b^2 = 12$

By using values of a and b in equation (i)

$$\frac{y^2}{4} - \frac{x^2}{12} = 1$$

or $3y^2 - x^2 = 12$

(d) When focus, eccentricity and centre are given

The method is explained with the help of the following example.

Example: Find the equation of hyperbola with centre $(2, 3)$ and transverse axis parallel to x-axis, such that a focus is $(6, 3)$ and eccentricity is $\sqrt{5}$.

Solution: $\because$ Centre is not at origin and transverse axis is parallel to x-axis

$\therefore$ equation of hyperbola will be

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \ldots\text{(i)}$$

We have

Centre $= (h, k) = (2, 3)$

and Focus $= (h + c, k) = (6, 3)$

i.e., $(2 + c, 3) = (6, 3)$

$\Longrightarrow$ $c = 4$

Now, $c = ae$

i.e., $a = \frac{4}{\sqrt{5}}$ $\qquad (\because e = \sqrt{5})$

We know that

$$c^2 = a^2 + b^2$$

i.e., $16 = \frac{16}{5} + b^2$

$\Longrightarrow$ $b^2 = \frac{80-16}{5}$

$\Longrightarrow$ $b^2 = \frac{64}{5}$

By using values of h, k, a^2 and b^2 in equation (i)

We get

$$\frac{(x-2)^2}{\frac{16}{5}} - \frac{(y-3)^2}{\frac{64}{5}} = 1$$

$$\Longrightarrow \frac{5(x-2)^2}{16} - \frac{5(y-3)^2}{64} = 1$$

(e) When focus, centre and directrix are given

The method is explained with the help of the following example.

Example: Find the equation of hyperbola whose centre is $(3,4)$ and transverse axis is parallel to y-axis, such that one focus is $(3,12)$ and one equation of directrix is $y=7$.

Solution: $\because$ Centre is not at origin and transverse axis parallel to y-axis.

$\therefore$ The equation of hyperbola will be

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \quad \ldots\text{(i)}$$

We have

Centre $=(h,k)=(3,4)$

and Focus $=(h,k+c)=(3,12)$

i.e., $k+c=12$

$\Longrightarrow$ $4+c=12$

i.e., $c=8$

and one equation of directrix is $y=7$

Comparing it with $y=\frac{a}{e}+k$

We get

$$\frac{a}{e}+4=7$$

$\Longrightarrow$ $a=3e$ …(ii)

Also, we know that

$c=ae$

i.e., $ae=8$

By using $a=3e$ from equation (ii)

we get

$3e^2=8$

$\Longrightarrow$ $e^2=\frac{8}{3}$

$\Longrightarrow$ $e=\sqrt{\frac{8}{3}}$

So, equation (ii) becomes $a=3\sqrt{\frac{8}{3}}$

i.e., $a^2=24$

We know that

$$c^2=a^2+b^2$$

i.e., $64 = 24 + b^2$

$\Rightarrow$ $b^2 = 40$

By using values of h, k, a^2 and b^2 in equation (i)

we get

$$\frac{(y-4)^2}{24} - \frac{(x-3)^2}{40} = 1$$

9.10.3 Convert a given equation to the standard form of equation of a hyperbola, find its elements and sketch the graph

As we have studied in section 9.10 that the general equation of hyperbola when transverse and conjugate axis are parallel to coordinate axes is

$$Ax^2 + By^2 + Gx + Fy + C = 0$$

Where A and B are non-zero and have opposite sign. Also, A, B, G, F and C are real numbers.

This general equation can be converted into standard forms by the method of completing square which will be explained in the following examples.

Technique for graphing hyperbolas

Graphs of hyperbolas from their standard equations can be drawn by using the following steps.

1. Determine whether the transverse axis is along or parallel to *x*-axis or *y*-axis which can be determined by checking the sign of x^2-term or y^2-term. In case of positive x^2-term, the transverse axis will be along or parallel to *x*-axis.
   In case of positive y^2-term, the transverse axis will be along or parallel to y-axis.
2. Determine the values of *a* and *b* and draw a rectangle extending a units on either side of the centre along the transverse axis and *b* units on either side of the centre along the conjugate axis.
3. Draw the asymptotes along the diagonals of the rectangle.
4. Using the rectangle and the asymptotes as guide draw the graph of hyperbola.

Example 1. Find the eccentricity, foci, vertices and directrices of hyperbola $9x^2 - 16y^2 - 144 = 0$. Also draw its graph.

Solution: First of all we convert the given equation into the standard form.

Given hyperbola: $9x^2 - 16y^2 - 144 = 0$

or $9x^2 - 16y^2 = 144$

Dividing both sides by 144

We get

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Here, centre is origin and the transverse axis is along x-axis with

$a^2 = 16$ and $b^2 = 9$

So, $a = 4$ and $b = 3$

We know that

$c^2 = a^2 + b^2$

i.e., $c^2 = 16 + 9$

$\Rightarrow$ $c^2 = 25$ or $c = 5$

Now, $c = ae$

i.e., $5 = 4e$ $\Rightarrow$ $e = \frac{5}{4}$

$\because$ Major axis is along x-axis.

$\therefore$ coordinates of foci $= (\pm c, 0)$

$= (\pm 5, 0)$

and coordinates of vertices $= (\pm a, 0)$

$= (\pm 4, 0)$

Equation of directrices will be

$$x = \pm \frac{a}{e}$$

or $x = \pm \frac{16}{5}$ i.e., $x = \pm \frac{4}{\frac{5}{4}}$

Graph of Hyperbola

Standard form of given hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

By using the steps of drawing graph, we sketch the graph as show in Fig. 9.65.

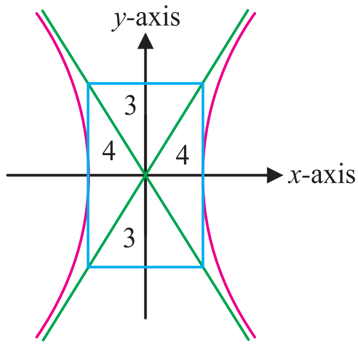


Fig 9.65

Example 2. Find centre, foci, eccentricity and vertices of hyperbola

$16y^2 - 9x^2 + 36x + 64y - 116 = 0$. Also draw its graph.

Solution: We first convert the given hyperbola in standard form.

Given hyperbola: $16y^2 - 9x^2 + 36x + 64y - 116 = 0$

By re-arranging the terms

we get $(16y^2 + 64y) - (9x^2 - 36x) = 116$

or $16(y^2 + 4y) - 9(x^2 - 4x) = 116$

or $16(y^2 + 4y + 4) - 9(x^2 - 4x + 4) = 116 + 64 - 36$

or $16(y + 2)^2 - 9(x - 2)^2 = 144$

Dividing both sides by 144

we get

$$\frac{(y+2)^2}{9} - \frac{(x-2)^2}{16} = 1$$

Comparing this equation with

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

We get $h = 2, k = -2, a^2 = 9$ and $b^2 = 16$.

and transverse axis is parallel to y-axis.

We know that

$c^2 = a^2 + b^2$

i.e., $c^2 = 9 + 16 = 25$

So, $c = 25, a = 3$ and $b = 4$

Now, $c = ae$

i.e., $5 = 3e \quad \Rightarrow \quad e = \frac{5}{3}$

Now, centre $= (h, k) = (2, -2)$

Foci $= (h, k \pm c) = (2, -2 \pm 5)$

So, Foci are $(2, 3)$ and $(2, -7)$

and vertices $= (h, k \pm a) = (2, -2 \pm 3)$

So, vertices are $(2, 1)$ and $(2, -5)$

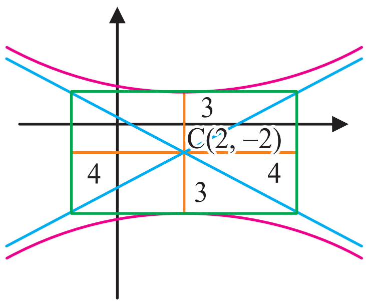


Fig 9.66

Graph of Hyperbola

By using the steps of drawing graph, we draw the graph as shown in the figure 9.66.

Exercise 9.5

1. Find the equation of the hyperbola with centre at the origin satisfying the following conditions.

   (i) Transverse and conjugate axes are 16 and 12 respectively. Also, transverse axis is along y-axis.

(ii) Hyperbola passes through $\left(\frac{3\sqrt{17}}{4}, 1\right)$ and $(3, 0)$ with transverse axis along x-axis.

(iii) Transverse axis of length 8 units and along y-axis where eccentricity is $\sqrt{5}$.

(iv) Transverse axis along x-axis with latus rectum $= 10$ units and eccentricity $= \frac{3}{2}$.

(v) Focus $(5, 0)$, directrix $x = 2$.

(vi) Eccentricity $= 3$ and focus $(8, 0)$.

(vii) Eccentricity $= 2$ and vertex $= (0, 4)$.

2. Find equation of the hyperbola with centre $(1, 3)$ and satisfying the following condition.

(i) Focus is $(2, 3)$ and eccentricity is $\sqrt{3}$, whereas transverse axis is parallel to x-axis.

(ii) Focus is $(4, 5)$ and an equation of directrix is $y = 1$ where transverse axis is parallel to y-axis.

3. Find eccentricity, foci, vertices and latus rectum of each of the following. Also, draw graph.

(i) $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (ii) $\frac{y^2}{5} - \frac{x^2}{4} = 1$

(iii) $9x^2 - y^2 + 1 = 0$ (iv) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

4. Find centre, foci, eccentricity, vertices and equations of directrices. Also draw the graph.

(i) $\frac{(x-5)^2}{9} - \frac{(y+3)^2}{16} = 1$ (ii) $\frac{(y-4)^2}{36} - \frac{(x+5)^2}{64} = 1$

(iii) $9x^2 - 4y^2 + 36x + 8y - 4 = 0$

(iv) $25x^2 - 150x - 9y^2 + 72y + 306 = 0$

5. Find equation of rectangular hyperbola with centre at origin whose vertices are $(\pm 4, 0)$ and find equation of its conjugate hyperbola. Also, find equations of asymptotes of the rectangular hyperbola.

6. Find the eccentricity of a hyperbola whose latus rectum is double the transverse axis.

7. Show that the eccentricities e_1 and e_2 of the two conjugate hyperbolas satisfy the relation $e_1^2 + e_2^2 = e_1^2 e_2^2$.

9.11 Equation of Tangent and Normal of a Hyperbola

In this section, we will discuss about the tangent and normal to a hyperbola along with their conditions and equations.

9.11.1 Recognize tangent and normal to a hyperbola

In the figure line l is tangent to the hyperbola as the line touches the hyperbola at a single point, whereas line m is the normal to the hyperbola as it is perpendicular to the tangent at the point of contact P.

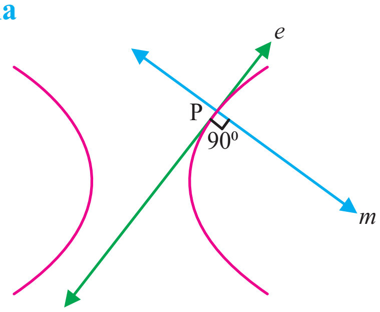


Fig 9.67

9.11.2 Find

- **points of intersection of a hyperbola with a line including the condition of tangency,**
- **the equation of a tangent in slope form.**

Consider a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ …(i)

and a line $y = mx + c$ …(ii)

Solving both equations simultaneously we get

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

$\Rightarrow$ $b^2x^2 - a^2m^2x^2 - 2a^2cmx - a^2c^2 = a^2b^2$

or $(b^2 - a^2m^2)x^2 - 2a^2cmx - a^2c^2 - a^2b^2 = 0$

Here, $\Delta = 4a^4c^2m^2 + 4(a^2c^2 + a^2b^2)(b^2 - a^2m^2)$

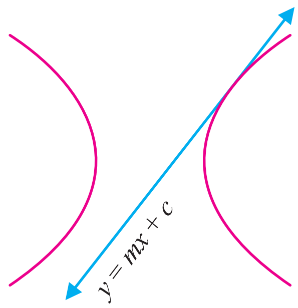


Fig 9.68

By quadratic formula

$$x = \frac{2a^2cm \pm \sqrt{\Delta}}{2a}$$

By using this value of x in equation (ii), we will get value of y, so we will get point of intersection.

The given line will be tangent, if $\Delta = 0$

i.e., $4a^4c^2m^2 + 4a^2b^2c^2 - 4a^4c^2m^2 + 4a^2b^4 - 4a^4b^2m^2 = 0$

$\Rightarrow$ $4a^2b^2(c^2 + b^2 - a^2m^2) = 0$

or $c^2 = a^2m^2 - b^2$

$\Rightarrow$ $c = \pm\sqrt{a^2m^2 - b^2}$

This is the condition of tangency when given line is tangent to the hyperbola.

By using this value of m in equation (ii)

we get, $y = mx \pm \sqrt{a^2m^2 - b^2}$

This is the equation of tangent to the given hyperbola in slope form.

Example 1. For what value of k, will the line $y = kx + 1$ be tangent to the hyperbola $3x^2 - 4y^2 = 12$?

Solution: We have

Hyperbola: $3x^2 - 4y^2 = 12$ ...(i)

and line: $y = kx + 1$ ...(ii)

Solving simultaneously,

we get $3x^2 - 4(kx + 1)^2 = 12$

$\Rightarrow$ $3x^2 - 4k^2x^2 - 8kx - 4 - 12 = 0$ ...(iii)

$(3 - 4k^2)x^2 - 8kx - 16 = 0$

$\Delta = 64k^2 - 4(3 - 4k^2)(-16)$

$= 64(k^2 + 3 - 4k^2)$

$= 64(3 - 3k^2)$

The given line will be tangent to the given hyperbola

if $\Delta = 0$

i.e., $64(3 - 3k^2) = 0$

$\Rightarrow$ $k^2 = 1$

$\Rightarrow$ $k = \pm 1$

This is the required value of k.

Example 2. Find the equation of tangent to the hyperbola $2x^2 - 3y^2 = 6$ whose slope is 2.

Solution: We have

Slope $= m = 2$

and hyperbola: $2x^2 - 3y^2 = 6$

i.e., $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$

Here $a^2 = 3$ and $b^2 = 2$

We know that the equation of tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

By using values, we get

$y = 2x \pm \sqrt{12 - 2}$

$y = 2x \pm \sqrt{10}$

This is the required equation of tangent.

9.11.3 Find the equation of a tangent and a normal to a hyperbola at a point

Consider a hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ ...(i)

Let $P(x_1, y_1)$ be a point of this hyperbola i.e., $$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \qquad \ldots\text{(ii)}$$

Differentiating equation (i) with respect to x, we get

$$\frac{2x}{a^2} - \frac{2y\frac{dy}{dx}}{b^2} = 0$$

$$\Longrightarrow \frac{dy}{dx} = \frac{b^2}{2y} \times \frac{2x}{a^2} = \frac{b^2x}{a^2y}$$

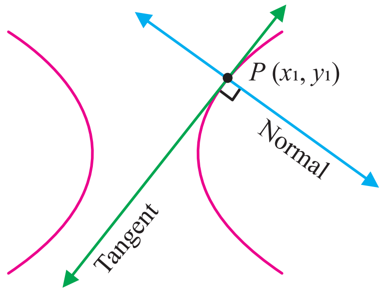


Fig 9.69

Now slope of tangent to the given hyperbola at (x_1, y_1) is

$$m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{b^2x_1}{a^2y_1}$$

By point slope form, the equation of tangent will be

$$y - y_1 = m(x - x_1)$$

i.e., $$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$\Longrightarrow \quad a^2yy_1 - a^2y_1^2 = b^2xx_1 - b^2x_1^2$$

$$\Longrightarrow \quad b^2xx_1 - a^2yy_1 - b^2x_1^2 + a^2y_1^2 = 0$$

Dividing both sides by a^2b^2

we get,

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) = 0$$

i.e., $$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0 \qquad \text{(Using equation (ii)}$$

or $$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

This is the equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1)

$\because$ Normal is perpendicular to the tangent at the point of contact

$\therefore$ Slope of normal $= m' = -\frac{a^2y_1}{b^2x_1}$

By point-slope form the equation of normal will be

$$y - y_1 = m'(x - x_1)$$

i.e., $$y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1)$$

$\Rightarrow \quad b^2x_1y - b^2x_1y_1 = -a^2xy_1 + a^2x_1y_1$

Dividing both sides by x_1y_1

we get,

$$\frac{b^2y}{y_1} - b^2 = -\frac{a^2x}{x_1} + a^2$$

$$\Rightarrow \frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

This is the equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1).

Example 1. Find the equations of tangent and normal to $\frac{y^2}{4} - \frac{x^2}{5} = 1$ at $(\sqrt{5}, 2\sqrt{2})$.

Solution: We have

Hyperbola: $\frac{y^2}{4} - \frac{x^2}{5} = 1$

Differentiating w.r.t x

$$\frac{2y\frac{dy}{dx}}{4} - \frac{2x}{5} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{5} \times \frac{2}{y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x}{5y}$$

Now slope of tangent at $(\sqrt{5}, 2\sqrt{2}) = m = \left(\frac{dy}{dx}\right)_{(\sqrt{5}, 2\sqrt{2})}$

i.e., $\quad m = \frac{4\sqrt{5}}{5(2\sqrt{2})}$

$$= \sqrt{\frac{2}{5}}$$

By point slope form, the equation of tangent will be

$$y - y_1 = m(x - x_1)$$

i.e., $\quad y - 2\sqrt{2} = \sqrt{\frac{2}{5}}(x - \sqrt{5})$

$\Rightarrow \quad \sqrt{5}y - 2\sqrt{10} = \sqrt{2}x - \sqrt{10}$

$\Rightarrow \quad \sqrt{2}x - \sqrt{5}y + \sqrt{10} = 0$

$\because \quad$ Normal is perpendicular to the tangent at the point of contact

∴ Slope of normal at $(\sqrt{5}, 2\sqrt{2}) = m' = -\frac{1}{m}$

i.e., $m' = -\sqrt{\frac{5}{2}}$

By point-slope form the equation of normal will be

$$y - y_1 = m'(x - x_1)$$

i.e., $y - 2\sqrt{2} = -\sqrt{\frac{5}{2}}(x - \sqrt{5})$

$\Rightarrow$ $\sqrt{2}y - 4 = -\sqrt{5}x + 5$

$\Rightarrow$ $\sqrt{5}x + \sqrt{2}y - 9 = 0$

9.12 Translation and Rotation of Axes

Translation and rotation of axes are the transformations which are commonly used to simplify the equation of a curve and to bring conics in standard forms. We discuss these concepts in detail as under.

9.12.1 Define translation and rotation of axes and demonstrate through examples

The concept of translation is of a transformation in which the location of the geometrical shape is changed but its size, shape or orientation remains same.

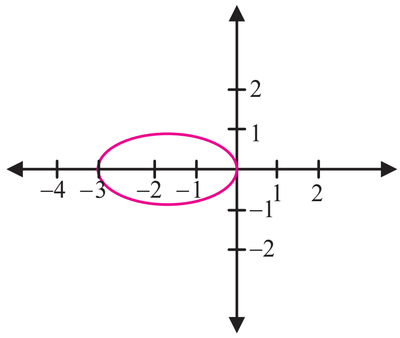


Fig. 9.70

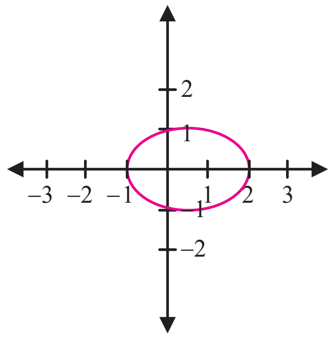


Fig. 9.71

For example, an ellipse in Fig. 9.70 has been translated 2 units to the right as shown in Fig. 9.71.

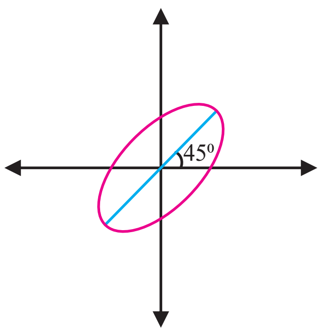


Fig. 9.72

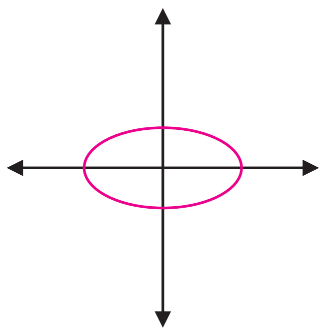


Fig. 9.73

The concept of rotation is also of a transformation in which the location of the geometrical shape is rotated around a fixed point but its size and shape are not changed, for example an ellipse in Fig.9.72 has been rotated 45° clockwise as shown in Fig. 9.73.

Definition: A translation of axes is a transformation between two rectangular coordinate systems in which the origins O and O′ are at different locations but the corresponding axes are parallel and have the same directions as shown in the figure 9.74.

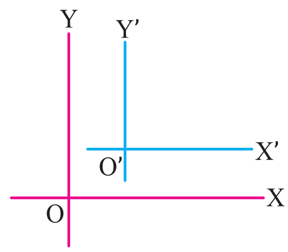


Fig 9.74

Definition: A rotation of axes in the plane is a transformation in which the axes OX and OY of one rectangular system are rotated about the origin O through an angle θ to locate the corresponding axes OX′ and OY′ of other coordinate system as shown in the figure 9.75.

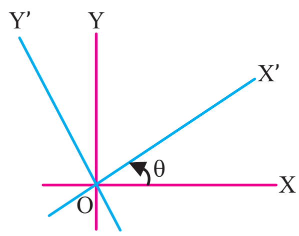


Fig 9.75

9.12.2 Find the equations of transformation for

- translation of axes,
- rotation of axes.

- **Equations of transformation for translation of axes**

In order to obtain the equations of transformation for translation of axes we have translated the axes of an xy −coordinate system to get a new $x'y'$ − coordinate system whose origin O′ is at the point (h, k) as shown in Fig. 9.76.

As a result, a point P in the plane will have both (x, y)- coordinates and (x', y')-coordinates as shown in the Fig. 9.77. These coordinates are related by

$$x = x' + h\ , \qquad y = y' + k$$

or $$x' = x - h\ , \qquad y' = y - k$$

These equations are called the equations of transformation for the translation of axes.

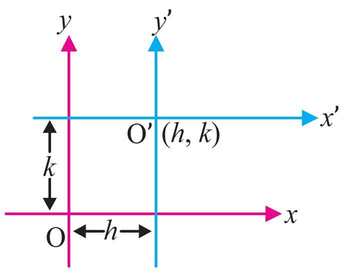


Fig. 9.76

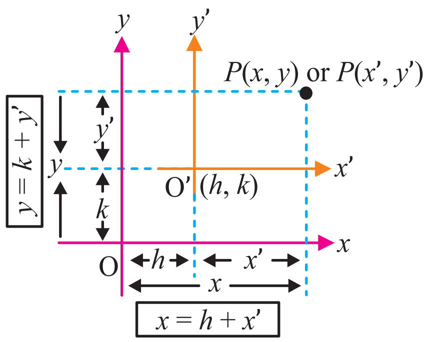


Fig. 9.77

- **Equations of transformation for rotation of axes**

In order to get the equations of transformation for rotation of axes, the axes of an $xy-$ coordinate system have been rotated about the origin through an angle θ to produce a new $x'y'-$ coordinate system as shown in the Fig. 9.78.

As a result, any point P in the plane will have both $P(x, y)$- coordinates and $P(x', y')$-coordinates as shown in the Fig. 9.79.

In order to relate these coordinates, we suppose r as the distance from the common origin to the point P and let α be the angle of $\overline{\text{OP}}$ from x'-axis as shown in the Fig.9.79.

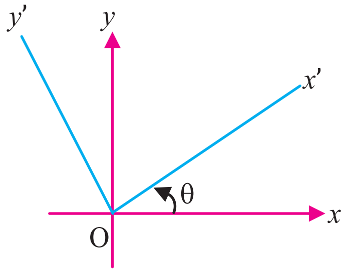


Fig. 9.78

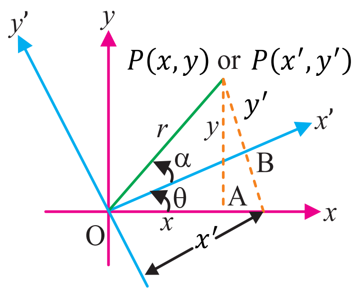


Fig. 9.79

From figure 9.79, in ΔOAP

$$\cos(\theta + \alpha) = \frac{x}{r} \text{ and } \sin(\theta + \alpha) = \frac{y}{r}$$

or $\quad x = r\cos(\theta + \alpha)$ ...(i)

or $\quad y = r\ \sin(\theta + \alpha)$ ...(ii)

In ΔOBP

$\quad x' = r\cos\alpha$ ...(iii)

and $\quad y' = r\sin\alpha$ ...(iv)

Using trigonometric identities equation (i) and equation (ii) become

$$x = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$$

and $\quad y = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha$

By using equation (iii) and (iv)

we get

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\}$$

These equations are called the equations of transformation for rotation of axes.

9.12.3 Find the transformed equation by using translation or rotation of axes

The method of finding the transformed equation by using translation or rotation of axes is explained with the help of the following examples.

Example 1. Find the transformed equation of parabola $(y+5)^2 = 4(x-3)$ when axes are translated with new origin $(3,-5)$.

Solution: Given parabola is $(y+5)^2 = 4(x-3)$ ...(i)

Shifting the origin to $(3,-5)$ and keeping the axes in parallel position.

Let (X,Y) be the new coordinates of any point $P(x,y)$ after shifting the origin.

By equations of transformation

$$x' = x - h \text{ and } y' = y - k$$

Here $(x',y') = (X,Y)$

and $(h,k) = (3,-5)$

So, we get

$$X = x - 3 \text{ and } Y = y + 5$$

So, equation (i) becomes

$$Y^2 = 4$$

This is the required transformed equation.

Example 2. Find the transformed equation of $5x^2 - 6xy + 5y^2 - 8 = 0$ when the axes are rotated through an angle of $45°$.

Solution: Given equation is $5x^2 - 6xy + 5y^2 - 8 = 0$ ...(i)

Now, we rotate the axes about the origin through an angle of $\theta = 45°$

Let (X,Y) be the new coordinates of any point $P(x,y)$ after rotation

By equations of transformation, we have

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\}$$

Here $(x',y') = (X,Y)$

and $\theta = 45°$

So, we get

$$x = X\cos 45°\, Y\sin 45° = \frac{X-Y}{\sqrt{2}}$$

and

$$y = X\sin 45° + Y\cos 45° = \frac{X+Y}{\sqrt{2}}$$

Substituting these values in equation (i), we get

$$5\left(\frac{X-Y}{\sqrt{2}}\right)^2 - 6\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + 5\left(\frac{X+Y}{\sqrt{2}}\right)^2 - 8 = 0$$

$$\Rightarrow \frac{5}{2}(X^2 - 2XY + Y^2) - 3(X^2 - Y^2) + \frac{5}{2}(X^2 + 2XY + Y^2) - 8 = 0$$

$$\Rightarrow \frac{5}{2}(2X^2 + 2Y^2) - 3X^2 + 3Y^2 - 8 = 0$$

$$\Rightarrow \quad 2X^2 + 8Y^2 - 8 = 0$$

or $\quad X^2 + 4y^2 = 4$

This is the required transformed equation.

9.12.4 Find new origin and new axes referred to old origin and old axes

Let O be the origin of *xy*-coordinate system as shown in the figure 9.80.

A new XY-coordinate system is introduced with new origin $O'(h, k)$. This system is translated h units in the x-direction and k units in the y-direction and then rotated anticlockwise by θ radians as shown in the figure.

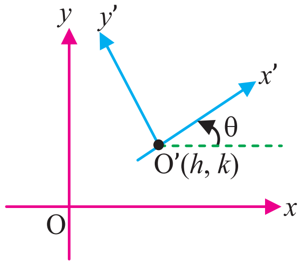


Fig 9.80

The relations among x, y, X, Y and θ are given below

$$x = (X + h)\cos\theta - (Y + k)\sin\theta$$

and $\quad y = (X + h)\sin\theta + (Y + k)\cos\theta$

Example 1. Find new origin in O' and new axes (X-axis and Y-axis) with respect to xy-coordinate system if it is translated 5 units to the right, 3 units down and rotated $\frac{\pi}{4}$ radius anticlockwise.

Solution: Here $\quad h = 5$ and $k = -3$

So, new origin $= (h, k) = (5, -3)$

Here inclination of X-axis $= \theta = \frac{\pi}{4}$

So, slope of X-axis $= \tan 45°$

$= 1$

By point slope form equation of X-axis will be

$$y - (-3) = 1(x - 5)$$

$$\Rightarrow \quad y + 3 = x - 5$$

$$\Rightarrow \quad x - y - 8 = 0$$

Now, inclination of Y-axis $= \theta = \frac{\pi}{4} + \frac{\pi}{2}$

or $\quad \theta = \frac{3\pi}{4}$

Its slope $= \tan\theta$

$= \tan\frac{3\pi}{4} = -1$

By point-slope form, the equation of Y-axis will be

$$y - (-3) = -1(x - 5)$$

$\Rightarrow \quad y + 3 = -x + 5$

$\Rightarrow \quad x + y - 2 = 0$

So new origin $= (5, -3)$,

Equation of X-axis is: $x - y - 8 = 0$

and equation of Y-axis is: $x + y - 2 = 0$

Example 2. Find new coordinates of $P(4, 5)$ if new origin is $(2, 3)$ and XY-coordinate system is rotated with $\frac{\pi}{4}$ radians anticlockwise from *xy*-coordinate system.

Solution: Here $\quad (x, y) = (4, 5)$

$(h, k) = (2, 3)$

and $\quad \theta = \frac{\pi}{4}$

By the equation of transformation

$x = (X + h)\cos\theta - (Y + k)\sin\theta$ and $\quad y = (X + h)\sin\theta + (Y + k)\cos\theta$

i.e., $\quad 4 = \frac{(X+2)}{\sqrt{2}} - \frac{(Y+3)}{\sqrt{2}}$ $\quad$ i.e., $\quad 5 = \frac{(X+2)}{\sqrt{2}} + \frac{(Y+3)}{\sqrt{2}}$

$\Rightarrow \quad 4\sqrt{2} = X - Y - 1 \quad \ldots(\text{i})$ $\quad \Rightarrow \quad 5\sqrt{2} = X + Y + 5 \quad \ldots(\text{ii})$

Adding equation (i) and (ii), we get

$9\sqrt{2} = 2X + 4$

$\Rightarrow \quad X = \frac{9\sqrt{2}-4}{2}$

By using this value of X in equation (ii), we get

$5\sqrt{2} = \frac{9\sqrt{2}-4}{2} + Y + 5$

$\Rightarrow \quad 5\sqrt{2} - 5 - \frac{(9\sqrt{2}-4)}{2} = y$

$\Rightarrow \quad y = \frac{\sqrt{2}-6}{2}$

So, new coordinates of $P(4, 5)$ are $\left(\frac{9\sqrt{2}-4}{2}, \frac{\sqrt{2}-6}{2}\right)$

9.12.5 Find the angle through which the axes be rotated about the origin so that the product term *xy* is removed from the transformed equations

If we remove *xy*-term from the second degree equation in x and y then the equation is reduced to familiar form of equation of conic.

The following theorem tells how to determine an appropriate rotation of axes to eliminate the *xy*-term of a second degree equation in x and y.

Theorem: If the equation $Ax^2 + By^2 + Hxy + Gx + Fy + C = 0$ is such that $H \neq 0$ and if an XY-coordinate system is obtained by rotating the xy-axes through an angle θ satisfying.

$$\cot 2\theta = \frac{A-B}{H}$$

then in XY-coordinates, the given equation will have the form

$$A'x^2 + B'y^2 + G'x + F'y + C' = 0$$

Example: Identify and sketch the curve $xy = 1$.

Solution: We have $xy = 1$ …(i)

Comparing given equation with $Ax^2 + By^2 + Hxy + Gx + Fy + C = 0$, we get

$A = 0, B = 0$ and $H = 1$

Now, $\cot 2\theta = \frac{A-B}{H} = 0$

$\Rightarrow \quad 2\theta = \frac{\pi}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{4} = 45°$

By the equation of transformations

$x = X\cos\theta - Y\sin\theta$ and

i.e., $x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}$

$y = X\sin\theta + Y\cos\theta$

$\Rightarrow \quad y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$

By substituting these values in equation (i)

We get,

$$\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) = 1$$

$$\Rightarrow \quad \frac{X^2}{2} - \frac{Y^2}{2} = 1$$

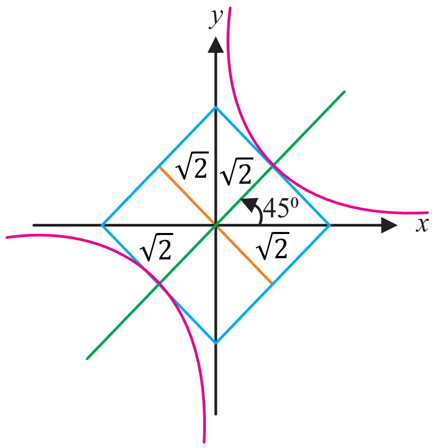


Fig 9.81

This is the equation of rectangular hyperbola with centre at origin and rotation of 45°.

Here $a^2 = 2$ and $b^2 = 2$

So, $c^2 = 4 \Rightarrow c = 2$

Here vertices are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ in XY-coordinate system.

The graph is sketched as shown in Fig. 9.81.

Exercise 9.6

1. For what value of k, the line $y = 2kx$ will be tangent to $2x^2 - 5y^2 = 10$.

2. Find the condition when the line $y = mx + c$ is tangent to $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

3. Find the equation of tangent to the hyperbola $3x^2 - 4y^2 = 12$ when slope is 3.

4. Find the equation of tangent and normal to $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ at (x_1, y_1).

5. Find the equation of tangent and normal to $\frac{x^2}{5} - \frac{y^2}{7} = 1$ at $(2\sqrt{5}, \sqrt{7})$.

6. Find the transformed equation of $\frac{(x-6)^2}{25} + \frac{(y+7)^2}{16} = 1$ when axes are translated with new origin $(6, -7)$.

7. If xy-axes are rotated through given angle θ then find the new coordinates of given point P

(i) $(2, 3), \theta = 60°$ (ii) $(6, 7), \theta = 45°$ (iii) $(-4, 6), \theta = 30°$

8. Find new origin O' and new XY-axes with respect to xy-coordinate system if it is translated 6 units to the left, 5 units up and rotated $\frac{\pi}{6}$ radians anticlockwise.

9. Find new coordinates of $P(4, 5)$ if new origin is $(1, 2)$ and XY-coordinate system is rotated with $\frac{\pi}{6}$ radians anticlockwise from xy-coordinate system.

10. Identity and sketch the curve $xy = 9$.

11. Through which angle the axes be rotated about origin so that the transformed equation of $9x^2 + 12xy + 4y^2 - x - y = 0$ does not contain the term involving XY.

Review Exercise 9

1. Tick the correct option.

(i) If the eccentricity is zero, then the conic is ----------------

(a) parabola (b) ellipse (c) circle (d) hyperbola

(ii) The focus of parabola $x^2 = -16y$ is ------------

(a) $(0, 0)$ (b) $(4, 0)$ (c) $(-4, 0)$ (d) $(0, -4)$

(iii) The latus rectum and vertex of $(y-3)^2 = -8(x+4)$ is ------------

(a) $-8, (3, -4)$ (b) $8, (3, -4)$ (c) $4, (-3, -4)$ (d) $8, (-4, 3)$

(iv) The equation of tangent at $(4, 6)$ to the parabola $y^2 = 9x$ is --------------

(a) $6y = \frac{9}{2}(x+4)$ (b) $6y = 9(x-4)$

(c) $4y = \frac{9}{2}(x+6)$ (d) $3x - 4y + 12 = 0$

(v) The latus rectum of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is -----------

(a) $\frac{5}{32}$ (b) $\frac{32}{5}$ (c) $\frac{50}{4}$ (d) None

(vi) The eccentricity of the conic $\frac{x^2}{5}+\frac{y^2}{4}=1$ is -----------

(a) $\sqrt{5}$ (b) $\frac{1}{\sqrt{5}}$ (c) 5 (d) None

(vii) The centre of ellipse $\frac{(x+5)^2}{10}+\frac{(y-3)^2}{20}=1$ is -----------

(a) $(\sqrt{10},\sqrt{20})$ (b) $(5,3)$ (c) $(-5,3)$ (d) None

(viii) The equation of directrix for the conic $\frac{x^2}{4}+\frac{y^2}{2}=1$ is -----------

(a) $x=\pm\frac{4}{\sqrt{2}}$ (b) $x=\pm\frac{\sqrt{5}}{4}$ (c) $x=\pm\frac{2}{\sqrt{5}}$ (d) $x=\pm 2\sqrt{2}$

(ix) $ax^2+by^2+gx+fy+c=0$ where a,b,g,f and c are real numbers that represents hyperbola if

(a) a and b are non-zero and of same sign

(b) a and b are non-zero and of different sign

(c) either $a=0$ or $b=0$ (d) $a=b=0$

(x) Auxiliary circle of ellipse $\frac{x^2}{6}+\frac{y^2}{5}=1$ is -----------

(a) $x^2+y^2=36$ (b) $x^2+y^2=25$

(c) $x^2+y^2=5$ (d) $x^2+y^2=6$

(xi) The equations of directrices for $\frac{(x-h)^2}{p^2}+\frac{(y-k)^2}{q^2}=1$ are ----------- where $q>p$

(a) $x=\pm\frac{p}{e}$ (b) $x-h=\pm\frac{q}{e}$

(c) $y-k=\pm\frac{q}{e}$ (d) $x-h=\pm\frac{p}{e}$

(xii) The vertices of hyperbola $\frac{x^2}{9}+\frac{y^2}{16}=1$ are -----------

(a) $(\pm 5,0)$ (b) $(0,\pm 5)$ (c) $(0,\pm 4)$ (d) $(\pm 4,0)$

(xiii) Conjugate hyperbola to $\frac{x^2}{5}-\frac{y^2}{6}=1$ is -----------

(a) $\frac{x^2}{5}-\frac{y^2}{6}=1$ (b) $\frac{y^2}{6}-\frac{x^2}{5}=1$ (c) $\frac{x^2}{6}-\frac{y^2}{5}=1$ (d) None

(xiv) The eccentricity of rectangular hyperbola is ------------

(a) 1 (b) 2 (c) $\sqrt{3}$ (d) $\sqrt{2}$

(xv) The equation of tangent to $\frac{x^2}{6}+\frac{y^2}{5}=1$ at $(\sqrt{6},0)$ is ------------

(a) $x=6$ (b) $x=\sqrt{6}$ (c) $y=\sqrt{6}$ (d) None

(xvi) For what value of k, $y=k$ is tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ is -----------

(a) ± 3 (b) ± 5 (c) $\pm\frac{7}{5}$ (d) None

(xvii) The equation of tangent to $\frac{x^2}{16}-\frac{y^2}{9}=1$ with slope 2 is ----------------

(a) $y=2x\pm\sqrt{23}$ (b) $y=2x\pm\sqrt{41}$

(c) $x=2y\pm\sqrt{23}$ (d) $y=2x\pm\sqrt{55}$

(xviii) The equation of $xy=c^2$ represents

(a) parabola (b) ellipse (c) hyperbola (d) circle

(xix) If origin is shifted to $(2,3)$ then coordinates of $(5,6)$ are -----------

(a) $(2,2)$ (b) $(3,3)$ (c) $(4,4)$ (d) None

(xx) If xy-coordinate system is rotated at angle of $\frac{\pi}{4}$ transformation for abscissa is

(a) $x=\frac{x'}{2}-\frac{y'}{\sqrt{2}}$ (b) $x=\frac{x'}{\sqrt{2}}-\frac{y'}{\sqrt{2}}$

(c) $x=\frac{x'}{\sqrt{3}}-\frac{y'}{\sqrt{3}}$ (d) None

2. Find the foci, vertices and directrices for the conic

(a) $\frac{(x-5)^2}{25}+\frac{(y+3)^2}{16}=1$ (b) $\frac{(x+4)^2}{9}-\frac{(y+7)^2}{16}=1$

3. Find the condition of tangency the line $y=x+c$ is tangent to the conic

(i) $y^2=10x$ (ii) $2x^2+3y^2=6$

(iii) $5x^2-7y^2=35$

4. Find transformed equation of $\frac{(x+5)^2}{7}-\frac{(y-3)^2}{5}=1$ when new origin is $(-5,3)$.

5. If xy-axes are rotated through angle θ, find coordinates of P if new coordinates is $(-2,7)$, $\theta=45°$.