Unit 1

REAL AND COMPLEX NUMBERS

Student Learning Outcomes (SLOs)

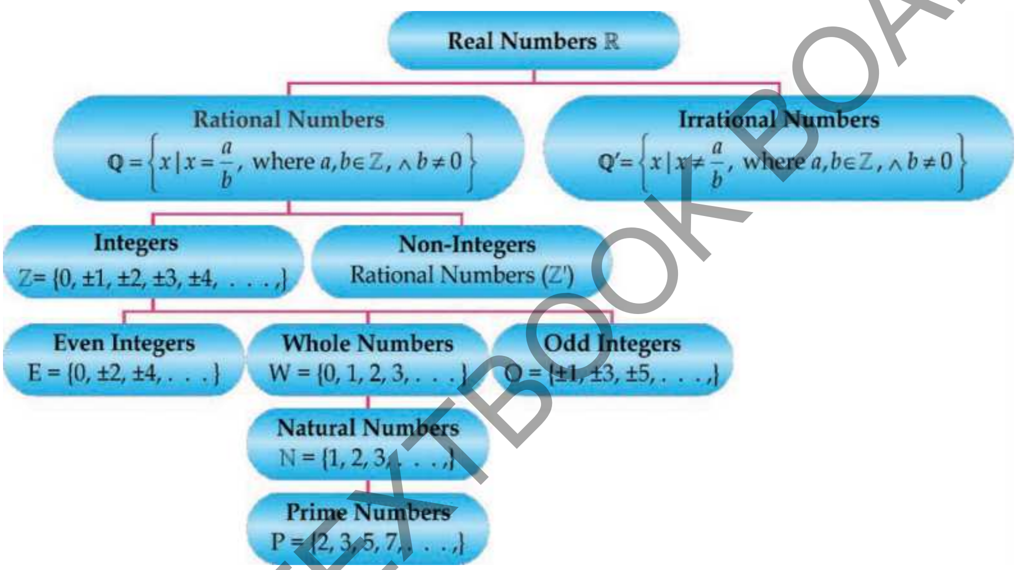
After completing this unit, students will be able to:

- Recall the set of real numbers as the union set of rational and irrational numbers.
- Represent real numbers on the number line.
- Demonstrate a number with terminating and non–terminating recurring decimal on the number line.
- Distinguish the decimal representation of rational and irrational numbers.
- Know the properties of real numbers
- Identify radicals and radicands.
- Differentiate between radical and exponential forms of an expression.
- Transform an expression given in radical form to an exponent form and vice versa.
- Recall base, exponent and value.
- Apply the laws of exponents to simplify expressions with real exponents.
- Elucidate, then define a complex number z represented by an expression of the form (a, b) or $z = a + i\,b$, where a is real and b is imaginary part and here $i = \sqrt{-1}$
- Recognize $\boldsymbol{a}$ as real part and $\boldsymbol{b}$ as imaginary part of $z = a + ib$ or $z = (a, b)$
- Define conjugate of a complex number
- Know the condition of equality of complex numbers.
- Carry–out basic operations (i.e. addition, subtraction, multiplication and division) on complex numbers.

Introduction

In pervious classes we have learned various kinds of numbers such as natural numbers (counting numbers), whole numbers, integers, rational numbers etc.

All these numbers are contained in the set of real numbers. Hence classification of real numbers is given below:



1.1 Real Numbers

1.1.1 Recall the set of real numbers as the union set of rational and irrational numbers.

The set of real numbers is the union of the set of rational and irrational numbers. i.e., $\mathbb{R} = Q \cup Q'$

We have already learned about rational and irrational numbers. Real numbers have many properties as the properties of rational numbers.

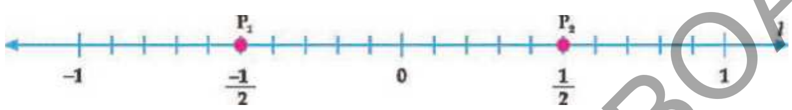
1.1.2 Represent Real Numbers on the Number Line

In the previous classes we have already studied whole numbers and integers and their representation on a number line. Similarly we can represent real numbers on number line.

Let us see the following examples.

Example 01 Represent the numbers $-\frac{1}{2}$ and $\frac{1}{2}$ on the number line l

Solution:

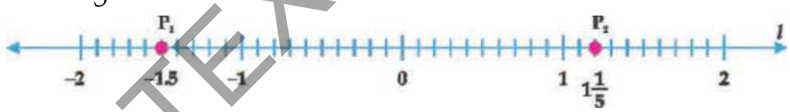


Thus, in the above figure the point P_1 represents number $-\frac{1}{2}$ and the point P_2 represents $\frac{1}{2}$.

Example 02 Represent –1.5 and $1\frac{1}{5}$ on the number line.

Solution:

Similar in the figure, point P_1 represents number –1.5 and P_2 represent number $1\frac{1}{5}$.



1.1.3 Demonstrate a Number with Terminating and Non–Terminating Recurring Decimal on the Number Line

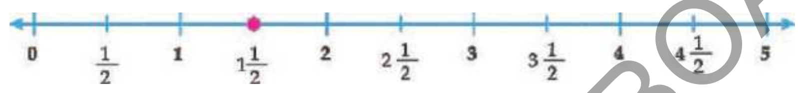
In order to locate a number with terminating and non–terminating recurring decimal on the number line, the points associated with the rational numbers $\frac{a}{b}$ and where a, b are positive integers, we sub–divide each unit length into b equal parts. Then the a^{th} point of division to the right of the origin represents $\frac{a}{b}$ and that to the left of the origin at the same distance represents $-\frac{a}{b}$.

Example 01

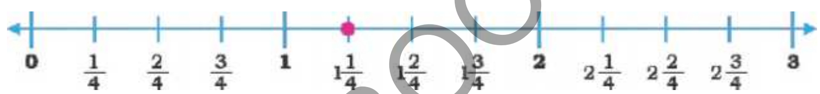
Show the following terminating decimal fractions on the number line.

i. $\frac{3}{2}$ ii. $\frac{5}{4}$

i. $\frac{3}{2}=1\frac{1}{2}$



ii. $\frac{5}{4}=1\frac{1}{4}$

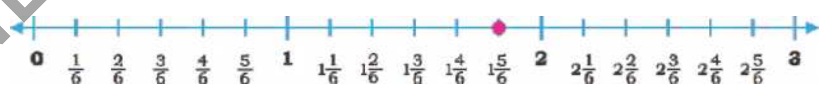


Example 02

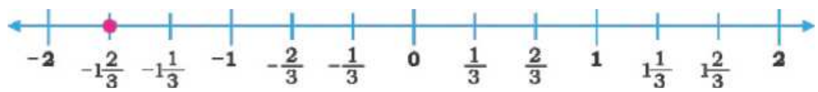
Show the following non terminating recurring decimal fractions on number line.

i. $\frac{11}{6}$ ii. $-\frac{5}{3}$

i. $\frac{11}{6}=1\frac{5}{6}$



iii. $-\frac{5}{3}=-1\frac{2}{3}$



1.1.4 Distinguish the Decimal Representation of Rational and Irrational Numbers.

When we represent rational numbers in the decimal form then two types of decimal fractions are possible i.e. terminating and non–terminating recurring decimal fractions, while Irrational numbers are represented as non- terminating non-recurring decimal fraction. We represent them in below table.

S.No	Number	Remarks
1.	$\frac{1}{2}=0.5$	Terminating decimal fraction
2.	$\frac{1}{4}=0.25$	Terminating decimal fraction
3.	$\frac{1}{3}=0.333...$	Non-terminating recurring decimal fraction
4.	$\frac{9}{11}=0.818181...$	Non-terminating recurring decimal fraction
5.	$\sqrt{2}=1.414213...$	Non-terminating non-recurring decimal fraction
6.	$\sqrt{3}=1.73205...$	Non-terminating non-recurring decimal fraction

Exercise 1.1

1. Identify the following numbers as rational and irrational numbers and also write each one in separate column.

(i) $\frac{1}{5}$ **(ii)** $\frac{\sqrt{2}}{8}$ **(iii)** $\frac{5}{\sqrt{6}}$ **(iv)** $\frac{2}{8}$ **(v)** $\frac{1}{\sqrt{3}}$ **(vi)** $\sqrt{8}$

(vii) e **(viii)** π **(ix)** $\sqrt{5}$ **(x)** $\frac{22}{3}$ **(xi)** $\frac{1}{\pi}$ **(xii)** $\frac{11}{12}$

2. Convert the following into decimal fractions. Also indicate them as terminating and non–terminating decimal fractions.

(i) $\frac{5}{8}$ **(ii)** $\frac{4}{18}$ **(iii)** $\frac{1}{15}$ **(iv)** $\frac{49}{8}$ **(v)** $\frac{207}{15}$ **(vi)** $\frac{50}{76}$

3. Represent the following rational numbers on number line.

(i) $\frac{8}{10}$ **(ii)** $-\frac{8}{10}$ **(iii)** $1\frac{1}{4}$ **(iv)** $-1\frac{1}{4}$ **(v)** $\frac{2}{3}$ **(vi)** $-\frac{2}{3}$

4. Can you make a list of all rational numbers between 1 and 2?

5. Give reason, why pi (π) is an irrational number?

6. Tick (✓) the correct statements.

(i) $\frac{5}{7}$ is an example of irrational number.

(ii) π is an irrational number.

(iii) 0.31591… is an example of non–terminating and non–repeating decimal fraction.

(iv) $0.12\bar{3}$ is an example of recurring decimal fraction.

(v) $\frac{1}{3}, \frac{2}{3}$ are lying between 0 and 1.

(vi) $\frac{1}{\sqrt{3}}$ is an example of rational number.

1.2 Properties of Real Numbers.

In real numbers there exist properties with respect to addition and multiplication. For real number a, b the sum is $a + b$ and product is written as $a.b$ or $a \times b$ or simply ab.

1.2.1 Know the Properties of Real Numbers

(a) Properties of Real Numbers with respect to Addition

(i) Closure Property:

Sum of any two real numbers is again a real number.

i.e. $\forall a, b \in \mathbb{R} \Rightarrow a + b \in \mathbb{R}$ is called closure property w.r.t addition.

e.g. (i) $5, 7 \in \mathbb{R} \Rightarrow 5 + 7 = 12 \in \mathbb{R}$

(ii) $\frac{4}{5}, \frac{3}{4} \in \mathbb{R} \Rightarrow \frac{4}{5} + \frac{3}{4} = \frac{16 + 15}{20} = \frac{31}{4} \in \mathbb{R}$

(ii) Commutative Property:

For any two real numbers a and b

$$a+b=b+a$$

is called commutative property w.r.t addition

e.g. (i) $3+7=7+3$ (ii) $\sqrt{5}+\sqrt{6}=\sqrt{6}+\sqrt{5}$.

(iii) Associative Property:

For any three real numbers a and b and c such that

$$(a+b)+c=a+(b+c)$$

is called associative property w.r.t addition.

e.g. $(4+5)+6=4+(5+6)$

(iv) Additive Identity:

There exists a number $0\in\mathbb{R}$ such that

$$a+0=a=0+a, \quad \forall a\in\mathbb{R}$$

'0' is called additive identity

e.g. $3+0=3=0+3, \ \frac{7}{8}+0=\frac{7}{8}=0+\frac{7}{8}$, etc

(v) Additive Inverse:

For each $a\in\mathbb{R}$, there exist $-a\in\mathbb{R}$ such that $a+(-a)=0=(-a)+a$

so, $-a$ and a are additive inverses of each other.

e.g. $6+(-6)=0=(-6)+6=0$

Here 6 and -6 are additive inverses of each other.

(b) Properties of Real Numbers with respect to Multiplication

(i) Closure Property:

The product of any two real numbers a and b is again a real number.

i.e., $a,b\in\mathbb{R}\Rightarrow ab\in\mathbb{R}$, is called closure properly w.r.t multiplication

e.g. (i) $5,7\in\mathbb{R}\Rightarrow(5)(7)=35\in\mathbb{R}$

(ii) $\frac{3}{5},\frac{6}{7}\in\mathbb{R}\Rightarrow\left(\frac{3}{5}\right)\left(\frac{6}{7}\right)=\frac{18}{35}\in\mathbb{R}$, etc

(ii) Commutative Property:

For any two real numbers a and b

$ab = ba$ is called commutative property w.r.t multiplication.

e.g. (i) $\sqrt{3}, \sqrt{5} \in \mathbb{R} \Rightarrow (\sqrt{3})(\sqrt{5}) = (\sqrt{5})(\sqrt{3})$

(ii) $3, 4 \in \mathbb{R} \Rightarrow 3\times4 = 4\times3$ etc.

(iii) Associative Property:

For any three real numbers a,b and c

$(ab)c = a(bc)$ is called associative property w.r.t multiplication.

e.g. (i) $4,5,6 \in \mathbb{R}$, then $(4\times5)\times6 = 4\times(5\times6)$,

(ii) $\frac{2}{5}, 4, \sqrt{3} \in \mathbb{R}$, then $(\frac{2}{5}\times4)\times\sqrt{3} = \frac{2}{5}\times(4\times\sqrt{3})$, etc.

(iv) Multiplicative Identity:

For any real number a there exist a number $1 \in \mathbb{R}$

$a\times1 = 1\times a = a$, '1' is called multiplicative identity.

e.g. $1\times3 = 3\times1 = 3$, $\frac{3}{5}\times1 = 1\times\frac{3}{5} = \frac{3}{5}$, etc.

(v) Multiplicative Inverse:

For each $a \in \mathbb{R} (a \neq 0)$ there exists an element $\frac{1}{a}$ or $a^{-1} \in \mathbb{R}$

$a\times\frac{1}{a} = \frac{1}{a}\times a = 1$, thus $\frac{1}{a}$ and a are the multiplicative inverses of each other.

e.g. $3\times\frac{1}{3} = 1 = \frac{1}{3}\times3$

Here 3 and $\frac{1}{3}$ are multiplicative inverses of each other.

(c) Distributive Property of Multiplication over Addition

For any three real numbers a,b,c such that

(i) $a(b+c) = ab+ac$, it is called Distributive Property of multiplication over addition. (Left Distributive Property)

(ii) $(a+b)c = ac + bc$, it is called distributive property of multiplication over addition. (Right Distributive Property)

e.g. $3(5+7) = 3\times5 + 3\times7$, (Left Distributive Property)

$(3+7)2 = 3\times2 + 7\times2$, (Right Distributive Property)

Note: $a(b-c) = ab-ac$ is the distributive property of multiplication over subtraction.

(d) Properties of Equality of Real Numbers

Following are the properties of equality of real numbers.

(i) **Reflexive Property**
If $a\in\mathbb{R}$ then $a = a$.

(ii) **Symmetric Property**
If $a,b\in\mathbb{R}$ then $a = b \Leftrightarrow b = a$.

(iii) **Transitive Property**
If $a,b,c\in\mathbb{R}$ then, $a = b$ and $b = c \Leftrightarrow a = c$.

(iv) **Additive Property**
If $a,b,c\in\mathbb{R}$then, $a = b \Leftrightarrow a+c = b+c$.

(v) **Multiplicative Property**
If $a,b,c\in\mathbb{R}$ such that, $a = b$ then $ac = bc$.

(vi) **Cancellation Property for Addition**
If $a,b,c\in\mathbb{R}$, if $a+c = b+c$ then $a = b$

(vii) **Cancellation property for multiplication**
If $a,b,c\in\mathbb{R}$ and $c \neq 0$ if $ac = bc$ then, $a = b$

(e) Properties of Inequalities of Real Numbers.

Following are the properties of inequalities of real numbers.

(i) **Trichotomy Property**
If $a,b,c\in\mathbb{R}$ then $a > b$ or $a < b$ or $a = b$.

(ii) **Transitive Property**
If $a,b,c\in\mathbb{R}$ then

(a) $a < b$ and $b < c \Rightarrow a < c$,

(b) $a > b$ and $b > c \Rightarrow a > c$.

(iii) Additive Property

If $a,b,c \in \mathbb{R}$ then

(a) $a < b \Rightarrow a+c < b+c,$

(b) $a > b \Rightarrow a+c > b+c.$

(iv) Multiplicative Property

If $a,b,c \in \mathbb{R}$ and $c > 0$, then

(a) $a > b \Rightarrow ac > bc,$

(b) $a < b \Rightarrow ac < bc,$

similarly, if $c < 0$ then,

(a) $a > b \Rightarrow ac < bc$

(b) $a < b \Rightarrow ac > bc$

(v) Reciprocative Property

If $a,b \in \mathbb{R}$ and a, b are of same sign then,

(a) If $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$ and if $\frac{1}{a} < \frac{1}{b} \Rightarrow a > b$

(b) If $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$ and if $\frac{1}{a} > \frac{1}{b} \Rightarrow a < b$

(vi) Cancellation property

If $a,b,c \in \mathbb{R}$

(a) $a+c > b+c \Rightarrow a > b$

(b) $a+c < b+c \Rightarrow a < b$

similarly, (a) $ac > bc \Rightarrow a > b$, where $c > 0$

(b) $ac < bc \Rightarrow a < b$, where $c > 0$

Exercise 1.2

1. Recognize the properties of real numbers used in the following:

(i) $\frac{1}{2}+\frac{2}{3}=\frac{2}{3}+\frac{1}{2}$

(ii) $\frac{4}{3}+\left(1\frac{1}{3}+\frac{2}{3}\right)=\left(\frac{4}{3}+1\frac{1}{3}\right)+\frac{2}{3}$

(iii) $9\times\left(\frac{10}{9}+\frac{20}{9}\right)=\left(9\times\frac{10}{9}\right)+\left(9\times\frac{20}{9}\right)$

(iv) $\left(\frac{4}{5}+\frac{5}{7}\right)\times\frac{7}{8}=\left(\frac{4}{5}\times\frac{7}{8}\right)+\left(\frac{5}{7}\times\frac{7}{8}\right)$

(v) $\left(\frac{7}{5}-\frac{3}{5}\right)\times\frac{10}{15}=\left(\frac{7}{5}\times\frac{10}{15}\right)-\left(\frac{3}{5}\times\frac{10}{15}\right)$

(vi) $\frac{d}{c}\times\frac{e}{f}=\frac{e}{f}\times\frac{d}{c}$

(vii) $11\times(15\times21)=(11\times15)\times21$

(viii) $\frac{2}{11}\times\frac{11}{2}=\frac{11}{2}\times\frac{2}{11}=1$

(ix) $\left(\frac{3}{5}\right)+\left(-\frac{3}{5}\right)=\left(-\frac{3}{5}\right)+\left(\frac{3}{5}\right)=0$

(x) $\left(\frac{a}{b}\right)\times\left(\frac{b}{a}\right)=\left(\frac{b}{a}\right)\times\left(\frac{a}{b}\right)=1$

(xi) $\frac{15}{10}\times\left(\frac{8}{5}-\frac{4}{10}\right)=\left(\frac{15}{10}\times\frac{8}{5}\right)-\left(\frac{15}{10}\times\frac{4}{10}\right)$

(xii) $\frac{\sqrt{2}}{3}\times\frac{3}{\sqrt{2}}=\frac{3}{\sqrt{2}}\times\frac{\sqrt{2}}{3}=1$

2. Fill the correct real number in the following to make the real numbers property correct.

(i) $\frac{\sqrt{2}}{5}+\frac{3}{\sqrt{6}}=\frac{\square}{\sqrt{6}}+\frac{\sqrt{2}}{5}$

(ii) $\frac{7}{10}+\left(\frac{70}{\square}+\frac{16}{33}\right)=\frac{7}{\square}+\left(\frac{\square}{10}+\frac{16}{\square}\right)$

(iii) $\frac{99}{50}\times\frac{50}{99}=\square$

(iv) $\left(\frac{59}{95}\right)\times\left(\frac{95}{59}\right)=\square$

(v) $(-21)+(\square)=0$

(vi) $\frac{5}{8}\times\left(\frac{2}{3}+\frac{5}{7}\right)=\left(\frac{\square}{\square}\times\frac{2}{3}\right)+\left(\frac{5}{8}\times\frac{\square}{\square}\right)$

3. Fill the following blanks to make the property correct/true.

(i) $5<8$ and $8<10\Rightarrow$ ____ $<$ ____

(ii) $10>8$ and $8>5\Rightarrow$ ____ $<$ ____

(iii) $3<6\Rightarrow 3+9<$ ___

(iv) $4<6\Rightarrow 4+8<$ ___

(v) $8>6\Rightarrow 6+8>$ ___

4. Fill the following blanks which make the property correct/true:

(i) $5 < 7 \Rightarrow 5 \times 12 <$ ___ $\times$ ___

(ii) $7 > 5 \Rightarrow 7 \times 12 >$ ___ $\times$ ___

(iii) $6 > 4 \Rightarrow 6 \times (-7)$ ____ $4 \times (-7)$

(iv) $2 < 8 \Rightarrow 2 \times (-4)$ ____ $8 \times (-4)$

5. Find the additive and multiplicative inverse of the following real numbers.

(i) 3 **(ii)** -7 **(iii)** 0.3 **(iv)** $\frac{-\sqrt{5}}{5}$ **(v)** $\frac{9}{\sqrt{12}}$ **(vi)** 0

1.3 Radicals and Radicands.

1.3.1 Identify radicals and radicands

Let $n \in Z^+$ (Set of Positive integers) and $n > 1$,

also let $a \in \mathbb{R}$, then for any positive real number x,

such that $x^2 = a \Rightarrow x = a^{\frac{1}{2}} \Rightarrow x = \sqrt{a}$ (square root of a)

similarly, $x^3 = a \Rightarrow x = a^{\frac{1}{3}} \Rightarrow x = \sqrt[3]{a}$ (cube root of a)

$x^4 = a \Rightarrow x = a^{\frac{1}{4}} \Rightarrow x = \sqrt[4]{a}$ (4th root of a)

In general, $x^n = a \Rightarrow x = a^{\frac{1}{n}} \Rightarrow x = \sqrt[n]{a}$ (nth root of a)

In $\sqrt[n]{a}$, 'a' is called radicand and 'n' is called the index.

> The symbol $\sqrt{}$ is called radical sign.

1.3.2 Differentiate between Radical and Exponential forms of an Expression

As we have studied that $x = \sqrt[n]{a}$ is in a radical form.

Similarly, $a^{\frac{1}{3}}, a^{\frac{2}{3}}, a^{\frac{3}{2}}, a^{\frac{1}{n}}, a^{\frac{m}{n}}$ are some examples of exponential form.

Remember that

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

Here, $\sqrt[n]{a}$ is in radical form and $a^{\frac{1}{n}}$ in exponential form.

Here are some properties of square root

For all $a, b \in \mathbb{R}^+ \wedge m, n \in Z$

Then,

(i) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

(ii) $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

(iii) $\dfrac{a}{\sqrt{a}} = \sqrt{a}$

(iv) $\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a$

(v) $\dfrac{\sqrt{a}}{\sqrt{a}} = 1$

(vi) $m\sqrt{a} \pm n\sqrt{a} = (m \pm n)\sqrt{a}$

(vii) $\sqrt{\left(\dfrac{a}{b}\right)^{-n}} = \sqrt{\left(\dfrac{b}{a}\right)^{n}}$

Similarly,

(i) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$

(ii) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

(iii) $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

(iv) $\sqrt[mn]{a^n} = a^{\frac{n}{nm}} = a^{\frac{1}{m}}$

(v) $\sqrt[mn]{a} = \sqrt[m]{a^{\frac{1}{n}}} = \sqrt[m]{\sqrt[n]{a}} = a^{\frac{1}{mn}}$

(vi) $\sqrt[n]{\sqrt[n]{a}} = \sqrt[n^2]{a} = a^{\frac{1}{n^2}}$

(vii) $\sqrt[n]{a^n} = a$

(viii) $\dfrac{\sqrt[n]{a^n}}{\sqrt[n]{a^n}} = 1$

1.3.3 Transform an Expression given in Radical Form to an Exponent Form and vice versa

The properties of radicals and exponential forms are very useful when we simplify the expressions involving radicals and exponents.

Example 01 Transform the following radical expressions into exponential forms.

(i) $\sqrt{\frac{2}{3}}$ (ii) $\sqrt[3]{18}$ (iii) $\sqrt[5]{\frac{5}{7}}$ (iv) $\sqrt[9]{\left(\frac{x}{y}\right)^2}$ (v) $\sqrt[4]{(ab)^3}$

Solutions:

(i) $\sqrt{\frac{2}{3}} = \left(\frac{2}{3}\right)^{\frac{1}{2}}$ (ii) $\sqrt[3]{18} = (18)^{\frac{1}{3}}$ (iii) $\sqrt[5]{\frac{5}{7}} = \left(\frac{5}{7}\right)^{\frac{1}{5}}$

(iv) $\sqrt[9]{\left(\frac{x}{y}\right)^2} = \left(\frac{x}{y}\right)^{\frac{2}{9}}$ (v) $\sqrt[4]{(ab)^3} = (ab)^{\frac{3}{4}}$.

Example 02 Transform the following exponential forms into radical expressions.

(i) $\left(\frac{5}{7}\right)^{\frac{1}{3}}$ (ii) $(12)^{\frac{n}{2}}$ (iii) $(-7)^{\frac{3}{4}}$ (iv) $\left(\frac{y}{x}\right)^{-\frac{2}{5}}$ (v) $\left(-\frac{x}{y}\right)^{\frac{m}{n}}$

Solutions:

(i) $\left(\frac{5}{7}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{5}{7}}$ (ii) $(12)^{\frac{n}{2}} = \sqrt{(12)^n}$ (iii) $(-7)^{\frac{3}{4}} = \sqrt[4]{(-7)^3}$

(iv) $\left(\frac{y}{x}\right)^{-\frac{2}{5}} = \sqrt[5]{\left(\frac{y}{x}\right)^{-2}} = \sqrt[5]{\left(\frac{x}{y}\right)^2}$ (v) $\left(-\frac{x}{y}\right)^{\frac{m}{n}} = \sqrt[n]{\left(-\frac{x}{y}\right)^m}$

Exercise 1.3

1. **Write the base and exponent of the following:**

(i) 3^4 (ii) $\left(\frac{1}{2}\right)^3$ (iii) $(-30)^{20}$

2. **Identify radicand and index in the following:**

(i) $\sqrt[3]{5}$ (ii) $\sqrt[4]{\frac{x}{y}}$ (iii) $\sqrt[5]{x^2yz}$

3. Transform the following into exponential forms.

(i) $\sqrt{\left(\frac{3}{4}\right)}$ (ii) $\sqrt{\left(\frac{x}{y}\right)^5}$ (iii) $\sqrt[3]{\left(\frac{y}{x}\right)^{-5}}$

(iv) $\sqrt[3]{(yz)^7}$ (v) $\sqrt[9]{27}$ (vi) $\sqrt[3]{(-64)^2}$

(vii) $\sqrt[3]{\left(\frac{1}{2}\right)^m}$ (viii) $\sqrt[5]{(xy)^3}$ (ix) $\sqrt[3]{\sqrt{\frac{4}{3}}}$

4. Transform the following into radical forms.

(i) $\left(5^3\right)^{\frac{1}{7}}$ (ii) $(ab^{-2})^{\frac{1}{3}}$ (iii) $\left[\left(\frac{5}{7}\right)^3\right]^{\frac{5}{7}}$

(iv) $\left(\frac{b}{a}\right)^{-\frac{m}{2}}$ (v) $\left[\left(\frac{11}{13}\right)\left(\frac{12}{13}\right)\right]^{\frac{1}{5}}$

1.4 Laws of Exponents/Indices:

Laws of exponents or indices are important in many fields of mathematics.

1.4.1 Recall Base, Exponent and value of Power

Consider an exponential form a^n here, **'a'** is called the base and **'n'** is called exponent or index i.e., read as a to the nth power. The result of a^n, where $a \in \mathbb{R}$ is called its value.

1.4.2 Apply the Laws of Exponents to Simplify Expressions with Real Exponents

The following laws of exponents are useful to simplify the expressions.

(i) Law of Product of Powers

(a) If $a, b \in \mathbb{R}$ and $x, y \in Z^+$

Then, $a^x \times a^y = a^{x+y}$

Some examples based on this law are given below:

(a) $a^2 \times a^3 = a^{2+3} = a^5$ (b) $3 \times 3^5 = 3^{1+5} \times 3^6 = 729$

(ii) Law of Power of Power

If $a \in \mathbb{R}$ and $x, y \in Z^+$, then $\left(a^x\right)^y = a^{xy}$

Some examples based on this law are given below:

(a) $(5^2)^4 = 5^{2\times 4} = 5^8$

(b) $\left\{\left(\frac{6}{11}\right)^4\right\}^3 = \left(\frac{6}{11}\right)^{4\times 3} = \left(\frac{6}{11}\right)^{12}$

(c) $\left\{\left(-\frac{3}{4}\right)^3\right\}^3 = \left(-\frac{3}{4}\right)^{3\times 3} = \left(-\frac{3}{4}\right)^9 = -\left(\frac{3}{4}\right)^9$

(iii) Law of Power of a Product

For all $a, b, \in \mathbb{R}$ and $n \in Z^+$,

Then, $(a\times b)^n = a^n \times b^n$

Following examples are based on this law:

(a) $(xy)^3 = x^3y^3$ (b) $\left\{\left[\frac{8}{9}\right]\left[\frac{7}{11}\right]\right\}^3 = \left(\frac{8}{9}\right)^3\left(\frac{7}{11}\right)^3$

(iv) Law of Power of a Quotient

For all $a, b, \in \mathbb{R}$ and $n \in Z^+$, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where $b \neq 0$

The following examples based on this law are given below:

(a) $\left(\frac{5}{8}\right)^3 = \frac{5^3}{8^3}$ (b) $\left(\frac{f}{g}\right)^4 = \frac{f^4}{g^4}$, $g\neq 0$

(v) Law of quotient of power

If $a \in \mathbb{R}$, $a \neq 0$ and $m, n \in Z^+$, then,

$$\frac{a^m}{a^n} = a^{m-n}, \text{ if } m > n$$

$$= \frac{1}{a^{n-m}}, \text{ if } n > m,$$

If $m = n$,

then, $a^{m-n} = a^{m-m} = a^0 = 1$

Similarly, $\frac{a^m}{a^n} = \frac{a^m}{a^m} = \frac{a^n}{a^n} = 1$

The following examples based on this law are given below:

(a) $\frac{3^5}{3^2} = 3^{5-2} = 3^3 = 27$

(b) $\frac{7^3}{7^5} = \frac{1}{7^{5-3}} = \frac{1}{7^2} = \frac{1}{49}$

Remember that:

$(-a)^n = a^n$, if n is an even exponent.
$= -a^n$, if n is an odd exponent.

Remember that:

If the power of a non-zero real number is zero then its value is equal to 1. For example: $3^0 = 1$.

Exercise 1.4

1. Simplify the following:

(i) $\frac{3^5}{3^2}$ (ii) $\frac{2^4\cdot 5^3}{10^2}$ (iii) $\frac{(a+b)^2\cdot(c+d)^3}{(a+b)\cdot(c+d)^2}$

2. Simplify using law of exponent:

(i) $\left(\frac{1}{3}\right)^4\times\left(\frac{1}{3}\right)^5$ (ii) $\left(\frac{3}{4}\right)^5\times\left(\frac{3}{4}\right)^2$ (iii) $\left(-\frac{4}{5}\right)^3\times\left(-\frac{4}{5}\right)^5$

(iv) $(-3\times 5^2)^3$ (v) $[3\times(-4)^2]^3$ (vi) $\left(-\frac{a}{bc}\right)^5\times\left(-\frac{a}{bc}\right)^4$

(vii) $\left(-\frac{c}{d}\right)^2\left(-\frac{c}{d}\right)^3\left(-\frac{c}{d}\right)^5$ (viii) $mn^2t^4n^3m^5t^7$ (ix) $a^2c^5b^2a^3c^3b^4a^4$

(x) $10x^4y^6x^2z^2y^3$ (xi) $\frac{3}{5}r^3s^2r^7t^2s^2$ (xii) $-\left(-\frac{l}{m}\right)^4\left(-\frac{l}{m}\right)^5\left(-\frac{l}{m}\right)^3$

3. Simplify using the law of exponent:

(i) $(5^2)^3$ (ii) $\{(xy)^3\}^5$ (iii) $\{(-4)^2\}^5$

(iv) $\{(-3)^3(-4)^2\}^3$ (v) $\left(\frac{b^2}{5}\right)^3$ (vi) $\left\{\left(-\frac{4}{9}\right)^2\right\}^3$

(vii) $\{(z^3)^2\}^4$ (viii) $\{(mm^2m^3m^4)^2\}^5$ (ix) $-[(-0.1)^2\,(-0.1)^3\,(-0.1)^4]^2$

1.5 Complex Numbers

1.5.1 Elucidate, then define a complex number z represented by an expression of the form (*a*,*b*) or *z* = *a* + *ib*, where a is real part and b is imaginary part.

We know that the square of real number is non-negative. So the solution of the equation $x^2+1=0$ does not exist in $\mathbb{R}$. To overcome this inadequacy of real number, mathematicians introduced a new number $\sqrt{-1}$, imaginary unit and denoted it by the letter i (iota) having the property that $i^2=-1$. Obviously i is not real number. It is a new mathematical entity that enables us to find the solution of every algebraic equation of the type $x^2+a=0$ where $a>0$. Numbers like $\sqrt{-1}=i, \sqrt{-5}=\sqrt{5}i, \sqrt{-49}=7i$ are called pure imaginary number.

Definition of Complex Number

A number of the form $a+ib$ where a and b are real numbers and i is an imaginary unit i.e. $i=\sqrt{-1}$ is called a complex number and it is denoted by z. e.g. $z=3+4i$ is an complex number.

The complex number $a+ib$ can be written in order pair form (a,b) such as $5+8i=(5,8)$.

1.5.2 Recognize *a* as real part and *b* as imaginary part of $z=a+ib$

In the complex number $z=a+ib$, "a" is the real part of complex number and "b" is the imaginary part of complex number. The real part of complex number is denoted by Re(z) and its imaginary part is denoted by Im(z).

Example Recognize real and imaginary parts for the given complex number.

$z=3-2i$

Here, Re(z)= a = 3 and Im(z)= b = –2

1.5.3 Define conjugate of a complex number

Conjugate of z is denoted by $\bar{z}$ i.e.,

If $z= a + ib$, then $\bar{z}= a-ib$ or If $z=(a,b)$, then $\bar{z}=(a,-b)$

and, if $z = a-ib$, then $\bar{z}= a+ib$ or and If $z=(a,-b)$, then $\bar{z}=(a,b)$

In conjugate we just change the sign of imaginary part.

Note: If any complex number z, $\overline{(\bar{z})} = z$

Example Find the conjugate of the following complex numbers.

(i) $3 + 4i$ (ii) $\left(-\frac{4}{5},\frac{5}{4}\right)$

Solutions:

(i) Let $z_1=3+4i$

then $\bar{z}_1=\overline{3+4i}$

$\bar{z}_1=3-4i$

(ii) Let $z_2=\left(-\frac{4}{5},-\frac{5}{4}\right)$

$\bar{z}_2=\overline{\left(-\frac{4}{5},-\frac{5}{4}\right)}$

$\bar{z}_2=\left(-\frac{4}{5},\frac{5}{4}\right)$

1.5.4 Know the condition of equality of complex numbers

Two complex numbers are said to be equal if and only if, they have same real parts and same imaginary parts. i.e.

$\forall a,b,c,d \in \mathbb{R}$, such that $a + ib = c + id$, iff $a = c$ and $b = d$.

Example 01 If $4x + 3yi = 16 + 9i$, find x and y.

Solution: Given that

$4x + 3yi = 16 + 9i \Rightarrow 4x = 16$ and $3y = 9$,

$\Rightarrow \frac{4x}{4} = \frac{16}{4}$ and $\frac{3y}{3} = \frac{9}{3}, \Rightarrow x = 4$ and $y = 3$.

Example 02 If $x^2 + iy^2 = 25 + i36$, find x and y.

Solution: Given that

$$x^2 + y^2 i = 25 + 36i \Rightarrow x^2 = 25 \text{ and } y^2 = 36,$$

$$x = \pm\sqrt{25} \text{ and } y = \pm\sqrt{36} \Rightarrow x = \pm 5 \text{ and } y = \pm 6$$

Exercise 1.5

1. **Write the following complex numbers in the form of $a+ib$ for the given ordered pairs.**

(i) (1,2) (ii) (2,2) (iii) (3,4)

(iv) (–1,1) (v) (–2,2) (vi) (–3,4)

2. **Identify real and imaginary parts for the following complex numbers.**

(i) $1+2i$ (ii) $9i+4$ (iii) $(-5,6)$

(iv) $-1-i$ (v) $\left(-\frac{3}{4}\right)-\left(-\frac{4}{5}\right)i$ (vi) $2i-1$

3. **Find the conjugate of the following complex numbers.**

(i) $3+2i$ (ii) $(4,9)$ (iii) $(-1,1)$

(iv) $1-i$ (v) $\left(-\frac{3}{4}\right)+\left(-\frac{4}{5}\right)i$ (vi) $3i+1$

4. **Verify that $\overline{(\overline{z})} = z$, for the following complex numbers.**

(i) $\left(\frac{4}{7}\right)+\left(\frac{9}{10}\right)i$ (ii) $\left(-\frac{9}{11}\right)+\left(\frac{10}{9}\right)i$ (iii) $\frac{1}{2}-3i$

(iv) $2+3i$ (v) $-2-3\left(-\frac{10}{9}\right)i$ (vi) $4x+3iy$

5. **Find the values of x and y, when**

(i) $x+yi=-5+5i$ (ii) $x^2+iy^2=\frac{16}{9}+\frac{9}{25}i$

(iii) $y^2+\frac{x}{3}i=121-\frac{9}{5}i$ (iv) $\frac{\sqrt{5}}{3}x-\frac{3}{\sqrt{2}}yi=\frac{6\sqrt{3}}{\sqrt{2}}+\frac{2\sqrt{2}}{9}i$

1.6 Basic Operations on Complex Numbers

1.6.1 Carry out Basic Operations (Addition, Subtraction, Multiplication and Division) on Complex Numbers

(i) Addition of complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers $\forall a, b, c, d \in \mathbb{R}$, then, their sum,

$z_1+z_2 = (a + ib) + (c + id)$

$= (a+ c) +i (b + d) = (a + c, b + d)$.

Remember that:

$(a,b)+(c,d)=(a+c,b+d)$

Example: If $z_1= 6+ 9i$ and $z_2= -1 + 2i$, find $z_1+ z_2$.

Solution: Given that $z_1= 6+ 9i = (6,9)$ and $z_2= -1 + 2i = (-1,2)$

we know that $z_1 + z_2 = (a + c) +i(b+d) = (a + c, b + d)$

$\therefore \quad z_1+ z_2 = (6, 9) + (-1, 2)$

$\Rightarrow \quad z_1+ z_2= (6 - 1, 9 +2)$

$\Rightarrow \quad z_1+ z_2= (5, 11)$

(ii) Subtraction of complex numbers.

Let $z_1= a + ib$ and $z_2= c + id$, $\forall a, b, c, d \in \mathbb{R}$,

then $z_1- z_2= (a + ib) - (c + id)$

$= (a -c) + i.(b-d) = (a -c, b-d)$

Example: If $z_1= - 7 + 2i$ and $z_2= 4 -9i$, find $z_1- z_2$.

Solution: Given that $z_1= - 7 + 2i = (-7,2)$ and $z_2= 4 -9i = (4,-9)$

we know that $z_1 - z_2 = (a - c, b - d)$

$\therefore \quad z_1- z_2 = (- 7 - 4, 2+9)$

$\Rightarrow \quad z_1- z_2= (-11, 11)$

Remember that:

$(a,b)-(c,d)=(a-c,b-d)$

(iii) Multiplication of complex numbers.

Let $z_1 = a + ib$ and $z_2 = c + id$, be any two complex numbers, $\forall a, b, c, d \in$

$$z_1 \cdot z_2 = (a+ib)(c+ib)$$
$$= c(a+ib) + di(a+ib)$$
$$= ac + bci + adi + bdi^2$$
$$= (ac-bd) + i(ad+bc) = (ac-bd, ad+bc) \qquad i^2 = -1$$

Remember that:

$$(a,b)(c,d) = (ac-bd, ad+bc)$$

Example If $z_1 = 3+4i = (3,4)$ and $z_2 = -3-4i = (-3,-4)$, find the product $z_1 z_2$.

Solution: Given that

$z_1 = 3+4i = (3,4)$ and $z_2 = -3-4i = (-3,-4)$

We know that $z_1 z_2 = (ac-bd, ad+bd)$

$\therefore \quad z_1 z_2 = (3,4)\cdot(-3,-4)$

$\Rightarrow \quad z_1 z_2 = (-9+16, -12-12) = (7,-24)$

(iv) Division of complex numbers

Let $z_1 = a+ib = (a,b)$ and $z_2 = c+id = (c,d)$, $z_2 \neq 0$.

Division of complex number z_1 by another complex number z_2 written as under

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$$
$$= \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$$
$$= \frac{(ac+bd) + i(bc-ad)}{c^2+d^2}$$
$$= \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right)$$
$$= \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$$

Remember that:

$$\frac{(a,b)}{(c,d)} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$$

Example 01 Simplify: $\dfrac{2+3i}{4+2i}$

Solution:

$$\frac{2+3i}{4+2i}$$
$$=\frac{2+3i}{4+2i}\times\frac{4-2i}{4-2i}$$
$$=\frac{(8+6)+i(12-4)}{(4)^2-(i2)^2}$$
$$=\frac{14+8i}{20}$$
$$=\frac{14}{20}+i\frac{8}{20}$$
$$=\frac{7}{10}+i\frac{4}{10}$$
$$=\left(\frac{7}{10},\frac{4}{10}\right)=\left(\frac{7}{10},\frac{2}{5}\right)$$ Hence simplified.

Example 02 Perform division of complex numbers using division formula. $(-1,3)\div(2,-4)$.

Solution: **Formula:** $\dfrac{z_1}{z_2}=\dfrac{(a,b)}{(c,d)}=\left(\dfrac{ac+bd}{c^2+d^2},\dfrac{bc-ad}{c^2+d^2}\right)$

$$\frac{(-1,3)}{(2,-4)}=\left(\frac{(-1)(2)+(3)(-4)}{2^2+(-4)^2},\frac{(3)(2)-(-1)(-4)}{2^2+(-4)^2}\right)$$
$$=\left(\frac{-2-12}{4+16},\frac{6-4}{4+16}\right)$$
$$=\left(\frac{-14}{20},\frac{2}{20}\right)$$
$$=\left(\frac{-7}{10},\frac{1}{10}\right)$$

Exercise 1.6

1. Perform the indicated operations of the following complex numbers.

(i) $(3,2)+(9,3)$

(ii) $\left(\frac{3}{2},\frac{2}{3}\right)+\left(\frac{2}{3},\frac{3}{2}\right)$

(iii) $\left(2xy,5y^2\right)-\left(\frac{1}{2}xy,6y^2\right)$

(iv) $(15,12)-(10,-9)$

(v) $\left(\frac{4}{5},\frac{8}{15}\right)-\left(\frac{4}{5},\frac{6}{10}\right)$

(vi) $(1,2)(1,-2)$

(vii) $(4,-5)(5,-4)$

(viii) $\left(\frac{\sqrt{3}}{2},\frac{\sqrt{6}}{\sqrt{2}}\right)\left(\frac{2\sqrt{3}}{4},\frac{2\sqrt{3}}{2\sqrt{2}}\right)$

(ix) $(3,-7)\div(3,2)$

(x) $(4,5)\div(2,-3)$

3. Simplify and write your answer in form of $a+ib$

(i) $\frac{-1}{1+i}$

(ii) $(1+i)^4$

(iii) $\frac{2-6i}{3-i}-\frac{4+i}{3+i}$

(iv) $\frac{1}{(2+3i)(1+i)}$

4. If $z_1=-4+6i$ and $z_2=2\frac{1}{2}-2i$, verify that

(i) $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

(ii) $\overline{z_1-z_2}=\overline{z_1}-\overline{z_2}$

5. If $z_1=2-5i$ and $z_2=2+3i$, verify that

(i) $\overline{z_1z_2}=\overline{z_1}\cdot\overline{z_2}$

(ii) $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\overline{z_1}}{\overline{z_2}}$

Review Exercise 1

1. Fill in the blanks.

(i) Multiplicative inverse of $\sqrt{5}$ is ____________.

(ii) $Q\cup Q'=$ ____________.

(iii) The additive identity in $\mathbb{R}$ is ____________.

(iv) 5+(6+7)=(5+6)+ ____________.

(v) 3+(−3)= ____________.

(vi) π is a ____________ number.

(vii) $\frac{22}{7}$ is a __________number.

(viii) The conjugate of $-3 + 5i$ is __________.

(ix) In $2i(3-i)$, the real part is __________.

(x) The product of two complex number (a,b) and (c,d)
i.e. $(a,b).(c.d)$ = __________.

2. Read the following sentences carefully and encircle 'T' in case of true and 'F' in case of false statement.

(i) $\mathbb{R}$ is closed under multiplication. T / F

(ii) if $x<y \wedge y<z \Rightarrow x<z$. T / F

(iii) $\forall x,y,z \in \mathbb{R}, x(y-z) = xy - xz$ T / F

(iv) The product of every two imaginary numbers is real. T / F

(v) The sum of two real numbers is a real number. T / F

3. Tick (✓) the correct answer.

(i) The additive inverse of $\sqrt{5}$ is

(a) $-\sqrt{5}$ (b) $\frac{1}{\sqrt{5}}$

(c) $\sqrt{-5}$ (d) -5

(ii) $(5i).(-2i)$ =

(a) -10 (b) 10

(c) $-10i$ (d) $10i$

(iii) 3(5+7)=3.5+3.7, name the property used

(a) Commutative (b) Associative

(c) Distributive (d) Closure

(iv) $\sqrt{-2} \times \sqrt{-2} =$

(a) 2 (b) -2

(c) $2i$ (d) $-2i$

Summary

- The set of real numbers is the union of set of rational and irrational numbers, i.e., $\mathbb{R} = Q \cup Q'$.
- There are two types of non–terminating decimal fractions i.e., non–recurring decimal fractions and recurring decimal fractions.
- Properties of real numbers w.r.t. "+" and "×"

(i) **Closure property:**

$a+b \in \mathbb{R}$ and $ab \in \mathbb{R}, \forall a,b \in \mathbb{R}$

(ii) **Associative property:**

$a+(b+c)=(a+b)+c$ and $a(bc)=(ab)c, \forall a,b,c\in\mathbb{R}$

(iii) **Commutative property:**

$a+b=b+a$ and $ab=ba, \forall a,b\in\mathbb{R}$

(iv) **Identities property:**

$a+0=a=0+a$ and $a\cdot 1=a=1\cdot a \ \forall a\in\mathbb{R}$

(v) **Distributive property:**

$a(b+c)=ab+ac$ or $(b+c)a=ba+ca, \forall a,b,c\in\mathbb{R}$

(vi) **Inverses property:**

$a+(-a)=0=-a+a$ and $a\times\frac{1}{a}=1=\frac{1}{a}\times 1, \forall a, \in\mathbb{R} \wedge a\neq 0$

- **Number line:** A line used for representing real number is called a number line.
- **Radical and Radicand:** In $\sqrt[n]{a}$, the $\sqrt{\ }$ is called radical sign a is called radicand.
- **Laws of Exponent:**

(i) If $a,b\in\mathbb{R}$ and $x,y\in Z^{+}$, then, $a^{x}\times a^{y}=a^{x+y}$.

(ii) If $a\in\mathbb{R}$ and $x,y\in Z^{+}$, then, $\left(a^{x}\right)^{y}=a^{xy}$

(iii) $\forall a,b\in\mathbb{R}$ and $n\in Z^{+}$, then, $(a\times b)^{n}=a^{n}\times b^{n}$

(iv) $\forall a,b\in\mathbb{R}$ and $n\in Z^{+}$, then, $\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}$, provided $b\neq 0$.

- **Complex Number:** $z=a+ib=(a,b)$ is called complex number, where 'a' is real part and 'b' is an imaginary part of z and $i=\sqrt{-1}$.
- **Operation on two complex numbers** $z_1=a+ib$ **and** $z_2=c+id$

$$z_1+z_2=(a+c)+i(b+d)$$

$$z_1-z_2=(a-c)+i(b-d)$$

$$z_1z_2=(ac-bd)+i(ad+bc)$$

$$\frac{z_1}{z_2}=\left(\frac{ac+bd}{c^2+d^2}\right)+i\left(\frac{bc-ad}{c^2+d^2}\right)$$