Unit 11

LINE BISECTORS AND ANGLE BISECTORS

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with their corollaries and apply them to solve allied problems.
- Any point on the right bisector of a line segment is equidistant from its end points.
- Any point equidistant from the points of a line segment is on its right bisector.
- The right bisectors of the sides of a triangle are concurrent.
- Any point on the bisector of an angle is equidistant from its arms.
- Any point inside an angle, equidistant from its arms, is on its bisector.
- The bisectors of the angles of a triangle are concurrent.

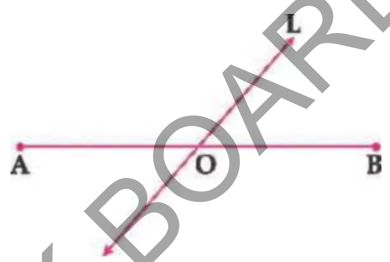
Introduction

We will discuss here the theorems and problems related to line bisector and angle bisector.

Definitions:

i) Bisector of a line segment.

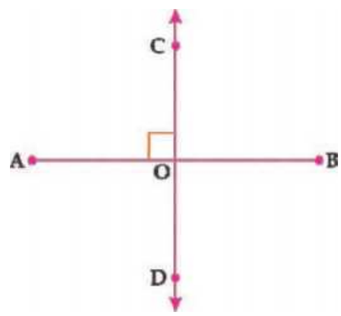
A line, ray or segment is called bisector which cuts another line segment into two equal parts.



For example, in the given figure, a bisector of a line segment AB is a line '**L**' that passes through the mid-point 'O' of the $\overline{AB}$.

ii) Right bisector of a line segment.

A right bisector of line segment can be defined as a line which bisects a line segment at 90 degrees.

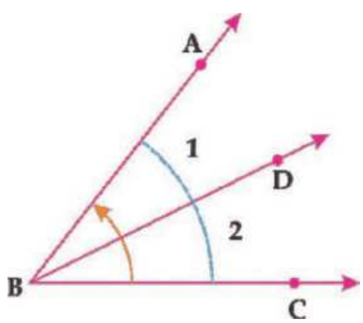


For example, in the given figure, $\overleftrightarrow{CD}$ is perpendicular to the line segment AB and it passes through its mid-point 'O'. Then $\overline{CD}$ is right bisector of $\overline{AB}$.

In the given figure, a line $\overleftrightarrow{CD}$ is a right bisector of $\overline{AB}$.

iii) Bisector of an angle.

A line or ray or line segment is called a **bisector of an angle** or **angle bisector,** if it divides the angle into two equal angles.

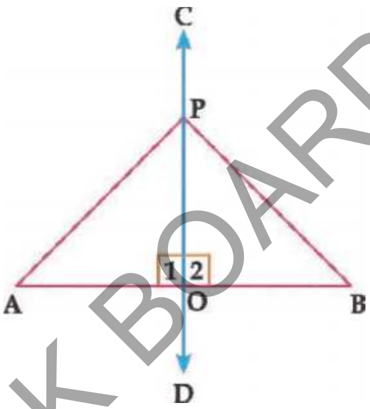


In the given figure, BD is an angle bisector of ∠CBA. BD divides ∠CBA into two equal angles ∠1 and ∠2 i.e. $\angle 1 \cong \angle 2$.

Theorem 11.1.1

Prove that:

Any point on the right bisector of a line segment is equidistant from its end points.



Given:

CD is the right bisector of $\overline{AB}$ intersecting it at O. P is any point on CD.

To Prove: $\overline{AP} \cong \overline{BP}$, i.e. P is equidistant from A and B.

Proof:

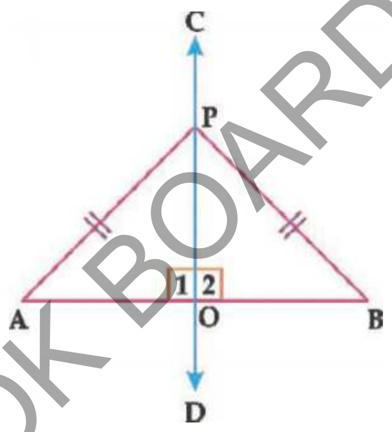
Statements	Reasons
In $\Delta AOP \leftrightarrow \Delta BOP$	
i. $\overline{AO} \cong \overline{OB}$	i. Given (O is the mid-point)
ii. $\angle 1 \cong \angle 2$	ii. Given ($\overline{CD} \perp \overline{AB}$ at O)
iii. $\overline{PO} \cong \overline{PO}$	iii. Common
$\therefore$ $\Delta AOP \leftrightarrow \Delta BOP$	S.A.S postulate
$\therefore$ $\overline{AP} \cong \overline{BP}$	By the congruence of Δs.
But P is an arbitrary point on CD	By assumption
Similarly any other point on CD is equidistant A and B. Hence, every point on the right bisector is equidistant from its end points.	By the above process.

Q.E.D

Theorem 11.1.2

Prove that:

Any point equidistant from end points of a line segment is on the right bisector of it. (Converse of the theorem 11.1)



Given:

A and B are two fixed points and P is a moving point such that $\overline{PA} \cong \overline{PB}$

To Prove:

P lies on the right bisector of $\overline{AB}$.

Construction:

Bisect $\overline{AB}$ at O. Join points P and O.

Proof:

Statements	Reasons
In $\Delta POA \leftrightarrow \Delta POB$	
i. $\overline{AO} \cong \overline{OB}$	i. Construction
ii. $\overline{PA} \cong \overline{PB}$	ii. Given
iii. $\overline{PO} \cong \overline{PO}$	iii. Common
$\Delta POA \cong \Delta BOP$	S.S.S $\cong$ S.S.S
$\angle 1 \cong \angle 2$	By the congruence of Δs.
But $\angle 1$ and $\angle 2$ are supplementary angles.	AB is a line (supplement postulate)
Each $\angle 1$ and $\angle 2$ is right angle.	If two supplementary angles are equal in measure each is right angle.
Thus $\overline{PO}$ is the right bisector of $\overline{AB}$.	$\overline{PO} \perp \overline{AB}$ and $\overline{AO} \cong \overline{BO}$
Thus every point equidistant from points A and B is on the right bisector of $\overline{AB}$.	We can prove by the above process.

Q.E.D

Exercise 11.1

1. Prove that the point of intersection of the right bisector of any two sides of a triangle is equidistance from all the vertices of the triangle.
2. Prove that the centre of the circle is on the right bisectors of each of its chords.
3. Where will be the centre of a circle passing through three non-collinear points and why?
4. If two circles intersect each other at points A and B then prove that the line passing through their centres will be the right bisector of $\overline{AB}$.
5. Three markets A, B and C are not on the same line. The business men of these markets want to construct a Masjid at such a place which is equidistant from these markets. After deciding the place of Masjid, prove that this place is equidistant from all the three markets.

Theorem 11.1.3

Prove that:

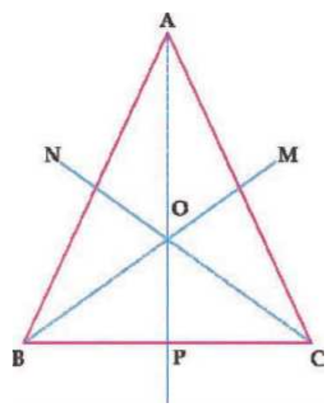
The right bisectors of the sides of a triangle are concurrent.

Given:

A triangle ABC

To Prove:

The right bisectors of the sides are concurrent.



Construction:

Draw $\overline{NO}$, $\overline{MO}$, the right bisectors of $\overline{AB}$ and $\overline{AC}$ meeting in O. Bisect $\overline{BC}$ at P. Draw $\overline{OP}$, $\overline{OA}$, $\overline{OB}$, $\overline{OC}$.

Proof:

Statements	Reasons
$\overline{NO}$ is right bisector of $\overline{AB}$	Construction
$\therefore \quad \overline{AO} \cong \overline{OB}$	
Similarly, $\overline{AO} \cong \overline{OC}$	$\overline{MO}$ is right bisector of $\overline{AC}$.
$\therefore \quad \overline{OB} \cong \overline{OC}$	Each is congruent to $\overline{AO}$.
P is the mid-point of $\overline{BC}$.	Construction
$\therefore$ $\overline{OP}$ is the right bisector of $\overline{BC}$	By theorem 11.1.2
Hence right bisector of the sides of a triangle are concurrent.	All of them meet in one point.

Q.E.D

Exercise 11.2

1. Prove that in an acute triangle the circumcenter falls in the interior of the triangle.

2. Prove that the right bisectors of the four sides of an isosceles trapezium are concurrent.

3. Prove that the altitudes of a triangle are concurrent.

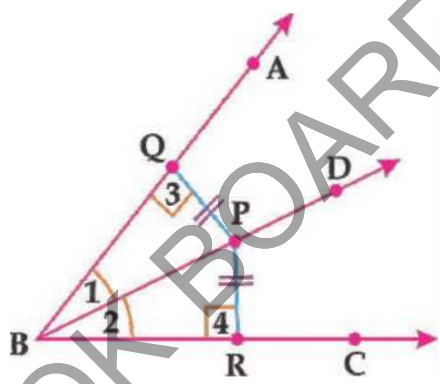
Theorem 11.1.4

Prove that:

Any point on the bisector of an angle is equidistant from its arms.

Given:

$\overrightarrow{BD}$ is the bisector of $\angle ABC$. P is any point on $\overrightarrow{BD}$. $\overline{PQ}$ and $\overline{PR}$ are perpendiculars on $\overrightarrow{BA}$ and $\overrightarrow{BC}$ respectively.



To prove:

$\overline{PQ} \cong \overline{PR}$ (i.e. point P is equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$)

Proof:

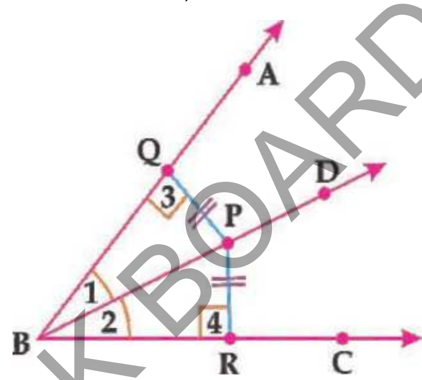
Statements	Reasons
In $\Delta PQB \leftrightarrow \Delta PRB$	
i. $\angle 3 \cong \angle 4$	i. Each is a right angle
ii. $\angle 1 \cong \angle 2$	ii. BD is bisector (Given)
iii. $\overline{BP} \cong \overline{BP}$	iii. Common
$\Delta PQB \cong \Delta PRB$	A.A.S≅ A.A.S
$\overline{PQ} \cong \overline{PR}$	By the congruence of Δs.
(i.e. P is equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$)	

Q.E.D

Theorem 11.1.5

Prove that:

Any point inside an angle, equidistant from its arms, is on the bisector of it. (Converse of Theorem 11.4)



Given:

P is any point of BD equidistant from the arms BA and BC of $\angle ABC$, i.e. $\overline{PQ} \cong \overline{PR}$ and $\overline{PQ} \perp BA$ and $\overline{PR} \perp BC$.

To Prove:

$\angle 1 \cong \angle 2$, i.e. BD is bisector of $\angle ABC$.

Proof:

Statements	Reasons
In $\Delta PQB \leftrightarrow \Delta PRB$	Corresponding in right Δs
i. $\angle 3 \cong \angle 4$	i. Each is a right angle
ii. $\overline{PQ} \cong \overline{PR}$	ii. Given
iii. $\overline{BP} \cong \overline{BP}$	iii. Common hypotenuse
$\Delta PQB \cong \Delta PRB$	In right triangles H.S≅ H.S
$\angle 1 \cong \angle 2$,	By the congruence of triangles.
(i.e. BD is the bisector of $\angle ABC$)	

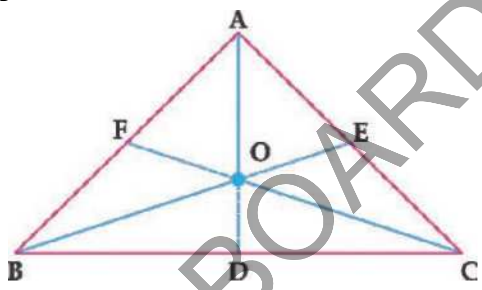
Q.E.D

Theorem 11.1.6

Prove that:

The bisectors of the angle of a triangle are concurrent.

Given:



In ΔABC, $\overline{BE}$ and $\overline{CF}$ are the bisectors of $\angle B$ and $\angle C$ respectively which intersect each other at point 'O'.

To Prove:

The bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent.

Construction:

Draw $\overline{OF} \perp \overline{AB}$ and $\overline{OD} \perp \overline{BC}$.

Proof:

Statements	Reasons
In correspondence $\overline{OD} \cong \overline{OF}$ ... (i)	A point on bisector of an angle is equidistant from its arm.
Similarly $\overline{OD} \cong \overline{OE}$ ... (ii)	
$\overline{OE} \cong \overline{OF}$	From (i) and (ii)
So, the point O is on the bisector of $\angle A$.	
Also the point O is on the bisectors of $\angle ABC$ and $\angle BCA$	Given
Thus, the bisectors of $\angle A, \angle B$ and $\angle C$ are concurrent at O.	

Q.E.D

Exercise 11.3

1. Two isosceles triangles have a common base, prove that the line joining vertices bisects the common base at right angle.
2. If the bisector of an angle of a triangle bisects the opposite side, prove that triangle is an isosceles. A
3. In an isosceles ΔABC, $m\overline{AB} = m\overline{AC}$. Prove that the perpendiculars from the vertices B and C to their opposite sides are equal.
4. Median of a triangle is a line segment joining the midpoint of side to the opposite vertex. Prove that medians of an equilateral triangle are congruent.

Review Exercise 11

1. Prove that, if two altitudes of a triangle are congruent, the triangle is an isosceles.
2. Prove that, a point in the interior of a triangle is an equidistant from all the three sides' lies on the bisector of all the three angles of the triangle.
3. Write **'T'** for True and **'F'** for False in front of each of the following statements
   (i) Bisection of side means, we divide the given side into two parts.
   (ii) In a right angled an isosceles triangle each angle on the base is of 45°.
   (iii) Triangle of congruent sides has congruent angles.
4. Choose the correct option:

(i) There are __________ acute angles in an acute angled triangle.
(a) One (b) Two (c) Three (d) None

(ii) An point equidistant from the end points of a line segment is on the __________ of it.
(a) Right bisector (b) Perpendicular
(c) Centre (d) Mid-point

(iii) __________ of the sides of an acute angled triangle intersect each other inside the triangle.
(a) Perpendicular (b) The right bisector
(c) Obtuse (d) Acute

(v)The bisector of the angles of a triangle are __________.
(a) Concurrent (b) Collinear
(c) Do not interest (d) Unequal

Summary

- A bisector of a line segment divides the line segment into two equal parts
- Right bisector cuts the line segment into two equal parts at 90°.
- Any point on the right bisector of a line segment is equidistant from its end points.
- Any point is equidistant from the points of a line segment is on its right bisector.
- The right bisectors of the sides of a triangle are concurrent.
- Any point on the bisector of an angle is equidistant from its arms.
- Any point inside an angle, equidistant from its arms, is on its bisector.
- The bisectors of the angles of a triangle are concurrent.