Unit 6

LINEAR EQUATION AND INEQUALITIES

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Recall linear equations in one variable.
- Solve linear equations with rational coefficients.
- Reduce equations involving radicals, to simple linear form and find their solutions.
- Define absolute values.
- Solve the equations, involving absolute values in one variable.
- Define inequalities (>, <) and (≥, ≤).
- Recognize properties of inequalities (i.e., trichotomy, transitive, additive and multiplicative).
- Solve linear inequalities with rational coefficients.

6.1 Linear Equations

6.1.1 Recall Linear Equation in one Variable:

If symbol of equality "=" is involved in an open sentence then such sentence is called an **equation**. Linear equations with one variable i.e. $ax+b=0$, $a \neq 0$, are equations where variable has an exponent "1" which is typically not shown.

6.1.2 Solve linear equations with Rational Coefficients:

The value of the unknown (variable) for which the given equation becomes true is called a solution or root of the equation.

Example 01 Solve: $3x-1=5$

Solution: $3x-1=5$

$\Rightarrow \quad 3x = 5+1$

$\Rightarrow \quad x = \frac{6}{3}$

$\Rightarrow \quad x = 2$

Thus, the solution set is $\{2\}$

Example 02 Solve: $\frac{2}{3}(x+3) = 3 + \frac{5x}{9}$

Solution: $\frac{2}{3}(x+3) = 3 + \frac{5x}{9}$

$9\times\frac{2}{3}(x+3) = 9\times 3 + 9\times\frac{5x}{9}$ (Multiplying both sides by 9),

$\Rightarrow \quad 3\times 2(x+3) = 27 + 5x$

$\Rightarrow \quad 6(x+3) = 27 + 5x$

$\Rightarrow \quad 6x + 18 = 27 + 5x$

$\Rightarrow \quad 6x - 5x = 27 - 18$

$\Rightarrow \quad x = 9$

Thus, the solution set is $\{9\}$.

Example 03 Age of father is 13 times the age of his son. It will be only five times after four years. Find their present ages.

Solution: Let present age of son = x years,

and present age of father = $13x$ years,

According to given condition,

$\therefore \quad 13x + 4 = 5(x+4)$

$\Rightarrow \quad 13x + 4 = 5x + 20$

$\Rightarrow \quad 13x - 5x + 4 = 5x - 5x + 20$ (Subtracting $5x$ from both sides)

$\Rightarrow \quad 8x + 4 = 20$

$\Rightarrow \quad 8x + 4 - 4 = 20 - 4$ (Subtracting 4 from both sides)

$\Rightarrow \quad 8x = 16$

$\Rightarrow \quad \dfrac{8x}{8} = \dfrac{16}{8}$

$\Rightarrow \quad x = 2$

Hence present age of father = 13×2 = 26 years
and present age of son = 2 years.

Example 04 When 16 is added to $\frac{1}{3}$ of number the result is $2\frac{1}{3}$ of the original number. Find the number?

Solution: Let x be the number, the according to the given condition:

$16 + \dfrac{1}{3}x = 2\dfrac{1}{3}x$

$\Rightarrow \quad 16x + \dfrac{1}{3}x = \dfrac{7}{3}x$

$\Rightarrow \quad 16 = \dfrac{7}{3}x - \dfrac{1}{3}x$

$\Rightarrow \quad 16 = \left(\dfrac{7}{3} - \dfrac{1}{3}\right)x$

$\Rightarrow \quad 16 = \left(\dfrac{7-1}{3}\right)x$

$\Rightarrow \quad 16 = \dfrac{6}{3}x$

$\Rightarrow \quad 16 \times 3 = 6x$

$\Rightarrow \quad \dfrac{48}{6} = x \qquad \Rightarrow x = 8$

6.1.3 Reduce Equations involving radicals to Simple linear Form and find their solutions.

Equations involving radical expression of the variable are called radical equation.

For example, $\left(3\sqrt{t} - \sqrt{t+1} = 2\right)$ and $\sqrt{x} = 8$ are radical equations.

Solution of radical equation is explained with help of the following examples

Example 01 Solve: $\sqrt{2x+11}=\sqrt{3x+7}$

Solution: $\sqrt{2x+11}=\sqrt{3x+7}$

Squaring on both the sides, we have,

$\therefore \quad \left(\sqrt{2x+11}\right)^2=\left(\sqrt{3x+7}\right)^2$

$\Rightarrow \quad 2x+11=3x+7$

$\Rightarrow \quad 2x-3x=7-11$

$\Rightarrow \quad -x=-4$

$\Rightarrow \quad x=4$

Verification: Put $x=4$ in the given equation, then,

$\sqrt{2(4)+11}=\sqrt{3(4)+7}$

or $\quad \sqrt{8+11}=\sqrt{12+7}$

or $\quad \sqrt{19}=\sqrt{19}$

Thus, solution set is $\{4\}$.

Note: Sometimes the obtained root from radical equation does not satisfy the original equation , it is called extraneous root.

Exercise 6.1

1. Solve the following equations

(i) $\frac{1}{4}x=5$ (ii) $\frac{x}{4}=-3$ (iii) $-5=\frac{-x}{6}$

(iv) $\frac{-x}{8}=-5$ (v) $y-\frac{2}{5}=-\frac{1}{3}$ (vi) $2y-\frac{3}{5}=\frac{1}{2}$

(vii) $\frac{2x-4}{5}=\frac{5x-12}{4}$ (viii) $\frac{3x}{5}+7=\frac{2x}{3}$ (ix) $\frac{3x}{5}+7=\frac{2x}{3}+\frac{4x}{5}$

(x) $\frac{6}{2x-5}-\frac{4}{x-3}=0$ (xi) $\frac{7x-4}{15}=\frac{7x+4}{10}$

(xii) $\frac{3x-2}{10}=\frac{7x-3}{15}-2$ (xiii) $\frac{12x-3}{12}=\frac{12x+3}{8}$

(xiv) $\frac{1}{4}x+x=-3+\frac{1}{2}x$ (xv) $\frac{1}{3}+2m=m-\frac{3}{2}$

2. When 25 added to a number, the result is halved; the answer is 3 times the original number. What is the number?
3. When a number is added to 4, the result is equal to subtracting 10 from 3 times of it. What is the number?
4. Bilal is 6 year older than Ali, Five years from now the sum of their age will be 40. How old are both of them.
5. **Find the Solution set of the following equations and also verify the answer:**

(i) $6+\sqrt{x}=7$ (ii) $\sqrt{x-9}=1$ (iii) $\sqrt{\frac{y}{4}}-2=3$

(iv) $\sqrt{4x+5}=\sqrt{3x-7}$ (v) $\frac{\sqrt{3y+12}}{7}=3$ (vi) $\sqrt{x}+9=7$

(vii) $\sqrt{25y-50}=\sqrt{y-2}$ (viii) $\sqrt{x}-8=1$ (ix) $10\sqrt{x+20}=100$

6.2 Equations involving absolute values

6.2.1 Define absolute values

The absolute value of a real number x is denoted by $|x|$, is the distance of x from zero, either from left or from right of zero. That is why absolute value is never negative.

If x is real number, then absolute value or modulus value of x is denoted by $|x|$, is defined as under:

$$|x| = \begin{cases} x, \text{ when } x > 0 \\ 0, \text{ when } x = 0 \\ -x, \text{ when } x < 0 \end{cases}$$

Example $|-5|=5, |+7|=7, \left|-\frac{1}{2}\right|=\frac{1}{2}, |0|=0$ and so on.

Note: The absolute value of a number is always non negative.

6.2.2. Solve the Equations, involving absolute values in one variable

Example 01 Find the solution set of $|5x-3|-2=3$

Solution: Given that

$$|5x-3|-2=3$$

$$\Rightarrow |5x-3|=5$$

By the definition of modulus, we have,

$5x-3=5$ or $5x-3=-5$

$\Rightarrow 5x=5+3$ or $5x=-5+3$

$\Rightarrow 5x=8$ or $5x=-2$

$\Rightarrow x=\frac{8}{5}$ or $x=-\frac{2}{5}$

Thus, the solution set is $\left\{\frac{8}{5}\ ,\ -\frac{2}{5}\right\}$

Example 02 Find the solution set of $|5x-3|+7=3$

Solution: Given that

$$|5x-3|+7=3$$

$$\Rightarrow |5x-3|=-4$$

The modulus of a real number can never be negative

$\therefore$ Solution set $=\{\ \}$

Example 03 Solve $|5x-3|-2=3$, where, $x\in W$.

Solution: Given that,

$$|5x-3|-2=3$$

$$\Rightarrow |5x-3|=5$$

By the definition of modulus, we have,

so, $5x-3=5$ or $5x-3=-5$

$\Rightarrow 5x=5+3$ or $5x=-5+3$

$\Rightarrow 5x=8$ or $5x=-2$

$\Rightarrow x=\frac{8}{5}$ or $x=-\frac{2}{5}$

$-\frac{2}{5}$ and $\frac{8}{5}\notin W$

Thus, the solution set is $\{\ \}$

Example 04 Solve $|2y-5|+2=7$

Solution: Given that $|2y-5|+2=7$

$\Rightarrow \quad |2y-5|=7-2$

$\Rightarrow \quad |2y-5|=5$

By definition of modulus we have,

so, $2y-5=5$ or $2y-5=-5$

$\Rightarrow \quad 2y=5+5$ or $2y=-5+5$

$\Rightarrow \quad 2y=10$ or $2y=0$

$\Rightarrow \quad y=\frac{10}{2}$ or $y=\frac{0}{2}$

$\Rightarrow \quad y=5$ or $y=0$

Thus, The solution set is {5, 0}.

Exercise 6.2

Find the solution set of the following equation

1. $|2x+1|=6$
2. $|5x-12|=7$, where $x\in$ W
3. $\left|\frac{2x}{7}\right|=12$
4. $\left|\frac{2x+1}{3}\right|=8$
5. $|5x-3|-8=4$, where $x\in$ N
6. $\left|\frac{5x+1}{7}\right|-3=8$
7. $\left|\frac{2x+3}{4}\right|+2=7$
8. $\left|\frac{3x+6}{12}\right|+1=3$, where $x\in$ Z
9. $\frac{3}{2}=|7x+8|$
10. $\left|\frac{2x-3}{5}\right|-12=5$

6.3 Linear inequalities

A linear algebraic expression which contain of sign of inequality is called linear inequality or inequation.

6.3.1 Define inequalities (>, <) and (≥ , ≤).

The following relational operators are called inequilities.

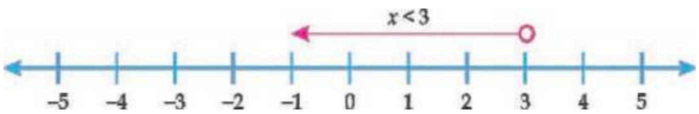
'<' means less than,

'>' means greater than,

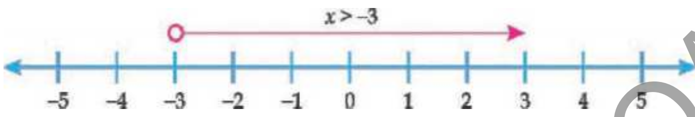
'≤' means less than or equal to,

'≥' means greater than or equal to.

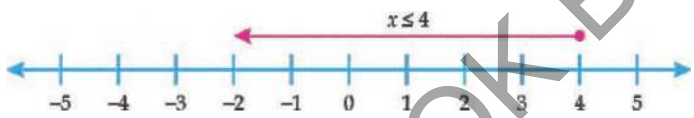
Example 01 $x < 3$



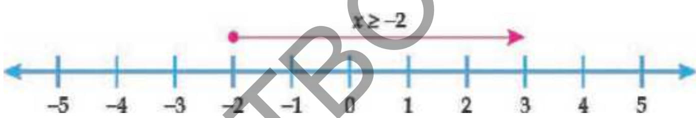
Example 02 $x > -3$



Example 03 $x \leq 4$



Example 04 $x \geq -2$



Note: Hollow circle 'o' shows that number is not included and dark circle '●' shows that number is included.

6.3.2 Recognize properties of inequalities (trichotomy, transitive, additive, multiplicative).

The following are some important properties of inequalities.

(i) Trichotomy Property:

For any two real numbers a and b, one and only one statement of the following is always true.

$a < b, a = b$ or $a > b$

(ii) Transitive Property:

For any three real numbers a ,b and c

If $a < b$ and $b < c \Rightarrow a < c$

and $a > b$ and $b > c \Rightarrow a > c$

(iii) Additive Property:

For any three real numbers

if $a > b$ then $a+c > b+c, \quad \forall\ a,b,c \in \mathbb{R}$

or if $a < b$ then $a+c < b+c, \quad \forall\ a,b,c \in \mathbb{R}$

(iv) Multiplicative Property:

(a) If $a > b$ then $ac > bc, \quad \forall a,b,c \in \mathbb{R}$ and $c > 0$

or If $a < b$ then $ac < bc, \quad \forall a,b,c \in \mathbb{R}$ and $c > 0$

(b) If $a > b$ then $ac < bc, \quad \forall a,b,c \in \mathbb{R}$ and $c < 0$

or If $a < b$ then $ac > bc, \quad \forall a,b,c \in \mathbb{R}$ and $c < 0$

6.4 Solving linear Inequalities

6.4.1 Solving linear inequalities with rational coefficients.

The following examples will help us understand the solution and show on the number line.

Example 01 Fine the solution set of $3x + 1 < 7 \quad \forall x \in \mathrm{W}$ and show on the number line

Solution: Given that

$$3x + 1 < 7 \qquad \forall x \in \mathrm{W}$$

$$3x - 1 < 7 - 1$$

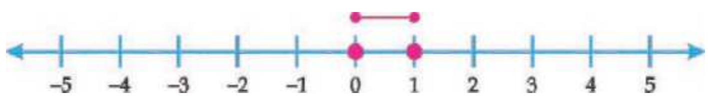
$$3x < 6$$

$$x < \frac{6}{3}$$

$$x < 2$$

Therefore, the solution set is $\{x \mid x \in \mathrm{W} \text{ and } x < 2\} = \{0,1\}$

The Solution is represented on the number line as under:

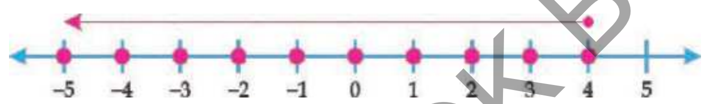


Example 02 Find the solution set of $x-11\leq 9-4x\ \forall x\in Z$ and show on the number line

Solution: Given that $x-11\leq 9-4x\ \ \forall x\in Z$

$$x-11\leq 9-4x$$
$$x+4x\leq 9+11$$
$$5x\leq 20$$
$$x\leq \frac{20}{5}$$
$$x\leq 4$$

Therefore, the solution set is $\{x\,|\,x\in \text{Z and } x\leq 4\}=\{\cdots,-2,-1,0,1,2,3,4\}$



Example 03 Find the solution set of $2x+5>7\ \ x\in$ also express it on number line.

Solution: Given that

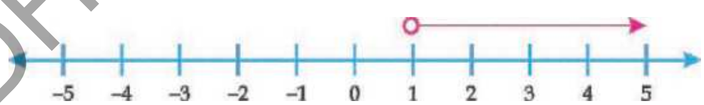
$$2x+5>7\quad x\in$$

This inequality can be expressed as under:

$$2x>7-5$$
or $\quad 2x>2$
or $\quad x>1$

Thus, the solution set is $\{x\,|\,x\in \quad \wedge x>1\}$

The solution on number line is represented as under:



Example 03 Find the solution of $-6<2x+1<11,\ \forall x\in Z$. Also express it on number line.

Solution: Given that $-6<2x+1<11,\ \forall x\in Z$.

This inequality can be expressed as under:

$-6<(2x+1)$ and $2x+1<11$

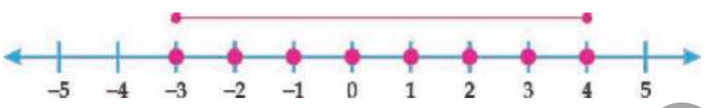
or $\quad -6-1<2x$ and $2x+1<11-1$

or $\quad -7<2x$ and $2x<10$

or $\frac{-7}{2} < x$ and $x < 5$

Thus, the solution set is $\{x | x \in Z \wedge -\frac{7}{2} < x < 5\} = \{-3, -2, -1, 0, 1, 2, 3, 4\}$

The solution on number line is represented as under:



Example 04 Ayesha scored 78,72 and 86 on the first three out of four tests. What score must be recorded on the fourth test to have average at least of 80?

Solution: let score of the fourth test be x so that.

$$\frac{78+72+86+x}{4} \geq 80$$

$$78+72+86+x \geq 320$$

$$236+x \geq 320$$

$$x \geq 320-236$$

$$x \geq 84$$

Ayesha must score 84 on the fourth test to maintain average of 80.

Exercise 6.3

1. **Find the solution sets of the following inequalities and represent number line**

   (i) $2x-7 > 6+x \quad \forall x \in \mathbb{N}$ (ii) $7x-6 > 3x+10, \quad \forall x \in \mathbb{R}$

   (iii) $\frac{y+5}{20} < \frac{25-4y}{10}, \forall y \in \mathbb{N}$ (iv) $|2x+3| < x+2, \forall y \in \mathbb{N}$

   (v) $|2y+8| < 11, \forall y \in \mathbb{R}$ (vi) $5(2y-3) > 6(y-8), \forall y \in \mathbb{R}$

2. Ali scored 66 and 72 marks respectively. For his two Tests, what is the lowest mark he must have scored for his third test If an average score of at least 75 is required to qualify for a bonus prize

3. Seven less than three times the sum of a number and 5 is at least 11, Find all the number that satisfy this condition

Review Exercise 6

1. True and false questions

Read the following sentences carefully and encircle 'T' in case of true and 'F' in case of false statement.

(i) $ay+b=0$, where $a=0$ is linear equation. T/F

(ii) The solution set of $3y-2<7 \wedge y\in \mathbb{N}$ is $\{4, 5, 6, \ldots\}$ T/F

(iii) The solution set of $\sqrt{y}+1=3$ is $\{4\}$. T/F

(iv) The solution set of $|4y|=8$ is $\{2, -2\}$. T/F

(v) The solution set of $-2\leq x\leq 2, x\in$ is $\{-2,0,2\}$. T/F

2. Fill in the blanks.

(i) The solution set of $2y=-y$ is____________.

(ii) The solution set of $\sqrt{y+5}=5$ is____________.

(iii) The solution set of $|x|-4=0$ is ____________.

(iv) The solution set of $\sqrt{x+5}+2=4$ is____________.

(v) The solution set of $0<y+2<5$ when $y\in\mathbb{R}$ is____________.

3. Tick (✓) the correct answer

(i) The solution set of linear equation in one variable has

(a) One element. (b) Two elements.

(c) Three element. (d) More than one elements

(ii) $|-20|$

(a) $=20$ (b) <20

(c) $=-20$ (d) >20

(iii) $x\leq 4$ means

(a) $x<4$ (b) $x=4$

(c) $x<4$ or $x=4$ (d) $x>4$ or $x=4$

(iv) The solution set of $\sqrt{y}=10$ is

(a) {100} (b) {10}

(c) {−10} (d) {−10,10}

(v) $\sqrt{y+4}+2=8$ is a

(a) Linear equation (b) Radical equation

(c) Cubic equation (d) Quadratic equation

(vi) The solution set of $5 - 3y = -7$ is

(a) { −4} (b) {1, 4}
(c) { 4} (d) { 12 }

(vii) The solution set of $\sqrt{5y+5} + 5 = 10$ is

(a) { ± 4 } (b) { 5 }
(c) { 4 } (d) { −4 }

(viii) The solution set of $\left|\frac{5y}{3}\right| = 5$ is

(a) { 3 } (b) { −5, 5 }
(c) { 3, −3 } (d) { −3 }

(ix) The solution set of $|-y| = 0$ is

(a) { 1 } (b) { −1 }
(c) { 0 } (d) { }

(x) If $x > 0$, $y > 0$ and $x - y < 0$, then?

(a) $x < y$ (b) $x + y < 0$
(c) $x > y$ (d) $y - x < 0$

Summary

- An equation of the form $ax + b = 0$, where $a,b \in \mathbb{R}$ and $a \neq 0$ is called a linear equation.
- An equation having radical sign is called a radical equation (Irrational equation). In case, the radical equation have extraneous roots, then verification of the solution is essential.
- If $x \in \mathbb{R}$ then, $|x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x & < 0 \end{cases}$
- If $x,y \in \mathbb{R}$ then

  (i) $|x| \geq 0$ (ii) $|-x| = |x|$ (iii) $|xy| = |x| \cdot |y|$

  (iv) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ (v) $|x| = b$ then $x = b$ or $x = -b$
- For inequality, we use $<, >, \leq, \geq$.
- A linear algebraic expression which contain the sign of inequality is called linear inequality or inequation.
- Properties of inequalities:

  (i) $a < b$ or $a = b$ or $a > b$, $\forall a,b \in$ (Trichotomy)

  (ii) $a > b$ and $b > c \Rightarrow a > c$, $\forall a,b,c \in$ (Transitive)

  (i) $a > b, c > 0 \Rightarrow ac > bc$ and $\dfrac{a}{c} > \dfrac{b}{c}$, $\forall a,b,c \in$ (Multiplication and Division Properties)