Unit 8

QUADRATIC EQUATION

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Solve a quadratic equation in one variable by
  - Factorization,
  - Completing the squares.
- Use method of completing the squares to drive the quadratic formula.
- Use quadratic formula to solve quadratic equations.
- Solve equations, reducible to quadratic form, of the type $ax^4 + bx^2 + c = 0$, Quartic or Bi-quadratic equations.
- Solve the equations of the type $ap(x) + \frac{c}{p(x)} = b$, where a, b and c are rational numbers.
- Solve the reciprocal equations of the type $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$, where a, b and c are rational numbers.
- Solve the exponential equations in which the variable occurs in exponents.
- Solve the equations of the type $(x+a)(x+b)(x+c)(x+d) = k$, where $a + b = c + d$ k $\neq 0$.
- Solve the equations of the type:
  - $\sqrt{(ax + b)} = \text{c}x + d$.
  - $\sqrt{(x + a)} + \sqrt{(x + b)} = \sqrt{(x + c)}$
  - $\sqrt{(x^2 + px + m)} + \sqrt{(x^2 + px + n)} = q$.

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8.1 Quadratic Equations and their Solutions

8.1.1 Elucidate, then define Quadratic Equation in it Standard Form

A polynomial equation with degree 2 is called Quadratic Equation.

The standard form of Quadratic Equation is $ax^2+bx+c=0$, where $a \neq 0$ and $a, b, c \in \mathbb{R}$. In this form a is the coefficient of x^2, b is the coefficient of x and c is the constant term.

In $ax^2+bx+c=0$, if $a=0$, then it reduces to linear equation i.e., $bx+c=0$ and if $b=0$ then it reduces to the pure quadratic form i.e., $ax^2+c=0$

Following are the examples of quadratic equations.

(i) $4x^2+4x+1=0$, (Quadratic equation is in the standard form)

(ii) $x^2-4=0$, (Pure quadratic equation)

8.1.2 Solve a quadratic equation in one variable by

- **Factorization**
- **Completing the square**

Here we consider two methods, for the solution of the quadratic equation.

(a) Method of factorization (b) Method of completing the square.

(a) Method of Factorization

Example 01 Solve: (i) $x^2+2x-15=0$ (ii) $2x^2-5x=12$

Solution (i): $x^2+2x-15=0$

$\Rightarrow \quad x^2+5x-3x-15=0$

$\Rightarrow \quad x(x+5)-3(x+5)=0$

$\Rightarrow \quad (x-3)(x+5)=0$

$\Rightarrow \quad x-3=0 \quad \text{or} \quad x+5=0$

$\Rightarrow \quad x=3 \quad \text{or} \quad x=-5$

Thus, the solution set is $\{-5,3\}$.

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Solution (ii):

$$2x^2 - 5x = 12$$
$$\Rightarrow \quad 2x^2 - 5x - 12 = 0$$
$$\Rightarrow \quad 2x^2 - 8x + 3x - 12 = 0$$
$$\Rightarrow \quad 2x(x-4) + 3(x-4) = 0$$
$$\Rightarrow \quad (x-4)(2x+3) = 0$$
$$\Rightarrow \quad x - 4 = 0 \quad \text{or} \quad 2x + 3 = 0$$
$$\Rightarrow \quad x = 4 \quad \text{or} \quad x = \frac{-3}{2}$$

Thus, the solution set is $\left\{-\frac{3}{2}, 4\right\}$

Example 02 Solve the pure Quadratic equation $4m^2 - 1 = 0$ form m by factorization method:

Solution: $4m^2 - 1 = 0$

$$\Rightarrow \quad (2m)^2 - (1)^2 = 0$$
$$\Rightarrow \quad (2m-1)(2m-1) = 0 \qquad \left[a^2 - b^2 = (a-b)(a+b)\right]$$
$$\Rightarrow \quad 2m - 1 = 0 \quad \text{or} \quad 2m + 1 = 0$$

i.e $\Rightarrow \quad 2m = 1 \quad \text{or} \quad 2m = -1$

$$\Rightarrow \quad m = \frac{1}{2} \quad \text{or} \quad m = -\frac{1}{2}$$

Thus, the solution set $= \left\{-\frac{1}{2}, \frac{1}{2}\right\}$.

(b) Method of Completing Square.

Method is explained as under:

(i) Write the equation in the standard form i.e. $ax^2+bx+c = 0$

(ii) Divide both the sides of the equation by leading coefficient of x^2 if in case it is not 1.

(iii) Shift the constant term to the R.H.S.

(iv) Add $\left(\frac{\text{coefficient of } x}{2}\right)^2$ on both sides

(v) Write the L.H.S. of the equations as a perfect square and then simplify the R.H.S.

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(vi) Take the square root of both the sides of the given equation. Solve the resulting equation to find the solution of the equation and then write solution set.

Example 01 Solve $2x^2+8x-1=0$

Solution: $2x^2+8x-1=0$

$\Rightarrow \quad 2x^2+8x=1 \quad ...(i)$

$\Rightarrow \quad x^2+4x=\frac{1}{2} \quad ...(ii)$ [By dividing equation (i) by 2]

By adding $\left[\frac{1}{2}\times 4\right]^2=4$ on both the sides in equation (ii)

we get,

$$x^2+4x+4=\frac{1}{2}+4$$

$$\Rightarrow x^2+2(2)x+(2)^2=\frac{1}{2}+(2)^2$$

$$\Rightarrow (x+2)^2=\frac{1}{2}+4$$

$$\Rightarrow (x+2)^2=\frac{9}{2}$$

$$\Rightarrow x+2=\pm\frac{3}{\sqrt{2}}$$

$\Rightarrow x+2=\frac{3}{\sqrt{2}}$	$x+2=-\frac{3}{\sqrt{2}}$
$\Rightarrow x+2=\frac{3\sqrt{2}}{2}$	$x+2=\frac{-3\sqrt{2}}{2}$
$\Rightarrow x=-2+\frac{3\sqrt{2}}{2}$	$x=-2-\frac{3\sqrt{2}}{2}$
$\Rightarrow x=\frac{-4+3\sqrt{2}}{2}$	$x=\frac{-4+3\sqrt{2}}{2}$

Thus, the solution set is $\left\{\frac{-4+3\sqrt{2}}{2}, \frac{-4-3\sqrt{2}}{2}\right\}$.

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Exercise 8.1

1. **Solve the following quadratic equations by factorization method:**
   (i) $x^2+5x+6=0$ (ii) $6x^2-x-1=0$ (iii) $x^2-11x+30=0$
   (iv) $x^2-2x=0$ (v) $x^2-2x-15=0$ (vi) $12x^2-41x+24=0$
   (vii) $(x-5)^2-9=0$ (viii) $(3x+4)^2-16=0$
2. **Solve each of the following by completing the square method:**
   (i) $x^2+6x+1=0$ (ii) $(3x+2)(x+2)=6-2(x+1)$. (iii) $3x^2-8x=-1$
   (iv) $24x^2=-10x+21$ (v) $2(x^2-3)-3x=2(x+3)$ (vi) $2x^2+4x-1=0$
3. **The equation $3x^2+bx-8=0$ has 2 as one of its roots.**
   (i) What is the value of b?
   (ii) What is the other root of the equation?

8.2 Quadratic Formula

For equation $ax^2+bx+c=0, a\neq 0$ we use the following quadratic formula to solve it i.e. $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ the formula is known as Quadratic formula.

8.2.1 Use method of completing the square to derive the Quadratic Formula.

Write the Quadratic equation in the standard form

i.e $\quad ax^2+bx+c=0 \; . \; . \; . \; \text{(i)}$

By dividing a on both sides of equation (i), we get

$$\frac{ax^2}{a}+\frac{bx}{a}+\frac{c}{a}=\frac{0}{a}$$

$$\therefore \quad x^2+\frac{bx}{a}+\frac{c}{a}=0$$

By shifting constant term $\frac{c}{a}$ to R.H.S

$$\therefore \quad x^2+\frac{bx}{a}=-\frac{c}{a} \; . \; . \; . \; \text{(ii)}$$

By, adding $\left(\dfrac{b}{2a}\right)^2$ on both sides of equation (ii)

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$$\therefore \quad x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\Rightarrow \quad x^2+2\left(\frac{b}{2a}\right)x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2=\frac{-4ac+b^2}{4a^2}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

8.2.2 Use of Quadratic Formula to solve Quadratic Equations.

Example 01 Solve by using quadratic formula

(i) $2x^2-5x-3=0$ (ii) $x^2+x+1=0$

Solution (i): $2x^2-5x-3=0$

Here, $a=2, b=-5$ and $c=-3$

By using quadratic formula

$$\because \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(-3)}}{4}$$

$$\Rightarrow \quad x=\frac{5\pm\sqrt{25-(-24)}}{4}$$

$$\Rightarrow \quad x=\frac{5\pm\sqrt{49}}{4}$$

$$\Rightarrow \quad x=\frac{5\pm7}{4}$$

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$\Rightarrow \quad x = \frac{5+7}{4} \qquad\qquad x = \frac{5-7}{4}$

$\Rightarrow \quad x = \frac{12}{4} \qquad\qquad x = \frac{-2}{4}$

$\Rightarrow \quad x = 3 \qquad\qquad x = -\frac{1}{2}$

so, the roots are 3 and $-\frac{1}{2}$

Thus, the solution set is $\left\{3, -\frac{1}{2}\right\}$.

Solution (ii): $x^2 + x + 1 = 0$

Here, $a = 1$, $b = 1$ and $c = 1$

By using quadratic formula

$$\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)}}{2(1)}$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2},$$

Thus, the solution set $= \left\{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right\}$

Exercise 8.2

Solve the following equations by using the Quadratic Formula:

(i) $x^2 - 2x = 15$	(ii) $10x^2 + 19x - 15 = 0$	(iii) $x^2 = -x + 1$
(iv) $2x = 9 - 3x^2$	(v) $9x^2 = 12x - 49$	(vi) $\frac{1}{2}x^2 + \frac{3}{4}x - 1 = 0$
(vii) $3x^2 - 2x + 2 = 0$	(viii) $6x^2 - x - 1 = 0$	(ix) $4x^2 - 10x = 0$
(x) $x^2 - 1 = 0$	(xi) $x^2 - 6x + 9 = 0$	(xii) $\frac{1}{x+4} - \frac{1}{x-4} = 4$

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8.3 Equations Reducible to Quadratic Form

There are various types of equations which are not quadratic, but can be reduced to the quadratic form by taking suitable substitution.

8.3.1 Solve equation reducible to quadratic form of the type $ax^4 + bx^2 + c = 0, a \neq 0$ i.e., quartic or bi-quadratic equation.

Consider the equation $ax^4 + bx^2 + c = 0$, it is quartic or bi-quadratic equation and can be reduced in the quadratic equation form, $ay^2 + by + c = 0$, where $y = x^2$. The method is explained by the following example.

Example Solve the quartic equation $4x^4 - 25x^2 + 36 = 0$

Solution: $4x^4 - 25x^2 + 36 = 0 \quad . \; . \; . \; \text{(i)}$

This equation can be written as:

$(x^2)^2 - 25x^2 + 36 = 0 \quad . \; . \; . \; \text{(ii)}$

By putting $y = x^2$ in equation (ii),

we have,

$4y^2 - 25x^2 + 36 = 0$

Here, $a = 4, b = -25$ and $c = 36$

$$\because \quad y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

$$\therefore \quad y = \frac{-(-25) \pm \sqrt{(-25)^2 - 4(4)(36)}}{2(4)}$$

$$y = \frac{25 \pm \sqrt{625 - 576}}{2(4)} = \frac{25 \pm \sqrt{49}}{8} = \frac{25 \pm 7}{8}$$

i.e., $y = \dfrac{25 + 7}{8}$

$y = \dfrac{32}{8} = 4$

but, $y = x^2$, then

$x^2 = 4$

$y = \dfrac{25 - 7}{8}$

$y = \dfrac{18}{8} = \dfrac{9}{4}$

$x^2 = \dfrac{9}{4}$

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$x = \pm 2$ | $x = \pm \frac{3}{2}$

Thus, the solution set $= \left\{\pm 2\ ,\ \pm \frac{3}{2}\right\}$

8.3.2 Solve equation of the type $ap(x)+\frac{b}{p(x)}=c$ where a,b and c are real numbers, $a\neq 0$, where $p(x)$ is an algebraic expression

Example 01 Solve $8\sqrt{x+3} - \frac{1}{\sqrt{x+3}} = 2$

Solution: $8\sqrt{x+3} - \frac{1}{\sqrt{x+3}} = 2 \ldots$ (i)

Let $p(x) = y = \sqrt{x+3}$ put this value in equation (i), we have,

$\Rightarrow \quad 8y - \frac{1}{y} = 2$

$\Rightarrow \quad 8y^2 - 1 = 2y$

$\Rightarrow \quad 8y^2 - 2y - 1 = 0$

$\Rightarrow \quad 8y^2 - 4y + 2y - 1 = 0$

$\Rightarrow \quad 4y(2y-1) + 1(2y-1) = 0$

$\Rightarrow \quad (4y+1)(2y-1) = 0$

$\Rightarrow \quad 4y+1 = 0$ | $2y-1 = 0$

$\Rightarrow \quad y = \frac{-1}{4}$ | $y = \frac{1}{2}$,

when $y = -\frac{1}{4}$, | $y = \frac{1}{2}$, then,

$\sqrt{x+3} = -\frac{1}{4}$ | $\sqrt{x+3} = \frac{1}{2}$

i.e Squaring the on side | i.e Squaring on both side

$\Rightarrow \quad x+3 = \frac{1}{16}$ | $x+3 = \frac{1}{4}$

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$\Rightarrow \quad x = \frac{1}{16} - 3$

$\Rightarrow \quad x = \frac{1-48}{16}$

$\Rightarrow \quad x = -\frac{47}{16}$

$x = \frac{1}{4} - 3$

$x = \frac{1-12}{4}$

$x = -\frac{11}{4}$

Verification:

$$8\sqrt{x+3} - \frac{1}{\sqrt{x+3}} = 2$$

By putting $x = -\frac{47}{16}$ in above equation

$$8\sqrt{\frac{-47}{16} + 3} - \frac{1}{\sqrt{\frac{-47}{16} + 3}} = 2$$

$$8\sqrt{\frac{-47+48}{16}} - \frac{1}{\sqrt{\frac{-47+48}{16}}} = 2$$

$$8\sqrt{\frac{1}{16}} - \frac{1}{\sqrt{\frac{1}{16}}} = 2$$

$$8\left(\frac{1}{4}\right) - \frac{1}{\left(\frac{1}{4}\right)} = 2$$

$$2 - 4 = 2$$

$$-2 \neq 2$$

Verified

By putting $x = -\frac{11}{4}$ in above equation

$$8\sqrt{\frac{-11}{4} + 3} - \frac{1}{\sqrt{\frac{-11}{4} + 3}} = 2$$

$$8\sqrt{\frac{-11+12}{4}} - \frac{1}{\sqrt{\frac{-11+12}{4}}} = 2$$

$$8\sqrt{\frac{1}{4}} - \frac{1}{\sqrt{\frac{1}{4}}} = 2$$

$$8\left(\frac{1}{2}\right) - \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$$4 - 2 = 2$$

$$2 = 2$$

Non verified

Thus, the solution set= $\left\{-\frac{11}{4}\right\}$.

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8.3.3 Solve of reciprocal equation of the type $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$, where a,b and c are rational numbers.

A polynomial equation is said to be a reciprocal equation, if it remain un-changed when x is replaced by $\frac{1}{x}$.

The method for solving reciprocal equation of the type $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$, where a, b, c are rational numbers given below through an example.

Example 01 Solve: $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0$

Solution: $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0 \ldots$ (i)

Let $x+\frac{1}{x}=y$, then $x^2+\frac{1}{x^2}=y^2-2$, so, equation (i) becomes as under,

$2\left(y^2-2\right)-9y+14=0$

$\Rightarrow \quad 2y^2-9y+10=0$

$\Rightarrow \quad 2y^2-4y-5y+10=0$

$\Rightarrow \quad 2y(y-2)-5(y-2)=0$

$\Rightarrow \quad (y-2)(2y-5)=0$

i.e. $y-2=0$

$\Rightarrow \quad y=2$

when $y=x+\frac{1}{x}$, then

we get $x+\frac{1}{x}=2$

$\Rightarrow \quad x^2+1=2x$

$\Rightarrow \quad x^2-2x+1=0$

$2y-5=0$

$y=\frac{5}{2}$

using $y=x+\frac{1}{x}$, then

we get $x+\frac{1}{x}=\frac{5}{2}$

$\Rightarrow \quad \frac{x^2+1}{x}=\frac{5}{2}$

$\Rightarrow \quad 2x^2+2=5x$

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$\Rightarrow \quad (x-1)^2 = 0$

$\Rightarrow \quad x-1 = 0$

$\Rightarrow \quad x = 1$

$\Rightarrow \quad 2x^2 - 5x + 2 = 0$

$\Rightarrow \quad 2x^2 - 4x - x + 2 = 0$

$\Rightarrow \quad 2x(x-2) - 1(x-2) = 0$

$\Rightarrow \quad (x-2)(2x-1) = 0$

$\Rightarrow \quad x-2 = 0$

$\Rightarrow \quad x = 2$

$\Rightarrow \quad 2x-1 = 0$

$\Rightarrow \quad x = \dfrac{1}{2}$

Thus, the solution set $= \left\{\dfrac{1}{2}, 2, 1\right\}$.

8.3.4 Solve the Exponential Equations in which the Variables occurs in Exponents

Equations in which the variable occur in exponent, are called exponential equations. Solution of such type of equation is illustrated through an example given below.

Example 01 solve $7^{1+x} + 7^{1-x} = 50$

Solution: $7^{1+x} + 7^{1-x} = 50$ . . . (i)

$7.7^x + 7.7^{-x} - 50 = 0$ (Splitting power)

$\Rightarrow \quad 7.7^x + \dfrac{7}{7^x} = 50$ . . . (ii)

Let $\quad y = 7^x$

$\therefore$ above equation (ii) reduces as under,

$7y + \dfrac{7}{y} - 50 = 0,$

$\Rightarrow \quad 7y^2 + 7 - 50y = 0$

$\Rightarrow \quad 7y^2 - 50y + 7 = 0$

$\Rightarrow \quad 7y^2 - 49y - y + 7 = 0$ (Factorizing)

$\Rightarrow \quad 7y(y-7) - 1(y-7) = 0$

$\Rightarrow \quad (7y-1)(y-7) = 0$

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$\Rightarrow \quad 7y-1=0$	$y-7=0$
$\Rightarrow \quad y=\frac{1}{7}$	$\Rightarrow \quad y=7$
then $7^x=7^{-1}$	$y=7$ then $7^x=7^1$
$\Rightarrow \quad x=-1$	$x=1$

Thus, the solution set $=\{-1,1\}$.

8.3.5 Solve the Equations of the type $(x+a)(x+b)(x+c)(x+d)=k$, where, $a+b=c+d$ and the constant $k\neq 0$.

Example 01 $(x+1)(x+2)(x+3)(x+4)=48$

Solution: $(x+1)(x+2)(x+3)(x+4)=48$

By rearranging

$(x+4)(x+3)(x+2)(x+1)=48$

$(x^2+4x+x+4)(x^2+2x+3x+6)=48$

$(x^2+5x+4)(x^2+5x+6)=48 \ldots$ (i)

Let $x^2+5x=t$

By substituting in equation (i)

$\Rightarrow \quad (t+4)(t+6)=48$

$\Rightarrow \quad t^2+4t+6t+24=48$

$\Rightarrow \quad t^2+10t-24=0$

$\Rightarrow \quad t^2+12t-2t-24=0$

$\Rightarrow \quad t(t+12)-2(t+12)=0$

$\Rightarrow \quad (t+12)(t-2)=0$

Either,

$\Rightarrow \quad t+12=0$	$t-2=0$
$\Rightarrow \quad t=-12$	$t=2$
Substituting in equation (ii)	Substituting in equation (ii)
$\Rightarrow \quad x^2+5x=-12$	$x^2+5x=2$
$\Rightarrow \quad x^2+5x+12=0$	$x^2+5x-2=0$

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Here: a=1, b=5 and c=12,

$\Rightarrow \quad x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$,

$\Rightarrow \quad x=\dfrac{-5\pm\sqrt{25-48}}{2}$

$\Rightarrow \quad x=\dfrac{-5\pm\sqrt{-23}}{2}$

$\Rightarrow \quad x=\dfrac{-5\pm i\sqrt{23}}{2}$

Solution set is $\left\{\dfrac{-5\pm i\sqrt{23}}{2},\dfrac{-5\pm\sqrt{23}}{2}\right\}$

Here: a=1, b= 5 and c= -2,

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\dfrac{-5\pm\sqrt{25+8}}{2}$

$x=\dfrac{-5\pm\sqrt{33}}{2}$

Exercise 8.3

Solve the following equations:

1. $x^4-8x^2-9=0$
2. $x^4-3x^2-4=0$
3. $12x^4-11x^2+2=0$
4. $\dfrac{2x+3}{x+1}+6\left(\dfrac{x+1}{2x+3}\right)=7$
5. $\sqrt{\dfrac{2x^2+1}{x^2+1}}+6\sqrt{\dfrac{x^2+1}{2x^2+1}}=5$
6. $5^{x+1}+5^{2-x}=5^3+1$
7. $5^{2x}-5^{x+3}+125=5^x$
8. $2\left(\dfrac{x}{x+1}\right)^2-5\left(\dfrac{x}{x+1}\right)+2=x\neq-1$
9. $2\left(\dfrac{x}{x-1}\right)^2-4\left(\dfrac{x}{x-1}\right)+4=0$
10. $9^{x+2}-6\cdot 3^{x+1}+1=0$
11. $2x^2+2^{-x+6}-20=0$
12. $(x-1)(x+5)(x+8)(x+2)=880$
13. $(x-1)(x+2)(x+3)(x+4)=120$
14. $7^{1+x}+7^{1-x}=50$
15. $(x-2)(x+1)(x+3)(x-4)=24$

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8.4 Radical Equations

Equations involving radical expression of the variables are called Radical Equations. Solution of the radical equations must be verified as it may have extraneous root.

8.4.1 Solution of the equations of the type:

Type (i) $\sqrt{ax+b} = cx+d$ Type (ii) $\sqrt{x+a} + \sqrt{x+b} = \sqrt{x+c}$

Type (iii) $\sqrt{x^2+px+m} + \sqrt{x^2+px+n} = q$

Type (i): $\sqrt{ax+b} = cx+d$

Example **Solve** $\sqrt{217-x} = x-7$

Solution: $\sqrt{217-x} = x-7$

Squaring on both sides, we have,

$\left(\sqrt{217-x}\right)^2 = (x-7)^2$

$\Rightarrow \quad 217-x=(x-7)^2$

$\Rightarrow \quad 217-x=x^2-14x+49$

$\Rightarrow \quad x^2-13x-168=0$

$\Rightarrow \quad x^2-21x+8x-168=0$

$\Rightarrow \quad x(x-21)+8(x-21)=0$

$\Rightarrow \quad (x-21)(x+8)=0$

i.e. $x=21$ *or* $x=-8$

verification : $x=21$	verification : $x=-8$
$\sqrt{217-x} = x-7$	$\sqrt{217-x} = x-7$
$\therefore \quad \sqrt{217-21} = 21-7$	$\therefore \quad \sqrt{217-(-8)} = -8-7$
$\Rightarrow \quad \sqrt{196} = 14$	$\Rightarrow \quad \sqrt{225} = 15$
$\Rightarrow \quad 14 = 14$	$\Rightarrow \quad 15 \neq -15$
verified	not verified

Thus, the solution set $=\{21\}$.

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Type (ii) : $\sqrt{x+a} + \sqrt{x+b} = \sqrt{x+c}$

Example **Solve:** $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

Solution: $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

Squaring on both sides, we get,

$(\sqrt{x+7} + \sqrt{x+2})^2 = (\sqrt{6x+13})^2$

$\Rightarrow (\sqrt{x+7})^2 + 2(\sqrt{x+7})(\sqrt{x+2}) + ((\sqrt{x+2})^2 = 6x+13$

$\Rightarrow x+7+2\sqrt{(x+7)(x+2x)} + x+2 = 6x+13,$

$\Rightarrow 2x+7+2+2\sqrt{(x^2+7x+2x+14} = 6x+13$

$\Rightarrow 2\sqrt{x^2+9x+14} = 4x+4$

$\Rightarrow \sqrt{x^2+9x+14} = 2x+2$

$\Rightarrow \sqrt{x^2+9x+14} = 2(x+1)$

Again Squaring on both sides

$\left(\sqrt{x^2+9x+14}\right)^2 = [2(x+1)]^2$

$\Rightarrow x^2+9x+14 = 4(x+1)^2$

$\Rightarrow x^2+9x+14 = 4(x^2+2x+1)$

$\Rightarrow x^2+9x+14 = 4x^2+8x+4$

$\Rightarrow 3x^2-x-10 = 0$

$\Rightarrow 3x^2+5x-6x-10 = 0$

$\Rightarrow x(3x+5)-2(3x+5) = 0$

$\Rightarrow (x-2)(3x+5) = 0$

Either

$3x+5 = 0$	$x-2 = 0$
i.e. $x = -\frac{5}{3}$ or	$x=2$

Verification:

$x = -\frac{5}{3}$	$x=2$
$\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$	$\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13},$

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$$\sqrt{-\frac{5}{3}+7}+\sqrt{-\frac{5}{3}+2}=\sqrt{6\left(-\frac{5}{3}\right)+13}$$

$$\Rightarrow \quad \sqrt{\frac{16}{3}}+\sqrt{\frac{1}{3}}=\sqrt{3}$$

$$\Rightarrow \quad \frac{4}{\sqrt{3}}+\frac{1}{\sqrt{3}}=\sqrt{3}$$

$$\Rightarrow \quad \frac{5}{\sqrt{3}}\neq\sqrt{3}$$

Not verified.

$$\sqrt{2+7}+\sqrt{2+2}=\sqrt{6(2)+13},$$

$$\Rightarrow \quad \sqrt{9}+\sqrt{4}=\sqrt{25}$$

$$\Rightarrow \quad 3+2=5$$

$$\Rightarrow \quad 5=5$$

Verified.

Since $-\frac{5}{3}$ is an extraneous root, therefore the solution is $x=5$,
Thus, the solution set = {5}.

Type (iii): $\sqrt{x^2+px+m}+\sqrt{x^2+px+n}=q$

Example 01 Solve: $\sqrt{x^2-3x+21}-\sqrt{x^2-3x+5}=2$

Solution: Put $y=x^2-3x$, in the given equation, we have,

$$\sqrt{x^2-3x+21}-\sqrt{x^2-3x+5}=2$$

$$\sqrt{y+21}-\sqrt{y+5}=2$$

$$\sqrt{y+21}=2+\sqrt{y+5}$$

Squaring on both sides,

$$y+21=(2)^2+4\sqrt{y+5}+(\sqrt{y+5})^2$$

$$\Rightarrow \quad y+21=(2)^2+4\sqrt{y+5}+\left(\sqrt{y+5}\right)^2$$

$$\Rightarrow \quad y+21=4+4\sqrt{y+5}+y+5$$

$$\Rightarrow \quad 4\sqrt{y+5}=y+21-4-y-5$$

$$\Rightarrow \quad 4\sqrt{y+5}=12$$

$$\Rightarrow \quad \sqrt{y+5}=3$$

again squaring on both the sides, we have,

$$\Rightarrow \quad y+5=9$$

$$\Rightarrow \quad y=4,$$

Put $y=4$ in the substitution $y=x^2-3x$, we have,

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$4 = x^2 - 3x,$

$\Rightarrow \quad x^2 - 3x - 4 = 0,$

$\Rightarrow \quad x^2 - 4x + x - 4 = 0,$

$\Rightarrow \quad x(x-4) + 1(x-4) = 0,$

$\Rightarrow \quad (x-4)(x+1) = 0,$

i.e., $x - 4 = 0$ $\qquad$ $x + 1 = 0$

$\Rightarrow \quad x = 4,$ $\qquad$ $x = -1,$

Verification :

For $x = -1$

$$\sqrt{x^2-3x+21} - \sqrt{x^2-3x+5} = 2$$
$$\sqrt{(-1)^2-3(-1)+21} - \sqrt{(-1)^2-3(-1)+5} = 2$$
$$\sqrt{1+3+21} - \sqrt{1+3+5} = 2$$
$$\sqrt{25} - \sqrt{9} = 2$$
$$5 - 3 = 2$$
$$2 = 2$$

For $x = 4$

$$\sqrt{x^2-3x+21} - \sqrt{x^2-3x+5} = 2$$
$$\sqrt{(4)^2-3(4)+21} - \sqrt{(4)^2-3(4)+5} = 2$$
$$\sqrt{16+12+21} - \sqrt{16-12+5} = 2$$
$$\sqrt{25} - \sqrt{9} = 2$$
$$5 - 3 = 2$$
$$2 = 2$$

Hence both the roots satisfied the given equation. Thus, the solution set is$\{-1,4\}$.

Exercise 8.4

Solve the fowling equations

1. $x + \sqrt{x+5} = 7$
2. $\sqrt{x-2} = 8 - x$
3. $\sqrt{7-5x} + \sqrt{13-5x} = 3\sqrt{4-2x}$
4. $\sqrt{x+2} + \sqrt{x+7} = \sqrt{6x+13}$
5. $\sqrt{x^2-3x+36} - \sqrt{x^2-3x+9} = 3$
6. $\sqrt{x^2+3x+32} + \sqrt{x^2+3x+5} = 9$

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Review Exercise 8

1. Fill in the blanks

(i) A Polynomial equation in which degree of variables is _____________ called quadratic equation.

(ii) Standard form of quadratic equation is _____________ .

(iii) $3^x + 3^{2x} = 1$ is called _____________ equation.

(iv) Solution of $3^x = 9$ is_____________.

(v) Solution of $ax^2 + bx + c = 0$ is _____________ .

2. Tick (✓) the correct answer

(i) Degree of quadratic equation is

(a) 1 (b) 2 (c) 3 (d) 4

(ii) Standard form of Quadratic Equation is

(a) $ax^2 + bx + c = 0, a \neq 0$ (b) $ax^2 + c = 0, a \neq 0$

(c) $ax^2 + bx = 0, a \neq 0$ (d) $ax^3 + bx^2 + c = 0, a \neq 0$

(iii) The Quadratic Formula for $ax^2 + bx + c = 0$ is

(a) $\dfrac{-b - \sqrt{b^2 - 4ac}}{2}$ (b) $\dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

(c) $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (d) $\dfrac{-b \pm \sqrt{b^2 + 4ac}}{2a}$

(iv) Solution set of $x^2 + 10x + 24 = 0$ is

(a) $\{-6, -4\}$ (b) $\{-6, 4\}$

(c) $\{6, 4\}$ (d) $\{6, -4\}$

(v) How many maximum roots of Quadratic Equation are

(a) 2 (b) 3 (c) 1 (d) 4

(vi) Two linear factors of $x^2 - 15x + 56$ are

(a) $(x-7)$ and $(x+8)$ (b) $(x+7)$ and $(x-8)$

(c) $(x-7)$ and $(x-8)$ (d) $(x+7)$ and $(x+8)$

(vii) Polynomial equation, which remains unchanged when x is replaced by $\dfrac{1}{x}$ is called a/ an

(a) Exponential equation (b) Reciprocal equation

(c) Radical equation (d) none of these

(viii) An equation of the type of $3^x + 3^{2-x} + 6 = 0$ is a/an

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(a) Exponential equation (b) Radical equation
(c) Reciprocal equation (d) none of these

(ix) The solution set of equation $4x^2 - 16 = 0$ is
(a) $\{\underline{+}\,4\}$ (b) $\{4\}$ (c) $\{\underline{+}2\}$ (d) none of these

(x) An equation of the form $2x^4 - 3x^3 + 7x^2 - 3x + 2 = 0$ is called a/an
(a) Reciprocal equation (b) Radical equation
(c) Exponential equation (d) none of these

3. True and false questions

Read the following sentences carefully and en-circle 'T' in case of true and 'F' in case of false statement.

(i) Every quadratic equation can be solved by factorization. T/F
(ii) Every Quartic equation has two roots. T/F
(iii) Every cubic equation has three roots. T/F
(iv) $ax^2 + bx + c = 0$ is called the quadratic equation in x if a=0 and b, c are real numbers. T/F
(viii) Extraneous root satisfy the equation. T/F
(ix) Extraneous roots do not satisfy the equation. T/F
(x) In the Quadratic Equation the highest exponent of the variable is two. T/F

Summary

- A Polynomial equation in which degree of a variable is 2, called quadratic equation.
- $ax^2 + bx + c = 0$, $a \neq 0$, a, b, c are real numbers is called standard form of quadratic equation.
- Formula for quadratic equation $ax^2 + bx + c = 0, a \neq 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- In exponential equations, variables occur in exponents.
- An equation involving expression under the radical sign is called a radical equation.

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