Unit 4

Rational and Circular Motion

Teaching Periods 11 **Weightage %** 8

Amusement Park, Hill park, Karachi

In this unit student should be able to:

- Define angular displacement, Angular Velocity and Angular acceleration and express angular displacement in radians.
- Solve problems by using $S = r\theta$ and $v = r\omega$
- Describe the qualitatively motion in curved path due to perpendicular Force
- Derive and use centripetal acceleration $a = r\omega^2, a = v^2/r$.
- Solve problems using centripetal Force $F = mr\omega^2$, $F = mv^2/r$
- Describe situations in which the centripetal acceleration is caused by tension force, a Frictional force, a gravitational force, or a Normal Force.
- Explain when a vehicle travels round a banked curved at specified speed for the banking angle, the horizontal component of the normal force on vehicle causes the centripetal acceleration.
- Describe the equation $\tan\theta = v^2/rg$, relating banking angle $\theta$to the speed V of vehicle and the radius curvature r.
- Define the term orbital velocity and drive relationship between orbital velocity, gravitational constant, mass and radius of orbit.
- Define the moment of inertia.
- Use the formula of moment of inertia of various bodies for solving problems.
- Define the angular momentum
- Explain the law of conservation of momentum.
- Define the Torque as the cross product of force and moment arm.
- Derive a relation between torque, moment of inertia and angular acceleration.

4.1 Kinematics of Angular Motion

The ***kinematics*** of rotational motion describes the relationships among rotation angle, angular velocity, angular acceleration, and time. Relationship between angle of rotation, angular velocity, and angular acceleration to their equivalents in linear *kinematics*.
we derive all kinematic equations for rotational motion under constant acceleration:

4.1.1 Rotational Quantities

Rotational motion happens when the body itself is spinning. Examples the earth spinning on its axis and the turning shaft of an electric motor. We see a turning wheel motion, but we need a different system of measurement.
There are three basic systems of defining angle measurement.
The revolution is one complete rotation of a body. This unit of measurement of rotational motion is the number of rotations. its unit is the revolution (rev).

Angular Displacement

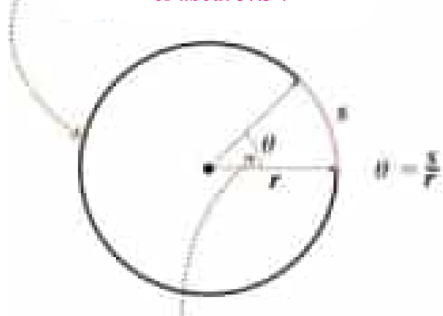
It is defined as the angle with its vertex at the center of a circle whose sides cut off an arc on the circle equal to its radius figure 4.1. where s = r and θ = 1 rad, then

$\theta = \frac{s}{r}$ …………… (4.1)

Where
θ = angle determined by s and r i.e. angular displacement
s = length of arc of the circle
r = radius of the circle

Angle θ is measured in radians is defined as **the ratio of two lengths: the length of the arc and the radius of a circle.** Since the length units in the ratio cancel, the radian is a dimensionless unit. A useful relationship is 2π rad equals one revolution. Therefore, 1 rev = 360^0 = 2π rad

The full circumference is 2π*r*, so 1 revolution is 2π radians. That makes 1 radian 360°/2π or about 57.3°.



Angle in radians is the ration of arcs *s* to radius *r*. θ = *s*/*r*. Here θ is a little less than 1 radian.

Fig: 4.1

A second system of angular measurement divides the circle of rotation into 360 degrees (360^0 = 1 rev). One degree is $1/360^0$ of a complete revolution.
The radian (rad), which is approximately 57.3^0 or exactly (360 / 2π), is a third angular unit of measurement.

$$1^o = 2\pi / 360^o = 0.0175 \text{ rad}$$

Angular Velocity

Any object which is rotating about an axis, every point on the object has the same angular velocity. The tangential velocity of any point is proportional to its distance from the axis of rotation.

Angular velocity is the rate of change of angular displacement and denoted by

$\omega_{average} = \frac{\Delta\theta}{\Delta t}$ .... (4.2)

ω = Angular Velocity
θ = Angular Displacement
t = time

This angular velocity is generally measured in **radians per second**, but angles are actually dimensionless. The SI unit for angular velocity is s^{-1} but it usually writes as $rads^{-1}$. Angular velocity is considered to be a vector quantity, with direction along the axis of rotation in the right – hand rule The instantaneous angular velocity, which is defined as the instantaneous rate of change of the angular displacement with respect to time, can then be written as

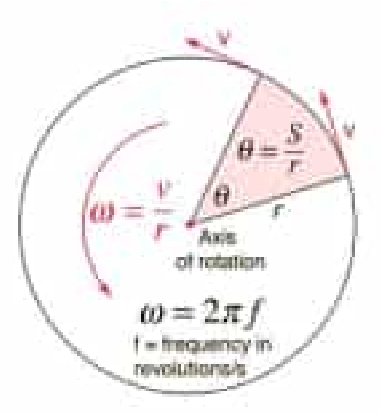


Fig: 4.2

$\omega_{inst} = \lim_{t\to 0}\frac{\Delta\theta}{\Delta t}$ .... (4.3)

The angular velocity is also referred as angular frequency.

$\omega = \frac{2\pi}{t} = 2\pi f$ .... (4.4) As f = 1/t

Another practical unit of angular velocity is revolution per minute (rpm) , usually used in industrial rotatory machines and odometer of vehicles.

$$1\ rpm = \left(\frac{2\pi}{60}\right)\ rad/s$$

Right -hand rule for angular quantities

Lets consider a circular disk rotating counter clockwise as shown in figure 4.3a, in result the direction of angular velocity vector will be outward direction. In similar way when disk start to rotate clockwise then the direction of angular velocity vector will be inward as represented in figure 4.3b.

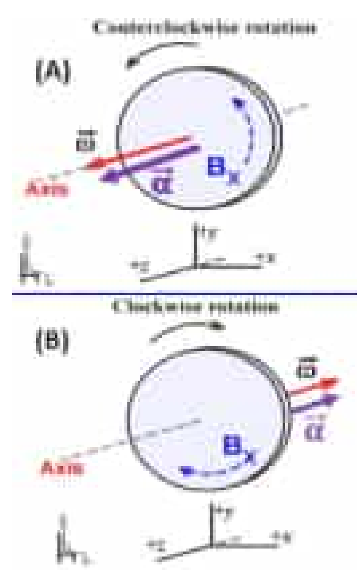


Fig: 4.3

Angular Acceleration

In rotational motion, changing the rate of rotation involves a change in angular velocity and results in an angular acceleration. For uniformly accelerated rotational motion, angular acceleration is defined as the rate of change of angular velocity. That is

$\alpha = \frac{\Delta\omega}{t}$ .... (4.5)

α = Angular acceleration

$\Delta\omega$ = Change in angular velocity
t = time
It's unit is rad/s^2.
The instantaneous angular acceleration, which is defined as the **instantaneous rate of change of the angular acceleration with respect to time.**

$$\alpha_{inst} = \lim_{t\to 0}\frac{\Delta\omega}{t} \text{ .... } (4.6)$$

4.1.2 Relationship between Linear and Angular quantities

Consider an object revolving in circle, which involve angular as well as linear motions i.e. stadium race ground, moving in circular laps but counted in linear terms. The equations of both motions as under:

$\theta = s/r$ ——————————————— (i)
$v = s/t$ ——————————————— (ii)
$\omega = \theta / t$ ——————————————— (iii)

Therefore, combining and substituting s / r for θ in eq (iii), we obtain

$$\omega = \frac{\frac{s}{r}}{t}$$

$$\omega\,(r) = \left(\frac{\frac{s}{r}}{t}\right)(r) \qquad \text{Multiply both sides by r}$$

$$\omega\, r = \frac{s}{t} \qquad \text{whereas } s/t = v$$

$V = r\omega$ .... (iv)
V = linear velocity of point on circle also known as tangential velocity
r = radius
ω = angular speed.

Similarly
If V = r ω is divide both sides by t then

$$\frac{V}{t} = r\,\frac{\omega}{t}$$

$a = r\alpha$ .... (v)
a = Linear or tangential acceleration
α = angular acceleration
r = radius

Self-Assessment Questions:
1. How does angular velocity differ from linear velocity? How are their units related?
2. Describe the difference between clockwise and counterclockwise rotations in terms of angular displacement and angular velocity.
3. Can angular acceleration be negative? If so, what does a negative angular acceleration indicate about the object's motion?

Worked Example 4.1

The platter of the hard drive of a computer rotates at 7300 rpm (a) What is the angular velocity of the platter? (b) if the reading head of the drive is located 3.1 cm from the rotation axis , what is the linear speed of the point on the platter just below it? (c) If a single bit requires 0.55 µm of length along the direction of motion, how many bits per second can the writing head write when it is 3.1 cm from the axis?

Data:

rev = f = 7300 rpm

ω = ?

r = 3.1 cm

v = ?

size of bit = 0.55 µm

Size of bits / time = ?

Solution:

Step 1: (a)

$$f = \frac{7300\ rev/min}{60\ s/min} = 121.7\ \frac{rev}{s} = 121.7\ Hz$$

The angular velocity is

$$\omega = 2\pi f = 2\pi \times 121.7 = 764.5\ \frac{rad}{s}$$

Step 2: (b) The linear speed of point 3.1 cm out from the axis is given by

$$v = r\omega$$

$$v = 0.031 \times 764.5 = 23.7\frac{m}{s}$$

Step 3: (c): Each bit requires 0.55×10^{-6} m , so at a speed 23.7 m/s , the number of bits passing the head per second is

$$= \frac{23.7\frac{m}{s}}{0.55 \times 10^{-6}\frac{m}{bit}} = 43\ \times 10^{6}\ bits\ per\ second$$

$$or\ 43\ megabits\ /s$$

Linear and Rotational Equations of motion

The angular equations for constant angular acceleration are analogous to kinematic equations, their comparison is given as under:

Table: 4.1(Put this table in Do You Know)

Linear Motion	Rotational Motion	
$\boldsymbol{S = Vt}$	$\theta = \frac{s}{r}\ or\ \theta = w\ t$	
$\boldsymbol{S = V_i\, t + \frac{1}{2}\, at^2}$	$\theta = \omega_i\, t + \frac{1}{2}\alpha t^2$	a constant α constant

$V_{avg} = \frac{V_f + V_i}{2}$	$\omega_{avg} = \frac{\omega_f + \omega_i}{2}$	
$V_f = V_i + at$	$\omega_f = \omega_i + \alpha t$	a constant α constant
$2aS = V_f^2 - V_i^2$	$2\alpha\theta = \omega_f^2 - \omega_i^2$	a constant α constant

Worked Example 4.2

A motor cycle wheel turns 3620 times while being ridden for 6.50 minutes. What is the angular speed in rev/min (rpm)?

Data:

t = 6.50 min

no of revolutions = 3620 rev

ω = ?

Solution:

Step 1: formula

$$\omega = \frac{\theta}{t}$$

$$= \frac{3620\ rev}{6.5\ min} = 557\ revolutions\ per\ minutes$$

ω = 557 rpm

4.2.1 Centripetal Force

An object moving in a circle is experiencing an acceleration, though it's moving along perimeter of circle, its velocity is changing subsequently an acceleration. According to Newton's second law of motion, an object experiences an acceleration must have force behind it.

The direction of the net force is in the same direction of acceleration, which is directed to support i.e. center or inward acceleration. The centripetal means center seeking.

The force which causes the acceleration is directed towards the center of the circle and is called a centripetal force.

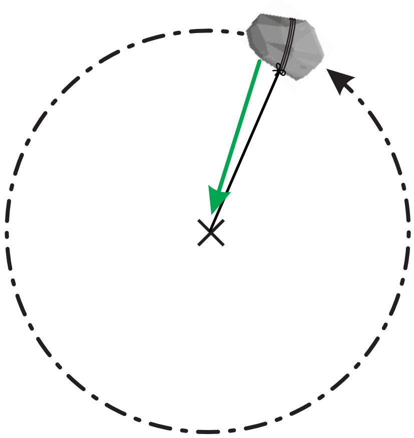


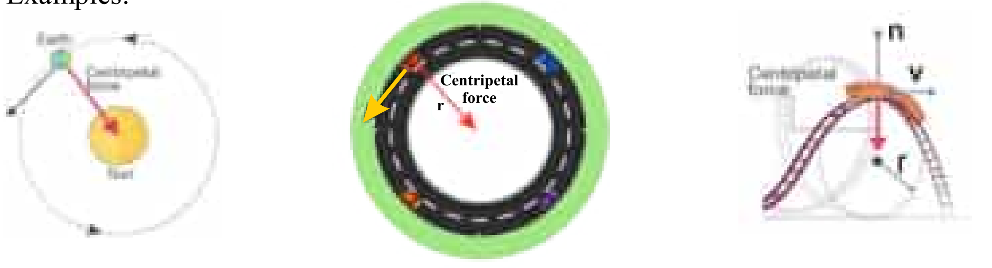
Fig: 4.4

Centripetal force is exerted towards center of the circle. if the string break, however, there would no longer be a centripetal force acting on stone, which would fly off tangent to the circle.

The Centripetal Force Formula is given as the product of mass and tangential velocity squared, divided by the radius that implies that on doubling the tangential velocity, the centripetal force will be quadrupled. Mathematically it is written as:

$F_c = \frac{mV^2}{r}$ ……………(4.7)

Some Examples:



(a) Planets orbiting Sun (b) Turning car at roundabout (c) Roller coaster over loop

Fig: 4.5

4.2.2 Centripetal Acceleration.

Figure 4.6 shows an object moving in a circular path at constant speed. The direction of the instantaneous velocity is shown at two points along the path. Acceleration is in the direction of the change in velocity, which points directly toward the center of rotation i.e. the center of the circular path. This is labeled with the vector diagram in the figure. We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the **centripetal acceleration**; centripetal means "toward the center" or "center seeking."

Centripetal acceleration, the acceleration of an object moving in a circle, directed towards the center.

Derivation for centripetal acceleration

Following the figure 4.6, observe that the tringle formed by the velocity vectors and the one formed by the radii r and ΔS are similar. Both the tringles ABC and PQR are isosceles triangle i.e. triangles having two sides identical. The two equal sides of the velocity vector triangles are the speeds $v_1 = v_2 = v$. Using the properties of two identical tringles we get

$\frac{\Delta v}{v} = \frac{\Delta S}{r}$

rearranging for Δv, we get

$\Delta v = \frac{v}{r}\,\Delta S$ Dividing both sides,

$\frac{\Delta v}{\Delta t} = \frac{v}{r}\frac{\Delta S}{\Delta t}$

$\frac{\Delta v}{\Delta t} = a_c$ and $\frac{\Delta S}{\Delta t} = v$ then, the magnitude of centripetal acceleration is

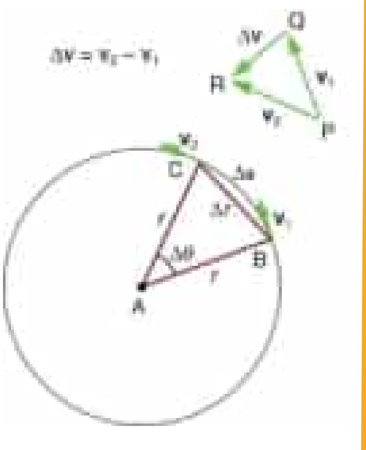


Fig: 4.6

$$\boldsymbol{a_c = \frac{v^2}{r}} \text{ ……………} \textbf{(4.8)}$$

It is useful to show centripetal acceleration in terms of angular velocity. Substituting V = r ω into centripetal acceleration formula, we get

$$a_c = \frac{(r\,\omega)^2}{r} = r\,\omega^2$$

$\alpha_c = r\,\omega^2$ .... (4.9)

We can express the magnitude of centripetal acceleration using either of two equations.

Self-Assessment Questions:

1. In what direction does centripetal acceleration point for an object moving in a circular path?
2. Is centripetal force a conservative force? Why or why not?

Worked Example 4.3

What is the magnitude of force and the centripetal acceleration of a car having mass of 300 kg following a curve of radius 500 m at a speed of 100 km/h? also compare the acceleration with that due to gravity for this fairly gentle curve taken at highway speed.

Data:

m=300kg

r = 500 m

V = 100 km/h = 27.8 m/s

a_c = ?

Solution: Step 1: formula

$$a_c = \frac{v^2}{r}$$

$$a_c = \frac{(27.8)^2}{500}$$

a_c= 1.54 m/s^2

Step 2: To calculate the magnitude of force

$$\mathbf{F = ma_c = mv^2/r}$$

$$F = ma_c = (300)(27.8)^2/500$$

$$\mathbf{F = 462N}$$

To compare this centripetal acceleration with acceleration due to gravity by taking ratio, we get

$$\frac{a_c}{g} = \frac{1.54}{9.8}\,g = 0.157\,g$$

0.157 g is noticeable gravity impact especially if you don't wear seat belt.

4.2.4 Centripetal acceleration caused by Tension force.

An acceleration must be produced by a force. Any force or combination of forces can cause a centripetal or radial acceleration. For examples are the tension in the rope constraint the motion of the ball, the force of Earth's gravity on the Moon to keep them in orbit, friction between roller skates, a rink floor, and a banked roadway's force on a car. Any net force causing uniform circular motion is called a centripetal force. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration.

If a ball is tied to the end of a string and whirling in a circle, the ball accelerates towards the center of the circle as shown in figure 4.7. The centripetal force which causes the inward acceleration is from the tension in the string caused by the person's hand pulling the string plus the weight of the ball also having its implications. The centripetal force on the object equals the tension of the string plus the weight of the ball, both acting toward the center of the vertical circle. Mathematically:

$F_c = F_t + F_w$

$F_t = \frac{m\,v^2}{r} - mg$

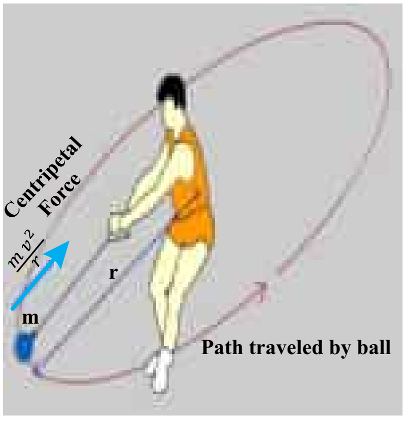


Fig: 4.8

The centripetal force on the object is equal to the difference between the tension of the string and the weight of the object. The tension is exerted inward toward the center of the vertical circle, while the weight is directed away from the center of the vertical circle. Mathematically:

$F_c = F_t - F_w$

$F_t = \frac{m\,v^2}{r} - mg$ …………… (4.10)

If the string breaks there is no longer a resultant force acting on the ball, so it will continue its motion in a straight line at constant speed.

4.2.5 Forces acting on Banked Curve

A banked curve is a curve that has its surface at angle with respect to the ground on which the curve is positioned as shown in figure 4.8. The reason for **banking curves is to decrease the moving object depends on the force of friction.** On a curve that is not banked, a car traveling along that curve will experience a force of static friction that will point towards the center of the circular pathway restricted by the moving car. This frictional force will be responsible for creating centripetal acceleration, which in turn will allow the car to move along the curve. On a banked curve however, the normal force acting on the object such as a car; will act at an angle with the horizontal, and that will create a component normal force that acts along the x axis. This component normal force will now be responsible for creating the centripetal acceleration required to move the car along the curve. Therefore, for every single angle, there exists a velocity for which no friction is required at all to move the object along the curve. This means that the car will be able to turn even under the most slippery conditions (ice or water).

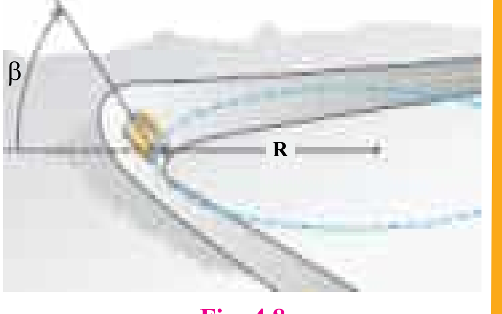


Fig: 4.8

4.2.6 Banking dependence on angle and speed of vehicle.

Banked Curves

Let us suppose banked curves, where the slope of the road helps you assign the curve figure 4.10. The greater the angle θ, the faster and easily you can take the curve. Race tracks for bikes as well as cars, for example, often have steeply banked curves. In an "ideally banked curve," the angle θ is such that you can assign the curve at a certain speed without the support of friction between the tires and the road. We will derive an expression for θ for an ideally banked curve. For ideal banking, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force N in the horizontal and vertical directions must equal the centripetal force and the weight of the car,

A free -body diagram is shown in figure 4.9 for a car on a frictionless banked curve. If the angle θ is ideal for the speed and radius r, then the net external force equals the necessary centripetal force. The only two external forces acting on the car are its weight and the normal force of the road N. (A frictionless surface can only exert a force perpendicular to the surface—that is, a normal force.) These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude mv^2/r. As it is a crucial force and is horizontal, we use a coordinate system with vertical and horizontal axes. Only the normal force has a horizontal component, so this must equal the centripetal force, that is,

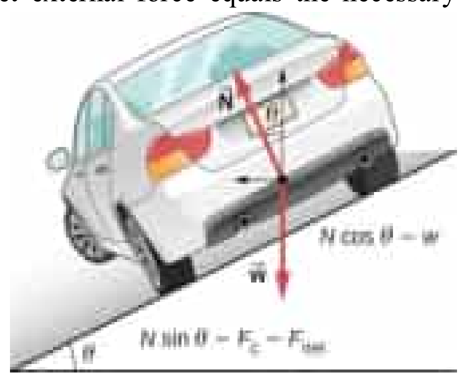


Fig: 4.9

$$N\ Sin\ \theta = \frac{mv^2}{r}$$

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From figure 4.10, we note that the vertical component of the normal force is $N\cos\theta$, and the only other vertical force is the car's weight. These must be equal in magnitude; thus,

$$N\ Cos\ \theta = mg$$

Now we can combine these two equations to eliminate N and get an expression for $\theta\theta$, as desired. Solving the second equation for $N = mg/(cos\theta)$ and substituting this into the first yields

$$mg\ \frac{Sin\ \theta}{Cos\ \theta} = \frac{mv^2}{r}$$

$$mg \tan\theta = \frac{mv^2}{r}$$

$$\tan\theta = \frac{v^2}{rg}$$

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right) \text{ .... } (4.11)$$

This expression can be understood by considering how θ depends on v and r. A large θ is obtained for a large v and a small r. That is, roads must be steeply banked for high speeds and sharp curves. Friction helps, because it allows you to take the curve at greater or lower speed than if the curve were frictionless. Note that θ does not depend on the mass of the vehicle.

Worked Example 4.4

Curves on some test tracks and race courses, such as M-1 Islamabad – Lahore Motorway, are very steeply banked. This banking, with the support of tire friction and very stable car configurations, allows the curves to be taken at very high speed. To illustrate, calculate the speed at which a $100\,\text{m}$ radius curve banked at $30°$ should be driven if the road were frictionless.

Approach

We first note that all terms in the expression for the ideal angle of a banked curve except for speed are known; thus, we need only rearrange it so that speed appears on the left-hand side and then substitute known quantities.

Solution:

Step 1:

Data:

r = 100 m

θ = 300

v = ?

Step 2:

$$\tan\theta = \frac{v^2}{rg}$$

$$v = \sqrt{r\ g\ \tan\theta}$$

Step 3:

$$v = \sqrt{100\times\ 9.8\times\ \tan 30}$$

v = 23.8 m/s

4.3 Orbital Velocity

Orbital velocity is the speed required to achieve orbit around a heavenly body, such as a planet or a star. This requires traveling at a sustained speed that:

- Aligns with the heavenly body's rotational velocity
- Is fast enough to counteract the force of gravity pulling the orbiting object toward the body's surface
- An airplane can travel in the sky but it does not travel at a velocity fast enough to sustain orbit around the earth. This means that once the airplane's engines are turned off, the plane will slow down and be pulled back down to earth, via the force of gravity. By contrast, a satellite (such as the one that powers your phone's GPS or the one that transmits a Direct TV signal) does not need to expend fuel to maintain its orbit around the earth. This is because such satellites travel at a velocity that overrides the force of gravity.

4.3.1 Velocity, Radius

Newton's laws of motion are governing motion of all objects. The same laws which govern the motion of objects on earth also extended to the heavenly bodies to govern the motion of planets, moons, and other satellites. Already we had discussed the bodies in circular motion.

Orbital Speed

Suppose a satellite with mass $M_{satelite}$ orbiting a central body with mass $M_{Central.}$ The central body could be a planet, the moon, the sun or any other heavenly body which may be capable of causing reasonable acceleration over a less massive body nearly. When the satellite is moving in a circular motion, then the net centripetal force acting upon this orbiting satellite is

$$Fc = \frac{M_{satelite} V^2}{R}$$

This net centripetal force is the resultant of the gravitational force which attracts the satellite towards the central body

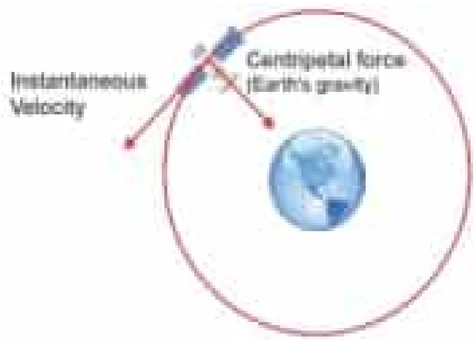


Fig: 4.10
Orbital velocity of satellite

$$FG = \frac{G\, M_{satellite} M_{central}}{R^2}$$

Since $F_c = F_G$, then

$$\frac{M_{satellite} V^2}{R} = \frac{G\, M_{satellite} M_{central}}{R^2}$$

$$V^2 = \frac{G\, M_{central}}{R}$$

$$V = \sqrt{\frac{G\, M_{central}}{R}} \quad \ldots\ldots\ldots\ldots\ldots (4.12)$$

G = Gravitational Constant = $6.67 \times 10^{-11}\ Nm^2 / kg^2$

$M_{central}$ = Mass of earth = 5.98×10^{24} kg

R = distance from the center of object to the center of earth

- This means that all satellites, whatever their mass, will travel at the same speed v in a particular orbit radius r
- Recall that since the direction of a planet orbiting in circular motion is constantly changing, it has centripetal acceleration

Time Period and Orbital Radius

Sine a planet or a satellite is traveling in circular motion when in orbit, its orbital time periods T to travel the circumference of the orbit $2\pi r$, the linear speed is

$$V = \frac{2\pi R}{T}$$

Substituting V

$$V^2 = \left(\frac{2\pi R}{T}\right)^2 = \frac{G\, M_{central}}{R} \qquad \text{Squaring both sides}$$

$$T^2 = \frac{4\pi^2 R^3}{GM_{central}} \quad \ldots\ldots\ldots\ldots\ldots (4.13)$$

Where:

T = Time period of the orbit (s)
R = orbital radius (m)
G = Newton's Gravitational Constant
M = mass of the object being orbited (kg)

The equation shows that the orbital period T is related to the radius r of the orbit. This is also known as Kepler's third law

Self-Assessment Questions:

1. What factors influence the orbital velocity of an object around a celestial body, such as a planet or a star?
2. Compare the orbital velocity of a satellite around earth and moon. How do their orbital velocities differ?

Worked Example 4.5

The International Space Station orbits at an altitude of 400 km above the surface of the Earth. What is the space station's orbital velocity?

Solution:

Step 1:

The orbital velocity depends on the distance from the center of mass of the Earth to the space station. This distance is the sum of the radius of the Earth and the distance from the space station to the surface:

r = (6.38 x 106 m) + (400 km)

r = 6380000 + 400000 m

r = 6780000 m

Step 2:

The orbital velocity can be found using the formula:

$$V = \sqrt{\frac{G\,M_{central}}{R}}$$

Step 3:

$$V = \sqrt{\frac{6.673\times10^{-11}\times5.98\times10^{24}}{678000}}$$

V=7672 m/s

The orbital velocity of the International Space Station is **7672 m/s.**

4.4 Moment of Inertia

Moment of inertia is the property of the body by virtue of it resists angular acceleration, which is the sum of the products of the mass of each particle in the body with the square of its distance from the axis of rotation.

Or

simply it can be described as a quantity that adopts the amount of torque necessary for a specific angular acceleration in a rotational axis. Moment of Inertia is also known as the rotational inertia or angular mass. The higher the moment of inertia, the more resistant a body is to angular rotation.

DO YOU KNOW?

A large - diameter cylinder has a greater rotational inertia than one of smaller diameter but has equal mass.

A body is usually made from several small particles forming the entire mass. The mass moment of inertia depends on the distribution of each individual mass concerning the perpendicular distance to the axis of rotation.

Mathematically, the moment of inertia can be expressed in terms of its individual masses as the sum of the product of each individual mass and the squared perpendicular distance to the axis of rotation.

$I = \Sigma\, m\, r^2$ …………… (4.14)

I = Moment of Inertia

m = Mass

r =distance to axis of rotation

The moment of inertia is measured in kilogram square meters ($kg{\cdot}m^2$), its dimensional formula is $[M^1L^2T^0]$.

The moment of inertia depends upon the following factors:

- Shape and size of the body
- The density of body
- Axis of rotation (distribution of mass relative to the axis)

4.4.2 Rotational Inertia of a two particle system

Consider a rigid body containing of two particles of mass m connected by a rod of length L with negligible mass.

(a) Two particles each at perpendicular distance ½ L from the axis of rotation.

For two particles each at perpendicular distance ½ L from the axis of rotation, we have:

$$I = \Sigma\, m\, r^2 = (m)\left(\frac{1}{2}L\right)^2 + (m)\left(\frac{1}{2}L\right)^2$$

$$I = \frac{1}{2}mL^2 \text{ …………… (4.15)}$$

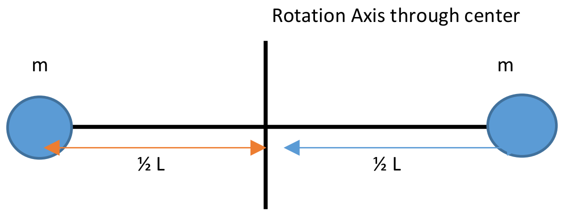


Fig: 4.11

(b) Rotational inertia I of the body about an axis through left end of rod and parallel to the first axis.

Here it is simple to find I using either method. The first is as in part (a) we have done. The perpendicular distance r is zero for the particle on left and L for the particle on the right. We have:

$$I = m\,(0)^2 + m\,L^2$$

$$I = mL^2 \text{ …………… (4.16)}$$

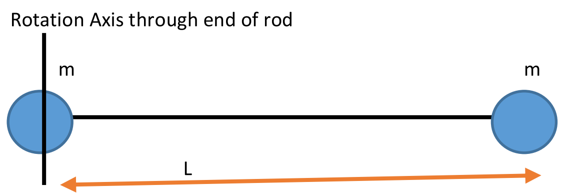


Fig: 4.12

Second Method:

As I_c about an axis through the center of mass and because the axis here is parallel to that center of axis, we can apply the parallel- axis theorem.

$$I = I_c + Mh^2$$

$I = \frac{1}{2}mL^2 + (2m)\left(\frac{1}{2}L\right)^2$

$I = mL^2$

Moment of Inertia of Various bodies:

Consider a solid cylinder with mass **M**, and radius **R**. The moment of inertia of a solid cylinder rotating about its central axis is given by the formula:

$\mathbf{I} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{M}\,\mathbf{R}^{\mathbf{2}}$ …………… (4.17)

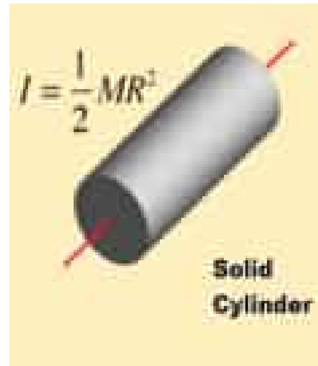


Fig: 4.13

Let's assume we have a solid cylinder with a mass of 2 kg and a radius of 0.5 meters. The moment of inertia would be:

$I = \frac{1}{2}(2kg)(0.5m)^2$

$\mathbf{I = 0.5\ kg.m^2}$

the moment of inertia of the solid cylinder is $\mathbf{0.5\ kg.m^2}$

Moment of Inertia of a Hollow Cylinder:

Consider a hollow cylinder with mass **M**, inner radius **a**, and outer radius **b**. The moment of inertia of a hollow cylinder rotating about its central axis is given by the formula:

$I = \frac{1}{2}m(a^2 + b^2)$ …………… (4.18)

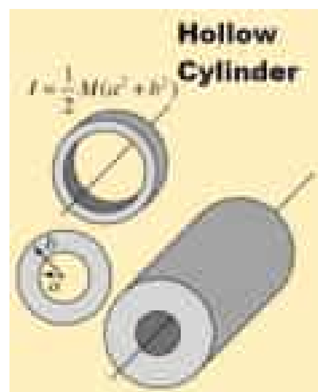


Fig: 4.14

Let's consider we have a hollow cylinder with a mass of 3 kg, an inner radius of 0.4 meters, and an outer radius of 0.6 meters. The moment of inertia would be:

$I = \frac{1}{2}(3kg)[(0.4)^2 + (0.6)^2]$

$\mathbf{I = 1.2\ kg.m^2}$

The moment of inertia of the hollow cylinder is $\mathbf{1.2\ kg.m^2}$

Moment of Inertia of a Sphere:

Consider a solid sphere with mass **M** and radius **R**. The moment of inertia of a solid sphere rotating about its center is given by the formula:

$\mathbf{I} = \frac{\mathbf{2}}{\mathbf{5}}\mathbf{M}\,\mathbf{R}^{\mathbf{2}}$…………… (4.19)

Fig: 4.15

Let's assume we have a solid sphere with a mass of 1 kg and a radius of 0.3 meters. The moment of inertia would be:

$$I = \frac{2}{5}\,(1\text{kg})\,(0.3\text{m})^2$$

$$\mathbf{I = 0.036\ kg.m^2}$$

The moment of inertia of the solid sphere is $\mathbf{0.036\ kg.m^2}$

These examples demonstrate how to calculate the moment of inertia for different bodies. However, it's important to note that the moment of inertia can vary depending on the axis of rotation and the specific mass distribution within the object.

Self-Assessment Questions:

1. How does moment of inertia differ from mass? What role does mass play in moment of inertia?
2. How does the moment of inertia of an object change when its axis of rotation is shifted closer to or farther from its center of mass?

DO YOU KNOW?

Even though the two cylinders are of equal mass, the hollow cylinder has more rotational inertia because its mass is concentrated farther from its axis of rotation.

4.5 Angular Momentum

Rotating bodies show the same reluctance i.e. inertia to a change in their angular velocity as compared to bodies moving in a straight line do to a change in their linear velocity. This is due to the angular momentum of the object. If you try to get on a bicycle and try to balance without a kickstand you are probably going to fall off. But once you start pedaling, these wheels pick up angular momentum. They are going to resist change, thereby balancing gets easier.

4.5.1 Angular Momentum of a point particle.

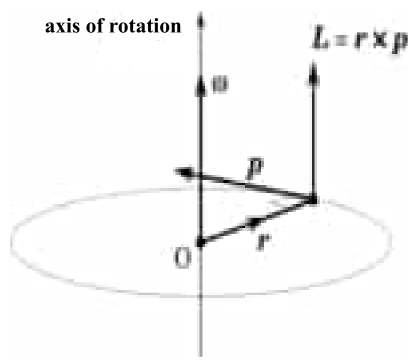


Fig: 4.16

Suppose a point particle of mass m in Cartesian coordinates at position r with respect to origin O, is having momentum p as shown in figure 4.16. The point particle is accelerating around a fixed point e.g. Earth revolving around the Sun.

The angular momentum $\vec{L}$, of a point particle is the vector product of its position / moment arm and linear momentum.

$\vec{L} = \vec{r} \times \vec{p}$ …………… (4.20)

The magnitude of angular momentum is

$\vec{L} = rp\,Sin\theta$ as $p = mv$

Where θ is angle between position and momentum vectors.

Its direction is perpendicular to both position and linear momentum and determined by right-hand rule.

$\vec{L} = mvr\,Sin\,\theta$ For $\theta = 90^0$

$\vec{L} = mvr$

The maximum magnitude of angular momentum.

4.5.2 Law of conservation of angular momentum

The law of conservation of angular momentum states that when no resultant external torque acts on a body, its angular momentum remains constant.

In figure 4.17, consider a particle in uniform circular motion due to a force acting on the particle that is centripetal force required to deflect the particle to keep it in circular path.

(The central force is a force whose line of action passes through a fixed point.) The centripetal force always points towards the center of the circle so it produces no torque about an origin at the center of the circle.

The law of conservation of angular momentum then ensures that the angular momentum of the particle is constant. The angular momentum is constant in magnitude ($mr\omega^2$ remains fixed) and constant in direction (the motion is confined to the plane of rotation).

It is worth emphasizing that the above discussion relies on the origin being at the center of the circle. In Figure 6 a different origin O has been chosen, on the axis of rotation but out of the plane of rotation. In this case angular momentum is

$\vec{L} = r \times p$

$\vec{L} = (r_{\parallel} + r_{\perp}) \times p$

$\vec{L} = (r_{\parallel} \times p) + (r_{\perp} \times p)$

that the angular momentum has a component $L_{\perp} = \boldsymbol{r}_{\parallel} \times \boldsymbol{p}$ which is perpendicular to the axis of rotation and is not conserved. This is not a problem because, *relative to the origin of Figure 6*, the particle experiences a torque $\boldsymbol{\tau}_{\perp} = \boldsymbol{r}_{\parallel} \times \boldsymbol{F}$ where $\boldsymbol{F}$ is the *centripetal force* acting along $-\boldsymbol{r}_{\perp}$.

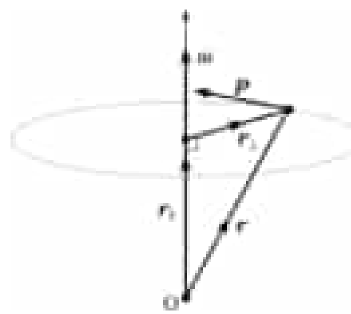


Fig: 4.17

Moreover, because $\boldsymbol{r}_{\parallel}$ is constant we have

$$\frac{dL_{\perp}}{dt} = r_{\parallel} \times \frac{dp}{dt} = r_{\parallel} \times F = \tau_{\perp}$$

so the rate of change of $\boldsymbol{L}_{\perp}$ is supported by the existence of $\tau_{\perp}$. At the same time, the component of angular momentum parallel to the axis of rotation remains constant because there is no torque in that direction.

$$\frac{dL}{dt} = \tau$$

$$\frac{dL}{dt} = 0$$

where $\tau = 0$ (integrating both sides)

$L = constant$

Hence, the angular momentum of the particle is conserved if the net torque acting on it is zero.

Self-Assessment Questions:

1. How is angular momentum different from linear momentum? Explain the key differences between the two.
2. How does angular momentum change when an object's moment of inertia is altered, but its angular velocity remains constant? Provide an example to illustrate this concept.

Worked Example 4.6

A basketball spinning on the finger of an athlete has angular velocity ω = 120.0 rad/s. The moment of inertia of a sphere that is hollow, where M is the mass and R is the radius. If the basketball has a weight of 0.6000 kg and has a radius of 0.1200 m, what is the angular momentum of this basketball?

Solution:

Data:

$\omega = 120$ rad /s
m = 0.60 kg
r = 0.120 m
L = ?

We can find the angular momentum of the basketball by using the moment of inertia of a sphere that is hollow, and the formula. The angular momentum will be:

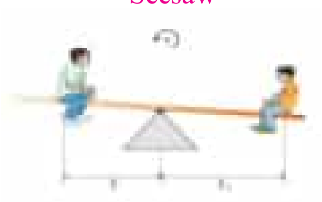
$L = I\omega$
$I = 2/3\ MR^2$
$I = 0.66 \times 0.60 \times (0.12)^2$
$I = 5.76 \times 10^{-3}$
$L = 5.76 \times 10^{-3} \times 120$
$L = 0.6912$ kg.m^2/s

The angular momentum of the basketball that is spinning will be 0.6912 kg. m^2/s.

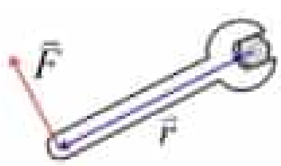
4.6 Torque

Torque is an important quantity for describing the dynamics of a rotating rigid body. There are many applications of torque in our world. We all have an intuition about torque, as when we use a large wrench to unscrew a stubborn bolt. Torque is at work in unseen ways, as when we press on the accelerator in a car, causing the engine to put additional torque on the engine. Or every time we move our bodies from a standing position, we apply a torque to our limbs. In physics, torque is simply the tendency of a force to turn or twist. Example of torque.

Seesaw



Wrench



Door Opening

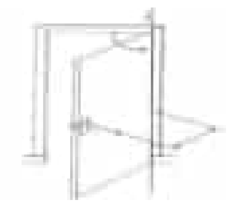


Fig: 4.18

4.6.1 Torque or Moment of Force

Torque is produced when a force is applied to an object produce a rotation. **This turning effect of the force about the axis of rotation is called torque.** torque is also known as moment of force.

Consider a body which can rotate about O (axis of rotation). A force F acts on other end whose position vector with respect to O is r as shown in figure 4.19 The distance from the pivot point to the point where the force acts is called the moment arm. The force is in x-y plane, so resolved to act accordingly. Torque is defined as **the cross product of moment arm and force.**

$\vec{\tau} = \vec{r} \times \vec{F}$.............. (4.21)

$|\tau| = r.F \sin\theta$

$|\tau| = r\,(F \sin\theta)$

$|\tau| = F\,.(r \sin\theta)$

Torque is positive if directed outward from paper and negative inward to paper.

Torque is always perpendicular to the plane r and F. Thus clockwise torque is negative and counter-clockwise is positive. The SI unit of torque is kgm^2/s^2 and dimesionally it can be written as $[ML^2T^{-2}]$.

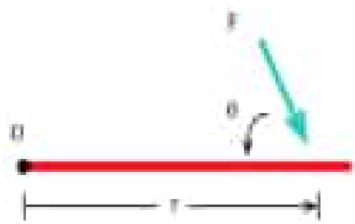


Fig: 4.19

Self-Assessment Questions:

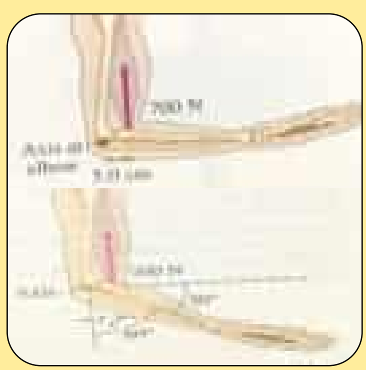
1. What causes torque to occur in a system? How is torque different from force?
2. How does the distance between the point of rotation (fulcrum) and the point of application of force affect the torque?

DO YOU KNOW?

You imagine pushing a door to open it. Your push causes the door to rotate about its hinges. How hard you need to push depends on the distance you are from the hinges. The closer you are to the hinges, the harder it is to push. This is what happens when you try to push open a door on the wrong side. The torque you created on the door is smaller than it would have been had you pushed the correct side.

Worked Example 4.7

The biceps muscle exerts a vertical force on the lower arm, bent as shown in figure. For each case, calculate the torque about the axis of rotation through the elbow joint, assuming the muscle is attached 5 cm from the elbow as shown.



Data:

F = 700 N

r⊥ = 5 cm = 0.05 m

Solution: Step 1:

(a) τ = r⊥ F = 700 × 0.05 = 35 N.m

Step 2:

(b) τ = r⊥ F = r Sin θ F

τ = (0.05) Sin 60° × 700

τ = 30 N.m

The arm exerts less torque at this angle than when it is at 90°. Weight machines at gyms are designed on these parameters.

4.6.2 Derivation a relation torques, moment of inertia and angular acceleration.

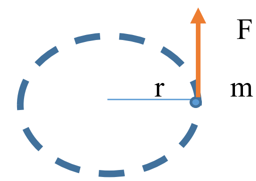


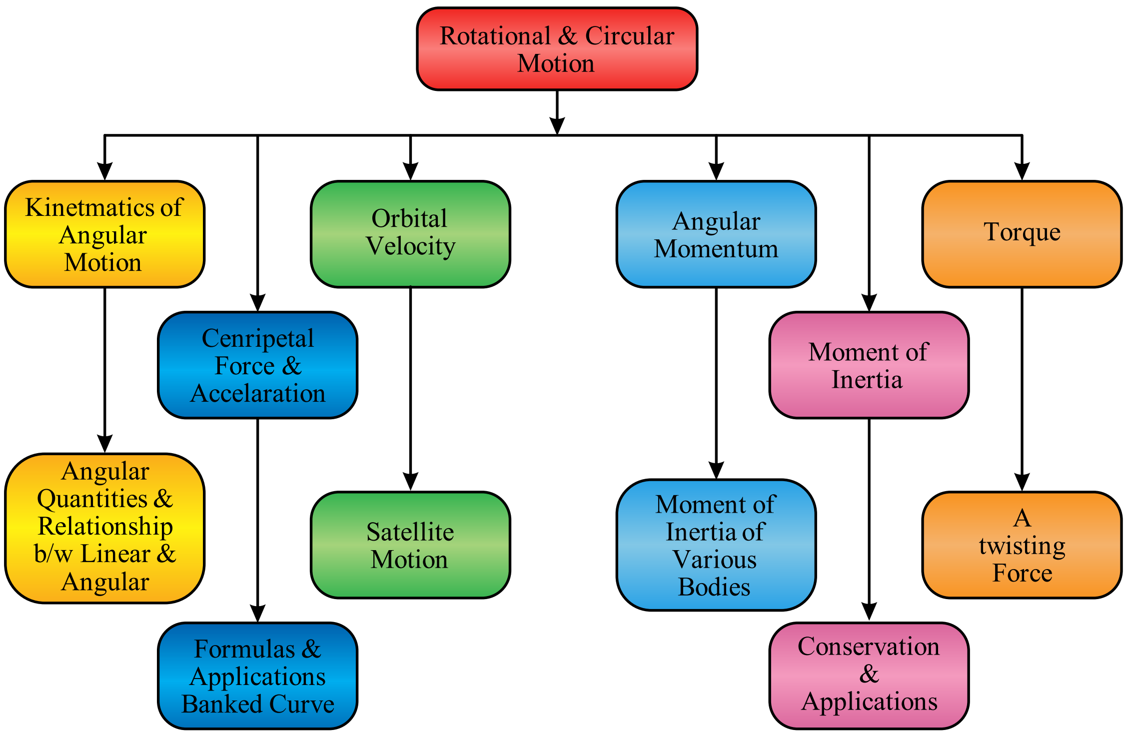
Fig: 4.20

Consider a particle of mass m rotating in a circle of radius r at the end of a string whose mass is negligible as compared to mass of string. Assume that a single force F acts on mass as shown in figure: 4.20.
The torque gives rise to the angular acceleration is $\tau = r F$.
Newton's second law of motion is $F = m a$,
Tangential linear acceleration $\boldsymbol{\alpha_{tan}} = \mathbf{r}\,\alpha$
Equation … can be rewritten as
$\vec{F} = m\, r\, \alpha$ Multiplying both sides with r
$r\vec{F} = m r^2 \alpha$
$\tau = I\alpha$
$\tau \propto \alpha$
We have a direct relation between the angular acceleration and the applied torque where as mr^2 is representing rotational inertia which is called moment of inertia.
Now let us consider a rotating rigid object, such as wheel rotating about an axis through its center, which could be an axle. We can think of the wheel as containing of many particles located at various distances from the axis of rotation. We can apply eq…. to each particle of the object, and then sum over all the particles. The sum of the various torques is just the total torque, we get
$\Sigma\tau = (\Sigma\, mr^2)\,\alpha$
If each particle is assigned a number (1,2,3, 4….), then
$I = \Sigma\, mr^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^3 + \ldots\ldots\ldots\ldots$
By combining equations we get
$\boldsymbol{\Sigma\,\tau = I\,\alpha}$ **……(4.22)**
This is rotational equivalent of Newton's Second law. It is valid for the rotation of a rigid bodies about a fixed axis.

DO YOU KNOW?

Just as planets fall around the sun, stars fall around the centers of galaxies. Those with insufficient tangential speeds are pulled into, and are gobbled up by, the galactic nucleus – usually a black hole.

CONCEPT MAP


Rotational & Circular Motion
Kinetmatics of Angular Motion
Cenripetal Force & Accelaration
Orbital Velocity
Angular Momentum
Moment of Inertia
Torque
Angular Quantities & Relationship b/w Linear & Angular
Formulas & Applications Banked Curve
Satellite Motion
Moment of Inertia of Various Bodies
Conservation & Applications
A twisting Force

SUMMARY

- Angular displacement is the ratio of two lengths: the length of the arc and the radius of a circle.
- Angular velocity is the rate of change of angular displacement
- Angular acceleration is the rate of change of angular velocity.
- The force which causes the acceleration is directed towards the center of the circle and is called a centripetal force.
- The acceleration of an object moving in a circle, directed towards the center.
- A curve in a road that is sloping in a manner that helps a vehicle negotiate the curve.
- Orbital velocity is the speed required to achieve orbit around a celestial body, such as a planet or a star.Moment of inertia is the property of the body by virtue of it resists angular acceleration, which is the sum of the products of the mass of each particle in the body with the square of its distance from the axis of rotation.
- The moment of inertia of a body about an axis parallel to the body passing through its center is equal to the sum of moment of inertia of the body about the axis passing through the center and product of the mass of the body times the square of the distance between the two axes.
- The angular momentum, of a point particle is the vector product of its position / moment arm and linear momentum.
- The law of conservation of angular momentum states that when no resultant external torque acts on a body, its angular momentum remains constant.
- The turning effect of the force about the axis of rotation is called torque.
- Static torque is which does not produce an angular acceleration.
- Dynamic torque is that in which angular acceleration is produced.

EXERCISE

Section (A): Multiple Choice Questions (MCQs)

1. One radian is about
   a) 25^{O}
   b) 37^{O}
   c) 45^{O}
   d) 57^{O}
2. Wheel turns with constant angular speed then:
   a) each point on its rim moves with constant velocity
   b) each point on its rim moves with constant acceleration
   c) the wheel turns through equal angles in equal times
   d) the angle through which the wheel turns in each second increases as time goes on
   e) the angle through which the wheel turns in each second decreases as time goes on
3. The rotational inertia of a wheel about its axle does not depend upon its
   a) diameter
   b) mass
   c) distribution of mass
   d) speed of rotation
4. A force with at given magnitude is to be applied to a wheel. The torque can be maximized by
   a) applying the force near the axle, radially outward from the axle.
   b) applying the force near the rim, radially outward.
   c) applying the force near the axle, parallel to a tangent to the wheel.
   d) applying the force at the rim, tangent to the rim
5. An object rotating about a fixed axis, I is its rotational inertia and α is its angular acceleration. Its
   a) is the definition of torque
   b) is the definition of rotational inertia
   c) is definition of angular acceleration
   d) follows directly from Newton's second law.
6. The angular momentum vector of Earth about its rotation axis, due to its daily rotation is, directed
   a) tangent to the equator towards east
   b) tangent to the equator towards the west
   c) north
   d) towards the Sun.
7. A stone of 2 kg is tied to a 0.50 m long string and swung around a circle at angular velocity of 12 rad/s . The net torque on the stone about the center of the circle is
   a) 0 N.m
   b) 6 N.m
   c) 12 N.m
   d) 72 N.m
8. A man, with his arms at his sides, is spinning on a light frictionless turntable . When he extends his arms
   a) his angular velocity increases
   b) his angular velocity remains same
   c) his rotational inertia decreases
   d) his angular momentum remains the same.

9. A space station revolves around the earth as a satellite, 100 km above the Earth's surface. What is the net force on an astronaut at rest inside the space station?
   a) equal to her weight on earth
   b) a little less than her weight on earth
   c) less than half her weight on earth
   d) zero (she is weightless)
10. If the external torque acting on a body is zero, then its
   a) angular momentum is zero
   b) angular momentum is conserved
   c) angular acceleration is maximum
   d) rotational motion is maximum

Section (B): Structured Questions

CRQ:

1. For an isolated rotating body, what is the relation between angular velocity and radius?
2. When the moment of inertia of a rotating body is halved, then what will be the effect on angular velocity?
3. Compare kinematics equation of linear motion and circular motion
4. Can a small force ever exert a greater torque than a larger force? Give reason.
5. Give two real world applications of angular momentum
6. Derive relationship between torque and angular acceleration.
7. List the moment of inertia dependent factors.

ERQ

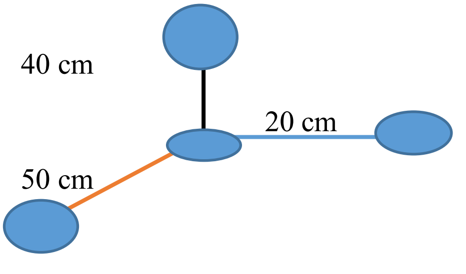
1. State and explain the law of conservation of angular momentum. Give two examples to illustrate it.
2. Discuss forces action on banked curve and derive relation between curve angle and velocity?
3. Derive the formula for centripetal acceleration using fundamental principles and equations, illustrating each step of the derivation
4. Derive the formula for the moment of inertia of a uniform rod or a solid sphere. Clearly illustrate each step of the derivation.
5. Define torque and explain how does it differ from force in linear motion?

Numericals:

1. A car mechanic applies a force of 800 N to a wrench for the purpose of loosening a bolt. He applies the force which is perpendicular to the arm of the wrench. The distance from the bolt to the mechanic's hand is 0.40 m. Find out the magnitude of the torque applied?
   (320 N .m)
2. A car accelerates uniformly from rest and reaches a speed of 22 m/s in 9 s If the diameter of a tire is 58 cm, find
   (a) the number of revolutions the tire makes during this motion, assuming no slipping, and
   (b) the final rotational speed of the tire in revolutions per second. **(55 rev, 12 rad /s)**

3. An ordinary workshop grindstone has a radius of 7.5 cm and rotates 6500 rev/min.
   (a) Calculate the magnitude of centripetal acceleration at its edge in m/s^2 and convert it into multiples of g.
   (b) What is the linear speed of a point on its edge?
   (3.47×10^4 m/s^2, 3.55×10^3g, 51 m/s)
4. A satellite is orbiting the Earth with an orbital velocity of 3200 m/s. What is the orbital radius? **(3.897×10^7m)**
5. A satellite wishes to orbit the earth at a height of 100 km (approximately 60 miles) above the surface of the earth. Determine the speed, acceleration and orbital period of the satellite. (Given: $M_{earth} = 5.98 \times 10^{24}$ kg, $R_{earth} = 6.37 \times 10^6$ m)
   (7.85×10^3 m/s, 9.53 m/s^2, 1.44 hours)
6. A thin disk with a 0.3m diameter and a total moment of inertia of 0.45kg·m^2 is rotating about its center of mass. There are three rocks with masses of 0.2kg on the outer part of the disk. Find the total moment of inertia of the system.? **(0.464kg. m^2)**
7. What is the ideal banking angle for a gentle turn of 1.20 km radius on a highway with a 105 km/h speed limit (about 65 mi/h), assuming everyone travels at the limit? **(4.14°)**
8. A 1 500-kg car moving on a flat, horizontal road negotiates a curve as shown in Figure 4.21. If the radius of the curve is 35.0 m and the coefficient of static friction between the tires and dry pavement is 0.523, find the maximum speed the car can have and still make the turn successfully **(13.5 m/s)**

9. A system of points shown in figure 4.22. Each particle has same mass of 0.3 kg and they all lie in the same plane. What is the moment of inertia of the system about given axis? **(0.135 kg m^2)**



10. (a) What is the angular momentum of a 2.9 kg uniform cylindrical grinding wheel of radius 20 cm when rotating 1550 rpm? (b) How much torque is required to stop it in 6 s? **(9.5 kg·m^2/s^2, - 1.6 N.m)**
11. Determine the angular momentum of the Earth (a) about its rotation axis (Assume the Earth as uniform sphere), and (b) in its orbit around the Sun (Take Earth as a particle orbiting the Sun). The Earth has mass 6×10^{24} kg and radius 6.4×10^6m, and is 1.5×10^8 km from the Sun. **(7.3×10^{33} kg·m^2/s, 2.7×10^{40} kg m^2/s)**